Interpolation of Nevanlinna functions by rationals with poles on the real line*

A. Bultheel[†], P. González-Vera[‡], E. Hendriksen[§], O. Njåstad[¶]

Abstract

Let μ be a measure on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Its Nevanlinna transform is defined as $\Omega_\mu(z) = \int D(t,z)d\mu(t)$ with $D(t,z) = i(1+tz)/(z-t)$. It is a holomorphic function in the upper half plane with nonnegative real part. In this paper we construct rational approximants $\Omega_n(z)$ of degree n, which belong to the same class and are such that for an arbitrary sequence of points $\alpha_k \in \hat{\mathbb{R}}$, $k = 1, 2, \ldots$, we have Hermite interpolation at these points for the boundary functions of Ω_μ and Ω_n. This means that the nontangential limits of $\Omega_\mu - \Omega_n$ and an appropriate number of its derivatives vanish in the given points α_k.

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1 Hamburger moment problems and rational interpolation

Let μ be a finite positive measure on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Consider a sequence $A = \{\alpha_k\}_{k\geq 1}$ with $\alpha_k \in \hat{\mathbb{R}} \setminus \{0\}$. We define $\mathcal{L}_n$ to be the space of rational functions of degree n whose poles are in A, i.e., if Π_n is the set of polynomials of degree n at most, then

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \Pi_n, \pi_n = \prod_{k=1}^{n} \left(1 - \frac{z}{\alpha_k}\right) \right\}.$$

Note that if all $\alpha_k = \infty$, then $\mathcal{L}_n = \Pi_n$. To highlight this fact, we choose as a natural set of basis functions

$$b_0 = 1, \quad b_n(z) = Z_1(z)Z_2(z)\cdots Z_n(z), \quad n = 1, 2, \ldots$$

with

$$Z_k(z) = \frac{z}{1 - z/\alpha_k}, \quad k = 0, 1, \ldots$$

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[†]Department of Computer Science, K.U.Leuven, Belgium
[‡]Department Análisis Math., Univ. La Laguna, Tenerife, Spain
[§]Department of Math., Univ. of Amsterdam, The Netherlands
[¶]Department of Math. Sciences, Norwegian Univ. of Science and Technology, Trondheim, Norway

This is indeed a natural generalization since these basis functions reduce to the powers of z when all $\alpha_k = \infty$. Obviously

$$\mathcal{L}_n = \text{span}\{b_0, b_1, \ldots, b_n\}.$$

Suppose we know the generalized moments

$$m_k = \int b_k(t) d\mu(t), \quad k = 0, 1, \ldots$$

(here and in the sequel, all integrals are over $\hat{\mathbb{R}}$). To solve the moment problem means to find conditions for existence and uniqueness of a measure having these moments and if solutions exist to construct them. In the special case where all $\alpha_k = \infty$, this is the classical Hamburger moment problem [1, 11, 2, 16]. In the general case, this is almost the extended Hamburger moment problem as studied in [12, 13, 15]. The difference is that there b_k is replaced by π_k^{-1} and that there are only a finite number of different $\alpha_k \in \mathbb{R}$, say $\alpha_1, \ldots, \alpha_p$ which are cyclically repeated to give the sequence A. We refer to this as the cyclic case. Generalizations to the noncyclic case were considered in [9, 10, 5] which treat the trigonometric analog, i.e., where the real line is replaced by the unit circle. See also [14] where the poles are in a finite subset of $\mathbb{R}$. Some aspects of the problem in the case where the cyclic sequence of α_k's is replaced by an arbitrary sequence of real numbers are discussed in [4, 7, 6].

We remark that if we assume all the moments to be finite, then, since b_n has poles at all the points in $A_n = \{\alpha_1, \ldots, \alpha_n\}$, μ can not have a mass point in any of the α_k. In the classical polynomial case, all the poles are at ∞ and therefore it is then natural to take integrals over $\mathbb{R}$ rather than over $\hat{\mathbb{R}}$ because ∞ can not be a mass point. In general however, if none of the poles is infinite, a function in $\mathcal{L}_n$ will be finite at ∞ and a mass point at ∞ may well be possible, that is why we take integrals over $\hat{\mathbb{R}}$.

