Orthogonal rational functions on the unit circle

from the scalar to the matrix case

Adhemar Bultheel

Dept. Computer Science, K.U.Leuven
joint work with
P. González-Vera, E. Hendriksen, O. Njåstad

July 2004

Part 0: Motivation

1. The linear prediction problem
2. Krylov subspace iteration
3. Rational Szegő quadrature
4. For the rest of these lectures

Part 0: Motivation

1. The linear prediction problem
2. Krylov subspace iteration
3. Rational Szegő quadrature
4. For the rest of these lectures

Part 0: Motivation

1. The linear prediction problem
2. Krylov subspace iteration
3. Rational Szegő quadrature
4. For the rest of these lectures

Part 0: Motivation

1. The linear prediction problem
2. Krylov subspace iteration
3. Rational Szegő quadrature
4. For the rest of these lectures

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part I: The scalar case

5 Preliminaries

6 Fundamental spaces

7 Recurrence relations

8 Quadrature and interpolation

9 Favard theorem

10 Convergence

11 Szegő's problem

Outline of Part II: the matrix case

12 Hilbert modules

13 MORF and reproducing kernels

14 Recurrence

15 Interpolation and quadrature

16 Favard theorem

17 Szegő's problem

Outline of Part II: the matrix case

12 Hilbert modules

13 MORF and reproducing kernels

14 Recurrence

15 Interpolation and quadrature

16 Favard theorem

17 Szegő's problem

Outline of Part II: the matrix case

12 Hilbert modules

13 MORF and reproducing kernels

14 Recurrence

15 Interpolation and quadrature

16 Favard theorem

17 Szegő's problem

Outline of Part II: the matrix case

12 Hilbert modules

13 MORF and reproducing kernels

14 Recurrence

15 Interpolation and quadrature

16 Favard theorem

17 Szegő's problem

Outline of Part II: the matrix case

12 Hilbert modules

13 MORF and reproducing kernels

14 Recurrence

15 Interpolation and quadrature

16 Favard theorem

17 Szegő's problem

Outline of Part II: the matrix case

12 Hilbert modules

13 MORF and reproducing kernels

14 Recurrence

15 Interpolation and quadrature

16 Favard theorem

17 Szegő's problem

Part I

Motivation

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in $\mathrm{span}\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow L_2(\mathbb{T},\mu)$ with $\langle f,g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in $\mathrm{span}\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in $\mathrm{span}\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in span$\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in span$\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow L_2(\mathbb{T}, \mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)} d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T}, \mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in span$\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in span$\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow$ $L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in span$\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow$ $L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Stochastic process

- $\{y_n\}_{n\in\mathbb{Z}}$, sequence of stochastic variables
- $\mu_k = \mathrm{E}\{y_n \overline{y}_{n-k}\}$ stationary (time invariant)
- $\hat{y}_n = -\sum_{k=1}^{\infty} a_k y_{n-k}$ predict from past
- in span$\{y_k\}$ is $\mathrm{E}\{x\overline{y}\}$ an inner product
- $\Leftrightarrow$ $L_2(\mathbb{T},\mu)$ with $\langle f, g\rangle = \int f(t)\overline{g(t)}d\mu(t)$
- μ is spectral measure (μ_k are the trigonometric moments of μ)
- $y_k \leftrightarrow z^k$, $H_2^- = \mathrm{span}\{z^{-k} : k \geq 1\}$ = past of y_0
- minimize energy $\|e_n\|^2$, $e_n = y_n - \hat{y}_n = \sum_{k=0}^{\infty} a_k y_{n-k}$, $a_0 = 1$
- $\Leftrightarrow$ project 1 onto H_2^- in $L_2(\mathbb{T},\mu)$

Take n-dimensional subspaces of H_2^-

- project 1 onto $\mathrm{span}\{z^{-k}, k = 1, \ldots, n\} \Rightarrow \min_{p_n} \|p_n\|_\mu^2$, $p_n(z) = 1 + a_1/z + \ldots + a_n/z^n$
- But if predictor is $1 + a/z + a^2/z^2 + \cdots + a^n/z^n + \cdots = 1/(1 - a/z)$ and $|a|$ is close to 1, then coefficients will decay very slowly and need a predictor polynomial of a high degree.
- Better to look for $\min_{p_n} \|p_n\|_\mu^2$, $p_n(\infty) = 1$, $p_n \in \mathcal{L}_n$ with $\mathcal{L}_n = \mathrm{span}\{1, 1/(1 - a/z), 1/(1 - a/z)^2, \ldots, 1/(1 - a/z)^n\}$ or more generally $\mathcal{L}_n = \mathrm{span}\left\{1, \frac{1}{\pi_1}, \frac{1}{\pi_2}, \ldots, \frac{1}{\pi_n}\right\}$, $\pi_k(z) = \prod_{j=1}^k (1 - a_j/z)$
- ORF on $\mathbb{T}$

Take n-dimensional subspaces of H_2^-

- project 1 onto $\mathrm{span}\{z^{-k}, k = 1, \ldots, n\} \Rightarrow \min_{p_n} \|p_n\|_\mu^2$, $p_n(z) = 1 + a_1/z + \ldots + a_n/z^n$
- But if predictor is $1 + a/z + a^2/z^2 + \cdots + a^n/z^n + \cdots = 1/(1 - a/z)$ and $|a|$ is close to 1, then coefficients will decay very slowly and need a predictor polynomial of a high degree.
- Better to look for $\min_{p_n} \|p_n\|_\mu^2$, $p_n(\infty) = 1$, $p_n \in \mathcal{L}_n$ with $\mathcal{L}_n = \mathrm{span}\{1, 1/(1 - a/z), 1/(1 - a/z)^2, \ldots, 1/(1 - a/z)^n\}$ or more generally $\mathcal{L}_n = \mathrm{span}\left\{1, \frac{1}{\pi_1}, \frac{1}{\pi_2}, \ldots, \frac{1}{\pi_n}\right\}, \quad \pi_k(z) = \prod_{j=1}^k (1 - a_j/z)$
- ORF on $\mathbb{T}$

Take n-dimensional subspaces of H_2^-

- project 1 onto $\mathrm{span}\{z^{-k}, k = 1, \ldots, n\} \Rightarrow \min_{p_n} \|p_n\|_\mu^2$, $p_n(z) = 1 + a_1/z + \ldots + a_n/z^n$
- But if predictor is $1 + a/z + a^2/z^2 + \cdots + a^n/z^n + \cdots = 1/(1 - a/z)$ and $|a|$ is close to 1, then coefficients will decay very slowly and need a predictor polynomial of a high degree.
- Better to look for $\min_{p_n} \|p_n\|_\mu^2$, $p_n(\infty) = 1$, $p_n \in \mathcal{L}_n$ with $\mathcal{L}_n = \mathrm{span}\{1, 1/(1 - a/z), 1/(1 - a/z)^2, \ldots, 1/(1 - a/z)^n\}$ or more generally $\mathcal{L}_n = \mathrm{span}\left\{1, \frac{1}{\pi_1}, \frac{1}{\pi_2}, \ldots, \frac{1}{\pi_n}\right\}, \quad \pi_k(z) = \prod_{j=1}^k (1 - a_j/z)$
- ORF on $\mathbb{T}$

Take n-dimensional subspaces of H_2^-

- project 1 onto $\mathrm{span}\{z^{-k}, k = 1, \ldots, n\} \Rightarrow \min_{p_n} \|p_n\|_\mu^2$, $p_n(z) = 1 + a_1/z + \ldots + a_n/z^n$
- But if predictor is $1 + a/z + a^2/z^2 + \cdots + a^n/z^n + \cdots = 1/(1 - a/z)$ and $|a|$ is close to 1, then coefficients will decay very slowly and need a predictor polynomial of a high degree.
- Better to look for $\min_{p_n} \|p_n\|_\mu^2$, $p_n(\infty) = 1$, $p_n \in \mathcal{L}_n$ with $\mathcal{L}_n = \mathrm{span}\{1, 1/(1 - a/z), 1/(1 - a/z)^2, \ldots, 1/(1 - a/z)^n\}$ or more generally $\mathcal{L}_n = \mathrm{span}\left\{1, \frac{1}{\pi_1}, \frac{1}{\pi_2}, \ldots, \frac{1}{\pi_n}\right\}, \quad \pi_k(z) = \prod_{j=1}^k (1 - a_j/z)$
- ORF on $\mathbb{T}$

Computation of eigenvalues, solution of linear systems

- A is a Hilbert space operator (e.g. $A \in \mathbb{R}^{N\times N}$)
- Krylov subspace
  $\mathcal{K}_{n+1}(A, v) = \text{span}\{v_0, \ldots, v_n\}, \quad v_k = A^k v, \quad v \in \mathbb{R}^N$
- project problem on low dimensional subspace $\mathcal{K}_{n+1}(A, v)$
- need orthogonal basis for $\mathcal{K}_{n+1}(A, v)$: $\{q_0, \ldots, q_n\}$,
  $q_k = \phi_k(A)v$, ϕ_k a polynomial of degree k
- If A is symmetric then:
  $q_k^T q_l = \delta_{kl} \Leftrightarrow \langle \phi_k, \phi_l \rangle = \delta_{kl}$, with inner product defined as
  $\langle \phi_k, \phi_l \rangle = L(\phi_k \phi_l)$ and the linear functional L is defined by
  the moments $L(z^k) = m_k = v^T A^k v$.
- This is classical Lanczos (3-term recurrence relation for OPS)

Computation of eigenvalues, solution of linear systems

- A is a Hilbert space operator (e.g. $A \in \mathbb{R}^{N \times N}$)
- Krylov subspace
  $\mathcal{K}_{n+1}(A, v) = \text{span}\{v_0, \ldots, v_n\}, \quad v_k = A^k v, \quad v \in \mathbb{R}^N$
- project problem on low dimensional subspace $\mathcal{K}_{n+1}(A, v)$
- need orthogonal basis for $\mathcal{K}_{n+1}(A, v)$: $\{q_0, \ldots, q_n\}$,
  $q_k = \phi_k(A)v$, ϕ_k a polynomial of degree k
- If A is symmetric then:
  $q_k^T q_l = \delta_{kl} \Leftrightarrow \langle \phi_k, \phi_l \rangle = \delta_{kl}$, with inner product defined as
  $\langle \phi_k, \phi_l \rangle = L(\phi_k \phi_l)$ and the linear functional L is defined by
  the moments $L(z^k) = m_k = v^T A^k v$.
- This is classical Lanczos (3-term recurrence relation for OPS)

Computation of eigenvalues, solution of linear systems

- A is a Hilbert space operator (e.g. $A \in \mathbb{R}^{N \times N}$)
- Krylov subspace
  $\mathcal{K}_{n+1}(A, v) = \text{span}\{v_0, \ldots, v_n\}, \quad v_k = A^k v, \quad v \in \mathbb{R}^N$
- project problem on low dimensional subspace $\mathcal{K}_{n+1}(A, v)$
- need orthogonal basis for $\mathcal{K}_{n+1}(A, v)$: $\{q_0, \ldots, q_n\}$,
  $q_k = \phi_k(A)v$, ϕ_k a polynomial of degree k
- If A is symmetric then:
  $q_k^T q_l = \delta_{kl} \Leftrightarrow \langle \phi_k, \phi_l \rangle = \delta_{kl}$, with inner product defined as
  $\langle \phi_k, \phi_l \rangle = L(\phi_k \phi_l)$ and the linear functional L is defined by
  the moments $L(z^k) = m_k = v^T A^k v$.
- This is classical Lanczos (3-term recurrence relation for OPS)

Computation of eigenvalues, solution of linear systems

- A is a Hilbert space operator (e.g. $A \in \mathbb{R}^{N\times N}$)
- Krylov subspace
  $\mathcal{K}_{n+1}(A, v) = \text{span}\{v_0, \ldots, v_n\}, \quad v_k = A^k v, \quad v \in \mathbb{R}^N$
- project problem on low dimensional subspace $\mathcal{K}_{n+1}(A, v)$
- need orthogonal basis for $\mathcal{K}_{n+1}(A, v)$: $\{q_0, \ldots, q_n\}$,
  $q_k = \phi_k(A)v$, ϕ_k a polynomial of degree k
- If A is symmetric then:
  $q_k^T q_l = \delta_{kl} \Leftrightarrow \langle \phi_k, \phi_l \rangle = \delta_{kl}$, with inner product defined as $\langle \phi_k, \phi_l \rangle = L(\phi_k \phi_l)$ and the linear functional L is defined by the moments $L(z^k) = m_k = v^T A^k v$.
- This is classical Lanczos (3-term recurrence relation for OPS)

Computation of eigenvalues, solution of linear systems

- A is a Hilbert space operator (e.g. $A \in \mathbb{R}^{N \times N}$)
- Krylov subspace
  $\mathcal{K}_{n+1}(A, v) = \text{span}\{v_0, \ldots, v_n\}, \quad v_k = A^k v, \quad v \in \mathbb{R}^N$
- project problem on low dimensional subspace $\mathcal{K}_{n+1}(A, v)$
- need orthogonal basis for $\mathcal{K}_{n+1}(A, v)$: $\{q_0, \ldots, q_n\}$,
  $q_k = \phi_k(A)v$, ϕ_k a polynomial of degree k
- If A is symmetric then:
  $q_k^T q_l = \delta_{kl} \Leftrightarrow \langle \phi_k, \phi_l \rangle = \delta_{kl}$, with inner product defined as $\langle \phi_k, \phi_l \rangle = L(\phi_k \phi_l)$ and the linear functional L is defined by the moments $L(z^k) = m_k = v^T A^k v$.
- This is classical Lanczos (3-term recurrence relation for OPS)

Computation of eigenvalues, solution of linear systems

- A is a Hilbert space operator (e.g. $A \in \mathbb{R}^{N\times N}$)
- Krylov subspace
  $\mathcal{K}_{n+1}(A, v) = \text{span}\{v_0, \ldots, v_n\}, \quad v_k = A^k v, \quad v \in \mathbb{R}^N$
- project problem on low dimensional subspace $\mathcal{K}_{n+1}(A, v)$
- need orthogonal basis for $\mathcal{K}_{n+1}(A, v)$: $\{q_0, \ldots, q_n\}$,
  $q_k = \phi_k(A)v$, ϕ_k a polynomial of degree k
- If A is symmetric then:
  $q_k^T q_l = \delta_{kl} \Leftrightarrow \langle \phi_k, \phi_l \rangle = \delta_{kl}$, with inner product defined as $\langle \phi_k, \phi_l \rangle = L(\phi_k \phi_l)$ and the linear functional L is defined by the moments $L(z^k) = m_k = v^T A^k v$.
- This is classical Lanczos (3-term recurrence relation for OPS)

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step:
  $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step:
  $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step: $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step:
  $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step:
  $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step:
  $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

eigenvalues

- power method $\Rightarrow$ q_k converges to eigenvector for largest eigenvalue
- if eigenvalue near α is required: iterate with $B = (A - \alpha I)^{-1}$
- Rational Krylov (Ruhe) different α_k in every step:
  $v_k = (A - \alpha_k I)^{-1} v_{k-1}$
- or even $v_k = (A - \sigma_k I)(A - \alpha_k I)^{-1} v_{k-1}$
  α_k enforces influence of eigenspaces for eigenvalues near α_k
  σ_k reduces the influence of eigenspaces for eigenvalues near σ_k
- thus $v_k = r_k(A)v$ with $r_k(z) = p_k(z)/[(z - \alpha_1) \cdots (z - \alpha_n)]$
- thus $q_k = \phi_k(A)v$ with ϕ_k an orthogonal basis for space of rational functions.
- ORF on $\mathbb{R}$

Gaussian quadrature

- compute $\int_a^b f(x)w(x)dx$ by gaussian quadrature $\sum_{k=1}^n \lambda_k f(\xi_k)$
- exact for all f, polynomials of degree $2n-1$ if ξ_k are the n zeros of the nth orthogonal polynomial for weight $w(x)$ on $[a,b]$
- the $\lambda_k = 1/\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2$ are the Christoffel numbers
- but for a function of the form $f(x) = p_n(x)/\prod_{j=1}^{n/2}(x^2+a_j^2)$ it would be better to be exact in spaces of rational functions
- so take ξ_k to be the zeros of the nth orthogonal rational function for space of rational functions whose poles are (near) the poles of f
- ORF on $[a,b]$

Gaussian quadrature

- compute $\int_a^b f(x)w(x)dx$ by gaussian quadrature $\sum_{k=1}^n \lambda_k f(\xi_k)$
- exact for all f, polynomials of degree $2n-1$ if ξ_k are the n zeros of the nth orthogonal polynomial for weight $w(x)$ on $[a,b]$
- the $\lambda_k = 1/\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2$ are the Christoffel numbers
- but for a function of the form $f(x) = p_n(x)/\prod_{j=1}^{n/2}(x^2+a_j^2)$ it would be better to be exact in spaces of rational functions
- so take ξ_k to be the zeros of the nth orthogonal rational function for space of rational functions whose poles are (near) the poles of f
- ORF on $[a,b]$

Gaussian quadrature

- compute $\int_a^b f(x)w(x)dx$ by gaussian quadrature $\sum_{k=1}^n \lambda_k f(\xi_k)$
- exact for all f, polynomials of degree $2n-1$ if ξ_k are the n zeros of the nth orthogonal polynomial for weight $w(x)$ on $[a,b]$
- the $\lambda_k = 1/\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2$ are the Christoffel numbers
- but for a function of the form $f(x) = p_n(x)/\prod_{j=1}^{n/2}(x^2+a_j^2)$ it would be better to be exact in spaces of rational functions
- so take ξ_k to be the zeros of the nth orthogonal rational function for space of rational functions whose poles are (near) the poles of f
- ORF on $[a,b]$

Gaussian quadrature

- compute $\int_a^b f(x)w(x)dx$ by gaussian quadrature $\sum_{k=1}^n \lambda_k f(\xi_k)$
- exact for all f, polynomials of degree $2n-1$ if ξ_k are the n zeros of the nth orthogonal polynomial for weight $w(x)$ on $[a,b]$
- the $\lambda_k = 1/\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2$ are the Christoffel numbers
- but for a function of the form $f(x) = p_n(x)/\prod_{j=1}^{n/2}(x^2+a_j^2)$ it would be better to be exact in spaces of rational functions
- so take ξ_k to be the zeros of the nth orthogonal rational function for space of rational functions whose poles are (near) the poles of f
- ORF on $[a,b]$

Gaussian quadrature

- compute $\int_a^b f(x)w(x)dx$ by gaussian quadrature $\sum_{k=1}^n \lambda_k f(\xi_k)$
- exact for all f, polynomials of degree $2n-1$ if ξ_k are the n zeros of the nth orthogonal polynomial for weight $w(x)$ on $[a,b]$
- the $\lambda_k = 1/\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2$ are the Christoffel numbers
- but for a function of the form $f(x) = p_n(x)/\prod_{j=1}^{n/2}(x^2+a_j^2)$ it would be better to be exact in spaces of rational functions
- so take ξ_k to be the zeros of the nth orthogonal rational function for space of rational functions whose poles are (near) the poles of f
- ORF on $[a,b]$

Gaussian quadrature

- compute $\int_a^b f(x)w(x)dx$ by gaussian quadrature $\sum_{k=1}^n \lambda_k f(\xi_k)$
- exact for all f, polynomials of degree $2n-1$ if ξ_k are the n zeros of the nth orthogonal polynomial for weight $w(x)$ on $[a,b]$
- the $\lambda_k = 1/\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2$ are the Christoffel numbers
- but for a function of the form $f(x) = p_n(x)/\prod_{j=1}^{n/2}(x^2+a_j^2)$ it would be better to be exact in spaces of rational functions
- so take ξ_k to be the zeros of the nth orthogonal rational function for space of rational functions whose poles are (near) the poles of f
- ORF on $[a,b]$

ORF on $\mathbb{T}$

In the rest of these lectures we shall only consider orthogonal rational functions on the unit circle.

♠

The theory for the real line, the half-line or the interval is similar, but not quite the same.

Part II

The scalar case

Definitions and notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
  $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$
- parahermitian conjugate:
  $f_*(z) = \overline{f(1/\overline{z})} \Rightarrow \; f_*(t) = \overline{f(t)}$, $t \in \mathbb{T}$
- $L_2(\mu)$: $\langle f, g \rangle_\mu = \int f(t)\overline{g(t)} d\mu(t)$, $\quad (\int d\mu(t) = 1)$
- $d\mu = d\lambda = dt/(2i\pi t) =$ classical Lebesgue
- $H_2(\mu)$: analytic in $\mathbb{D}$ with boundary value in $L_2(\mu)$

Definitions and notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
  $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$
- parahermitian conjugate:
  $f_*(z) = \overline{f(1/\overline{z})} \Rightarrow \quad f_*(t) = \overline{f(t)},\ t \in \mathbb{T}$
- $L_2(\mu)$: $\langle f, g \rangle_\mu = \int f(t)\overline{g(t)}d\mu(t), \quad (\int d\mu(t) = 1)$
- $d\mu = d\lambda = dt/(2i\pi t) =$ classical Lebesgue
- $H_2(\mu)$: analytic in $\mathbb{D}$ with boundary value in $L_2(\mu)$

Definitions and notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
  $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$
- parahermitian conjugate:
  $f_*(z) = \overline{f(1/\overline{z})} \Rightarrow \quad f_*(t) = \overline{f(t)},\ t \in \mathbb{T}$
- $L_2(\mu)$: $\langle f, g \rangle_\mu = \int f(t)\overline{g(t)}d\mu(t), \quad (\int d\mu(t) = 1)$
- $d\mu = d\lambda = dt/(2i\pi t) =$ classical Lebesgue
- $H_2(\mu)$: analytic in $\mathbb{D}$ with boundary value in $L_2(\mu)$

Definitions and notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
  $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$
- parahermitian conjugate:
  $f_*(z) = \overline{f(1/\overline{z})} \Rightarrow \quad f_*(t) = \overline{f(t)},\ t \in \mathbb{T}$
- $L_2(\mu)$: $\langle f, g \rangle_\mu = \int f(t)\overline{g(t)} d\mu(t), \quad (\int d\mu(t) = 1)$
- $d\mu = d\lambda = dt/(2i\pi t) =$ classical Lebesgue
- $H_2(\mu)$: analytic in $\mathbb{D}$ with boundary value in $L_2(\mu)$

Definitions and notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
  $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$
- parahermitian conjugate:
  $f_*(z) = \overline{f(1/\overline{z})} \Rightarrow \quad f_*(t) = \overline{f(t)},\ t \in \mathbb{T}$
- $L_2(\mu)$: $\langle f, g \rangle_\mu = \int f(t)\overline{g(t)} d\mu(t), \quad (\int d\mu(t) = 1)$
- $d\mu = d\lambda = dt/(2i\pi t) =$ classical Lebesgue
- $H_2(\mu)$: analytic in $\mathbb{D}$ with boundary value in $L_2(\mu)$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- trigonometric moments $c_k = \int t^{-k} d\mu(t)$, $k \in \mathbb{Z}$
- Riesz-Herglotz kernel: $D(t,z) = \frac{t+z}{t-z}$
- Riesz-Herglotz transform: $\Omega_\mu(z) = ic + \int D(t,z) d\mu(t)$
  we normalize by assuming $\Omega_\mu(0) = 1$ (i.e. $c = 0$)
- Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
- $\Omega_\mu(z) = c_0 + 2\sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1)$
- $\lim_{z \to t \in \mathbb{T}} \mathrm{Re}\Omega_\mu(z) = \mu'(t)$
- Poisson kernel: $P(t,z) = \frac{1}{2}[D(t,z) + D_*(t,z)] = \frac{t(1-|z|^2)}{(t-z)(1-\bar{z}t)}$
  $P(t,z) = \frac{1-|z|^2}{|t-z|^2}$ if $t \in \mathbb{T}$ and $z \in \mathbb{D}$