We preferred to use the b_n to define the moments instead of the π_n^{-1} because this allows us to treat $\alpha_k = \infty$ in the same way as the other finite α_k. The polynomial situation of the classical Hamburger moment problem drops out as a special case.

For completeness, we mention that also a strong version of the Hamburger (and other) moment problem has been studied. In that case the power moments $\int t^k d\mu(t)$ are prescribed for $k \in \mathbb{Z}$. Neither this problem nor its rational generalization will be considered in this paper. Also the problem where all the α_k are in the upper half plane instead of on its boundary (i.e. the real line) has been considered in a series of papers, mainly in its trigonometric form. See for example [3, 4] for an introduction. Here we only discuss the boundary case.

What we do want to discuss in this paper is one aspect which shows up in the analysis of this kind of moment problem, namely the construction of rational interpolants for the Nevanlinna transform [1, p. 92]

$$\Omega_\mu(z) = \int D(t, z) d\mu(t), \quad D(t, z) = -i\frac{1 + tz}{t - z}.$$

This function is holomorphic in the upper half plane (in fact everywhere off the real line) and it maps the open upper half plane to the closed right half plane. We shall call the class of such functions the class $\mathcal{P}$ of positive real functions (they have a nonnegative real part in the upper half plane). These functions have a pseudo-analytic continuation to the lower half plane by the definition

$$f(z) = \overline{f(\overline{z})}.$$

The boundary values exist a.e. and one has

$$\lim_{y\downarrow 0} f(x+iy) = \lim_{h\downarrow 0} \frac{\mu(x-h, x+h)}{2h}$$

which can be used in an inversion formula for the Nevanlinna transform.

In the polynomial case, the Nevanlinna kernel $D(t,z)$ is often replaced by the simpler kernel $-i(t-z)^{-1}$. However, this is only possible if [1, p. 93]

$$\sup_{y\geq 1} |y\Omega(iy)| < \infty.$$

If μ has a mass point at ∞, this condition may not be satisfied and that is why we use here the more complicated kernel $D(t,z)$.

The rational interpolants Ω_n of degree n we are looking for will be obtained as the Nevanlinna transform of a discrete measure μ_n with n mass points

$$\Omega_n(z) = \int D(t,z) d\mu_n(t) = \sum_{k=1}^{n} \lambda_{nk} D(\xi_{nk}, z)$$

with $\xi_{nk} \in \hat{\mathbb{R}}$ and $\lambda_{nk} > 0$. The construction of such a measure will be based on quasi-orthogonal functions (just as in the polynomial or the cyclic case). However, if we want to exploit orthogonality, we need an inner product in $\mathcal{L} = \cup_{n=0}^{\infty} \mathcal{L}_n$. Now in the polynomial case or the cyclic case we have $\mathcal{R} = \mathcal{L}\cdot\mathcal{L} = \mathcal{L}$. Therefore we can define an inner product in $\mathcal{L}$ given the moments m_k, $k = 0, 1, \ldots$ However when we are in the noncyclic case, then $\mathcal{R} = \mathcal{L}\cdot\mathcal{L} \neq \mathcal{L}$. Thus to define an inner product in $\mathcal{L}$, we need to assume that all the integrals $\int f(t) d\mu(t)$ exist for $f \in \mathcal{R}$. This means that all the moments $m_{k,l} = \int b_k(t) b_l(t) d\mu_n(t)$, $k, l = 0, 1, \ldots$ should be defined. We call these the moments in $\mathcal{R}$ while the previous moments are the moments in $\mathcal{L}$. The measure μ_n that will be constructed shall fit a number of the moments in $\mathcal{R}$, namely

$$\int b_k(t) b_l(t) d\mu_n(t) = m_{kl} = \int b_k(t) b_l(t) d\mu(t), \quad k, l = 0, 1, \ldots, n-1.$$