Definitions and notation

- Schur class: $\mathcal{B} = \{f \in H(\mathbb{D}) : f(\mathbb{D}) \subset \mathbb{D}\}$
  Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
  $\Gamma \in \mathcal{B} \Leftrightarrow \Omega = (1-\Gamma)/(1+\Gamma) \in \mathcal{C}$
- Blaschke product: $B_n = \zeta_1 \cdots \zeta_n$
  $\{\alpha_k\}_{k\geq 0} \subset \mathbb{D}$, $\alpha_0 = 0$
  $\zeta_k(z) = z_k \dfrac{z-\alpha_k}{1-\overline{\alpha}_k z}$, $z_k = -\dfrac{|\alpha_k|}{\alpha_k}$ if $\alpha_k \neq 0$, otherwise 1.
- converges to nonzero function analytic in $\mathbb{D}$ iff
  $\sum_n (1-|\alpha_n|) < \infty$

Definitions and notation

- Schur class: $\mathcal{B} = \{f \in H(\mathbb{D}) : f(\mathbb{D}) \subset \mathbb{D}\}$
  Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
  $\Gamma \in \mathcal{B} \Leftrightarrow \Omega = (1-\Gamma)/(1+\Gamma) \in \mathcal{C}$
- Blaschke product: $B_n = \zeta_1 \cdots \zeta_n$
  $\{\alpha_k\}_{k\geq 0} \subset \mathbb{D}$, $\alpha_0 = 0$
  $\zeta_k(z) = z_k \dfrac{z-\alpha_k}{1-\overline{\alpha}_k z}$, $z_k = -\dfrac{|\alpha_k|}{\alpha_k}$ if $\alpha_k \neq 0$, otherwise 1.
- converges to nonzero function analytic in $\mathbb{D}$ iff
  $\sum_n (1-|\alpha_n|) < \infty$

Definitions and notation

- Schur class: $\mathcal{B} = \{f \in H(\mathbb{D}) : f(\mathbb{D}) \subset \mathbb{D}\}$
  Carathéodory class $\mathcal{C} = \{f \in H(\mathbb{D}) : \mathrm{Re} f(\mathbb{D}) > 0\}$
  $\Gamma \in \mathcal{B} \Leftrightarrow \Omega = (1-\Gamma)/(1+\Gamma) \in \mathcal{C}$
- Blaschke product: $B_n = \zeta_1 \cdots \zeta_n$
  $\{\alpha_k\}_{k\geq 0} \subset \mathbb{D}$, $\alpha_0 = 0$
  $\zeta_k(z) = z_k \dfrac{z-\alpha_k}{1-\overline{\alpha}_k z}$, $z_k = -\dfrac{|\alpha_k|}{\alpha_k}$ if $\alpha_k \neq 0$, otherwise 1.
- converges to nonzero function analytic in $\mathbb{D}$ iff
  $\sum_n (1-|\alpha_n|) < \infty$

Definitions and notation

- outer function in H_2: no poles, no zeros
  $\sigma(z) = c \exp\{\frac{1}{2} \int D(t,z) \log \mu'(t) d\lambda(t)\}$, $c \in \mathbb{T}$
- note $|\sigma(z)|^2 = \exp\{\int P(t,z) \log \mu'(t) d\lambda(t)\}$
  σ is outer spectral factor of μ'
- inner function: catches all zeros
  $U(z) = e^{i\gamma} B(z) S(z)$, $\gamma \in \mathbb{R}$, $S(z) = \exp\{-\int D(t,z) d\nu(t)\}$
- every $f \in H_2$ has an inner-outer factorization.

Definitions and notation

- outer function in H_2: no poles, no zeros
  $\sigma(z) = c \exp\{\frac{1}{2} \int D(t,z) \log \mu'(t) d\lambda(t)\}$, $c \in \mathbb{T}$
- note $|\sigma(z)|^2 = \exp\{\int P(t,z) \log \mu'(t) d\lambda(t)\}$
  σ is outer spectral factor of μ'
- inner function: catches all zeros
  $U(z) = e^{i\gamma} B(z) S(z)$, $\gamma \in \mathbb{R}$, $S(z) = \exp\{-\int D(t,z) d\nu(t)\}$
- every $f \in H_2$ has an inner-outer factorization.

Definitions and notation

- outer function in H_2: no poles, no zeros
  $\sigma(z) = c \exp\{\frac{1}{2} \int D(t,z) \log \mu'(t) d\lambda(t)\}$, $c \in \mathbb{T}$
- note $|\sigma(z)|^2 = \exp\{\int P(t,z) \log \mu'(t) d\lambda(t)\}$
  σ is outer spectral factor of μ'
- inner function: catches all zeros
  $U(z) = e^{i\gamma} B(z) S(z)$, $\gamma \in \mathbb{R}$, $S(z) = \exp\{-\int D(t,z) d\nu(t)\}$
- every $f \in H_2$ has an inner-outer factorization.

Definitions and notation

- outer function in H_2: no poles, no zeros
  $\sigma(z) = c \exp\{\frac{1}{2} \int D(t,z) \log \mu'(t) d\lambda(t)\}$, $c \in \mathbb{T}$
- note $|\sigma(z)|^2 = \exp\{\int P(t,z) \log \mu'(t) d\lambda(t)\}$
  σ is outer spectral factor of μ'
- inner function: catches all zeros
  $U(z) = e^{i\gamma} B(z) S(z)$, $\gamma \in \mathbb{R}$, $S(z) = \exp\{-\int D(t,z) d\nu(t)\}$
- every $f \in H_2$ has an inner-outer factorization.

The spaces $\mathcal{L}_n$

- $\{\alpha_k\} \subset \mathbb{D}$, $B_n(z)$ Blaschke products
- $\mathcal{L}_n = \operatorname{span}\{p_n/\pi_n : p_n \in \mathcal{P}_n; \pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z)\}$
- The $L_2(\mu)$ inner product in $\mathcal{L}_n$ is completely defined by the values $\{\Omega_\mu^{(k)}(\alpha_i) : i = 0, \ldots, d; k = 0, \ldots, m_i - 1\}$ where
  $d =$ the number of different α_k in $\{\alpha_0, \ldots, \alpha_n\}$
  $m_i =$ the number of times α_i shows up in the set $\{\alpha_0, \ldots, \alpha_n\}$
- proof: compute Gram matrix for the basis
  $\{1\} \cup \{1/(1-\overline{\alpha}_i z)^k : k = 0 \ldots, m_i - 1; i = 0, \ldots, d\}$

The spaces $\mathcal{L}_n$

- $\{\alpha_k\} \subset \mathbb{D}$, $B_n(z)$ Blaschke products
- $\mathcal{L}_n = \operatorname{span}\{p_n/\pi_n : p_n \in \mathcal{P}_n; \pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z)\}$
- The $L_2(\mu)$ inner product in $\mathcal{L}_n$ is completely defined by the values $\{\Omega_\mu^{(k)}(\alpha_i) : i = 0, \ldots, d; k = 0, \ldots, m_i - 1\}$ where
  $d =$ the number of different α_k in $\{\alpha_0, \ldots, \alpha_n\}$
  $m_i =$ the number of times α_i shows up in the set $\{\alpha_0, \ldots, \alpha_n\}$
- proof: compute Gram matrix for the basis
  $\{1\} \cup \{1/(1-\overline{\alpha}_i z)^k : k = 0 \ldots, m_i - 1; i = 0, \ldots, d\}$

The spaces $\mathcal{L}_n$

- $\{\alpha_k\} \subset \mathbb{D}$, $B_n(z)$ Blaschke products
- $\mathcal{L}_n = \operatorname{span}\{p_n/\pi_n : p_n \in \mathcal{P}_n; \pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z)\}$
- The $L_2(\mu)$ inner product in $\mathcal{L}_n$ is completely defined by the values $\{\Omega_\mu^{(k)}(\alpha_i) : i = 0, \ldots, d; k = 0, \ldots, m_i - 1\}$ where
  $d =$ the number of different α_k in $\{\alpha_0, \ldots, \alpha_n\}$
  $m_i =$ the number of times α_i shows up in the set $\{\alpha_0, \ldots, \alpha_n\}$
- proof: compute Gram matrix for the basis
  $\{1\} \cup \{1/(1-\overline{\alpha}_i z)^k : k = 0 \ldots, m_i - 1; i = 0, \ldots, d\}$

The spaces $\mathcal{L}_n$

- $\{\alpha_k\} \subset \mathbb{D}$, $B_n(z)$ Blaschke products
- $\mathcal{L}_n = \operatorname{span}\{p_n/\pi_n : p_n \in \mathcal{P}_n; \pi_n(z) = \prod_{k=1}^{n}(1-\overline{\alpha}_k z)\}$
- The $L_2(\mu)$ inner product in $\mathcal{L}_n$ is completely defined by the values $\{\Omega_\mu^{(k)}(\alpha_i) : i = 0, \ldots, d; k = 0, \ldots, m_i - 1\}$ where
  $d =$ the number of different α_k in $\{\alpha_0, \ldots, \alpha_n\}$
  $m_i =$ the number of times α_i shows up in the set $\{\alpha_0, \ldots, \alpha_n\}$
- proof: compute Gram matrix for the basis
  $\{1\} \cup \{1/(1-\overline{\alpha}_i z)^k : k = 0 \ldots, m_i - 1; i = 0, \ldots, d\}$

Reproducing kernel

- $\{e_0, \ldots, e_n\}$ orthonormal basis for $\mathcal{L}_n$ then $k_n(z,w) = \sum_{k=0}^{n} e_k(z)\overline{e_k(w)}$ is reproducing kernel: $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu = f(w)$, $\forall f \in \mathcal{L}_n$.
- $f \in \mathcal{L}_m$ with $m > n$, then $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu$ is the projection of f onto $\mathcal{L}_n$
- $k_n(z,w) = B_n(z)\overline{B_n(w)}k_n(1/\overline{w}, 1/\overline{z})$
- proof: If $\{e_k\}$ is an ON basis, then also $\{B_n e_{k*}\}$ is ON basis
- super star: $f \in \mathcal{L}_n$ then $f^* = B_n f_* \in \mathcal{L}_n$; note $B_{k*} = 1/B_k$

Reproducing kernel

- $\{e_0, \ldots, e_n\}$ orthonormal basis for $\mathcal{L}_n$ then $k_n(z,w) = \sum_{k=0}^{n} e_k(z)\overline{e_k(w)}$ is reproducing kernel: $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu = f(w)$, $\forall f \in \mathcal{L}_n$.
- $f \in \mathcal{L}_m$ with $m > n$, then $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu$ is the projection of f onto $\mathcal{L}_n$
- $k_n(z,w) = B_n(z)\overline{B_n(w)}k_n(1/\overline{w}, 1/\overline{z})$
- proof: If $\{e_k\}$ is an ON basis, then also $\{B_n e_{k*}\}$ is ON basis
- super star: $f \in \mathcal{L}_n$ then $f^* = B_n f_* \in \mathcal{L}_n$; note $B_{k*} = 1/B_k$

Reproducing kernel

- $\{e_0, \ldots, e_n\}$ orthonormal basis for $\mathcal{L}_n$ then $k_n(z,w) = \sum_{k=0}^{n} e_k(z)\overline{e_k(w)}$ is reproducing kernel: $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu = f(w)$, $\forall f \in \mathcal{L}_n$.
- $f \in \mathcal{L}_m$ with $m > n$, then $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu$ is the projection of f onto $\mathcal{L}_n$
- $k_n(z,w) = B_n(z)\overline{B_n(w)}k_n(1/\overline{w}, 1/\overline{z})$
- proof: If $\{e_k\}$ is an ON basis, then also $\{B_n e_{k*}\}$ is ON basis
- super star: $f \in \mathcal{L}_n$ then $f^* = B_n f_* \in \mathcal{L}_n$; note $B_{k*} = 1/B_k$

Reproducing kernel

- $\{e_0, \ldots, e_n\}$ orthonormal basis for $\mathcal{L}_n$ then $k_n(z,w) = \sum_{k=0}^{n} e_k(z)\overline{e_k(w)}$ is reproducing kernel: $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu = f(w)$, $\forall f \in \mathcal{L}_n$.
- $f \in \mathcal{L}_m$ with $m > n$, then $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu$ is the projection of f onto $\mathcal{L}_n$
- $k_n(z,w) = B_n(z)\overline{B_n(w)}k_n(1/\overline{w}, 1/\overline{z})$
- proof: If $\{e_k\}$ is an ON basis, then also $\{B_n e_{k*}\}$ is ON basis
- super star: $f \in \mathcal{L}_n$ then $f^* = B_n f_* \in \mathcal{L}_n$; note $B_{k*} = 1/B_k$

Reproducing kernel

- $\{e_0, \ldots, e_n\}$ orthonormal basis for $\mathcal{L}_n$ then $k_n(z,w) = \sum_{k=0}^{n} e_k(z)\overline{e_k(w)}$ is reproducing kernel: $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu = f(w)$, $\forall f \in \mathcal{L}_n$.
- $f \in \mathcal{L}_m$ with $m > n$, then $\langle f(\cdot), k_n(\cdot, w)\rangle_\mu$ is the projection of f onto $\mathcal{L}_n$
- $k_n(z,w) = B_n(z)\overline{B_n(w)}k_n(1/\overline{w}, 1/\overline{z})$
- proof: If $\{e_k\}$ is an ON basis, then also $\{B_n e_{k*}\}$ is ON basis
- super star: $f \in \mathcal{L}_n$ then $f^* = B_n f_* \in \mathcal{L}_n$; note $B_{k*} = 1/B_k$

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

$k_n(z, w) = B_n(z)\overline{B_n(w)}k_n(1/\overline{w}, 1/\overline{z})$ fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

$$k_n(z,w) = B_n(z)\overline{B_n(w)} \sum_0^n \phi_{k*}(z)\overline{\phi_{k*}(w)}$$ fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

$$k_n(z,w) = \sum_0^n \frac{B_n(z)}{B_k(z)}\phi_k^*(z)\overline{\frac{B_n(w)}{B_k(w)}\phi_k^*(w)}$$

fill in $w = \alpha_n$ QED

The ϕ_n

- ORF: $\mathcal{L}_{n-1} \perp_\mu \phi_n \in \mathcal{L}_n$
- $\phi_n = a_{nn}B_n + a_{n,n-1}B_{n-1} + \cdots$, assume $a_{nn} = \kappa_n > 0$
- $\phi_n^* = a_{nn} + \bar{a}_{n,n-1}B_n/B_{n-1} + \cdots \Rightarrow \phi_n^*(\alpha_n) = \kappa_n$
- thus $k_n(z, \alpha_n) = \kappa_n \phi_n^*(z)$ and $k_n(\alpha_n, \alpha_n) = \kappa_n^2$

Proof:

$$k_n(z,w) = \sum_0^n \frac{B_n(z)}{B_k(z)}\phi_k^*(z)\overline{\frac{B_n(w)}{B_k(w)}\phi_k^*(w)}$$

fill in $w = \alpha_n$ QED

Christoffel-Darboux relations

Theorem

If not $z = w = t \in \mathbb{T}$ *then*

$$\begin{aligned} k_n(z,w) &= \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}} \\ &= \frac{\phi^*_n(z)\overline{\phi^*_n(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} \end{aligned}$$

else $P(t,\alpha_{n+1})k_n(t,t) = t[\phi'_{n+1}(t)\overline{\phi_{n+1}(t)} - \phi^{*\prime}_{n+1}\overline{\phi^*_{n+1}(t)}]$

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

Skip proof

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- multiply with $B_{n+1}(z)\overline{B_{n+1}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- $[k_n(z,w) + \phi_{n+1}(z)\overline{\phi_{n+1}(w)}] - \zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}k_n(z,w) = \phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}$

Skip proof

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}[k_n(z,w) - \phi_n(z)\overline{\phi_n(w)}] = \phi_n^*(z)\overline{\phi_n^*(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}[k_n(z,w) - \phi_n(z)\overline{\phi_n(w)}] = \phi_n^*(z)\overline{\phi_n^*(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $n \to n-1$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}[k_n(z,w) - \phi_n(z)\overline{\phi_n(w)}] = \phi_n^*(z)\overline{\phi_n^*(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}[k_n(z,w) - \phi_n(z)\overline{\phi_n(w)}] = \phi_n^*(z)\overline{\phi_n^*(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}[k_n(z,w) - \phi_n(z)\overline{\phi_n(w)}] = \phi_n^*(z)\overline{\phi_n^*(w)}$
- QED

Proof of $k_n(z,w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}$

- $k_n(z,w) = B_n(z)\overline{B_n(w)}\sum_{k=0}^{n}\phi_{k*}(z)\overline{\phi_{k*}(w)}$
- hence: $\frac{k_{n+1}(z,w)}{B_{n+1}(z)\overline{B_{n+1}(w)}} - \frac{k_n(z,w)}{B_n(z)\overline{B_n(w)}} = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_{n+1}(z,w) - \frac{B_{n+1}(z)\overline{B_{n+1}(w)}}{B_n(z)\overline{B_n(w)}}k_n(z,w) = \phi_{(n+1)*}(z)\overline{\phi_{(n+1)*}(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)}$
- $k_n(z,w) - \zeta_n(z)\overline{\zeta_n(w)}[k_n(z,w) - \phi_n(z)\overline{\phi_n(w)}] = \phi_n^*(z)\overline{\phi_n^*(w)}$
- QED

Corollary $\phi_n^*(z) \neq 0$ in $\mathbb{D}$

$z \notin \mathbb{T}$:

- $0 < k_{n-1}(z,z) = \frac{|\phi_n^*(z)|^2 - |\phi_n(z)|^2}{1-|\zeta_n(z)|^2}$
- $|\zeta_n(z)| < 1$ in $\mathbb{D}$, and $|\zeta_n(z)| > 1$ in $\mathbb{E}$
- $|\phi_n^*(z)| > |\phi_n(z)|$ in $\mathbb{D} \Rightarrow \phi_n^*(z) \neq 0$ in $\mathbb{D}$
  $|\phi_n^*(z)| < |\phi_n(z)|$ in $\mathbb{E} \Rightarrow \phi_n(z) \neq 0$ in $\mathbb{E}$
- $|\phi_n(z)/\phi_n^*(z)| < 1$ in $\mathbb{D}$ and > 1 in $\mathbb{E}$

Corollary $\phi_n^*(z) \neq 0$ in $\mathbb{D}$

$z \notin \mathbb{T}$:

- $0 < k_{n-1}(z,z) = \frac{|\phi_n^*(z)|^2 - |\phi_n(z)|^2}{1-|\zeta_n(z)|^2}$
- $|\zeta_n(z)| < 1$ in $\mathbb{D}$, and $|\zeta_n(z)| > 1$ in $\mathbb{E}$
- $|\phi_n^*(z)| > |\phi_n(z)|$ in $\mathbb{D} \Rightarrow \phi_n^*(z) \neq 0$ in $\mathbb{D}$
  $|\phi_n^*(z)| < |\phi_n(z)|$ in $\mathbb{E} \Rightarrow \phi_n(z) \neq 0$ in $\mathbb{E}$
- $|\phi_n(z)/\phi_n^*(z)| < 1$ in $\mathbb{D}$ and > 1 in $\mathbb{E}$

Corollary $\phi_n^*(z) \neq 0$ in $\mathbb{D}$

$z \notin \mathbb{T}$:

- $0 < k_{n-1}(z,z) = \frac{|\phi_n^*(z)|^2 - |\phi_n(z)|^2}{1-|\zeta_n(z)|^2}$
- $|\zeta_n(z)| < 1$ in $\mathbb{D}$, and $|\zeta_n(z)| > 1$ in $\mathbb{E}$
- $|\phi_n^*(z)| > |\phi_n(z)|$ in $\mathbb{D} \Rightarrow \phi_n^*(z) \neq 0$ in $\mathbb{D}$
  $|\phi_n^*(z)| < |\phi_n(z)|$ in $\mathbb{E} \Rightarrow \phi_n(z) \neq 0$ in $\mathbb{E}$
- $|\phi_n(z)/\phi_n^*(z)| < 1$ in $\mathbb{D}$ and > 1 in $\mathbb{E}$

Corollary $\phi_n^*(z) \neq 0$ in $\mathbb{D}$

$z \notin \mathbb{T}$:

- $0 < k_{n-1}(z,z) = \frac{|\phi_n^*(z)|^2 - |\phi_n(z)|^2}{1-|\zeta_n(z)|^2}$
- $|\zeta_n(z)| < 1$ in $\mathbb{D}$, and $|\zeta_n(z)| > 1$ in $\mathbb{E}$
- $|\phi_n^*(z)| > |\phi_n(z)|$ in $\mathbb{D} \Rightarrow \phi_n^*(z) \neq 0$ in $\mathbb{D}$
  $|\phi_n^*(z)| < |\phi_n(z)|$ in $\mathbb{E} \Rightarrow \phi_n(z) \neq 0$ in $\mathbb{E}$
- $|\phi_n(z)/\phi_n^*(z)| < 1$ in $\mathbb{D}$ and > 1 in $\mathbb{E}$

Recurrence for kernels

Theorem

$$\begin{bmatrix} k_n^*(z,w) \\ k_n(z,w) \end{bmatrix} = t_n(z,w) \begin{bmatrix} k_{n-1}^*(z,w) \\ k_{n-1}(z,w) \end{bmatrix}$$

$$t_n(z,w) = c_n \begin{bmatrix} 1 & \overline{\rho}_n \\ \rho_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_n(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \overline{\gamma}_n \\ \gamma_n & 1 \end{bmatrix}$$

$$\begin{aligned} c_n &= (1-|\rho_n|^2)^{-1} \\ \rho_n &= \rho_n(w) = \overline{\phi_n(w)}/\phi_n^*(w) \in \mathbb{D} \\ \gamma_n &= \gamma_n(w) = -\zeta_n(w)\rho_n(w) \in \mathbb{D} \end{aligned}$$

Proof

▸ Skip proof

- CD: $(1-\zeta_n(z)\overline{\zeta_n(w)})k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}$
- $k_{n-1}(z,w) + \phi_n(z)\overline{\phi_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$

Proof

- CD: $(1-\zeta_n(z)\overline{\zeta_n(w)})k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}$
- $k_{n-1}(z,w) + \phi_n(z)\overline{\phi_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$

Proof

- CD: $(1-\zeta_n(z)\overline{\zeta_n(w)})k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}$
- $k_{n-1}(z,w) + \phi_n(z)\overline{\phi_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$

Proof

- CD: $(1-\zeta_n(z)\overline{\zeta_n(w)})k_{n-1}(z,w) = \phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}$
- $k_{n-1}(z,w) + \phi_n(z)\overline{\phi_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (1) $\quad k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- $\phi_n(z)\overline{\phi_n(w)} = k_n(z,w) - k_{n-1}(z,w)$
- $\phi_n^*(z)\phi_n(w) = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $\phi_n^*(z)\overline{\phi_n^*(w)}\frac{\phi_n(w)}{\phi_n^*(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- $\phi_n(z)\overline{\phi_n(w)} = k_n(z,w) - k_{n-1}(z,w)$
- $\phi_n^*(z)\phi_n(w) = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $\phi_n^*(z)\overline{\phi_n^*(w)}\frac{\phi_n(w)}{\phi_n^*(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- $\phi_n(z)\overline{\phi_n(w)} = k_n(z,w) - k_{n-1}(z,w)$
- superstar w.r.t. z
- $\phi_n^*(z)\phi_n(w) = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $\phi_n^*(z)\overline{\phi_n^*(w)}\frac{\phi_n(w)}{\phi_n^*(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- $\phi_n(z)\overline{\phi_n(w)} = k_n(z,w) - k_{n-1}(z,w)$
- $\phi_n^*(z)\phi_n(w) = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $\phi_n^*(z)\overline{\phi_n^*(w)}\frac{\phi_n(w)}{\phi_n^*(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- $\phi_n(z)\overline{\phi_n(w)} = k_n(z,w) - k_{n-1}(z,w)$
- $\phi_n^*(z)\phi_n(w) = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $\phi_n^*(z)\overline{\phi_n^*(w)}\frac{\phi_n(w)}{\phi_n^*(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- $\phi_n(z)\overline{\phi_n(w)} = k_n(z,w) - k_{n-1}(z,w)$
- $\phi_n^*(z)\phi_n(w) = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $\phi_n^*(z)\overline{\phi_n^*(w)}\frac{\phi_n(w)}{\phi_n^*(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + \phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- take superstar of this and eliminate $k_n(z,w)$ or $k_n^*(z,w)$
- QED

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- (1)$\times\overline{\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + \phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- take superstar of this and eliminate $k_n(z,w)$ or $k_n^*(z,w)$
- QED

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $k_n(z,w)\overline{\rho_n(w)} =$
  $\zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + \phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} =$
  $\zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- take superstar of this and eliminate $k_n(z,w)$ or $k_n^*(z,w)$
- QED

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $k_n(z,w)\overline{\rho_n(w)} =$
  $\zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + \phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} =$
  $\zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- take superstar of this and eliminate $k_n(z,w)$ or $k_n^*(z,w)$
- QED

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + \phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- take superstar of this and eliminate $k_n(z,w)$ or $k_n^*(z,w)$
- QED

Proof

- (1) $k_n(z,w) = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w) + \phi_n^*(z)\overline{\phi_n^*(w)}$
- (2) $\phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)} = k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + \phi_n^*(z)\overline{\phi_n^*(w)\rho_n(w)}$
- $k_n(z,w)\overline{\rho_n(w)} = \zeta_n(z)\overline{\zeta_n(w)}k_{n-1}(z,w)\overline{\rho_n(w)} + k_n^*(z,w) - \zeta_n(z)k_{n-1}^*(z,w)$
- take superstar of this and eliminate $k_n(z,w)$ or $k_n^*(z,w)$
- QED

Recurrence for ϕ_n

Theorem

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = \tilde{t}_n(z) \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix}$$

$$\tilde{t}_n(z) = \frac{\kappa_n}{\kappa_{n-1}} \begin{bmatrix} 1 & 0 \\ 0 & z_n \end{bmatrix} \begin{bmatrix} \varepsilon_n & \delta_n \\ \overline{\delta}_n & \overline{\varepsilon}_n \end{bmatrix} \begin{bmatrix} \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} & 0 \\ 0 & \frac{1-\overline{\alpha}_{n-1} z}{1-\overline{\alpha}_n z} \end{bmatrix}$$

$$\varepsilon_n = z_n \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-|\alpha_{n-1}|^2} \frac{\overline{\phi_n^*(\alpha_{n-1})}}{\kappa_n} \qquad \delta_n = \frac{1-\overline{\alpha}_n\alpha_{n-1}}{1-|\alpha_{n-1}|^2} \frac{\overline{\phi_n(\alpha_{n-1})}}{\kappa_n}$$

Proof

Skip proof

- CD:
$$k_{n-1}(z,\alpha_{n-1}) = \phi_{n-1}^*(z)\kappa_{n-1} = \frac{\phi_n^*(z)\overline{\phi_n^*(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n(\alpha_{n-1})}}{1-\zeta_n(z)\overline{\zeta_n(\alpha_{n-1})}}$$
- $\phi_{n-1}(z)\kappa_{n-1} = \frac{\phi_n^*(z)\overline{\phi_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n^*(\alpha_{n-1})}}{\overline{\zeta_n(\alpha_{n-1})} - \zeta_n(z)}$
- QED

Proof

- CD:
  $k_{n-1}(z,\alpha_{n-1}) = \phi^*_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi^*_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n(\alpha_{n-1})}}{1-\zeta_n(z)\overline{\zeta_n(\alpha_{n-1})}}$
- $\phi_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi^*_n(\alpha_{n-1})}}{\overline{\zeta_n(\alpha_{n-1})} - \zeta_n(z)}$
- QED

Proof

- CD:
  $k_{n-1}(z,\alpha_{n-1}) = \phi^*_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi^*_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n(\alpha_{n-1})}}{1-\zeta_n(z)\overline{\zeta_n(\alpha_{n-1})}}$
- take superstar
- $\phi_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi^*_n(\alpha_{n-1})}}{\overline{\zeta_n(\alpha_{n-1})} - \zeta_n(z)}$
- QED

Proof

- CD:
  $k_{n-1}(z,\alpha_{n-1}) = \phi^*_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi^*_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n(\alpha_{n-1})}}{1-\zeta_n(z)\overline{\zeta_n(\alpha_{n-1})}}$
- $\phi_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi^*_n(\alpha_{n-1})}}{\overline{\zeta_n(\alpha_{n-1})} - \zeta_n(z)}$
- QED

Proof

- CD:
  $k_{n-1}(z,\alpha_{n-1}) = \phi^*_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi^*_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n(\alpha_{n-1})}}{1-\zeta_n(z)\overline{\zeta_n(\alpha_{n-1})}}$
- $\phi_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi^*_n(\alpha_{n-1})}}{\overline{\zeta_n(\alpha_{n-1})} - \zeta_n(z)}$
- eliminate $\phi^*_n(z)$
- QED

Proof

- CD:
  $k_{n-1}(z,\alpha_{n-1}) = \phi^*_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi^*_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi_n(\alpha_{n-1})}}{1-\zeta_n(z)\overline{\zeta_n(\alpha_{n-1})}}$
- $\phi_{n-1}(z)\kappa_{n-1} = \frac{\phi^*_n(z)\overline{\phi_n(\alpha_{n-1})} - \phi_n(z)\overline{\phi^*_n(\alpha_{n-1})}}{\overline{\zeta_n(\alpha_{n-1})} - \zeta_n(z)}$
- QED

Alternatives

- $$\tilde{t}_n(z) = c_n \begin{bmatrix} \overline{z}_{n-1}\varepsilon_n & 0 \\ 0 & z_n\overline{\varepsilon}_n \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix}$$
  $$c_n = \frac{\kappa_n}{\kappa_{n-1}} \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}, \quad \lambda_n = \overline{z}_{n-1}\overline{\delta}_n/\overline{\varepsilon}_n \in \mathbb{D}$$
- $$\tilde{t}_n(z) = d_n \begin{bmatrix} \eta_n^1 & 0 \\ 0 & z_n\eta_n^2 \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix}$$
  $$d_n = \sqrt{\frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2}} \frac{1}{\sqrt{1-|\lambda_n|^2}} \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}, \ \eta_n^1, \eta_n^2 \in \mathbb{T}$$
  $$\lambda_n = \overline{z}_{n-1}\overline{\delta}_n/\overline{\varepsilon}_n = \eta_n^1 \frac{\overline{\phi_n(\alpha_{n-1})}}{\phi_n^*(\alpha_{n-1})} \in \mathbb{D}$$

Alternatives

- $$\tilde{t}_n(z) = c_n \begin{bmatrix} \overline{z}_{n-1}\varepsilon_n & 0 \\ 0 & z_n\overline{\varepsilon}_n \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix}$$
  $$c_n = \frac{\kappa_n}{\kappa_{n-1}} \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}, \quad \lambda_n = \overline{z}_{n-1}\overline{\delta}_n/\overline{\varepsilon}_n \in \mathbb{D}$$
- $$\tilde{t}_n(z) = d_n \begin{bmatrix} \eta_n^1 & 0 \\ 0 & z_n\eta_n^2 \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix}$$
  $$d_n = \sqrt{\frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2}} \frac{1}{\sqrt{1-|\lambda_n|^2}} \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}, \ \eta_n^1, \eta_n^2 \in \mathbb{T}$$
  $$\lambda_n = \overline{z}_{n-1}\overline{\delta}_n/\overline{\varepsilon}_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}} \in \mathbb{D}$$

The functions ψ_n

- def: $\psi_0 = 1, \quad \psi_n(z) = \int D(t,z)[\phi_n(t) - \phi_n(z)]d\mu(t), \quad n \geq 1$
- $\psi_n \in \mathcal{L}_n$
- exercise: $\int D(t,z)[f(t)/f(z) - 1]\phi_n(t)d\mu(t) = 0, \quad \forall f_* \in \mathcal{L}_{n-1}$
- corollary: for $n \geq 1$
  $\psi_n(z) = \int D(t,z)[\phi_n(t)f(t)/f(z) - \phi_n(z)]d\mu(t), \quad \forall f_* \in \mathcal{L}_{n-1}$
  $\psi_n^*(z) = \int D(t,z)[\phi_n^*(t)f(t)/f(z) - \phi_n^*(z)]d\mu(t),$
  $\forall f \in \mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n; f(\alpha_n) = 0\}$
- $[\psi_n, \ -\psi_n^*]^T$ satisfy the same recurrence relation as the $[\phi_n, \ \phi_n^*]^T$

The functions ψ_n

- def: $\psi_0 = 1, \quad \psi_n(z) = \int D(t,z)[\phi_n(t) - \phi_n(z)]d\mu(t), \quad n \geq 1$
- $\psi_n \in \mathcal{L}_n$
- exercise: $\int D(t,z)[f(t)/f(z) - 1]\phi_n(t)d\mu(t) = 0, \quad \forall f_* \in \mathcal{L}_{n-1}$
- corollary: for $n \geq 1$
  $\psi_n(z) = \int D(t,z)[\phi_n(t)f(t)/f(z) - \phi_n(z)]d\mu(t), \quad \forall f_* \in \mathcal{L}_{n-1}$
  $\psi_n^*(z) = \int D(t,z)[\phi_n^*(t)f(t)/f(z) - \phi_n^*(z)]d\mu(t),$
  $\forall f \in \mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n; f(\alpha_n) = 0\}$
- $[\psi_n, \ -\psi_n^*]^T$ satisfy the same recurrence relation as the $[\phi_n, \ \phi_n^*]^T$

The functions ψ_n

- def: $\psi_0 = 1, \;\; \psi_n(z) = \int D(t,z)[\phi_n(t) - \phi_n(z)]d\mu(t), \;\; n \geq 1$
- $\psi_n \in \mathcal{L}_n$
- exercise: $\int D(t,z)[f(t)/f(z) - 1]\phi_n(t)d\mu(t) = 0, \;\; \forall f_* \in \mathcal{L}_{n-1}$
- corollary: for $n \geq 1$
  $\psi_n(z) = \int D(t,z)[\phi_n(t)f(t)/f(z) - \phi_n(z)]d\mu(t), \;\; \forall f_* \in \mathcal{L}_{n-1}$
  $\psi_n^*(z) = \int D(t,z)[\phi_n^*(t)f(t)/f(z) - \phi_n^*(z)]d\mu(t),$
  $\forall f \in \mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n; f(\alpha_n) = 0\}$
- $[\psi_n, \; -\psi_n^*]^T$ satisfy the same recurrence relation as the $[\phi_n, \;\; \phi_n^*]^T$

The functions ψ_n

- def: $\psi_0 = 1, \quad \psi_n(z) = \int D(t,z)[\phi_n(t) - \phi_n(z)]d\mu(t), \quad n \geq 1$
- $\psi_n \in \mathcal{L}_n$
- exercise: $\int D(t,z)[f(t)/f(z) - 1]\phi_n(t)d\mu(t) = 0, \quad \forall f_* \in \mathcal{L}_{n-1}$
- corollary: for $n \geq 1$
  $\psi_n(z) = \int D(t,z)[\phi_n(t)f(t)/f(z) - \phi_n(z)]d\mu(t), \quad \forall f_* \in \mathcal{L}_{n-1}$
  $\psi_n^*(z) = \int D(t,z)[\phi_n^*(t)f(t)/f(z) - \phi_n^*(z)]d\mu(t),$
  $\forall f \in \mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n; f(\alpha_n) = 0\}$
- $[\psi_n, \ -\psi_n^*]^T$ satisfy the same recurrence relation as the $[\phi_n, \ \phi_n^*]^T$

The functions ψ_n

- def: $\psi_0 = 1, \quad \psi_n(z) = \int D(t,z)[\phi_n(t) - \phi_n(z)]d\mu(t), \quad n \geq 1$
- $\psi_n \in \mathcal{L}_n$
- exercise: $\int D(t,z)[f(t)/f(z) - 1]\phi_n(t)d\mu(t) = 0, \quad \forall f_* \in \mathcal{L}_{n-1}$
- corollary: for $n \geq 1$
  $\psi_n(z) = \int D(t,z)[\phi_n(t)f(t)/f(z) - \phi_n(z)]d\mu(t), \quad \forall f_* \in \mathcal{L}_{n-1}$
  $\psi_n^*(z) = \int D(t,z)[\phi_n^*(t)f(t)/f(z) - \phi_n^*(z)]d\mu(t),$
  $\forall f \in \mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n; f(\alpha_n) = 0\}$
- $[\psi_n, \ -\psi_n^*]^T$ satisfy the same recurrence relation as the $[\phi_n, \quad \phi_n^*]^T$

Recurrence

Theorem

$$\begin{bmatrix} \phi_n(z) & \psi_n(z) \\ \phi_n^*(z) & -\psi_n^*(z) \end{bmatrix} = \tilde{t}_n(z) \begin{bmatrix} \phi_{n-1}(z) & \psi_{n-1}(z) \\ \phi_{n-1}^*(z) & -\psi_{n-1}^*(z) \end{bmatrix}$$

Proof

Skip proof

- def: $\begin{bmatrix} \psi_{n-1}(z) \\ -\psi^*_{n-1}(z) \end{bmatrix} =$

$$\int D(t,z) \begin{bmatrix} \phi_{n-1}(t) \\ \frac{B_{n-1}(z)}{B_{n-1}(t)} \phi^*_{n-1}(t) \end{bmatrix} d\mu(t) - \begin{bmatrix} \phi_{n-1}(z) \\ \phi^*_{n-1}(z) \end{bmatrix} \int D(t,z) d\mu(t)$$

- RHS: $-\Omega_\mu(z) \begin{bmatrix} \phi_n(z) \\ \phi^*_n(z) \end{bmatrix} + \int D(t,z) Q(t,z) \begin{bmatrix} \phi_n(t) \\ \phi^*_n(t) \end{bmatrix}$
- $Q(t,z) = \frac{f(z)}{f(t)}$, $f(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \in \mathcal{L}_{(n-1)^*} \cap \mathcal{L}_n(\alpha_n)$
- thus RHS $= \begin{bmatrix} \psi_n(z) \\ -\psi^*_n(z) \end{bmatrix}$
- QED

Proof

- def: $\begin{bmatrix} \psi_{n-1}(z) \\ -\psi^*_{n-1}(z) \end{bmatrix} = \qquad (D(t,z)_{*z} = -D(t,z),\ t \in \mathbb{T})$

$$\int D(t,z) \begin{bmatrix} \phi_{n-1}(t) \\ \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \phi^*_{n-1}(t) \end{bmatrix} d\mu(t) - \begin{bmatrix} \phi_{n-1}(z) \\ \phi^*_{n-1}(z) \end{bmatrix} \int D(t,z) d\mu(t)$$

- RHS: $-\Omega_\mu(z) \begin{bmatrix} \phi_n(z) \\ \phi^*_n(z) \end{bmatrix} + \int D(t,z) Q(t,z) \begin{bmatrix} \phi_n(t) \\ \phi^*_n(t) \end{bmatrix}$
- $Q(t,z) = \frac{f(z)}{f(t)}$, $f(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_n(\alpha_n)$
- thus RHS $= \begin{bmatrix} \psi_n(z) \\ -\psi^*_n(z) \end{bmatrix}$
- QED

Proof

- def: $\begin{bmatrix} \psi_{n-1}(z) \\ -\psi^*_{n-1}(z) \end{bmatrix} =$

  $\int D(t,z) \begin{bmatrix} \phi_{n-1}(t) \\ \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \phi^*_{n-1}(t) \end{bmatrix} d\mu(t) - \begin{bmatrix} \phi_{n-1}(z) \\ \phi^*_{n-1}(z) \end{bmatrix} \int D(t,z) d\mu(t)$
- $\times \tilde{t}_n(z)$
- RHS: $-\Omega_\mu(z) \begin{bmatrix} \phi_n(z) \\ \phi^*_n(z) \end{bmatrix} + \int D(t,z) Q(t,z) \begin{bmatrix} \phi_n(t) \\ \phi^*_n(t) \end{bmatrix}$
- $Q(t,z) = \frac{f(z)}{f(t)}$, $f(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \in \mathcal{L}_{(n-1)^*} \cap \mathcal{L}_n(\alpha_n)$
- thus RHS $= \begin{bmatrix} \psi_n(z) \\ -\psi^*_n(z) \end{bmatrix}$
- QED

Proof

- def: $\begin{bmatrix} \psi_{n-1}(z) \\ -\psi^*_{n-1}(z) \end{bmatrix} =$
  $$\int D(t,z) \begin{bmatrix} \phi_{n-1}(t) \\ \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \phi^*_{n-1}(t) \end{bmatrix} d\mu(t) - \begin{bmatrix} \phi_{n-1}(z) \\ \phi^*_{n-1}(z) \end{bmatrix} \int D(t,z) d\mu(t)$$
- RHS: $-\Omega_\mu(z) \begin{bmatrix} \phi_n(z) \\ \phi^*_n(z) \end{bmatrix} + \int D(t,z) Q(t,z) \begin{bmatrix} \phi_n(t) \\ \phi^*_n(t) \end{bmatrix}$
- $Q(t,z) = \frac{f(z)}{f(t)}$, $f(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_n(\alpha_n)$
- thus RHS $= \begin{bmatrix} \psi_n(z) \\ -\psi^*_n(z) \end{bmatrix}$
- QED

Proof

- def: $\begin{bmatrix} \psi_{n-1}(z) \\ -\psi_{n-1}^*(z) \end{bmatrix} =$
  $$\int D(t,z) \begin{bmatrix} \phi_{n-1}(t) \\ \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \phi_{n-1}^*(t) \end{bmatrix} d\mu(t) - \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix} \int D(t,z) d\mu(t)$$
- RHS: $-\Omega_\mu(z) \begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} + \int D(t,z) Q(t,z) \begin{bmatrix} \phi_n(t) \\ \phi_n^*(t) \end{bmatrix}$
- $Q(t,z) = \frac{f(z)}{f(t)}$, $f(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_n(\alpha_n)$
- thus RHS $= \begin{bmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{bmatrix}$
- QED

Proof

- def: $\begin{bmatrix} \psi_{n-1}(z) \\ -\psi_{n-1}^*(z) \end{bmatrix} =$
  $$\int D(t,z) \begin{bmatrix} \phi_{n-1}(t) \\ \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \phi_{n-1}^*(t) \end{bmatrix} d\mu(t) - \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix} \int D(t,z) d\mu(t)$$
- RHS: $-\Omega_\mu(z) \begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} + \int D(t,z) Q(t,z) \begin{bmatrix} \phi_n(t) \\ \phi_n^*(t) \end{bmatrix}$
- $Q(t,z) = \frac{f(z)}{f(t)}$, $f(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z} \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_n(\alpha_n)$
- thus RHS $= \begin{bmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{bmatrix}$
- QED

Liouville-Ostrogradskii/Green

Theorem

$[x_n,\ x_n^+]^T$ *and* $[y_n,\ y_n^+]^T$ *2 solutions of* $\mathbf{v}_n = \tilde{t}_n \mathbf{v}_{n-1}$

$F_n(z,w) = x_n^+(z)y_n(w) - x_n(z)y_n^+(w)$

Liouville-Ostrogradskii

$$\frac{F_n(z,w)}{1-\frac{\zeta_n(z)}{\zeta_n(w)}} - B_n(w)\frac{F_0(z,w)}{1-\frac{\zeta_0(z)}{\zeta_0(w)}} = -\sum_{k=0}^{n-1} x_k(z)y_k^+(w)\frac{B_n(w)}{B_k(w)}$$

$G_n(z,w) = x_n^+(z)\overline{y_n^+(w)} - x_n(z)\overline{y_n(w)}$

Green

$$\frac{G_n(z,w)}{1-\zeta_n(z)\overline{\zeta_n(w)}} - \frac{G_0(z,w)}{1-\zeta_0(z)\overline{\zeta_0(w)}} = \sum_{k=0}^{n-1} x_k(z)\overline{y_k(w)}$$

Example: determinant formula

$(x, x^+) = (\phi, \phi^*)$, $(y, y^+) = (\psi, -\psi^*)$, and $z = w$ in L-O:
$\psi_n(z)\phi_n^*(z) + \psi_n^*(z)\phi_n(z) = 2B_n(z)P(z, \alpha_n)$, $P(t, z) =$ Poisson
Also obtained as determinant of joint recurrence for ϕ_n and ψ_n

♠

- (ψ_n, ψ_n^*) satisfies a recurrence like (ϕ_n, ϕ_n^*) except that λ_n is replaced by $-\lambda_n$.
- Are the ψ_n also ORF?
- Yes! with respect to ν with Riesz-Herglotz transform $1/\Omega_\mu$
- Needs Favard thm
- Needs quadrature

Example: determinant formula

$(x, x^+) = (\phi, \phi^*)$, $(y, y^+) = (\psi, -\psi^*)$, and $z = w$ in L-O:
$\psi_n(z)\phi_n^*(z) + \psi_n^*(z)\phi_n(z) = 2B_n(z)P(z, \alpha_n)$, $P(t, z) =$ Poisson
Also obtained as determinant of joint recurrence for ϕ_n and ψ_n

♠

- (ψ_n, ψ_n^*) satisfies a recurrence like (ϕ_n, ϕ_n^*) except that λ_n is replaced by $-\lambda_n$.
- Are the ψ_n also ORF?
- Yes! with respect to ν with Riesz-Herglotz transform $1/\Omega_\mu$
- Needs Favard thm
- Needs quadrature

Example: determinant formula

$(x, x^+) = (\phi, \phi^*)$, $(y, y^+) = (\psi, -\psi^*)$, and $z = w$ in L-O:
$\psi_n(z)\phi_n^*(z) + \psi_n^*(z)\phi_n(z) = 2B_n(z)P(z, \alpha_n)$, $P(t, z)$= Poisson
Also obtained as determinant of joint recurrence for ϕ_n and ψ_n

♠

- (ψ_n, ψ_n^*) satisfies a recurrence like (ϕ_n, ϕ_n^*) except that λ_n is replaced by $-\lambda_n$.
- Are the ψ_n also ORF?
- Yes! with respect to ν with Riesz-Herglotz transform $1/\Omega_\mu$
- Needs Favard thm
- Needs quadrature

Example: determinant formula

$(x, x^+) = (\phi, \phi^*)$, $(y, y^+) = (\psi, -\psi^*)$, and $z = w$ in L-O:
$\psi_n(z)\phi_n^*(z) + \psi_n^*(z)\phi_n(z) = 2B_n(z)P(z, \alpha_n)$, $P(t, z)$= Poisson
Also obtained as determinant of joint recurrence for ϕ_n and ψ_n