Note that this implies that it also fits the moments in $\mathcal{L}$

$$\int b_k(t) d\mu_n(t) = m_k = \int b_k(t) d\mu(t), \quad k = 0, 1, \ldots, n-1.$$

Although we do not want to solve the moment problem here, we mention that essentially by letting n tend to ∞ and by using Helly's theorems, it follows that an accumulation point of the sequence $\{\mu_n\}$ will be a solution of the moment problem in $\mathcal{R}$ and thus also of the moment problem in $\mathcal{L}$ [16, 8].

The interpolation properties that we shall give below generalize the polynomial case of Akhiezer [1, p. 22] and for the cyclic case in [14, 10]. Analogous results were obtained in the trigonometric case for the noncyclic case (using a different approach) in [6].

2 Orthogonal rational functions and quadrature

In order to construct the quadrature formulas we gave at the end of the previous section, we first have to introduce orthogonal rational functions. Suppose μ is a finite positive measure

on $\hat{\mathbb{R}}$. We suppose it to be normalized by $m_0 = \int d\mu(t) = 1$. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of points in $\hat{\mathbb{R}} \setminus \{0\}$ and define the basis functions b_k and $\mathcal{L}_n = \text{span}\{b_0, \ldots, b_n\}$ as before. We also introduce $\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_n = \{fg : f, g \in \mathcal{L}_n\}$. Furthermore, let $\mathcal{L} = \cup_{n=0}^{\infty} \mathcal{L}_n$, $\Pi_n = \text{span}\{1, z, \ldots, z^n\}$, $\Pi = \cup_{n=0}^{\infty} \Pi_n$ and $\mathcal{R} = \cup_{b=0}^{\infty} \mathcal{R}_n$. Note that if all $\alpha_k = \infty$, then $\mathcal{L}_n = \Pi_n$ and $\mathcal{R}_n = \Pi_{2n}$. Assume also that the inner product

$$\langle f, g \rangle = \int f(t)\overline{g(t)} d\mu(t)$$

is well defined for $f, g \in \mathcal{L}$. This means that we should have

$$\int |b_n(t)|^2 d\mu(t) < \infty, \quad \forall n = 0, 1, 2, \ldots$$

Let $\phi_0, \phi_1, \ldots$ be an orthonormal basis obtained by applying the Gram-Schmidt procedure on the basis $b_0, b_1, b_2, \ldots$. If

$$\phi_n(z) = \phi(0) + \cdots + \kappa_n' b_{n-1}(z) + \kappa_n b_n(z)$$

then we normalise ϕ_n by taking $\kappa_n > 0$. With these orthonormal functions we associate the functions of the second kind defined by

$$\psi_0 = iz, \quad \psi_n(z) = \int D(t, z)[\phi_n(t) - \phi_n(z)] d\mu(t), \quad n \geq 1.$$

Note that $\psi_n \in \mathcal{L}_n$ for $n \geq 1$.

Such an orthonormal function $\phi_n(z) = p_n(z)/\pi_n(z)$ is called regular if $p_n(\alpha_{n-1}) \neq 0$. If $\alpha_{n-1} = \infty$, then $p_n(\alpha_{n-1}) \neq 0$ means that the polynomial p_n has degree n (and not lower). We call the sequence $\{\phi_n\}$ regular if all ϕ_n in the sequence are regular.

Quasi-orthogonal functions are defined for $\tau \in \hat{\mathbb{R}}$ as

$$Q_n(z, \tau) = \phi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}(z), \quad k \geq 1.$$

If $\tau = \infty$, then this should be interpreted as

$$Q_n(z, \infty) = \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}(z).$$

The associated functions are given by

$$P_n(z, \tau) = \psi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \psi_{n-1}(z), \quad \tau \in \hat{\mathbb{R}}, \quad n \geq 1$$

with a similar interpretation for $\tau = \infty$.