♠

- (ψ_n, ψ_n^*) satisfies a recurrence like (ϕ_n, ϕ_n^*) except that λ_n is replaced by $-\lambda_n$.
- Are the ψ_n also ORF?
- Yes! with respect to ν with Riesz-Herglotz transform $1/\Omega_\mu$
- Needs Favard thm
- Needs quadrature

Example: determinant formula

$(x, x^+) = (\phi, \phi^*)$, $(y, y^+) = (\psi, -\psi^*)$, and $z = w$ in L-O:
$\psi_n(z)\phi_n^*(z) + \psi_n^*(z)\phi_n(z) = 2B_n(z)P(z, \alpha_n)$, $P(t, z)$= Poisson
Also obtained as determinant of joint recurrence for ϕ_n and ψ_n

♠

- (ψ_n, ψ_n^*) satisfies a recurrence like (ϕ_n, ϕ_n^*) except that λ_n is replaced by $-\lambda_n$.
- Are the ψ_n also ORF?
- Yes! with respect to ν with Riesz-Herglotz transform $1/\Omega_\mu$
- Needs Favard thm
- Needs quadrature

Example: determinant formula

$(x, x^+) = (\phi, \phi^*)$, $(y, y^+) = (\psi, -\psi^*)$, and $z = w$ in L-O:
$\psi_n(z)\phi_n^*(z) + \psi_n^*(z)\phi_n(z) = 2B_n(z)P(z, \alpha_n)$, $P(t, z) =$ Poisson
Also obtained as determinant of joint recurrence for ϕ_n and ψ_n

♠

- (ψ_n, ψ_n^*) satisfies a recurrence like (ϕ_n, ϕ_n^*) except that λ_n is replaced by $-\lambda_n$.
- Are the ψ_n also ORF?
- Yes! with respect to ν with Riesz-Herglotz transform $1/\Omega_\mu$
- Needs Favard thm
- Needs quadrature

Interpolatory quadrature

- problem: $I_\mu\{f\} := \int f(t)d\mu(t) \approx I_n\{f\} := \sum_{i=1}^n f(\xi_{ni})\lambda_{ni}$
- interpolation:
  $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_{p*}, g \in \mathcal{L}_q\} = \{\frac{\sum_{k=-p}^q c_k z^k}{\prod_{i=1}^p (t-\alpha_i)\prod_{j=1}^q (1-\overline{\alpha}_j t)}\}$,
  $p, q \geq 0,\ p+q = n-1,\ \dim\mathcal{L}_{p,q} = n$
- Lagrange: $f_n(t) = \sum_{k=1}^n L_{nk}(t)f(\xi_{nk}),\ f \in \mathcal{L}_{p,q}$
  $L_{nk}(\xi_{nl}) = \delta_{kl} \Rightarrow f(t) = f_n(t), t \in \{\xi_{nk}\}_{k=1}^n$
- $I_\mu\{f\} = I_n\{f\} = \sum_{k=1}^n \lambda_{nk} f(\xi_{nk}),\ \lambda_{nk} = \int L_{nk}(t)d\mu(t)$
- n-point formula exact in space of dimension n
- special choice of ξ_{nk} may give larger domain of exactness

Interpolatory quadrature

- problem: $I_\mu\{f\} := \int f(t)d\mu(t) \approx I_n\{f\} := \sum_{i=1}^n f(\xi_{ni})\lambda_{ni}$
- interpolation:
  $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_{p*}, g \in \mathcal{L}_q\} = \{\frac{\sum_{k=-p}^q c_k z^k}{\prod_{i=1}^p (t-\alpha_i)\prod_{j=1}^q (1-\overline{\alpha}_j t)}\},$
  $p, q \geq 0,\ p+q = n-1,\ \dim\mathcal{L}_{p,q} = n$
- Lagrange: $f_n(t) = \sum_{k=1}^n L_{nk}(t)f(\xi_{nk}),\ f \in \mathcal{L}_{p,q}$
  $L_{nk}(\xi_{nl}) = \delta_{kl} \Rightarrow f(t) = f_n(t), t \in \{\xi_{nk}\}_{k=1}^n$
- $I_\mu\{f\} = I_n\{f\} = \sum_{k=1}^n \lambda_{nk} f(\xi_{nk}),\ \lambda_{nk} = \int L_{nk}(t)d\mu(t)$
- n-point formula exact in space of dimension n
- special choice of ξ_{nk} may give larger domain of exactness

Interpolatory quadrature

- problem: $I_\mu\{f\} := \int f(t)d\mu(t) \approx I_n\{f\} := \sum_{i=1}^n f(\xi_{ni})\lambda_{ni}$
- interpolation:
  $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_{p*}, g \in \mathcal{L}_q\} = \{\frac{\sum_{k=-p}^q c_k z^k}{\prod_{i=1}^p (t-\alpha_i)\prod_{j=1}^q (1-\overline{\alpha}_j t)}\}$,
  $p, q \geq 0,\ p+q = n-1,\ \dim\mathcal{L}_{p,q} = n$
- Lagrange: $f_n(t) = \sum_{k=1}^n L_{nk}(t)f(\xi_{nk}),\ f \in \mathcal{L}_{p,q}$
  $L_{nk}(\xi_{nl}) = \delta_{kl} \Rightarrow f(t) = f_n(t), t \in \{\xi_{nk}\}_{k=1}^n$
- $I_\mu\{f\} = I_n\{f\} = \sum_{k=1}^n \lambda_{nk} f(\xi_{nk}),\ \lambda_{nk} = \int L_{nk}(t)d\mu(t)$
- n-point formula exact in space of dimension n
- special choice of ξ_{nk} may give larger domain of exactness

Interpolatory quadrature

- problem: $I_\mu\{f\} := \int f(t)d\mu(t) \approx I_n\{f\} := \sum_{i=1}^n f(\xi_{ni})\lambda_{ni}$
- interpolation:
  $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_{p*}, g \in \mathcal{L}_q\} = \{\frac{\sum_{k=-p}^q c_k z^k}{\prod_{i=1}^p (t-\alpha_i)\prod_{j=1}^q (1-\overline{\alpha}_j t)}\}$,
  $p, q \geq 0,\ p+q = n-1,\ \dim\mathcal{L}_{p,q} = n$
- Lagrange: $f_n(t) = \sum_{k=1}^n L_{nk}(t)f(\xi_{nk}),\ f \in \mathcal{L}_{p,q}$
  $L_{nk}(\xi_{nl}) = \delta_{kl} \Rightarrow f(t) = f_n(t), t \in \{\xi_{nk}\}_{k=1}^n$
- $I_\mu\{f\} = I_n\{f\} = \sum_{k=1}^n \lambda_{nk} f(\xi_{nk}),\ \lambda_{nk} = \int L_{nk}(t)d\mu(t)$
- n-point formula exact in space of dimension n
- special choice of ξ_{nk} may give larger domain of exactness

Interpolatory quadrature

- problem: $I_\mu\{f\} := \int f(t)d\mu(t) \approx I_n\{f\} := \sum_{i=1}^n f(\xi_{ni})\lambda_{ni}$
- interpolation:
  $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_{p*}, g \in \mathcal{L}_q\} = \{\frac{\sum_{k=-p}^q c_k z^k}{\prod_{i=1}^p (t-\alpha_i)\prod_{j=1}^q (1-\overline{\alpha}_j t)}\}$,
  $p, q \geq 0,\ p+q = n-1,\ \dim\mathcal{L}_{p,q} = n$
- Lagrange: $f_n(t) = \sum_{k=1}^n L_{nk}(t)f(\xi_{nk}),\ f \in \mathcal{L}_{p,q}$
  $L_{nk}(\xi_{nl}) = \delta_{kl} \Rightarrow f(t) = f_n(t), t \in \{\xi_{nk}\}_{k=1}^n$
- $I_\mu\{f\} = I_n\{f\} = \sum_{k=1}^n \lambda_{nk} f(\xi_{nk}),\ \lambda_{nk} = \int L_{nk}(t)d\mu(t)$
- n-point formula exact in space of dimension n
- special choice of ξ_{nk} may give larger domain of exactness

Interpolatory quadrature

- problem: $I_\mu\{f\} := \int f(t)d\mu(t) \approx I_n\{f\} := \sum_{i=1}^n f(\xi_{ni})\lambda_{ni}$
- interpolation:
  $\mathcal{L}_{p,q} = \{fg : f \in \mathcal{L}_{p*}, g \in \mathcal{L}_q\} = \{\frac{\sum_{k=-p}^q c_k z^k}{\prod_{i=1}^p (t-\alpha_i)\prod_{j=1}^q (1-\overline{\alpha}_j t)}\}$,
  $p, q \geq 0,\ p+q = n-1,\ \dim\mathcal{L}_{p,q} = n$
- Lagrange: $f_n(t) = \sum_{k=1}^n L_{nk}(t)f(\xi_{nk}),\ f \in \mathcal{L}_{p,q}$
  $L_{nk}(\xi_{nl}) = \delta_{kl} \Rightarrow f(t) = f_n(t), t \in \{\xi_{nk}\}_{k=1}^n$
- $I_\mu\{f\} = I_n\{f\} = \sum_{k=1}^n \lambda_{nk} f(\xi_{nk}),\ \lambda_{nk} = \int L_{nk}(t)d\mu(t)$
- n-point formula exact in space of dimension n
- special choice of ξ_{nk} may give larger domain of exactness

Para-orthogonal rational functions

Theorem

The function $Q_n(z) = \phi_n(z) + \tau\phi_n^*(z)$, $\tau \in \mathbb{T}$ *has n simple zeros on* $\mathbb{T}$. *It is para-orthogonal:* $Q_n \perp_\mu [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$ *while* $\langle 1, Q_n\rangle_\mu \neq 0 \neq \langle B_n, Q_n\rangle_\mu$, $n \geq 1$.

♠

note: $\mathcal{S} = \{\frac{(1-\overline{\alpha}_n z)(z-\alpha_n)p_{n-2}(z)}{\pi_n(z)} : p_{n-2} \in \mathcal{P}_{n-2}\}$

$\mathcal{S}_* = \{\frac{(1-\overline{\alpha}_n z)(z-\alpha_n)p_{n-2}(z)}{\pi_n^*(z)} : p_{n-2} \in \mathcal{P}_{n-2}\}$

Proof (simple zeros)

Skip proof

- $|\phi_n^*| > |\phi_n|$ in $\mathbb{D}$ and $|\phi_n^*| < |\phi_n|$ in $\mathbb{E}$.
  Thus zeros of $Q_n = \phi_n + \tau\phi_n^*$, $\tau \in \mathbb{T}$ are on $\mathbb{T}$.
- CD: $\frac{(1-\overline{\alpha}_n t)(t-\alpha_n)}{1-|\alpha_n|^2}[\phi_n'(t)\overline{\phi_n(t)} - (\phi_n^*)'(t)\overline{\phi_n^*(t)}] = k_n(t,t) > 0$
- $Q_n(t) = 0 \Rightarrow \frac{\phi_n}{\phi_n^*} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{\phi_n^*}}{\phi_n}$.
  $Q_n'(t) = 0 \Rightarrow \frac{\phi_n'}{(\phi_n^*)'} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{(\phi_n^*)'}}{\overline{\phi_n'}}$.
  Hence $\frac{\overline{\phi_n^*}}{\phi_n} = \frac{\phi_n'}{(\phi_n^*)'} \Rightarrow \phi_n'\overline{\phi_n} - (\phi_n^*)'\overline{\phi_n^*} = 0$
- Contradiction $\therefore$ zeros are simple and on $\mathbb{T}$

Proof (simple zeros)

- $|\phi_n^*| > |\phi_n|$ in $\mathbb{D}$ and $|\phi_n^*| < |\phi_n|$ in $\mathbb{E}$.
  Thus zeros of $Q_n = \phi_n + \tau\phi_n^*$, $\tau \in \mathbb{T}$ are on $\mathbb{T}$.
- CD: $\frac{(1-\overline{\alpha}_n t)(t-\alpha_n)}{1-|\alpha_n|^2}[\phi_n'(t)\overline{\phi_n(t)} - (\phi_n^*)'(t)\overline{\phi_n^*(t)}] = k_n(t,t) > 0$
- $Q_n(t) = 0 \Rightarrow \frac{\phi_n}{\phi_n^*} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{\phi_n^*}}{\phi_n}$.
  $Q_n'(t) = 0 \Rightarrow \frac{\phi_n'}{(\phi_n^*)'} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{(\phi_n^*)'}}{\overline{\phi_n'}}$.
  Hence $\frac{\overline{\phi_n^*}}{\phi_n} = \frac{\phi_n'}{(\phi_n^*)'} \Rightarrow \phi_n'\overline{\phi_n} - (\phi_n^*)'\overline{\phi_n^*} = 0$
- Contradiction $\therefore$ zeros are simple and on $\mathbb{T}$

Proof (simple zeros)

- $|\phi_n^*| > |\phi_n|$ in $\mathbb{D}$ and $|\phi_n^*| < |\phi_n|$ in $\mathbb{E}$.
  Thus zeros of $Q_n = \phi_n + \tau\phi_n^*$, $\tau \in \mathbb{T}$ are on $\mathbb{T}$.
- CD: $\frac{(1-\overline{\alpha}_n t)(t-\alpha_n)}{1-|\alpha_n|^2}[\phi_n'(t)\overline{\phi_n(t)} - (\phi_n^*)'(t)\overline{\phi_n^*(t)}] = k_n(t,t) > 0$
- $Q_n(t) = 0 \Rightarrow \frac{\phi_n}{\phi_n^*} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{\phi_n^*}}{\phi_n}$.
  $Q_n'(t) = 0 \Rightarrow \frac{\phi_n'}{(\phi_n^*)'} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{(\phi_n^*)'}}{\overline{\phi_n'}}$.
  Hence $\frac{\overline{\phi_n^*}}{\phi_n} = \frac{\phi_n'}{(\phi_n^*)'} \Rightarrow \phi_n'\overline{\phi_n} - (\phi_n^*)'\overline{\phi_n^*} = 0$
- Contradiction $\therefore$ zeros are simple and on $\mathbb{T}$

Proof (simple zeros)

- $|\phi_n^*| > |\phi_n|$ in $\mathbb{D}$ and $|\phi_n^*| < |\phi_n|$ in $\mathbb{E}$.
  Thus zeros of $Q_n = \phi_n + \tau\phi_n^*$, $\tau \in \mathbb{T}$ are on $\mathbb{T}$.
- CD: $\frac{(1-\overline{\alpha}_n t)(t-\alpha_n)}{1-|\alpha_n|^2}[\phi_n'(t)\overline{\phi_n(t)} - (\phi_n^*)'(t)\overline{\phi_n^*(t)}] = k_n(t,t) > 0$
- $Q_n(t) = 0 \Rightarrow \frac{\phi_n}{\phi_n^*} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{\phi_n^*}}{\phi_n}$.
  $Q_n'(t) = 0 \Rightarrow \frac{\phi_n'}{(\phi_n^*)'} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{(\phi_n^*)'}}{\overline{\phi_n'}}$.
  Hence $\frac{\overline{\phi_n^*}}{\phi_n} = \frac{\phi_n'}{(\phi_n^*)'} \Rightarrow \phi_n'\overline{\phi_n} - (\phi_n^*)'\overline{\phi_n^*} = 0$
- Contradiction $\therefore$ zeros are simple and on $\mathbb{T}$

Proof (simple zeros)

- $|\phi_n^*| > |\phi_n|$ in $\mathbb{D}$ and $|\phi_n^*| < |\phi_n|$ in $\mathbb{E}$.
  Thus zeros of $Q_n = \phi_n + \tau\phi_n^*$, $\tau \in \mathbb{T}$ are on $\mathbb{T}$.
- CD: $\frac{(1-\overline{\alpha}_n t)(t-\alpha_n)}{1-|\alpha_n|^2}[\phi_n'(t)\overline{\phi_n(t)} - (\phi_n^*)'(t)\overline{\phi_n^*(t)}] = k_n(t,t) > 0$
- $Q_n(t) = 0 \Rightarrow \frac{\phi_n}{\phi_n^*} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{\phi_n^*}}{\phi_n}$.
  $Q_n'(t) = 0 \Rightarrow \frac{\phi_n'}{(\phi_n^*)'} = -\tau = -\frac{1}{\overline{\tau}} = \frac{\overline{(\phi_n^*)'}}{\overline{\phi_n'}}$.
  Hence $\frac{\overline{\phi_n^*}}{\phi_n} = \frac{\phi_n'}{(\phi_n^*)'} \Rightarrow \phi_n'\overline{\phi_n} - (\phi_n^*)'\overline{\phi_n^*} = 0$
- Contradiction $\therefore$ zeros are simple and on $\mathbb{T}$

Proof (para-orthogonal)

Skip proof

- ¿ $\langle Q_n, f \rangle_\mu = 0 \; \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*, \; g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Proof (para-orthogonal)

- ¿ $\langle Q_n, f \rangle_\mu = 0 \ \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*,\ g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Proof (para-orthogonal)

- ¿ $\langle Q_n, f \rangle_\mu = 0 \ \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*,\ g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Proof (para-orthogonal)

- ¿ $\langle Q_n, f \rangle_\mu = 0 \; \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*,\ g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Proof (para-orthogonal)

- ¿ $\langle Q_n, f \rangle_\mu = 0 \ \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*,\ g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Proof (para-orthogonal)

- ¿ $\langle Q_n, f \rangle_\mu = 0 \ \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*,\ g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Proof (para-orthogonal)

- ¿ $\langle Q_n, f \rangle_\mu = 0 \; \forall f \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]$?
- $f \in \mathcal{L}_{n-1} \Rightarrow \langle \phi_n, f \rangle_\mu = 0$
- $f \in \mathcal{L}_n(\alpha_n) \Rightarrow f = B_n g_*,\ g \in \mathcal{L}_{n-1}$
- $\langle \phi_n^*, f \rangle_\mu = \langle B_n \phi_{n*}, B_n g_* \rangle_\mu = \langle \phi_{n*}, g_* \rangle_\mu = \langle g, \phi_n \rangle_\mu = 0$
- $\langle 1, Q_n \rangle_\mu = \tau \langle 1, \phi_n^* \rangle_\mu = \tau \langle \phi_n, B_n \rangle_\mu \neq 0$
  $\langle B_n, Q_n \rangle_\mu = \langle B_n, \phi_n \rangle_\mu \neq 0$
- QED

Interpolation/Integration error

- $R \in \mathcal{L}_{n-1,n-1}$, interpolant $R_n \in \mathcal{L}_{n-1}$ in the zeros ξ_{nk} of Q_n
- interpolation error: $E_n(z) = R(z) - R_n(z) = Q_n(z)s(z)$, $s \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]_*$
- The *integration* error will be zero if $\int Q_n(t)s(t)d\mu(t) = 0$
- This n-point formula is exact in $\mathcal{L}_{n-1,n-1}$ and can not be made exact in $\mathcal{L}_{n-1,n}$ or $\mathcal{L}_{n,n-1}$
- *Rational Szegő quadrature formula*

Interpolation/Integration error

- $R \in \mathcal{L}_{n-1,n-1}$, interpolant $R_n \in \mathcal{L}_{n-1}$ in the zeros ξ_{nk} of Q_n
- interpolation error: $E_n(z) = R(z) - R_n(z) = Q_n(z)s(z)$, $s \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]_*$
- The *integration* error will be zero if $\int Q_n(t)s(t)d\mu(t) = 0$
- This n-point formula is exact in $\mathcal{L}_{n-1,n-1}$ and can not be made exact in $\mathcal{L}_{n-1,n}$ or $\mathcal{L}_{n,n-1}$
- *Rational Szegő quadrature formula*

Interpolation/Integration error

- $R \in \mathcal{L}_{n-1,n-1}$, interpolant $R_n \in \mathcal{L}_{n-1}$ in the zeros ξ_{nk} of Q_n
- interpolation error: $E_n(z) = R(z) - R_n(z) = Q_n(z)s(z)$, $s \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]_*$
- proof: $\frac{p_{2n-2}}{\pi_{n-1}\pi^*_{n-1}} - \frac{p_{n-1}\pi^*_{n-1}}{\pi_{n-1}\pi^*_{n-1}} = \frac{\Pi(z-\xi_k)}{\pi_{n-1}(1-\overline{\alpha}_n z)} \frac{(1-\overline{\alpha}_n z)(z-\alpha_n)p_{n-2}}{\pi^*_{n-1}(z-\alpha_n)} = Q_n(z)s(z)$
- The *integration* error will be zero if $\int Q_n(t)s(t)d\mu(t) = 0$
- This n-point formula is exact in $\mathcal{L}_{n-1,n-1}$ and can not be made exact in $\mathcal{L}_{n-1,n}$ or $\mathcal{L}_{n,n-1}$
- *Rational Szegő quadrature formula*

Interpolation/Integration error

- $R \in \mathcal{L}_{n-1,n-1}$, interpolant $R_n \in \mathcal{L}_{n-1}$ in the zeros ξ_{nk} of Q_n
- interpolation error: $E_n(z) = R(z) - R_n(z) = Q_n(z)s(z)$, $s \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]_*$
- The *integration* error will be zero if $\int Q_n(t)s(t)d\mu(t) = 0$
- This n-point formula is exact in $\mathcal{L}_{n-1,n-1}$ and can not be made exact in $\mathcal{L}_{n-1,n}$ or $\mathcal{L}_{n,n-1}$
- *Rational Szegő quadrature formula*

Interpolation/Integration error

- $R \in \mathcal{L}_{n-1,n-1}$, interpolant $R_n \in \mathcal{L}_{n-1}$ in the zeros ξ_{nk} of Q_n
- interpolation error: $E_n(z) = R(z) - R_n(z) = Q_n(z)s(z)$, $s \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]_*$
- The *integration* error will be zero if $\int Q_n(t)s(t)d\mu(t) = 0$
- This n-point formula is exact in $\mathcal{L}_{n-1,n-1}$ and can not be made exact in $\mathcal{L}_{n-1,n}$ or $\mathcal{L}_{n,n-1}$
- *Rational Szegő quadrature formula*

Interpolation/Integration error

- $R \in \mathcal{L}_{n-1,n-1}$, interpolant $R_n \in \mathcal{L}_{n-1}$ in the zeros ξ_{nk} of Q_n
- interpolation error: $E_n(z) = R(z) - R_n(z) = Q_n(z)s(z)$, $s \in [\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)]_*$
- The *integration* error will be zero if $\int Q_n(t)s(t)d\mu(t) = 0$
- This n-point formula is exact in $\mathcal{L}_{n-1,n-1}$ and can not be made exact in $\mathcal{L}_{n-1,n}$ or $\mathcal{L}_{n,n-1}$
- *Rational Szegő quadrature formula*

Para-orthogonal functions of the second kind

- def: μ_n (discrete) measure of R-Szegő q.f.
- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t) = -\frac{P_n(z)}{Q_n(z)}$ then $P_n(z) = \int D(t,z)[Q_n(t)f(t)/f(z) - Q_n(z)]d\mu(t) = \psi_n(z) - \tau\psi_n^*(t)$ $f_* \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$
- proof: $n \geq 2$. Integrand $\in \mathcal{L}_{n-1,n-1} \Rightarrow \int [\cdots] d\mu(t) = \int [\cdots] d\mu_n(t)$
- $\therefore P_n(z)f(z) = -f(z)Q_n(z)I_n\{D(\cdot,z)\} = -f(z)Q_n(z)\Omega_n(z)$
- $n = 1$: exercise
- QED

Para-orthogonal functions of the second kind

- def: μ_n (discrete) measure of R-Szegő q.f.
- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t) = -\frac{P_n(z)}{Q_n(z)}$ then $P_n(z) = \int D(t,z)[Q_n(t)f(t)/f(z) - Q_n(z)]d\mu(t) = \psi_n(z) - \tau\psi_n^*(t)$ $f_* \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$
- proof: $n \geq 2$. Integrand $\in \mathcal{L}_{n-1,n-1} \Rightarrow \int [\cdots] d\mu(t) = \int [\cdots] d\mu_n(t)$
- $\therefore P_n(z)f(z) = -f(z)Q_n(z)I_n\{D(\cdot,z)\} = -f(z)Q_n(z)\Omega_n(z)$
- $n = 1$: exercise
- QED

Para-orthogonal functions of the second kind

- def: μ_n (discrete) measure of R-Szegő q.f.
- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t) = -\frac{P_n(z)}{Q_n(z)}$ then $P_n(z) = \int D(t,z)[Q_n(t)f(t)/f(z) - Q_n(z)]d\mu(t) = \psi_n(z) - \tau\psi_n^*(t)$ $f_* \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$
- proof: $n \geq 2$. Integrand $\in \mathcal{L}_{n-1,n-1} \Rightarrow \int[\cdots]d\mu(t) = \int[\cdots]d\mu_n(t)$
- $\therefore P_n(z)f(z) = -f(z)Q_n(z)I_n\{D(\cdot,z)\} = -f(z)Q_n(z)\Omega_n(z)$
- $n = 1$: exercise
- QED

Para-orthogonal functions of the second kind

- def: μ_n (discrete) measure of R-Szegő q.f.
- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t) = -\frac{P_n(z)}{Q_n(z)}$ then $P_n(z) = \int D(t,z)[Q_n(t)f(t)/f(z) - Q_n(z)]d\mu(t) = \psi_n(z) - \tau\psi_n^*(t)$ $f_* \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$
- proof: $n \geq 2$. Integrand $\in \mathcal{L}_{n-1,n-1} \Rightarrow \int[\cdots]d\mu(t) = \int[\cdots]d\mu_n(t)$
- $\therefore P_n(z)f(z) = -f(z)Q_n(z)I_n\{D(\cdot,z)\} = -f(z)Q_n(z)\Omega_n(z)$
- $n = 1$: exercise
- QED

Para-orthogonal functions of the second kind

- def: μ_n (discrete) measure of R-Szegő q.f.
- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t) = -\frac{P_n(z)}{Q_n(z)}$ then $P_n(z) = \int D(t,z)[Q_n(t)f(t)/f(z) - Q_n(z)]d\mu(t) = \psi_n(z) - \tau\psi_n^*(t)$ $f_* \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$
- proof: $n \geq 2$. Integrand $\in \mathcal{L}_{n-1,n-1} \Rightarrow \int[\cdots]d\mu(t) = \int[\cdots]d\mu_n(t)$
- $\therefore P_n(z)f(z) = -f(z)Q_n(z)I_n\{D(\cdot,z)\} = -f(z)Q_n(z)\Omega_n(z)$
- $n = 1$: exercise
- QED

Para-orthogonal functions of the second kind

- def: μ_n (discrete) measure of R-Szegő q.f.
- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t) = -\frac{P_n(z)}{Q_n(z)}$ then $P_n(z) = \int D(t,z)[Q_n(t)f(t)/f(z) - Q_n(z)]d\mu(t) = \psi_n(z) - \tau\psi_n^*(t)$ $f_* \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$
- proof: $n \geq 2$. Integrand $\in \mathcal{L}_{n-1,n-1} \Rightarrow \int[\cdots]d\mu(t) = \int[\cdots]d\mu_n(t)$
- $\therefore P_n(z)f(z) = -f(z)Q_n(z)I_n\{D(\cdot,z)\} = -f(z)Q_n(z)\Omega_n(z)$
- $n = 1$: exercise
- QED

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: $\Omega_n(z) = -\frac{P_n(z)}{Q_n(z)} = I_n\{D(\cdot,z)\} = \sum_{k=1}^{n} \lambda_{nk} D(\xi_{nk},z)$
- $(z-\xi_{nk})\Omega_n(z) = P_n(z)\frac{z-\xi_{nk}}{Q_n(z)-Q_n(\xi_{nk})} =$
  $\sum_{j\neq k} \lambda_{nj} D(\xi_{nj},z)(z-\xi_{nk}) + \lambda_{nk} D(\xi_{nk},z)(z-\xi_{nk})$
- take $z \longrightarrow \xi_{nk} \Rightarrow$ first formula

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: $\Omega_n(z) = -\frac{P_n(z)}{Q_n(z)} = I_n\{D(\cdot, z)\} = \sum_{k=1}^{n} \lambda_{nk} D(\xi_{nk}, z)$
- $(z - \xi_{nk})\Omega_n(z) = P_n(z) \frac{z-\xi_{nk}}{Q_n(z)-Q_n(\xi_{nk})} =$
  $\sum_{j\neq k} \lambda_{nj} D(\xi_{nj}, z)(z - \xi_{nk}) + \lambda_{nk} D(\xi_{nk}, z)(z - \xi_{nk})$
- take $z \longrightarrow \xi_{nk} \Rightarrow$ first formula

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: $\Omega_n(z) = -\frac{P_n(z)}{Q_n(z)} = I_n\{D(\cdot, z)\} = \sum_{k=1}^{n} \lambda_{nk} D(\xi_{nk}, z)$
- $(z - \xi_{nk})\Omega_n(z) = P_n(z) \frac{z-\xi_{nk}}{Q_n(z)-Q_n(\xi_{nk})} =$
  $\sum_{j\neq k} \lambda_{nj} D(\xi_{nj}, z)(z - \xi_{nk}) + \lambda_{nk} D(\xi_{nk}, z)(z - \xi_{nk})$
- take $z \longrightarrow \xi_{nk} \Rightarrow$ first formula

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: $\Omega_n(z) = -\frac{P_n(z)}{Q_n(z)} = I_n\{D(\cdot, z)\} = \sum_{k=1}^{n} \lambda_{nk} D(\xi_{nk}, z)$
- $(z - \xi_{nk})\Omega_n(z) = P_n(z) \frac{z-\xi_{nk}}{Q_n(z)-Q_n(\xi_{nk})} =$
  $\sum_{j\neq k} \lambda_{nj} D(\xi_{nj}, z)(z - \xi_{nk}) + \lambda_{nk} D(\xi_{nk}, z)(z - \xi_{nk})$
- take $z \to \xi_{nk} \Rightarrow$ first formula

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: (2nd formula)
- $Q_n(\xi_{nk}) = 0 \Rightarrow \tau = \frac{1}{\overline{\tau}} = -\frac{\phi_n(\xi_{nk})}{\phi_n^*(\xi_{nk})}$
- $Q_n'(\xi_{nk}) = \frac{P(\xi_{nk},\alpha_n)}{\xi_{nk}\overline{\phi_n(\xi_{nk})}} k_{n-1}(\xi_{nk},\xi_{nk})$
- $P_n(\xi_{nk}) = \frac{2P(\xi_{nk},\alpha_n)}{\overline{\phi_n(\xi_{nk})}}$
- QED

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: (2nd formula)
- $Q_n(\xi_{nk}) = 0 \Rightarrow \tau = \frac{1}{\overline{\tau}} = -\frac{\phi_n(\xi_{nk})}{\phi_n^*(\xi_{nk})}$
- $Q_n'(\xi_{nk}) = \frac{P(\xi_{nk},\alpha_n)}{\xi_{nk}\overline{\phi_n(\xi_{nk})}} k_{n-1}(\xi_{nk},\xi_{nk})$
- $P_n(\xi_{nk}) = \frac{2P(\xi_{nk},\alpha_n)}{\overline{\phi_n(\xi_{nk})}}$
- QED

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: (2nd formula)
- $Q_n(\xi_{nk}) = 0 \Rightarrow \tau = \frac{1}{\overline{\tau}} = -\frac{\phi_n(\xi_{nk})}{\phi_n^*(\xi_{nk})}$
- confluent CD: $\xi = \xi_{nk}$, $\phi_n^*(\xi) = B_n(\xi)\overline{\phi_n(\xi)}$
  $P(\xi,\alpha_n)k_{n-1}(\xi,\xi) = \xi[\phi_n'(\xi)\overline{\phi_n(\xi)} - (\phi_n^*)'(\xi)\overline{\phi_n^*(\xi)}]$
  $Q_n'(\xi) = \phi_n'(\xi) + \tau(\phi_n^*)'(\xi) = \frac{P(\xi,\alpha_n)}{\xi\overline{\phi_n(\xi)}} k_{n-1}(\xi,\xi)$
- $P_n(\xi_{nk}) = \frac{2P(\xi_{nk},\alpha_n)}{\overline{\phi_n(\xi_{nk})}}$
- QED

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: (2nd formula)
- $Q_n(\xi_{nk}) = 0 \Rightarrow \tau = \frac{1}{\overline{\tau}} = -\frac{\phi_n(\xi_{nk})}{\phi_n^*(\xi_{nk})}$
- $Q_n'(\xi_{nk}) = \frac{P(\xi_{nk},\alpha_n)}{\xi_{nk}\overline{\phi_n(\xi_{nk})}} k_{n-1}(\xi_{nk},\xi_{nk})$
- det. form. $\psi_n(\xi)\phi_n^*(\xi) + \psi_n^*(\xi)\phi_n(\xi) = 2B_n(\xi)P(\xi,\alpha_n)$
  $P_n(\xi) = \psi_n(\xi) - \tau\psi_n^*(\xi) = \psi_n(\xi) + \frac{\phi_n(\xi)}{\phi_n^*(\xi)}\psi_n^*(\xi)$
  $P_n(\xi) = \frac{\psi_n(\xi)\phi_n^*(\xi)+\phi_n(\xi)\psi_n^*(\xi)}{B_n(\xi)\overline{\phi_n(\xi)}}$
  $P_n(\xi_{nk}) = 2P(\xi_{nk},\alpha_n)/\overline{\phi_n(\xi_{nk})}$
- QED

The weights

- theorem: $\lambda_{nk} = \frac{1}{2\xi_{nk}} \frac{P_n(\xi_{nk})}{Q_n'(\xi_{nk})} = \frac{1}{k_{n-1}(\xi_{nk},\xi_{nk})} > 0$
- proof: (2nd formula)
- $Q_n(\xi_{nk}) = 0 \Rightarrow \tau = \frac{1}{\overline{\tau}} = -\frac{\phi_n(\xi_{nk})}{\phi_n^*(\xi_{nk})}$
- $Q_n'(\xi_{nk}) = \frac{P(\xi_{nk},\alpha_n)}{\xi_{nk}\overline{\phi_n(\xi_{nk})}} k_{n-1}(\xi_{nk},\xi_{nk})$
- $P_n(\xi_{nk}) = \frac{2P(\xi_{nk},\alpha_n)}{\overline{\phi_n(\xi_{nk})}}$
- QED

Interpolation

- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t)$, $\Omega_\mu(z) = \int D(t,z) d\mu(t)$
  $\Rightarrow \Omega_\mu(z) - \Omega_n(z) = \Omega_\mu(z) + \frac{P_n(z)}{Q_n(z)} = zB_{n-1}(z)h(z)$,
  $h \in H(\mathbb{D})$
- proof: The inner product is the same in $\mathcal{L}_{n-1}$
  $\therefore$ Gram matrices are the same, and these depend on the values of Ω in the zeros of zB_{n-1} (multiplicities counted)

Interpolation

- theorem: $\Omega_n(z) = \int D(t,z) d\mu_n(t)$, $\Omega_\mu(z) = \int D(t,z) d\mu(t)$
  $\Rightarrow \Omega_\mu(z) - \Omega_n(z) = \Omega_\mu(z) + \frac{P_n(z)}{Q_n(z)} = zB_{n-1}(z)h(z)$,
  $h \in H(\mathbb{D})$
- proof: The inner product is the same in $\mathcal{L}_{n-1}$
  $\therefore$ Gram matrices are the same, and these depend on the values of Ω in the zeros of zB_{n-1} (multiplicities counted)

Interpolation

- theorem:
  $\Omega_\mu(z)\phi_n(z) + \psi_n(z) = zB_{n-1}(z)h(z)$
  $\Omega_\mu(z)\phi_n^*(z) - \psi_n^*(z) = zB_n(z)g(z)$
  $h, g \in H(\mathbb{D})$
- proof: Cauchy-Stieltjes integral
  $\frac{\Omega_\mu(z)\phi_n(z)+\psi_n(z)}{B_{n-1}(z)} = \Omega_\mu(z)\frac{\phi_n(z)}{B_{n-1}(z)} + \frac{\psi_n(z)}{B_{n-1}(z)} =$
  $\int D(t,z)\frac{\phi_n(z)}{B_{n-1}(z)}d\mu(t) + \int D(t,z)[\frac{\phi_n(t)}{B_{n-1}(t)} - \frac{\phi_n(z)}{B_{n-1}(z)}]d\mu(t) =$
  $\int D(t,z)\frac{\phi_n(t)}{B_{n-1}(t)}d\mu(t) \in H(\mathbb{D})$
  Since $D(t,0) = 1$, the integral is $\langle \phi_n, B_{n-1}\rangle_\mu = 0$.
- Second formula is similar. QED

Interpolation

- theorem:
  $\Omega_\mu(z)\phi_n(z) + \psi_n(z) = zB_{n-1}(z)h(z)$
  $\Omega_\mu(z)\phi_n^*(z) - \psi_n^*(z) = zB_n(z)g(z)$
  $h, g \in H(\mathbb{D})$
- proof: Cauchy-Stieltjes integral
  $\frac{\Omega_\mu(z)\phi_n(z)+\psi_n(z)}{B_{n-1}(z)} = \Omega_\mu(z)\frac{\phi_n(z)}{B_{n-1}(z)} + \frac{\psi_n(z)}{B_{n-1}(z)} =$
  $\int D(t,z)\frac{\phi_n(z)}{B_{n-1}(z)}d\mu(t) + \int D(t,z)[\frac{\phi_n(t)}{B_{n-1}(t)} - \frac{\phi_n(z)}{B_{n-1}(z)}]d\mu(t) =$
  $\int D(t,z)\frac{\phi_n(t)}{B_{n-1}(t)}d\mu(t) \in H(\mathbb{D})$
  Since $D(t,0) = 1$, the integral is $\langle \phi_n, B_{n-1}\rangle_\mu = 0$.
- Second formula is similar. QED

Interpolation

- theorem:
  $\Omega_\mu(z)\phi_n(z) + \psi_n(z) = zB_{n-1}(z)h(z)$
  $\Omega_\mu(z)\phi_n^*(z) - \psi_n^*(z) = zB_n(z)g(z)$
  $h, g \in H(\mathbb{D})$
- proof: Cauchy-Stieltjes integral
  $\frac{\Omega_\mu(z)\phi_n(z)+\psi_n(z)}{B_{n-1}(z)} = \Omega_\mu(z)\frac{\phi_n(z)}{B_{n-1}(z)} + \frac{\psi_n(z)}{B_{n-1}(z)} =$
  $\int D(t,z)\frac{\phi_n(z)}{B_{n-1}(z)}d\mu(t) + \int D(t,z)[\frac{\phi_n(t)}{B_{n-1}(t)} - \frac{\phi_n(z)}{B_{n-1}(z)}]d\mu(t) =$
  $\int D(t,z)\frac{\phi_n(t)}{B_{n-1}(t)}d\mu(t) \in H(\mathbb{D})$
  Since $D(t,0) = 1$, the integral is $\langle \phi_n, B_{n-1}\rangle_\mu = 0$.
- Second formula is similar. QED

Inner product in $\mathcal{L}_n$

- theorem: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n}$,
  $d\nu_n(t) = w_n(t) d\lambda(t)$, $w_n(t) = \frac{P(t,\alpha_n)}{|\phi_n(t)|^2}$
- proof: $\int |\phi_n(t)|^2 w_n(t) d\lambda(t) = \int P(t, \alpha_n) d\lambda(t) = 1 = \|\phi_n\|_\mu^2$
- $\langle \phi_n, \phi_k \rangle_{w_n} = \int P(t, \alpha_n) \frac{\phi_{k*}(t)}{\phi_{n*}(t)} d\lambda(t) =$
  $\int P(t, \alpha_n) \frac{\phi_k^*(t) \frac{B_n(t)}{B_k(t)}}{\phi_n^*(t)} d\lambda(t) = 0$ (ratio $\in H(\mathbb{D} \cup \mathbb{T}) \Rightarrow$ Poisson integral and $\frac{B_n}{B_k}$ is zero in α_n for $n > k$)
- Thus ϕ_n is normalized and $\perp_\mu \mathcal{L}_{n-1}$.
  Run backward recurrence to define the previous ϕ_k uniquely.
  QED

Inner product in $\mathcal{L}_n$

- theorem: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n}$,
  $d\nu_n(t) = w_n(t) d\lambda(t)$, $w_n(t) = \frac{P(t,\alpha_n)}{|\phi_n(t)|^2}$
- proof: $\int |\phi_n(t)|^2 w_n(t) d\lambda(t) = \int P(t, \alpha_n) d\lambda(t) = 1 = \|\phi_n\|_\mu^2$
- $\langle \phi_n, \phi_k \rangle_{w_n} = \int P(t, \alpha_n) \frac{\phi_{k*}(t)}{\phi_{n*}(t)} d\lambda(t) =$
  $\int P(t, \alpha_n) \frac{\phi_k^*(t) \frac{B_n(t)}{B_k(t)}}{\phi_n^*(t)} d\lambda(t) = 0$ (ratio $\in H(\mathbb{D} \cup \mathbb{T}) \Rightarrow$ Poisson integral and $\frac{B_n}{B_k}$ is zero in α_n for $n > k$)
- Thus ϕ_n is normalized and $\perp_\mu \mathcal{L}_{n-1}$.
  Run backward recurrence to define the previous ϕ_k uniquely.
  QED

Inner product in $\mathcal{L}_n$

- theorem: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n}$,
  $d\nu_n(t) = w_n(t) d\lambda(t)$, $w_n(t) = \frac{P(t,\alpha_n)}{|\phi_n(t)|^2}$
- proof: $\int |\phi_n(t)|^2 w_n(t) d\lambda(t) = \int P(t, \alpha_n) d\lambda(t) = 1 = \|\phi_n\|_\mu^2$
- $\langle \phi_n, \phi_k \rangle_{w_n} = \int P(t, \alpha_n) \frac{\phi_{k*}(t)}{\phi_{n*}(t)} d\lambda(t) =$
  $\int P(t, \alpha_n) \frac{\phi_k^*(t) \frac{B_n(t)}{B_k(t)}}{\phi_n^*(t)} d\lambda(t) = 0$ (ratio $\in H(\mathbb{D} \cup \mathbb{T}) \Rightarrow$ Poisson integral and $\frac{B_n}{B_k}$ is zero in α_n for $n > k$)
- Thus ϕ_n is normalized and $\perp_\mu \mathcal{L}_{n-1}$.
  Run backward recurrence to define the previous ϕ_k uniquely.
  QED

Inner product in $\mathcal{L}_n$

- theorem: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n}$,
  $d\nu_n(t) = w_n(t) d\lambda(t)$, $w_n(t) = \frac{P(t,\alpha_n)}{|\phi_n(t)|^2}$
- proof: $\int |\phi_n(t)|^2 w_n(t) d\lambda(t) = \int P(t, \alpha_n) d\lambda(t) = 1 = \|\phi_n\|_\mu^2$
- $\langle \phi_n, \phi_k \rangle_{w_n} = \int P(t, \alpha_n) \frac{\phi_{k*}(t)}{\phi_{n*}(t)} d\lambda(t) =$
  $\int P(t, \alpha_n) \frac{\phi_k^*(t) \frac{B_n(t)}{B_k(t)}}{\phi_n^*(t)} d\lambda(t) = 0$ (ratio $\in H(\mathbb{D} \cup \mathbb{T}) \Rightarrow$ Poisson integral and $\frac{B_n}{B_k}$ is zero in α_n for $n > k$)
- Thus ϕ_n is normalized and $\perp_\mu \mathcal{L}_{n-1}$.
  Run backward recurrence to define the previous ϕ_k uniquely.
  QED

Density of the ORF

- theorem: $B_{-n} = 1/B_n$ then $\{B_n\}_{n\in\mathbb{Z}}$ complete in L_p for any $p \geq 1$ and in $C(\mathbb{T})$ iff $\sum(1-|\alpha_k|) = \infty$ iff $\{B_n\}_{n=0}^{\infty}$ is complete in H_p for any $p \geq 1$
- theorem:
  - $\sum(1-|\alpha_k|) = \infty$ then $\log \mu' \notin L_1 \Leftrightarrow \{B_n\}_{n\geq 0}$ complete in $L_2(\mu)$
  - $\log \mu' \in L_1$ then $\sum(1-|\alpha_k|) = \infty \Leftrightarrow \{B_n\}_{n\geq 0}$ complete in $H_2(\mu)$
  - $\sum(1-|\alpha_k|) = \infty$ then $\{B_n\}_{n\in\mathbb{Z}}$ complete in $L_2(\mu)$

Density of the ORF

- theorem: $B_{-n} = 1/B_n$ then $\{B_n\}_{n\in\mathbb{Z}}$ complete in L_p for any $p \geq 1$ and in $C(\mathbb{T})$ iff $\sum(1 - |\alpha_k|) = \infty$ iff $\{B_n\}_{n=0}^{\infty}$ is complete in H_p for any $p \geq 1$
- theorem:
  - $\sum(1 - |\alpha_k|) = \infty$ then $\log \mu' \notin L_1 \Leftrightarrow \{B_n\}_{n\geq 0}$ complete in $L_2(\mu)$
  - $\log \mu' \in L_1$ then $\sum(1 - |\alpha_k|) = \infty \Leftrightarrow \{B_n\}_{n\geq 0}$ complete in $H_2(\mu)$
  - $\sum(1 - |\alpha_k|) = \infty$ then $\{B_n\}_{n\in\mathbb{Z}}$ complete in $L_2(\mu)$

Favard theorem

- theorem: If the ϕ_k are generated by their recurrence relation with $\{\alpha_0 = 0, \alpha_1, \ldots\} \subset \mathbb{D}$, and all $\lambda_n \in \mathbb{D}$, and normalized such that $\phi_n^*(\alpha_n) = \kappa_n > 0$, then these ϕ_n will be a sequence of ORF for some probability measure on $\mathbb{T}$. If B_n diverges then μ is unique.
- proof: superstar ϕ_n^* is ok and the $\lambda_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}}$
- $\phi_n \perp_{w_n} \mathcal{L}_{n-1}$:
  $\langle \phi_n, \phi_m \rangle_{w_n} = \int \frac{\phi_{m*}(t)}{\phi_{n*}(t)} P(t, \alpha_n) d\lambda(t) = \left. \frac{B_n(z)\phi_m^*(z)}{B_m(z)\phi_n^*(z)} \right|_{z=\alpha_n} = \delta_{nm}$
- backward recurrence $\Rightarrow \phi_m$, $m < n$ ok
- $n \to \infty$, w_n uniformly bounded $\Rightarrow w_{n_k} d\lambda$ cvg weakly to $d\mu$
- B_n dvg $\Rightarrow$ complete in $C(\mathbb{T}) \Rightarrow \mu$ is unique (Riesz repr.)