Along the same lines as in the proof of [5] for the case of the unit circle, or as in [12] for the cyclic situation of the real line, one can prove

Theorem 2.1 *Let the sequence $\{\phi_n\}$ of orthonormal functions be regular. Then for each n there are infinitely many so called regular values $\tau \in \hat{\mathbb{R}}$ for which the quasi-orthogonal functions $Q_n(z, \tau)$ have precisely n simple zeros $\xi_{nk}(\tau)$, $k = 1, \ldots, n$ in $\hat{\mathbb{R}} \setminus \{0, \alpha_1, \ldots, \alpha_n\}$. Furthermore, if we define*

$$\lambda_{nk}(\tau) = \left[\sum_{i=0}^{n-1} |\phi_i(\xi_{nk}(\tau))|^2 \right]^{-1}, \quad k = 1, \ldots, n$$

and if $\mu_n(\cdot,\tau)$ is the discrete measure wich takes masses $\lambda_{nk}(\tau)$ at the points $\xi_{nk}(\tau)$, then

$$\int f(t)d\mu(t) = \int f(t)d\mu_n(t,\tau)$$

for all $f \in \mathcal{R}_{n-1} = \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$. If $\tau = 0$ is a regular value, then the formula holds for all $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$.

The theorem says that for general $\tau \in \hat{\mathbb{R}}$, the quadrature formula

$$\int f(t)d\mu_n(t,\tau) = \sum_{k=1}^{n} \lambda_{nk}(\tau) f(\xi_{nk}(\tau))$$

is exact in $\mathcal{R}_{n-1}$ or equivalently that the inner product in $\mathcal{L}_{n-1}$ with respect to μ and with respect to μ_n is the same.

3 Measures and interpolation

In this section we prove that equality of integrals with respect to two measures implies interpolation properties for their Nevanlinna transforms. This interpolation will hold at the points $\alpha_k \in \hat{\mathbb{R}}$. Since a Nevanlinna transform $\Omega_\mu(z)$ is defined by the integral $\int D(t,z)d\mu(t)$ for z off $\hat{\mathbb{R}}$, we have to consider boundary values for Ω_μ in these points α_k. These are obtained by taking nontangential limits. For simplicity, we shall assume we take limits along vertical lines and call them orthogonal limits.

We set by definition for $t \in \mathbb{R}$ (note the special arrow $\mapsto$ used in the limit)

$$\lim_{z \mapsto t} f(z) = \lim_{y \downarrow t} f(t+iy)$$

while for $t = \infty$, we set

$$\lim_{z \mapsto \infty} f(z) = \lim_{z \mapsto 0} f(1/z).$$

Furthermore we introduce the notation A_n to denote the set $A_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\tilde{A}_{n-1}$ for the sequence (including repetitions) $\tilde{A}_{n-1} = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1})$ while $\tilde{A}'_n$ adds the single point α_n to it: $\tilde{A}'_n = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n)$. By $\alpha^\#$ we indicate the multiplicity of $\alpha \in A_{n-1}$ in the sequence $\tilde{A}_{n-1}$, that is the number of times α occurs in the sequence $\tilde{A}_{n-1}$. At other places, the same notation is also used to indicate the multiplicity of $\alpha \in A_n$ in the sequence $\tilde{A}'_n$

Let μ be such that $\mathcal{L}_n \cdot \mathcal{L}_{n-1} \subset L^1(\mu)$ and denote its Nevanlinna transform as

$$\Omega_\mu(z) = \int D(t,z)d\mu(t).$$

Then we have

Lemma 3.1 *The integrals for functions in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$ are completely defined by the values*

$$\Omega_\mu(i) \quad \textit{and} \quad \lim_{w \mapsto \alpha} \Omega_\mu^{(k)}(w), \quad \alpha \in A_n, \quad k = 0, 1, \ldots, \alpha^\# - 1$$

where the superscript $^{(k)}$ denotes the kth derivative, $A_n = \{\alpha_1, \alpha_1, \ldots, \alpha_n\}$ and $\alpha^\#$ is the multiplicity of α in the sequence $\tilde{A}'_n = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n)$.