Favard theorem

- theorem: If the ϕ_k are generated by their recurrence relation with $\{\alpha_0 = 0, \alpha_1, \ldots\} \subset \mathbb{D}$, and all $\lambda_n \in \mathbb{D}$, and normalized such that $\phi_n^*(\alpha_n) = \kappa_n > 0$, then these ϕ_n will be a sequence of ORF for some probability measure on $\mathbb{T}$. If B_n diverges then μ is unique.
- proof: superstar ϕ_n^* is ok and the $\lambda_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}}$
- $\phi_n \perp_{w_n} \mathcal{L}_{n-1}$:
$$\langle \phi_n, \phi_m \rangle_{w_n} = \int \frac{\phi_{m*}(t)}{\phi_{n*}(t)} P(t, \alpha_n) d\lambda(t) = \left. \frac{B_n(z)\phi_m^*(z)}{B_m(z)\phi_n^*(z)} \right|_{z=\alpha_n} = \delta_{nm}$$
- backward recurrence $\Rightarrow \phi_m$, $m < n$ ok
- $n \to \infty$, w_n uniformly bounded $\Rightarrow w_{n_k} d\lambda$ cvg weakly to $d\mu$
- B_n dvg $\Rightarrow$ complete in $C(\mathbb{T}) \Rightarrow \mu$ is unique (Riesz repr.)

Favard theorem

- theorem: If the ϕ_k are generated by their recurrence relation with $\{\alpha_0 = 0, \alpha_1, \ldots\} \subset \mathbb{D}$, and all $\lambda_n \in \mathbb{D}$, and normalized such that $\phi_n^*(\alpha_n) = \kappa_n > 0$, then these ϕ_n will be a sequence of ORF for some probability measure on $\mathbb{T}$. If B_n diverges then μ is unique.
- proof: superstar ϕ_n^* is ok and the $\lambda_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}}$
- $\phi_n \perp_{w_n} \mathcal{L}_{n-1}$:
  $\langle \phi_n, \phi_m \rangle_{w_n} = \int \frac{\phi_{m*}(t)}{\phi_{n*}(t)} P(t, \alpha_n) d\lambda(t) = \left. \frac{B_n(z)\phi_m^*(z)}{B_m(z)\phi_n^*(z)} \right|_{z=\alpha_n} = \delta_{nm}$
- backward recurrence $\Rightarrow \phi_m$, $m < n$ ok
- $n \to \infty$, w_n uniformly bounded $\Rightarrow w_{n_k} d\lambda$ cvg weakly to $d\mu$
- B_n dvg $\Rightarrow$ complete in $C(\mathbb{T}) \Rightarrow \mu$ is unique (Riesz repr.)

Favard theorem

- theorem: If the ϕ_k are generated by their recurrence relation with $\{\alpha_0 = 0, \alpha_1, \ldots\} \subset \mathbb{D}$, and all $\lambda_n \in \mathbb{D}$, and normalized such that $\phi_n^*(\alpha_n) = \kappa_n > 0$, then these ϕ_n will be a sequence of ORF for some probability measure on $\mathbb{T}$. If B_n diverges then μ is unique.
- proof: superstar ϕ_n^* is ok and the $\lambda_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}}$
- $\phi_n \perp_{w_n} \mathcal{L}_{n-1}$:
  $\langle \phi_n, \phi_m \rangle_{w_n} = \int \frac{\phi_{m*}(t)}{\phi_{n*}(t)} P(t, \alpha_n) d\lambda(t) = \left. \frac{B_n(z)\phi_m^*(z)}{B_m(z)\phi_n^*(z)} \right|_{z=\alpha_n} = \delta_{nm}$
- backward recurrence $\Rightarrow \phi_m$, $m < n$ ok
- $n \to \infty$, w_n uniformly bounded $\Rightarrow w_{n_k} d\lambda$ cvg weakly to $d\mu$
- B_n dvg $\Rightarrow$ complete in $C(\mathbb{T}) \Rightarrow \mu$ is unique (Riesz repr.)

Favard theorem

- theorem: If the ϕ_k are generated by their recurrence relation with $\{\alpha_0 = 0, \alpha_1, \ldots\} \subset \mathbb{D}$, and all $\lambda_n \in \mathbb{D}$, and normalized such that $\phi_n^*(\alpha_n) = \kappa_n > 0$, then these ϕ_n will be a sequence of ORF for some probability measure on $\mathbb{T}$. If B_n diverges then μ is unique.
- proof: superstar ϕ_n^* is ok and the $\lambda_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}}$
- $\phi_n \perp_{w_n} \mathcal{L}_{n-1}$:
  $\langle \phi_n, \phi_m \rangle_{w_n} = \int \frac{\phi_{m*}(t)}{\phi_{n*}(t)} P(t, \alpha_n) d\lambda(t) = \left. \frac{B_n(z)\phi_m^*(z)}{B_m(z)\phi_n^*(z)} \right|_{z=\alpha_n} = \delta_{nm}$
- backward recurrence $\Rightarrow \phi_m$, $m < n$ ok
- $n \to \infty$, w_n uniformly bounded $\Rightarrow w_{n_k} d\lambda$ cvg weakly to $d\mu$
- B_n dvg $\Rightarrow$ complete in $C(\mathbb{T}) \Rightarrow \mu$ is unique (Riesz repr.)

Favard theorem

- theorem: If the ϕ_k are generated by their recurrence relation with $\{\alpha_0 = 0, \alpha_1, \ldots\} \subset \mathbb{D}$, and all $\lambda_n \in \mathbb{D}$, and normalized such that $\phi_n^*(\alpha_n) = \kappa_n > 0$, then these ϕ_n will be a sequence of ORF for some probability measure on $\mathbb{T}$. If B_n diverges then μ is unique.
- proof: superstar ϕ_n^* is ok and the $\lambda_n = \eta_n^1 \overline{\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}}$
- $\phi_n \perp_{w_n} \mathcal{L}_{n-1}$:
  $\langle \phi_n, \phi_m \rangle_{w_n} = \int \frac{\phi_{m*}(t)}{\phi_{n*}(t)} P(t, \alpha_n) d\lambda(t) = \left. \frac{B_n(z)\phi_m^*(z)}{B_m(z)\phi_n^*(z)} \right|_{z=\alpha_n} = \delta_{nm}$
- backward recurrence $\Rightarrow \phi_m$, $m < n$ ok
- $n \to \infty$, w_n uniformly bounded $\Rightarrow w_{n_k} d\lambda$ cvg weakly to $d\mu$
- B_n dvg $\Rightarrow$ complete in $C(\mathbb{T}) \Rightarrow \mu$ is unique (Riesz repr.)

OPVM = OPS for varying measures

- varying measure: $d\mu(t)/|\pi_n(t)|^2$, $\pi_n(z) = \prod_1^n (1 - \overline{\alpha}_k z)$
- OPVM: $\phi_{nk}(z) = v_{nk} z^k + \cdots, \; v_{nk} > 0, \quad k = 0, 1, \ldots$
- recurrence: $\phi_{nk} = e_{nk}[z\phi_{n,k-1}(z) + \overline{\lambda}_{nk}\phi^*_{n,k-1}(z)]$

  $f^*(z) = z^n f_*(z)$, $f \in \mathcal{P}_n$, $e_{nk} = \frac{1}{\sqrt{1-|\lambda_{nk}|^2}}$, $\lambda_{nk} = \frac{\overline{\phi_{nk}(0)}}{v_{nk}}$
- exercise: $f_{nk} = \frac{\phi_{nk}}{\pi_n}$, $k = 0, \ldots, n$ is orthonormal basis for $\mathcal{L}_n$,

  (GS on $\{\frac{z^k}{\pi_n} : k = 0, \ldots, n\}$)

  $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi^*_{nn}(\alpha_n)}\alpha_n\phi^*_{nn}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$

OPVM = OPS for varying measures

- varying measure: $d\mu(t)/|\pi_n(t)|^2$, $\pi_n(z) = \prod_1^n (1 - \overline{\alpha}_k z)$
- OPVM: $\phi_{nk}(z) = \nu_{nk} z^k + \cdots, \ \nu_{nk} > 0, \quad k = 0, 1, \ldots$
- recurrence: $\phi_{nk} = e_{nk}[z\phi_{n,k-1}(z) + \overline{\lambda}_{nk}\phi^{\bar{*}}_{n,k-1}(z)]$

  $f^{\bar{*}}(z) = z^n f_*(z)$, $f \in \mathcal{P}_n$, $e_{nk} = \frac{1}{\sqrt{1-|\lambda_{nk}|^2}}$, $\lambda_{nk} = \frac{\overline{\phi_{nk}(0)}}{\nu_{nk}}$
- exercise: $f_{nk} = \frac{\phi_{nk}}{\pi_n}$, $k = 0, \ldots, n$ is orthonormal basis for $\mathcal{L}_n$,

  (GS on $\{\frac{z^k}{\pi_n} : k = 0, \ldots, n\}$)

  $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi^{\bar{*}}_{nn}(\alpha_n)}\alpha_n \phi^{\bar{*}}_{nn}(z)}{(z-\alpha_n)\pi_n(z)} = c_n \phi_n(z)$

OPVM = OPS for varying measures

- varying measure: $d\mu(t)/|\pi_n(t)|^2$, $\pi_n(z) = \prod_1^n (1 - \overline{\alpha}_k z)$
- OPVM: $\phi_{nk}(z) = \nu_{nk} z^k + \cdots, \ \nu_{nk} > 0, \quad k = 0, 1, \ldots$
- recurrence: $\phi_{nk} = e_{nk}[z\phi_{n,k-1}(z) + \overline{\lambda}_{nk}\phi^{\bar{*}}_{n,k-1}(z)]$

  $f^{\bar{*}}(z) = z^n f_*(z)$, $f \in \mathcal{P}_n$, $e_{nk} = \frac{1}{\sqrt{1-|\lambda_{nk}|^2}}$, $\lambda_{nk} = \frac{\overline{\phi_{nk}(0)}}{\nu_{nk}}$
- exercise: $f_{nk} = \frac{\phi_{nk}}{\pi_n}$, $k = 0, \ldots, n$ is orthonormal basis for $\mathcal{L}_n$,

  (GS on $\{\frac{z^k}{\pi_n} : k = 0, \ldots, n\}$)

  $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi^{\bar{*}}_{nn}(\alpha_n)}\alpha_n\phi^{\bar{*}}_{nn}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$

OPVM = OPS for varying measures

- varying measure: $d\mu(t)/|\pi_n(t)|^2$, $\pi_n(z) = \prod_1^n (1 - \overline{\alpha}_k z)$
- OPVM: $\phi_{nk}(z) = \nu_{nk} z^k + \cdots, \ \nu_{nk} > 0, \quad k = 0, 1, \ldots$
- recurrence: $\phi_{nk} = e_{nk}[z\phi_{n,k-1}(z) + \overline{\lambda}_{nk}\phi^{\bar{*}}_{n,k-1}(z)]$

  $f^{\bar{*}}(z) = z^n f_*(z)$, $f \in \mathcal{P}_n$, $e_{nk} = \frac{1}{\sqrt{1-|\lambda_{nk}|^2}}$, $\lambda_{nk} = \frac{\overline{\phi_{nk}(0)}}{\nu_{nk}}$
- exercise: $f_{nk} = \frac{\phi_{nk}}{\pi_n}$, $k = 0, \ldots, n$ is orthonormal basis for $\mathcal{L}_n$,

  (GS on $\{\frac{z^k}{\pi_n} : k = 0, \ldots, n\}$)

  $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi^{\bar{*}}_{nn}(\alpha_n)}\alpha_n\phi^{\bar{*}}_{nn}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$

Conditions

- Szegő Condition: $\mu \in SZ \Leftrightarrow \log \mu' \in L_1$
- Erdős-Turán: $\mu \in ET \Leftrightarrow \mu' > 0 a.e.(\lambda)$
- Blaschke Divergence: $A = \{\alpha_k\} \in BD \Leftrightarrow \sum(1 - |\alpha_k|) = \infty$
- Compactly Contained: $A \in CC \Leftrightarrow |\alpha_k| \leq r < 1, k = 0, 1, \ldots$
- note:
  $\mu \in SZ \Rightarrow \mu \in ET$
  $A \in CC \Rightarrow A \in BD$

Conditions

- Szegő Condition: $\mu \in SZ \Leftrightarrow \log \mu' \in L_1$
- Erdős-Turán: $\mu \in ET \Leftrightarrow \mu' > 0 a.e.(\lambda)$
- Blaschke Divergence: $A = \{\alpha_k\} \in BD \Leftrightarrow \sum(1 - |\alpha_k|) = \infty$
- Compactly Contained: $A \in CC \Leftrightarrow |\alpha_k| \leq r < 1, k = 0, 1, \ldots$
- note:
  $\mu \in SZ \Rightarrow \mu \in ET$
  $A \in CC \Rightarrow A \in BD$

Conditions

- Szegő Condition: $\mu \in SZ \Leftrightarrow \log \mu' \in L_1$
- Erdős-Turán: $\mu \in ET \Leftrightarrow \mu' > 0 a.e.(\lambda)$
- Blaschke Divergence: $A = \{\alpha_k\} \in BD \Leftrightarrow \sum(1 - |\alpha_k|) = \infty$
- Compactly Contained: $A \in CC \Leftrightarrow |\alpha_k| \leq r < 1, k = 0, 1, \ldots$
- note:
  $\mu \in SZ \Rightarrow \mu \in ET$
  $A \in CC \Rightarrow A \in BD$

Conditions

- Szegő Condition: $\mu \in SZ \Leftrightarrow \log \mu' \in L_1$
- Erdős-Turán: $\mu \in ET \Leftrightarrow \mu' > 0 a.e.(\lambda)$
- Blaschke Divergence: $A = \{\alpha_k\} \in BD \Leftrightarrow \sum(1 - |\alpha_k|) = \infty$
- Compactly Contained: $A \in CC \Leftrightarrow |\alpha_k| \leq r < 1, k = 0, 1, \ldots$
- note:
  $\mu \in SZ \Rightarrow \mu \in ET$
  $A \in CC \Rightarrow A \in BD$

Conditions

- Szegő Condition: $\mu \in SZ \Leftrightarrow \log \mu' \in L_1$
- Erdős-Turán: $\mu \in ET \Leftrightarrow \mu' > 0 a.e.(\lambda)$
- Blaschke Divergence: $A = \{\alpha_k\} \in BD \Leftrightarrow \sum(1 - |\alpha_k|) = \infty$
- Compactly Contained: $A \in CC \Leftrightarrow |\alpha_k| \leq r < 1, k = 0, 1, \ldots$
- note:
  $\mu \in SZ \Rightarrow \mu \in ET$
  $A \in CC \Rightarrow A \in BD$

Theorem (Lopez-Lagomasino)

Theorem

If $\mu \in ET$ & $A \in BD$ then

- $\lim_{n\to\infty} \lambda_{n,n+k+1} = 0$
- $\lim_{n\to\infty} \frac{v_{n,n+k+1}}{v_{n,n+k}} = 1$
- $\lim_{n\to\infty} \frac{\phi_{n,n+k+1}(z)}{\phi_{n,n+k}(z)} = z$ *l.u.* $\mathbb{E}$
- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k+1}(z)}{\phi^*_{n,n+k}(z)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k}(z)}{\phi_{n,n+k}(z)} = 0$ *l.u.* $\mathbb{E}$

If $\mu \in SC$, $A \in CC$, σ is outer spectral factor of μ then

- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k}(z)}{\pi_n(z)} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$

Theorem (Lopez-Lagomasino)

Theorem

If $\mu \in ET$ & $A \in BD$ then

- $\lim_{n\to\infty} \lambda_{n,n+k+1} = 0$
- $\lim_{n\to\infty} \frac{v_{n,n+k+1}}{v_{n,n+k}} = 1$
- $\lim_{n\to\infty} \frac{\phi_{n,n+k+1}(z)}{\phi_{n,n+k}(z)} = z$ *l.u.* $\mathbb{E}$
- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k+1}(z)}{\phi^*_{n,n+k}(z)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k}(z)}{\phi_{n,n+k}(z)} = 0$ *l.u.* $\mathbb{E}$

If $\mu \in SC$, $A \in CC$, σ is outer spectral factor of μ then

- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k}(z)}{\pi_n(z)} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$

Theorem (Lopez-Lagomasino)

Theorem

If $\mu \in ET$ & $A \in BD$ then

- $\lim_{n\to\infty} \lambda_{n,n+k+1} = 0$
- $\lim_{n\to\infty} \frac{v_{n,n+k+1}}{v_{n,n+k}} = 1$
- $\lim_{n\to\infty} \frac{\phi_{n,n+k+1}(z)}{\phi_{n,n+k}(z)} = z$ *l.u.* $\mathbb{E}$
- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k+1}(z)}{\phi^*_{n,n+k}(z)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k}(z)}{\phi_{n,n+k}(z)} = 0$ *l.u.* $\mathbb{E}$

If $\mu \in SC$, $A \in CC$, σ is outer spectral factor of μ then

- $\lim_{n\to\infty} \frac{\phi^*_{n,n+k}(z)}{\pi_n(z)} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$

Theorem (Lopez-Lagomasino)

Theorem

If $\mu \in ET$ & $A \in BD$ then

- $\lim_{n\to\infty} \lambda_{n,n+k+1} = 0$
- $\lim_{n\to\infty} \frac{v_{n,n+k+1}}{v_{n,n+k}} = 1$
- $\lim_{n\to\infty} \frac{\phi_{n,n+k+1}(z)}{\phi_{n,n+k}(z)} = z$ *l.u.* $\mathbb{E}$
- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k+1}(z)}{\phi^{\bar{*}}_{n,n+k}(z)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k}(z)}{\phi_{n,n+k}(z)} = 0$ *l.u.* $\mathbb{E}$

If $\mu \in SC$, $A \in CC$, σ is outer spectral factor of μ then

- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k}(z)}{\pi_n(z)} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$

Theorem (Lopez-Lagomasino)

Theorem

If $\mu \in ET$ & $A \in BD$ then

- $\lim_{n\to\infty} \lambda_{n,n+k+1} = 0$
- $\lim_{n\to\infty} \frac{v_{n,n+k+1}}{v_{n,n+k}} = 1$
- $\lim_{n\to\infty} \frac{\phi_{n,n+k+1}(z)}{\phi_{n,n+k}(z)} = z$ *l.u.* $\mathbb{E}$
- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k+1}(z)}{\phi^{\bar{*}}_{n,n+k}(z)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k}(z)}{\phi_{n,n+k}(z)} = 0$ *l.u.* $\mathbb{E}$

If $\mu \in SC$, $A \in CC$, σ is outer spectral factor of μ then

- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k}(z)}{\pi_n(z)} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$

Theorem (Lopez-Lagomasino)

Theorem

If $\mu \in ET$ *&* $A \in BD$ *then*

- $\lim_{n\to\infty} \lambda_{n,n+k+1} = 0$
- $\lim_{n\to\infty} \frac{v_{n,n+k+1}}{v_{n,n+k}} = 1$
- $\lim_{n\to\infty} \frac{\phi_{n,n+k+1}(z)}{\phi_{n,n+k}(z)} = z$ *l.u.* $\mathbb{E}$
- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k+1}(z)}{\phi^{\bar{*}}_{n,n+k}(z)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k}(z)}{\phi_{n,n+k}(z)} = 0$ *l.u.* $\mathbb{E}$

If $\mu \in SC$, $A \in CC$, σ *is outer spectral factor of* μ *then*

- $\lim_{n\to\infty} \frac{\phi^{\bar{*}}_{n,n+k}(z)}{\pi_n(z)} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$

Convergence theorem

Theorem

If $\mu \in SZ$ *&* $A \in BD$ *then*

- $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$, *l.u.* $\mathbb{D}\times\mathbb{D}$
- $\lim_{n\to\infty} (1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$, *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$, *l.u.* $\mathbb{D}$

If $\mu \in SZ$ $\sigma(0) > 0$, $A \in CC$ *then*

- $\lim_{n\to\infty} \eta_n \phi_n^*(z)\frac{1-\overline{\alpha}_n z}{\sqrt{1-|\alpha_n|^2}} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$, $\eta_n = \frac{|\phi_n^*(0)|}{\phi_n^*(0)} \in \mathbb{T}$

Convergence theorem

Theorem

If $\mu \in SZ$ *&* $A \in BD$ *then*

- $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$, *l.u.* $\mathbb{D}\times\mathbb{D}$
- $\lim_{n\to\infty} (1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$, *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$, *l.u.* $\mathbb{D}$

If $\mu \in SZ$ $\sigma(0) > 0$, $A \in CC$ *then*

- $\lim_{n\to\infty} \eta_n \phi_n^*(z)\frac{1-\overline{\alpha}_n z}{\sqrt{1-|\alpha_n|^2}} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$, $\eta_n = \frac{|\phi_n^*(0)|}{\phi_n^*(0)} \in \mathbb{T}$

Convergence theorem

Theorem

If $\mu \in SZ$ *&* $A \in BD$ *then*

- $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$, *l.u.* $\mathbb{D}\times\mathbb{D}$
- $\lim_{n\to\infty} (1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$, *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$, *l.u.* $\mathbb{D}$

If $\mu \in SZ$ $\sigma(0) > 0$, $A \in CC$ *then*

- $\lim_{n\to\infty} \eta_n \phi_n^*(z)\frac{1-\overline{\alpha}_n z}{\sqrt{1-|\alpha_n|^2}} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$, $\eta_n = \frac{|\phi_n^*(0)|}{\phi_n^*(0)} \in \mathbb{T}$

Convergence theorem

Theorem

If $\mu \in SZ$ *&* $A \in BD$ *then*

- $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$, *l.u.* $\mathbb{D}\times\mathbb{D}$
- $\lim_{n\to\infty} (1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$, *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$, *l.u.* $\mathbb{D}$

If $\mu \in SZ$ $\sigma(0) > 0$, $A \in CC$ *then*

- $\lim_{n\to\infty} \eta_n \phi_n^*(z)\frac{1-\overline{\alpha}_n z}{\sqrt{1-|\alpha_n|^2}} = \frac{1}{\sigma(z)}$ *l.u.* $\mathbb{D}$, $\eta_n = \frac{|\phi_n^*(0)|}{\phi_n^*(0)} \in \mathbb{T}$

Proof $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$

Skip proof

- $k_n(z,w) = \sum_{k=0}^{n} f_{nk}(z)\overline{f_{nk}(w)} = \frac{\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)}}{\pi_n(z)\overline{\pi_n(w)}}$
- $\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)} = \frac{\phi^*_{nn}(z)\overline{\phi^*_{nn}(w)}-\phi_{nn}(z)\overline{\phi_{nn}(w)}}{1-\overline{w}z}$
- $\frac{k_n(z,w)}{f^*_{nn}(z)\overline{f^*_{nn}(w)}} = \frac{1}{1-\overline{w}z} - \frac{z\overline{w}}{1-\overline{w}z}\underbrace{\left(\frac{\phi_{nn}(z)}{\phi^*_{nn}(z)}\right)}_{\to 0}\underbrace{\overline{\left(\frac{\phi_{nn}(w)}{\phi^*_{nn}(w)}\right)}}_{\to 0}$
- $\lim_{n\to\infty}\eta_n f^*_{nn}(z) = \frac{1}{\sigma(z)}$
- QED

Proof $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$

- $k_n(z,w) = \sum_{k=0}^{n} f_{nk}(z)\overline{f_{nk}(w)} = \frac{\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)}}{\pi_n(z)\overline{\pi_n(w)}}$
- $\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)} = \frac{\phi^*_{nn}(z)\overline{\phi^*_{nn}(w)}-\phi_{nn}(z)\overline{\phi_{nn}(w)}}{1-\overline{w}z}$
- $\frac{k_n(z,w)}{f^*_{nn}(z)\overline{f^*_{nn}(w)}} = \frac{1}{1-\overline{w}z} - \frac{z\overline{w}}{1-\overline{w}z}\underbrace{\left(\frac{\phi_{nn}(z)}{\phi^*_{nn}(z)}\right)}_{\to 0}\overline{\underbrace{\left(\frac{\phi_{nn}(w)}{\phi^*_{nn}(w)}\right)}_{\to 0}}$
- $\lim_{n\to\infty}\eta_n f^*_{nn}(z) = \frac{1}{\sigma(z)}$
- QED

Proof $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$

- $k_n(z,w) = \sum_{k=0}^{n} f_{nk}(z)\overline{f_{nk}(w)} = \frac{\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)}}{\pi_n(z)\overline{\pi_n(w)}}$
- $\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)} = \frac{\phi^*_{nn}(z)\overline{\phi^*_{nn}(w)} - \phi_{nn}(z)\overline{\phi_{nn}(w)}}{1-\overline{w}z}$
- $\frac{k_n(z,w)}{f^*_{nn}(z)\overline{f^*_{nn}(w)}} = \frac{1}{1-\overline{w}z} - \frac{z\overline{w}}{1-\overline{w}z}\underbrace{\left(\frac{\phi_{nn}(z)}{\phi^*_{nn}(z)}\right)}_{\to 0}\overline{\underbrace{\left(\frac{\phi_{nn}(w)}{\phi^*_{nn}(w)}\right)}_{\to 0}}$
- $\lim_{n\to\infty}\eta_n f^*_{nn}(z) = \frac{1}{\sigma(z)}$
- QED

Proof $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$

- $k_n(z,w) = \sum_{k=0}^{n} f_{nk}(z)\overline{f_{nk}(w)} = \frac{\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)}}{\pi_n(z)\overline{\pi_n(w)}}$
- $\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)} = \frac{\phi^*_{nn}(z)\overline{\phi^*_{nn}(w)} - \phi_{nn}(z)\overline{\phi_{nn}(w)}}{1-\overline{w}z}$
- $\frac{k_n(z,w)}{f^*_{nn}(z)\overline{f^*_{nn}(w)}} = \frac{1}{1-\overline{w}z} - \frac{z\overline{w}}{1-\overline{w}z}\underbrace{\left(\frac{\phi_{nn}(z)}{\phi^*_{nn}(z)}\right)}_{\to 0}\overline{\underbrace{\left(\frac{\phi_{nn}(w)}{\phi^*_{nn}(w)}\right)}_{\to 0}}$
- $\lim_{n\to\infty}\eta_n f^*_{nn}(z) = \frac{1}{\sigma(z)}$
- QED

Proof $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$

- $k_n(z,w) = \sum_{k=0}^{n} f_{nk}(z)\overline{f_{nk}(w)} = \frac{\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)}}{\pi_n(z)\overline{\pi_n(w)}}$
- $\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)} = \frac{\phi^*_{nn}(z)\overline{\phi^*_{nn}(w)} - \phi_{nn}(z)\overline{\phi_{nn}(w)}}{1-\overline{w}z}$
- $\frac{k_n(z,w)}{f^*_{nn}(z)\overline{f^*_{nn}(w)}} = \frac{1}{1-\overline{w}z} - \frac{z\overline{w}}{1-\overline{w}z}\underbrace{\left(\frac{\phi_{nn}(z)}{\phi^*_{nn}(z)}\right)}_{\to 0}\overline{\underbrace{\left(\frac{\phi_{nn}(w)}{\phi^*_{nn}(w)}\right)}_{\to 0}}$
- $\lim_{n\to\infty}\eta_n f^*_{nn}(z) = \frac{1}{\sigma(z)}$
- QED

Proof $\lim_{n\to\infty} k_n(z,w) = \frac{1}{\overline{\sigma(w)}(1-\overline{w}z)\sigma(z)}$

- $k_n(z,w) = \sum_{k=0}^{n} f_{nk}(z)\overline{f_{nk}(w)} = \frac{\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)}}{\pi_n(z)\overline{\pi_n(w)}}$
- $\sum_{k=0}^{n}\phi_{nk}(z)\overline{\phi_{nk}(w)} = \frac{\phi^*_{nn}(z)\overline{\phi^*_{nn}(w)} - \phi_{nn}(z)\overline{\phi_{nn}(w)}}{1-\overline{w}z}$
- $\frac{k_n(z,w)}{f^*_{nn}(z)\overline{f^*_{nn}(w)}} = \frac{1}{1-\overline{w}z} - \frac{z\overline{w}}{1-\overline{w}z}\underbrace{\left(\frac{\phi_{nn}(z)}{\phi^*_{nn}(z)}\right)}_{\to 0}\overline{\underbrace{\left(\frac{\phi_{nn}(w)}{\phi^*_{nn}(w)}\right)}_{\to 0}}$
- $\lim_{n\to\infty}\eta_n f^*_{nn}(z) = \frac{1}{\sigma(z)}$
- QED

Proof $\lim_{n\to\infty}(1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$

Skip proof

- $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^*(\alpha_n)}\alpha_n\phi_{nn}^*(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$
- $(1-\overline{\alpha}_n z)\phi_n^*(z) = \frac{1}{\overline{c}_n}\underbrace{\frac{\phi_{nn}^*(z)}{\pi_n(z)}}_{\to c/\sigma(z)}\left[1-\overline{\eta}_n\overline{\alpha}_n z\underbrace{\frac{\phi_{nn}(z)}{\phi_{nn}^*(z)}}_{\to 0}\right]$
- for $z=0$ same expression and divide.
- QED

Proof $\lim_{n\to\infty}(1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$

- $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\bar{\phi}^*_{nn}(\alpha_n)}\alpha_n\bar{\phi}^*_{nn}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$
- $(1-\overline{\alpha}_n z)\phi_n^*(z) = \frac{1}{\overline{c}_n}\underbrace{\frac{\bar{\phi}^*_{nn}(z)}{\pi_n(z)}}_{\to c/\sigma(z)}\left[1-\overline{\eta}_n\overline{\alpha}_n z\underbrace{\frac{\phi_{nn}(z)}{\bar{\phi}^*_{nn}(z)}}_{\to 0}\right]$
- for $z=0$ same expression and divide.
- QED

Proof $\lim_{n\to\infty}(1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$

- $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}\alpha_n\phi_{nn}^{\bar{*}}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$
- $(1-\overline{\alpha}_n z)\phi_n^*(z) = \frac{1}{\overline{c}_n}\underbrace{\frac{\phi_{nn}^{\bar{*}}(z)}{\pi_n(z)}}_{\to c/\sigma(z)}\left[1-\overline{\eta}_n\overline{\alpha}_n z\underbrace{\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}_{\to 0}\right]$
- for $z=0$ same expression and divide.
- QED

Proof $\lim_{n\to\infty}(1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$

- $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}\alpha_n\phi_{nn}^{\bar{*}}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$
- $(1-\overline{\alpha}_n z)\phi_n^*(z) = \frac{1}{\overline{c}_n}\underbrace{\frac{\phi_{nn}^{\bar{*}}(z)}{\pi_n(z)}}_{\to c/\sigma(z)}\left[1-\overline{\eta}_n\overline{\alpha}_n z\underbrace{\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}_{\to 0}\right]$
- for $z=0$ same expression and divide.
- QED

Proof $\lim_{n\to\infty}(1-\overline{\alpha}_n z)\frac{\phi_n^*(z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)}$

- $h_n(z) = \frac{z\phi_{nn}(z) - \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}\alpha_n\phi_{nn}^{\bar{*}}(z)}{(z-\alpha_n)\pi_n(z)} = c_n\phi_n(z)$
- $(1-\overline{\alpha}_n z)\phi_n^*(z) = \frac{1}{\overline{c}_n}\underbrace{\frac{\phi_{nn}^{\bar{*}}(z)}{\pi_n(z)}}_{\to c/\sigma(z)}\left[1-\overline{\eta}_n\overline{\alpha}_n z\underbrace{\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}_{\to 0}\right]$
- for $z=0$ same expression and divide.
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

Skip proof

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Proof $\lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0$

- $\lim_{n\to\infty} k_n(z,z) = \sum_{k=0}^{\infty} |\phi_n(z)|^2$ converges $\Rightarrow$ (1)
- $\lim_{n\to\infty} \frac{\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)} = \frac{\sigma(z)}{\sigma(0)}$. and $|1-\overline{\alpha}_n z| > 0$ l.u. $\mathbb{D}$
- Thus $\frac{\phi_n^*(0)}{\phi_n^*(z)}$ is uniformly bounded. $\Rightarrow$ (2)
- CD: $\frac{|\phi_n^*(0)|^2}{1-|\alpha_n|^2} = \underbrace{k_{n-1}(0,0)}_{\to 1/|\sigma(0)|^2} + \underbrace{\frac{|\phi_n(0)|^2}{1-|\alpha_n|^2}}_{\to 0}$
- $\therefore \frac{|\phi_n^*(0)|}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{\sigma(0)}$
- QED

Ratio asymptotics

- If $\mu \notin SZ$ then $\sigma(z)$ may not exist.
- The $\sigma(z)$ will drop from the previous relations if we take ratios.

♠

exercise: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n} \ \forall f, g \in \mathcal{L}_n$ when

$d\nu_n(t) = \frac{v_n^2}{|k_n(t,0)|^2} d\lambda(t)$

Ratio asymptotics

- If $\mu \notin SZ$ then $\sigma(z)$ may not exist.
- The $\sigma(z)$ will drop from the previous relations if we take ratios.

♠

exercise: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n} \ \forall f, g \in \mathcal{L}_n$ when

$d\nu_n(t) = \frac{v_n^2}{|k_n(t,0)|^2} d\lambda(t)$

Ratio asymptotics

- If $\mu \notin SZ$ then $\sigma(z)$ may not exist.
- The $\sigma(z)$ will drop from the previous relations if we take ratios.

♠

exercise: $\langle f, g \rangle_\mu = \langle f, g \rangle_{\nu_n} \ \forall f, g \in \mathcal{L}_n$ when

$d\nu_n(t) = \frac{v_n^2}{|k_n(t,0)|^2} d\lambda(t)$

The kernels

Theorem

If $\mu \in ET$ *and* $A \in BD$ *then*

- $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_n(z,0)} = 0$ *l.u.* $\mathbb{D}$

The kernels

Theorem

If $\mu \in ET$ *and* $A \in BD$ *then*

- $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$ *l.u.* $\mathbb{D}$
- $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_n(z,0)} = 0$ *l.u.* $\mathbb{D}$

proof $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$

▸ Skip proof

- $\langle f,g\rangle_\mu = \langle f,g\rangle_{\frac{v_n^2}{|k_n(t,0)|^2}}$, $g_n(t) = \frac{k_n(t,0)}{k_{n+1}(t,0)} - \frac{v_n}{v_{n+1}} \Rightarrow$
  $\int |g_n(t)|^2 d\lambda(t) = 2\frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Poisson:
  $\int P(t,z) g_n(t) d\lambda(t) = g_n(z) \Rightarrow |g_n(z)|^2 \leq M^2 \int |g_n(t)|^2 d\lambda$
- $|g_n(z)|^2 \leq 2M^2 \frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Need to prove that $v_n/v_{n+1} \to 1$

proof $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$

- $\langle f,g\rangle_\mu = \langle f,g\rangle_{\frac{v_n^2}{|k_n(t,0)|^2}}$, $g_n(t) = \frac{k_n(t,0)}{k_{n+1}(t,0)} - \frac{v_n}{v_{n+1}} \Rightarrow$
  $\int |g_n(t)|^2 d\lambda(t) = 2\frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Poisson:
  $\int P(t,z) g_n(t) d\lambda(t) = g_n(z) \Rightarrow |g_n(z)|^2 \leq M^2 \int |g_n(t)|^2 d\lambda$
- $|g_n(z)|^2 \leq 2M^2 \frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Need to prove that $v_n/v_{n+1} \to 1$

proof $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$

- $\langle f,g\rangle_\mu = \langle f,g\rangle_{\frac{v_n^2}{|k_n(t,0)|^2}}$, $g_n(t) = \frac{k_n(t,0)}{k_{n+1}(t,0)} - \frac{v_n}{v_{n+1}} \Rightarrow$
  $\int |g_n(t)|^2 d\lambda(t) = 2\frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Poisson:
  $\int P(t,z) g_n(t) d\lambda(t) = g_n(z) \Rightarrow |g_n(z)|^2 \leq M^2 \int |g_n(t)|^2 d\lambda$
- $|g_n(z)|^2 \leq 2M^2 \frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Need to prove that $v_n/v_{n+1} \to 1$

proof $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$

- $\langle f,g\rangle_\mu = \langle f,g\rangle_{\frac{v_n^2}{|k_n(t,0)|^2}}$, $g_n(t) = \frac{k_n(t,0)}{k_{n+1}(t,0)} - \frac{v_n}{v_{n+1}} \Rightarrow$
  $\int |g_n(t)|^2 d\lambda(t) = 2\frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Poisson:
  $\int P(t,z) g_n(t) d\lambda(t) = g_n(z) \Rightarrow |g_n(z)|^2 \leq M^2 \int |g_n(t)|^2 d\lambda$
- $|g_n(z)|^2 \leq 2M^2 \frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Need to prove that $v_n/v_{n+1} \to 1$

proof $\lim_{n\to\infty} \frac{k_n(z,0)}{k_{n+1}(z,0)} = 1$

- $\langle f,g\rangle_\mu = \langle f,g\rangle_{\frac{v_n^2}{|k_n(t,0)|^2}}$, $g_n(t) = \frac{k_n(t,0)}{k_{n+1}(t,0)} - \frac{v_n}{v_{n+1}} \Rightarrow$
  $\int |g_n(t)|^2 d\lambda(t) = 2\frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Poisson:
  $\int P(t,z) g_n(t) d\lambda(t) = g_n(z) \Rightarrow |g_n(z)|^2 \leq M^2 \int |g_n(t)|^2 d\lambda$
- $|g_n(z)|^2 \leq 2M^2 \frac{v_n^2}{v_{n+1}^2}(1 - \frac{v_n}{v_{n+1}})$
- Need to prove that $v_n/v_{n+1} \to 1$

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_{n+1}(z,0)} = 1$

- recall $k_n(0,0) = v_n^2$
- $\frac{k_{n-1}(0,0)}{k_n(0,0)} \leq 1$
- CD: $\frac{k_{n-1}(0,0)}{k_n(0,0)} = \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\alpha_n|^2|\phi_n(0)|^2} \geq \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\phi_n(0)|^2} = 1$
- QED

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_{n+1}(z,0)} = 1$

- recall $k_n(0,0) = v_n^2$
- $\frac{k_{n-1}(0,0)}{k_n(0,0)} \leq 1$
- CD: $\frac{k_{n-1}(0,0)}{k_n(0,0)} = \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\alpha_n|^2|\phi_n(0)|^2} \geq \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\phi_n(0)|^2} = 1$
- QED

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_{n+1}(z,0)} = 1$

- recall $k_n(0,0) = v_n^2$
- $\frac{k_{n-1}(0,0)}{k_n(0,0)} \leq 1$
- CD: $\frac{k_{n-1}(0,0)}{k_n(0,0)} = \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\alpha_n|^2|\phi_n(0)|^2} \geq \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\phi_n(0)|^2} = 1$
- QED

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_{n+1}(z,0)} = 1$

- recall $k_n(0,0) = v_n^2$
- $\frac{k_{n-1}(0,0)}{k_n(0,0)} \leq 1$
- CD: $\frac{k_{n-1}(0,0)}{k_n(0,0)} = \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\alpha_n|^2|\phi_n(0)|^2} \geq \frac{|\phi_n^*(0)|^2 - |\phi_n(0)|^2}{|\phi_n^*(0)|^2 - |\phi_n(0)|^2} = 1$
- QED

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_n(z,0)} = 0$

- $k_n(z,0) = \frac{\eta_n \nu_n \bar{\phi}_{nn}^*(z)}{\pi_n(z)} \Rightarrow k_n^*(z,0) = \frac{\overline{\eta}_n \nu_n \phi_{nn}(z)}{\pi_n(z)}$
- $\frac{k_n^*(z,0)}{k_n(z,0)} = \frac{\overline{\eta}_n}{\eta_n} \underbrace{\frac{\bar{\phi}_{nn}^*(z)}{\phi_{nn}(z)}}_{\to 0}$
- QED

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_n(z,0)} = 0$

- $k_n(z,0) = \frac{\eta_n \nu_n \bar{\phi}_{nn}^*(z)}{\pi_n(z)} \Rightarrow k_n^*(z,0) = \frac{\overline{\eta}_n \nu_n \phi_{nn}(z)}{\pi_n(z)}$
- $\frac{k_n^*(z,0)}{k_n(z,0)} = \frac{\overline{\eta}_n}{\eta_n} \underbrace{\frac{\bar{\phi}_{nn}^*(z)}{\phi_{nn}(z)}}_{\to 0}$
- QED

proof $\lim_{n\to\infty} \frac{k_n^*(z,0)}{k_n(z,0)} = 0$

- $k_n(z,0) = \frac{\eta_n \nu_n \phi_{nn}^{\bar{*}}(z)}{\pi_n(z)} \Rightarrow k_n^*(z,0) = \frac{\overline{\eta}_n \nu_n \phi_{nn}(z)}{\pi_n(z)}$
- $\frac{k_n^*(z,0)}{k_n(z,0)} = \frac{\overline{\eta}_n}{\eta_n} \underbrace{\frac{\phi_{nn}^{\bar{*}}(z)}{\phi_{nn}(z)}}_{\to 0}$
- QED

The ϕ_n

Theorem

If $\mu \in ET$ *and* $A \in CC$ *then*

- $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$ *l.u.* $\mathbb{D}$

If $\Phi_n^*(z) = \frac{\phi_n^*(z)}{\phi_n^*(0)}$ *then*

- $\lim_{n\to\infty} \frac{\Phi_{n+1}^*(z)(1-\overline{\alpha}_{n+1}z)}{\Phi_n^*(z)(1-\overline{\alpha}_n z)} = 1$ *l.u.* $\mathbb{D}$

The ϕ_n

Theorem

If $\mu \in ET$ *and* $A \in CC$ *then*

- $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$ *l.u.* $\mathbb{D}$

If $\Phi_n^*(z) = \frac{\phi_n^*(z)}{\phi_n^*(0)}$ *then*

- $\lim_{n\to\infty} \frac{\Phi_{n+1}^*(z)(1-\overline{\alpha}_{n+1}z)}{\Phi_n^*(z)(1-\overline{\alpha}_n z)} = 1$ *l.u.* $\mathbb{D}$

Proof $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$

Skip proof

- $\phi_n(z) = c_n \frac{z\phi_{nn}(z) - \tau_n\alpha_n\bar{\phi}^*_{nn}(a)}{(z-\alpha_n)\pi_n(z)}$, $\tau_n = \frac{\phi_{nn}(\alpha_n)}{\bar{\phi}^*_{nn}(\alpha_n)}$
- $\phi_n^*(z) = \overline{c}_n \frac{\bar{\phi}^*_{nn}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)}{(1-\overline{\alpha}_n z)\pi_n(z)}$
- $\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\phi_{nn}(z) - \tau_n\alpha_n\bar{\phi}^*_{nn}(z)}{\bar{\phi}^*_{nn}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)} \frac{1}{\zeta_n(z)}$
- $\frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\frac{\phi_{nn}(z)}{\bar{\phi}^*_{nn}(z)} - \tau_n\alpha_n}{1-\overline{\tau}_n\overline{\alpha}_n z\frac{\phi_{nn}(z)}{\bar{\phi}^*_{nn}(z)}}$, $\frac{\phi_{nn}(z)}{\bar{\phi}^*_{nn}(z)} \to 0$ and $\tau_n \to 0$ $(A \in CC)$
- $0 < m \leq |zeta_n(z)| \leq M < \infty \Rightarrow (2)$, $|\lambda_n| = |\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}| \to 0$
- QED

Proof $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$

- $\phi_n(z) = c_n \frac{z\phi_{nn}(z) - \tau_n\alpha_n\bar{\phi}_{nn}^*(a)}{(z-\alpha_n)\pi_n(z)}$, $\tau_n = \frac{\phi_{nn}(\alpha_n)}{\bar{\phi}_{nn}^*(\alpha_n)}$
- $\phi_n^*(z) = \overline{c}_n \frac{\bar{\phi}_{nn}^*(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)}{(1-\overline{\alpha}_n z)\pi_n(z)}$
- $\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\phi_{nn}(z) - \tau_n\alpha_n\bar{\phi}_{nn}^*(z)}{\bar{\phi}_{nn}^*(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)} \frac{1}{\zeta_n(z)}$
- $\frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\frac{\phi_{nn}(z)}{\bar{\phi}_{nn}^*(z)} - \tau_n\alpha_n}{1-\overline{\tau}_n\overline{\alpha}_n z\frac{\phi_{nn}(z)}{\bar{\phi}_{nn}^*(z)}}$, $\frac{\phi_{nn}(z)}{\bar{\phi}_{nn}^*(z)} \to 0$ and $\tau_n \to 0$ $(A \in CC)$
- $0 < m \leq |zeta_n(z)| \leq M < \infty \Rightarrow (2)$, $|\lambda_n| = |\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}| \to 0$
- QED

Proof $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$

- $\phi_n(z) = c_n \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(a)}{(z-\alpha_n)\pi_n(z)}$, $\tau_n = \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}$
- $\phi_n^*(z) = \overline{c}_n \frac{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)}{(1-\overline{\alpha}_n z)\pi_n(z)}$
- $\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(z)}{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)} \frac{1}{\zeta_n(z)}$
- $\frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} - \tau_n\alpha_n}{1-\overline{\tau}_n\overline{\alpha}_n z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}$, $\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} \to 0$ and $\tau_n \to 0$ $(A \in CC)$
- $0 < m \leq |zeta_n(z)| \leq M < \infty \Rightarrow (2)$, $|\lambda_n| = |\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}| \to 0$
- QED

Proof $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$

- $\phi_n(z) = c_n \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(a)}{(z-\alpha_n)\pi_n(z)}$, $\tau_n = \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}$
- $\phi_n^*(z) = \overline{c}_n \frac{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)}{(1-\overline{\alpha}_n z)\pi_n(z)}$
- $\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(z)}{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)} \frac{1}{\zeta_n(z)}$
- $\frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} - \tau_n\alpha_n}{1-\overline{\tau}_n\overline{\alpha}_n z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}$, $\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} \to 0$ and $\tau_n \to 0$ $(A \in CC)$
- $0 < m \leq |zeta_n(z)| \leq M < \infty \Rightarrow (2)$, $|\lambda_n| = |\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}| \to 0$
- QED

Proof $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$

- $\phi_n(z) = c_n \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(a)}{(z-\alpha_n)\pi_n(z)}$, $\tau_n = \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}$
- $\phi_n^*(z) = \overline{c}_n \frac{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)}{(1-\overline{\alpha}_n z)\pi_n(z)}$
- $\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(z)}{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)} \frac{1}{\zeta_n(z)}$
- $\frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} - \tau_n\alpha_n}{1-\overline{\tau}_n\overline{\alpha}_n z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}$, $\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} \to 0$ and $\tau_n \to 0$ $(A \in CC)$
- $0 < m \leq |zeta_n(z)| \leq M < \infty \Rightarrow (2)$, $|\lambda_n| = |\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}| \to 0$
- QED

Proof $\lim_{n\to\infty} \frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = \lim_{n\to\infty} \lambda_n = 0$