Proof. These integrals involve the computation of integrals of the form $\int b_k(t)b_l(t)d\mu(t)$. By partial fraction decomposition of the integrand, it is seen that besides $\Omega_\mu(i) = \int d\mu(t)$ these will be completely defined in terms of integrals of the form

$$\int \frac{d\mu(t)}{(t-\alpha)^{k+1}}, \quad k = 0, 1, \ldots, \alpha^\# - 1 \tag{3.1}$$

where α ranges over all finite elements of A_n and if $\alpha = \infty$ is also in A_n, then we should add integrals of the form

$$\int t^{k+1} d\mu(t), \quad k = 0, 1, \ldots, \infty^\# - 1. \tag{3.2}$$

For α finite, it is easily seen that taking the kth derivative with respect to the variable z, we have

$$\partial_z^k \Omega_\mu(z) = \int \partial_z^k D(t,z) d\mu(t) = -i(-1)^k k! \int \frac{d\mu(t)}{(t-z)^{k+1}}. \tag{3.3}$$

If $\alpha \in \mathbb{R}$ and $\epsilon > 0$, then

$$\frac{1}{|t-\alpha|} - \frac{1}{|t-\alpha-i\epsilon|} \leq \Big|\frac{1}{t-\alpha} - \frac{1}{t-\alpha-i\epsilon}\Big| = \frac{\epsilon}{|t-\alpha||t-\alpha-i\epsilon|}.$$

For $t \in \hat{\mathbb{R}}$, it holds that $|t-\alpha-i\epsilon| \geq \epsilon$. Hence the previous expression is bounded by $|t-\alpha|^{-1}$ and therefore we have for z in the upper half plane and $t \in \hat{\mathbb{R}}$ that $|t-z|^{-1} \leq 2|t-\alpha|^{-1}$. Now we can apply the dominated convergence theorem when taking the orthogonal limit $z \mapsto \alpha$ in (3.3). So we see that there is a one-to-one correspondence between the generalized moments (3.1) and the values $\lim_{z\mapsto\alpha} \Omega_\mu^{(k)}(\alpha)$, $k = 0, 1, \ldots, \alpha^\# - 1$.

By a similar argument and recalling that $\lim_{z\mapsto\infty} \Omega_\mu(z) = \lim_{z\mapsto 0} \Omega_\mu(1/z)$, it is found that the moments (3.2) are defined by the values $\lim_{z\mapsto\infty} \Omega_\mu^{(k)}(z)$, $k = 0, 1, \ldots, \infty^\# - 1$. □

Using this lemma in combination with Theorem 2.1, immediately gives the following theorem.

Theorem 3.2 *With the conditions and notation of Theorem 2.1, we define*

$$\Omega_\mu(z) = \int D(t,z) d\mu(t) \quad \textit{and} \quad \Omega_n(z,\tau) = \int D(t,z) d\mu_n(t,\tau).$$

Then

$$\Omega_\mu(i) = \Omega_n(i,\tau) \quad \textit{and} \quad \lim_{w\mapsto\alpha}[\Omega_\mu^{(k)}(w) - \Omega_n^{(k)}(w,\tau)] = 0, \quad \alpha \in A_{n-1}, \quad k = 0, 1, \ldots, \alpha^\# - 1$$

with $A_{n-1} = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ *and* $\alpha^\#$ *the multiplicity of* α *in the sequence* $\tilde{A}_{n-1} = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1})$.
If $\tau = 0$ *is a regular value, then with* $\Omega_n(z) = \Omega_n(z,0)$

$$\Omega_\mu(i) = \Omega_n(i) \quad \textit{and} \quad \lim_{w\mapsto\alpha}[\Omega_\mu^{(k)}(w) - \Omega_n^{(k)}(w)] = 0, \quad \alpha \in A_n, \quad k = 0, 1, \ldots, \alpha^\# - 1$$

with $A_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *and* $\alpha^\#$ *the multiplicity of* α *in the sequence* $\tilde{A}'_n = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n)$.

Proof. Because by Theorem 2.1, the inner product in $\mathcal{L}_{n-1}$ is the same when using the measure μ or the measure μ_n. Hence by the previous lemma, we have the interpolation properties as stated.

When $\tau = 0$ is a regular value, then the quadrature formulas are exact in $\mathcal{L}_{n-1} \cdot \mathcal{L}_n$ and we have the extra interpolation as described above. □

4 An alternative expression for $\Omega_n(z,\tau)$

By its definition we have the following expression for the Nevanlinna function $\Omega_n(z,\tau)$

$$\Omega_n(z,\tau) = \sum_{k=1}^{n} \lambda_{nk}(\tau) D(\xi_{nk}(\tau), z) = i \sum_{k=1}^{n} \lambda_{nk}(\tau) \frac{1+\xi_{nk}(\tau)z}{z-\xi_{nk}(\tau)}.$$

It is however possible to give a simple expression in terms of the quasi-orthogonal functions and their associates.

To reach this goal, we first give the following lemma.

Lemma 4.1 *Let ϕ_n be the orthonormal functions and ψ_n the associated functions of the second kind. Then for $n > 0$ and for any function f such that (as a function of t), $D(t,z)[f(t)-f(z)] \in \mathcal{L}_{n-1}$ we have*

$$\psi_n(z) f(z) = \int D(t,z)[\phi_n(t) f(t) - \phi_n(z) f(z)] d\mu(t).$$

Proof. This expression for ψ_n corresponds to the original definition if it holds that

$$\int D(t,z)\phi_n(t)[f(t)-f(z)]d\mu(t) = 0.$$

If $D(t,z)[f(t)-f(z)] \in \mathcal{L}_{n-1}$, then this is zero because $\phi_n \perp \mathcal{L}_{n-1}$. □

We have as a corollary the following result.

Lemma 4.2 *Let $Q_n(z,\tau)$ be the quasi-orthogonal functions and $P_n(z,\tau)$ the associated functions. Then for $n \geq 2$*

$$\frac{Z_{n-1}(z)}{Z_n(z)} P_n(z,\tau) = \int D(t,z) \left[\frac{Z_{n-1}(t)}{Z_n(t)} Q_n(t,\tau) - \frac{Z_{n-1}(z)}{Z_n(z)} Q_n(z,\tau) \right] d\mu(t).$$

Proof. The right hand side is

$$\int D(t,z) \left[\frac{Z_{n-1}(t)}{Z_n(t)} \phi_n(t) - \frac{Z_{n-1}(z)}{Z_n(z)} \phi_n(z) \right] d\mu(t)$$
$$+\tau \int D(t,z)[\phi_{n-1}(t) - \phi_{n-1}(z)] d\mu(t).$$

The second term equals $\psi_{n-1}(z)$ by definition. The first term turns out to be

$$\frac{Z_{n-1}(z)}{Z_n(z)} \psi_n(z)$$

by the previous lemma when we use $f(z) = Z_{n-1}(z)/Z_n(z)$. Thus the lemma is proved. □

We can now prove the expression for $\Omega_\mu(z,\tau)$.

Theorem 4.3 *Let $\mu_n(\cdot,\tau)$ be the discrete measure of the quadrature formula, as it was constructed in the previous sections. Denote by $Q_n(z,\tau)$ and $P_n(z,\tau)$ the quasi-orthogonal functions and the associated functions of the second kind. Then for $n \geq 2$*

$$\Omega_n(z,\tau) = \int D(t,z) d\mu(t) = -\frac{P_n(z,\tau)}{Q_n(z,\tau)}.$$

Proof. By the previous lemma

$$\frac{Z_{n-1}(z)}{Z_n(z)}P_n(z,\tau) = \int D(t,z)\left[\frac{Z_{n-1}(t)}{Z_n(t)}Q_n(t,\tau) - \frac{Z_{n-1}(z)}{Z_n(z)}Q_n(z,\tau)\right]d\mu(t).$$

It is immediately checked that the integrand is in $\mathcal{R}_{n-1}$ and therefore we can replace $d\mu$ by $d\mu_n$. Furthermore, recalling that $Q_n(\xi_{nk}(\tau),\tau) = 0$ by definition, we get

$$\frac{Z_{n-1}(z)}{Z_n(z)}P_n(z,\tau) = -\left[\sum_{k=1}^{n} D(\xi_{nk}(\tau),z)\lambda_{nk}(\tau)\right]\frac{Z_{n-1}(z)}{Z_n(z)}Q_n(z,\tau).$$

In the square brackets, we recognize $\Omega_n(z,\tau)$ so that the required equality follows. □

Note that for the case when $\tau = 0$ is a regular value, we have

$$\Omega_n(z,0) = -\frac{\psi_n(z)}{\phi_n(z)}, \quad n \geq 2.$$

To conclude, we check how the polynomial case can be recovered from our general formulation. So assume that all $\alpha_k = \infty$. Theorem 3.2 says that in that case

$$\Omega_\mu(i) = \Omega_n(i,\tau) \quad \text{and} \quad \lim_{z\mapsto\infty}[\Omega_\mu^{(k)}(z) - \Omega_n^{(k)}(z,\tau)] = 0, \quad k = 0,1,\ldots,2n-3.$$

Hence that we have

$$\left\{-\frac{P_n(z,\tau)}{Q_n(z,\tau)} - i\left[s_0 + \frac{s_1}{z} + \frac{s_2}{z^2} + \cdots + \frac{s_{2n-3}}{z^{2n-3}}\right]\right\} z^{2n-3} \to 0, \quad z \to \infty$$

where

$$s_0 = m_1, \quad s_k = m_{k-1} + m_{k+1} \quad k \geq 1, \quad m_k = \int t^k d\mu(t).$$

This can be motivated as follows.

$$D(t,z) = -i\frac{1+tz}{t-z} = i\left[t + \frac{1+t^2}{z} + \frac{(1+t^2)t}{z^2} + \cdots + \frac{(1+t^2)t^{l-1}}{z^l}\right] + O\left(\frac{1}{z^{l+1}}\right), \quad z \to \infty.$$

Thus,

$$\Omega_\mu(t) = \int D(t,z)d\mu(t) \sim i\left[s_0 + \frac{s_1}{z} + \frac{s_2}{z^2} + \cdots + \frac{s_l}{z^l} + \cdots\right].$$

However, because the quadrature formula is exact for all polynomials in Π_{2n-2}, it follows that the first $2n-3$ coefficients in these expansions of Ω_n and Ω_μ at ∞ are the same.

We should note that in Akhiezer [1, p. 22], not the kernel $D(t,z)$ is used but the simpler kernel $(t-z)^{-1}$. Therefore (and this can also be seen from the previous formulas) one has that $\int(t-z)^{-1}d\mu_n(t,\tau)$ is a rational function where the coefficients of the asymptotic expansion at ∞ matches the coefficients in the expansion of $\int(t-z)^{-1}d\mu(t)$ up to the coefficient in $z^{-(2n-2)}$. Note that in the polynomial case $\tau = 0$ is a regular value for each n. Thus we can choose $\tau = 0$ and we get in that case correspondence up to the coefficient in $z^{-(2n-1)}$. With our choice of the kernel $D(t,z)$, this extra interpolation property at ∞ is replaced by an interpolation property at i.

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