- $\phi_n(z) = c_n \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(a)}{(z-\alpha_n)\pi_n(z)}$, $\tau_n = \frac{\phi_{nn}(\alpha_n)}{\phi_{nn}^{\bar{*}}(\alpha_n)}$
- $\phi_n^*(z) = \overline{c}_n \frac{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)}{(1-\overline{\alpha}_n z)\pi_n(z)}$
- $\frac{\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\phi_{nn}(z) - \tau_n\alpha_n\phi_{nn}^{\bar{*}}(z)}{\phi_{nn}^{\bar{*}}(z) - \overline{\tau}_n\overline{\alpha}_n z\phi_{nn}(z)} \frac{1}{\zeta_n(z)}$
- $\frac{\zeta_n(z)\phi_n(z)}{\phi_n^*(z)} = \frac{c_n}{\overline{c}_n} \frac{z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} - \tau_n\alpha_n}{1-\overline{\tau}_n\overline{\alpha}_n z\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)}}$, $\frac{\phi_{nn}(z)}{\phi_{nn}^{\bar{*}}(z)} \to 0$ and $\tau_n \to 0$ $(A \in CC)$
- $0 < m \leq |zeta_n(z)| \leq M < \infty \Rightarrow (2)$, $|\lambda_n| = |\frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}| \to 0$
- QED

Optimization problem in reproducing kernel Hilbert space

Theorem

In RKHS with kernel $k(z,w)$, min of $\|f\|_\mu^2$ with $f(w)=1$ is obtained for $f(z)=K(z)=\frac{k(z,w)}{k(w,w)}$ and the minimum is $\frac{1}{k(w,w)}$.

proof: $\|K(z)\|_\mu^2 = \frac{1}{k(w,w)}$

$\forall f$ with $f(w)=1$ and $\|f\|_\mu^2 = m$:

$0 \leq \|f-K\|_\mu^2 = \|f\|_\mu^2 + \frac{1}{k(w,w)} - 2\mathrm{Re}\,\langle f, K\rangle_\mu = m - \frac{1}{k(w,w)}$

Thus $m \geq \frac{1}{k(w,w)}$ QED

Optimization problem in reproducing kernel Hilbert space

Theorem

In RKHS with kernel $k(z,w)$, min of $\|f\|_\mu^2$ with $f(w)=1$ is obtained for $f(z)=K(z)=\frac{k(z,w)}{k(w,w)}$ and the minimum is $\frac{1}{k(w,w)}$.

proof: $\|K(z)\|_\mu^2 = \frac{1}{k(w,w)}$

$\forall f$ with $f(w)=1$ and $\|f\|_\mu^2 = m$:

$0 \leq \|f-K\|_\mu^2 = \|f\|_\mu^2 + \frac{1}{k(w,w)} - 2\mathrm{Re}\,\langle f,K\rangle_\mu = m - \frac{1}{k(w,w)}$

Thus $m \geq \frac{1}{k(w,w)}$ QED

Optimization problem in reproducing kernel Hilbert space

Theorem

In RKHS with kernel $k(z,w)$, min of $\|f\|_\mu^2$ with $f(w)=1$ is obtained for $f(z)=K(z)=\frac{k(z,w)}{k(w,w)}$ and the minimum is $\frac{1}{k(w,w)}$.

proof: $\|K(z)\|_\mu^2 = \frac{1}{k(w,w)}$

$\forall f$ with $f(w)=1$ and $\|f\|_\mu^2 = m$:

$0 \leq \|f-K\|_\mu^2 = \|f\|_\mu^2 + \frac{1}{k(w,w)} - 2\mathrm{Re}\,\langle f,K\rangle_\mu = m - \frac{1}{k(w,w)}$

Thus $m \geq \frac{1}{k(w,w)}$ QED

Optimization problem in reproducing kernel Hilbert space

Theorem

In RKHS with kernel $k(z,w)$, min of $\|f\|_\mu^2$ with $f(w)=1$ is obtained for $f(z)=K(z)=\frac{k(z,w)}{k(w,w)}$ and the minimum is $\frac{1}{k(w,w)}$.

proof: $\|K(z)\|_\mu^2 = \frac{1}{k(w,w)}$

$\forall f$ with $f(w)=1$ and $\|f\|_\mu^2 = m$:

$0 \leq \|f-K\|_\mu^2 = \|f\|_\mu^2 + \frac{1}{k(w,w)} - 2\mathrm{Re}\,\langle f, K\rangle_\mu = m - \frac{1}{k(w,w)}$

Thus $m \geq \frac{1}{k(w,w)}$ QED

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n, f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \longrightarrow \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\longrightarrow \frac{\sigma(0)}{\sigma(z)}$, pr. error $\longrightarrow |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n, f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \longrightarrow \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\longrightarrow \frac{\sigma(0)}{\sigma(z)}$, pr. error $\longrightarrow |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z, w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n; f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \to \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\to \frac{\sigma(0)}{\sigma(z)}$, pr. error $\to |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n; f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \longrightarrow \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\longrightarrow \frac{\sigma(0)}{\sigma(z)}$, pr. error $\longrightarrow |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n; f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \longrightarrow \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\longrightarrow \frac{\sigma(0)}{\sigma(z)}$, pr. error $\longrightarrow |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n; f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \to \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\to \frac{\sigma(0)}{\sigma(z)}$, pr. error $\to |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n; f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \longrightarrow \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\longrightarrow \frac{\sigma(0)}{\sigma(z)}$, pr. error $\longrightarrow |\sigma(0)|^2 \neq 0$ (remote past)

Linear prediction

- Recall: find min $\|f\|_\mu^2$ with $f(0) = 1$ in $H_2(\mu)$
- Polynomial case: RK for $H_2(\mu)$ is $s(z,w) = \frac{1}{(1-\overline{w}z)\sigma(z)\overline{\sigma(w)}}$
- If $\mu \in SZ$ and $A \in BD$ then $\frac{s(z,0)}{s(0,0)} = \frac{\sigma(0)}{\sigma(z)}$
- However min error is $\frac{1}{s(0,0)} = |\sigma(0)|^2 = 0$ only if $|\sigma(0)|^2 = \exp\{\int \log \mu'(t) d\lambda(t)\} = 0$ or $\log \mu' \notin L_1$
- Finite dim problem: $L_2(\mu)$-min over $\mathcal{L}_n \subset H_2$
- $\min_{f \in \mathcal{L}_n; f(w)=1} \|f\|_\mu^2 \Rightarrow f(z) = \frac{k_n(z,w)}{k_n(w,w)}$
- $\mu \in SZ$ & $A \in BD \Rightarrow \frac{k_n(z,w)}{k_n(w,w)} \longrightarrow \frac{(1-|w|^2)\sigma(w)}{(1-\overline{w}z)\sigma(z)}$
- If $w = 0$ then $\longrightarrow \frac{\sigma(0)}{\sigma(z)}$, pr. error $\longrightarrow |\sigma(0)|^2 \neq 0$ (remote past)

$L_2(\mu)$ convergence

Theorem

$\mu \in SZ$, $w \in \mathbb{D}$, $s_w(z) = s(z, w)$.
Then $\min_{f \in \mathcal{L}_n} \|f - s_w\|_\mu^2 = s_w(w) - k_n(w, w)$. *Minimizer is* $k_n(z, w)$.
If $A \in BD$: $\|k_n(\cdot, w) - s_w\|_\mu \to 0$.

proof:

$\|s_w - f\|_\mu^2 = \|s_w\|_\mu^2 + \|f\|_\mu^2 - 2\mathrm{Re}(f(w))$
If $f(w) = a$: $\min_a \{ \min_{f \in \mathcal{L}_n, f(w)=a} \|f\|_\mu^2 - 2\mathrm{Re}(a) \}$

$\Rightarrow \min_a |a|^2 / k_n(w, w) - 2\mathrm{Re}(a) \Rightarrow a = k_n(w, w)$.

$A \in BD \Rightarrow \mathcal{L}_n$ dense in $H_2(\mu)$, hence error $\to 0$.

$L_2(\mu)$ convergence

Theorem

$\mu \in SZ$, $w \in \mathbb{D}$, $s_w(z) = s(z, w)$.
Then $\min_{f \in \mathcal{L}_n} \|f - s_w\|_\mu^2 = s_w(w) - k_n(w, w)$. *Minimizer is* $k_n(z, w)$.
If $A \in BD$: $\|k_n(\cdot, w) - s_w\|_\mu \to 0$.

proof:

Skip proof

$\|s_w - f\|_\mu^2 = \|s_w\|_\mu^2 + \|f\|_\mu^2 - 2\mathrm{Re}(f(w))$
If $f(w) = a$: $\min_a \{ \min_{f \in \mathcal{L}_n, f(w)=a} \|f\|_\mu^2 - 2\mathrm{Re}(a) \}$
$\Rightarrow \min_a |a|^2 / k_n(w, w) - 2\mathrm{Re}(a) \Rightarrow a = k_n(w, w)$.
$A \in BD \Rightarrow \mathcal{L}_n$ dense in $H_2(\mu)$, hence error $\to 0$.

$L_2(\mu)$ convergence

Theorem

$\mu \in SZ$, $w \in \mathbb{D}$, $s_w(z) = s(z, w)$.
Then $\min_{f \in \mathcal{L}_n} \|f - s_w\|_\mu^2 = s_w(w) - k_n(w, w)$. *Minimizer is* $k_n(z, w)$.
If $A \in BD$: $\|k_n(\cdot, w) - s_w\|_\mu \to 0$.

proof:

$\|s_w - f\|_\mu^2 = \|s_w\|_\mu^2 + \|f\|_\mu^2 - 2\mathrm{Re}(f(w))$
If $f(w) = a$: $\min_a \{ \min_{f \in \mathcal{L}_n, f(w)=a} \|f\|_\mu^2 - 2\mathrm{Re}(a) \}$

$\Rightarrow \min_a |a|^2 / k_n(w, w) - 2\mathrm{Re}(a) \Rightarrow a = k_n(w, w)$.

$A \in BD \Rightarrow \mathcal{L}_n$ dense in $H_2(\mu)$, hence error $\to 0$.

$L_2(\mu)$ convergence

Theorem

$\mu \in SZ$, $w \in \mathbb{D}$, $s_w(z) = s(z, w)$.
Then $\min_{f \in \mathcal{L}_n} \|f - s_w\|_\mu^2 = s_w(w) - k_n(w, w)$. *Minimizer is* $k_n(z, w)$.
If $A \in BD$: $\|k_n(\cdot, w) - s_w\|_\mu \to 0$.

proof:

$\|s_w - f\|_\mu^2 = \|s_w\|_\mu^2 + \|f\|_\mu^2 - 2\mathrm{Re}(f(w))$
If $f(w) = a$: $\min_a \{ \min_{f \in \mathcal{L}_n, f(w)=a} \|f\|_\mu^2 - 2\mathrm{Re}(a) \}$

$\Rightarrow \min_a |a|^2 / k_n(w, w) - 2\mathrm{Re}(a) \Rightarrow a = k_n(w, w)$.

$A \in BD \Rightarrow \mathcal{L}_n$ dense in $H_2(\mu)$, hence error $\to 0$.

Vector valued prediction

- $\min \mathrm{E}[\sum_n |e_n|^2] = \min \|f\|_\mu^2$
  $e_n = y_n + a_1 y_{n-1} + a_2 y_{n-1} + \cdots, \quad f = 1 + a_1 z + a_2 z^2 + \cdots$
- If the y_n are vectors, then the a_k are square matrices.
  Thus the predictor is a matrix valued function.
  Also μ is matrix valued measure.
- We need a matrix valued version of $L_2(\mu)$, $H_2(\mu)$, ORF,...
- This is the subject of the next part.

Vector valued prediction

- $\min \mathrm{E}[\sum_n |e_n|^2] = \min \|f\|_\mu^2$
  $e_n = y_n + a_1 y_{n-1} + a_2 y_{n-1} + \cdots, \quad f = 1 + a_1 z + a_2 z^2 + \cdots$
- If the y_n are vectors, then the a_k are square matrices.
  Thus the predictor is a matrix valued function.
  Also μ is matrix valued measure.
- We need a matrix valued version of $L_2(\mu)$, $H_2(\mu)$, ORF,...
- This is the subject of the next part.

Vector valued prediction

- $\min \mathrm{E}[\sum_n |e_n|^2] = \min \|f\|_\mu^2$
  $e_n = y_n + a_1 y_{n-1} + a_2 y_{n-1} + \cdots, \quad f = 1 + a_1 z + a_2 z^2 + \cdots$
- If the y_n are vectors, then the a_k are square matrices.
  Thus the predictor is a matrix valued function.
  Also μ is matrix valued measure.
- We need a matrix valued version of $L_2(\mu)$, $H_2(\mu)$, ORF,...
- This is the subject of the next part.

Vector valued prediction

- $\min \mathrm{E}[\sum_n |e_n|^2] = \min \|f\|_\mu^2$
  $e_n = y_n + a_1 y_{n-1} + a_2 y_{n-1} + \cdots, \quad f = 1 + a_1 z + a_2 z^2 + \cdots$
- If the y_n are vectors, then the a_k are square matrices.
  Thus the predictor is a matrix valued function.
  Also μ is matrix valued measure.
- We need a matrix valued version of $L_2(\mu)$, $H_2(\mu)$, ORF,...
- This is the subject of the next part.

Part III

The matrix case

Preliminaries

- $A \in \mathbb{C}^{N\times N}$: $A > 0$, $A \geq 0$ positive (semi) definite.
  $A \geq B \Leftrightarrow A - B \geq 0$
- A^H Hermitian conjugate, $\operatorname{tr} A$ trace $(\frac{1}{N}\sum a_{ii})$
- $A > 0$ then $A = S^2$, $S = A^{1/2}$ Hermitian square root;
  $A = LL^H$, $L = A^{L/2}$ left square root
  $A = R^H R$, $R = A^{R/2}$ right square root
- Beware some $(A^{-1})^{L/2}$ = some $(A^{R/2})^{-1}$. $\therefore$ $A^{-L/2}$ is ambiguous.
- $f_*(z) = [f(1/\bar{z})]^H$

Preliminaries

- $A \in \mathbb{C}^{N\times N}$: $A > 0$, $A \geq 0$ positive (semi) definite.
  $A \geq B \Leftrightarrow A - B \geq 0$
- A^H Hermitian conjugate, $\operatorname{tr} A$ trace $(\frac{1}{N}\sum a_{ii})$
- $A > 0$ then $A = S^2$, $S = A^{1/2}$ Hermitian square root;
  $A = LL^H$, $L = A^{L/2}$ left square root
  $A = R^H R$, $R = A^{R/2}$ right square root
- Beware some $(A^{-1})^{L/2}$ = some $(A^{R/2})^{-1}$. $\therefore$ $A^{-L/2}$ is ambiguous.
- $f_*(z) = [f(1/\bar{z})]^H$

Preliminaries

- $A \in \mathbb{C}^{N\times N}$: $A > 0$, $A \geq 0$ positive (semi) definite.
  $A \geq B \Leftrightarrow A - B \geq 0$
- A^H Hermitian conjugate, $\operatorname{tr} A$ trace $(\frac{1}{N}\sum a_{ii})$
- $A > 0$ then $A = S^2$, $S = A^{1/2}$ Hermitian square root;
  $A = LL^H$, $L = A^{L/2}$ left square root
  $A = R^H R$, $R = A^{R/2}$ right square root
- Beware some $(A^{-1})^{L/2}$ = some $(A^{R/2})^{-1}$. $\therefore$ $A^{-L/2}$ is ambiguous.
- $f_*(z) = [f(1/\bar{z})]^H$

Preliminaries

- $A \in \mathbb{C}^{N\times N}$: $A > 0$, $A \geq 0$ positive (semi) definite.
  $A \geq B \Leftrightarrow A - B \geq 0$
- A^H Hermitian conjugate, $\operatorname{tr} A$ trace $(\frac{1}{N}\sum a_{ii})$
- $A > 0$ then $A = S^2$, $S = A^{1/2}$ Hermitian square root;
  $A = LL^H$, $L = A^{L/2}$ left square root
  $A = R^H R$, $R = A^{R/2}$ right square root
- Beware some $(A^{-1})^{L/2} =$ some $(A^{R/2})^{-1}$. $\therefore$ $A^{-L/2}$ is ambiguous.
- $f_*(z) = [f(1/\overline{z})]^H$

Preliminaries

- $A \in \mathbb{C}^{N\times N}$: $A > 0$, $A \geq 0$ positive (semi) definite.
  $A \geq B \Leftrightarrow A - B \geq 0$
- A^H Hermitian conjugate, $\operatorname{tr} A$ trace $(\frac{1}{N}\sum a_{ii})$
- $A > 0$ then $A = S^2$, $S = A^{1/2}$ Hermitian square root;
  $A = LL^H$, $L = A^{L/2}$ left square root
  $A = R^H R$, $R = A^{R/2}$ right square root
- Beware some $(A^{-1})^{L/2} =$ some $(A^{R/2})^{-1}$. $\therefore$ $A^{-L/2}$ is ambiguous.
- $f_*(z) = [f(1/\overline{z})]^H$

Measures

- μ nonneg def measure on $\mathbb{T}$ = matrix with complex valued scalar measures as entries, but the whole matrix takes pos def values on any Borel set of $\mathbb{T}$
- $0 \leq M \leq \mathrm{tr}\,(M)I$. Set $\tau = \mathrm{tr}\,(\mu)$ then $\tau(E) = 0 \Rightarrow \mu(E) = 0$. μ abs. cnt. wrt. τ: trace derivative $\mu'_\tau = [d\mu_{ij}/d\tau]_{i,j}$; $\mu(E) = \int_E \mu'_\tau d\tau$. Note $0 \leq \mu'_\tau \leq I$. Assume $\int d\mu = I$
- $\langle\langle f, g\rangle\rangle_{\ell,\mu} := \int f d\mu g^H := \int (f \mu'_\tau g^H) d\tau$
  $\langle\langle f, g\rangle\rangle_{r,\mu} := \int f^H d\mu g := \int (f^H \mu'_\tau g) d\tau$

Measures

- μ nonneg def measure on $\mathbb{T}$ = matrix with complex valued scalar measures as entries, but the whole matrix takes pos def values on any Borel set of $\mathbb{T}$
- $0 \leq M \leq \mathrm{tr}\,(M)I$. Set $\tau = \mathrm{tr}\,(\mu)$ then $\tau(E) = 0 \Rightarrow \mu(E) = 0$. μ abs. cnt. wrt. τ: trace derivative $\mu'_\tau = [d\mu_{ij}/d\tau]_{i,j}$; $\mu(E) = \int_E \mu'_\tau d\tau$. Note $0 \leq \mu'_\tau \leq I$. Assume $\int d\mu = I$
- $\langle\langle f, g\rangle\rangle_{\ell,\mu} := \int f d\mu g^H := \int (f \mu'_\tau g^H) d\tau$
  $\langle\langle f, g\rangle\rangle_{r,\mu} := \int f^H d\mu g := \int (f^H \mu'_\tau g) d\tau$

Measures

- μ nonneg def measure on $\mathbb{T}$ = matrix with complex valued scalar measures as entries, but the whole matrix takes pos def values on any Borel set of $\mathbb{T}$
- $0 \leq M \leq \mathrm{tr}\,(M)I$. Set $\tau = \mathrm{tr}\,(\mu)$ then $\tau(E) = 0 \Rightarrow \mu(E) = 0$. μ abs. cnt. wrt. τ: trace derivative $\mu'_\tau = [d\mu_{ij}/d\tau]_{i,j}$; $\mu(E) = \int_E \mu'_\tau d\tau$. Note $0 \leq \mu'_\tau \leq I$. Assume $\int d\mu = I$
- $\langle\langle f, g\rangle\rangle_{\ell,\mu} := \int f d\mu g^H := \int (f \mu'_\tau g^H) d\tau$
  $\langle\langle f, g\rangle\rangle_{r,\mu} := \int f^H d\mu g := \int (f^H \mu'_\tau g) d\tau$

(left) Hilbert module

- $\langle\langle af + bg, h\rangle\rangle_{\ell,\mu} = a\langle\langle f, h\rangle\rangle_{\ell,\mu} + b\langle\langle g, h\rangle\rangle_{\ell,\mu}$
  $\langle\langle f, g\rangle\rangle_{\ell,\mu} = [\langle\langle g, f\rangle\rangle_{\ell,\mu}]^H$
  $\langle\langle f, f\rangle\rangle_{\ell,\mu} \geq 0$ and $\operatorname{tr}\langle\langle f, f\rangle\rangle_{\ell,\mu} = 0 \Leftrightarrow f = 0$
- Pre-Hilbert space. Need some topology. Scalar inner product and scalar norm.
- $\langle f, g\rangle_{\ell,\mu} = \operatorname{tr}\langle\langle f, g\rangle\rangle_{\ell,\mu} = \int \operatorname{tr}(f\mu'_\tau h^H)d\tau$
  Norm: $|f|_{\ell,\mu} = [\langle f, f\rangle_{\ell,\mu}]^{1/2}$.
- $\mathbb{L}_2(\mu)$ Hilbert space, with $\langle\cdot,\cdot\rangle_{\ell,\mu}$, complete wrt $|f|_{\ell,\mu}$,
  Hilbert module wrt $\langle\langle f, g\rangle\rangle_{\ell,\mu}$

(left) Hilbert module

- $\langle\langle af + bg, h\rangle\rangle_{\ell,\mu} = a\langle\langle f, h\rangle\rangle_{\ell,\mu} + b\langle\langle g, h\rangle\rangle_{\ell,\mu}$
  $\langle\langle f, g\rangle\rangle_{\ell,\mu} = [\langle\langle g, f\rangle\rangle_{\ell,\mu}]^H$
  $\langle\langle f, f\rangle\rangle_{\ell,\mu} \geq 0$ and $\operatorname{tr}\langle\langle f, f\rangle\rangle_{\ell,\mu} = 0 \Leftrightarrow f = 0$
- Pre-Hilbert space. Need some topology. Scalar inner product and scalar norm.
- $\langle f, g\rangle_{\ell,\mu} = \operatorname{tr}\langle\langle f, g\rangle\rangle_{\ell,\mu} = \int \operatorname{tr}(f\mu'_\tau h^H)d\tau$
  Norm: $|f|_{\ell,\mu} = [\langle f, f\rangle_{\ell,\mu}]^{1/2}$.
- $\mathbb{L}_2(\mu)$ Hilbert space, with $\langle\cdot,\cdot\rangle_{\ell,\mu}$, complete wrt $|f|_{\ell,\mu}$,
  Hilbert module wrt $\langle\langle f, g\rangle\rangle_{\ell,\mu}$

(left) Hilbert module

- $\langle\langle af + bg, h\rangle\rangle_{\ell,\mu} = a\langle\langle f, h\rangle\rangle_{\ell,\mu} + b\langle\langle g, h\rangle\rangle_{\ell,\mu}$
  $\langle\langle f, g\rangle\rangle_{\ell,\mu} = [\langle\langle g, f\rangle\rangle_{\ell,\mu}]^H$
  $\langle\langle f, f\rangle\rangle_{\ell,\mu} \geq 0$ and $\operatorname{tr}\langle\langle f, f\rangle\rangle_{\ell,\mu} = 0 \Leftrightarrow f = 0$
- Pre-Hilbert space. Need some topology. Scalar inner product and scalar norm.
- $\langle f, g\rangle_{\ell,\mu} = \operatorname{tr}\langle\langle f, g\rangle\rangle_{\ell,\mu} = \int \operatorname{tr}(f\mu'_\tau h^H)d\tau$
  Norm: $|f|_{\ell,\mu} = [\langle f, f\rangle_{\ell,\mu}]^{1/2}$.
- $\mathbb{L}_2(\mu)$ Hilbert space, with $\langle\cdot,\cdot\rangle_{\ell,\mu}$, complete wrt $|f|_{\ell,\mu}$, Hilbert module wrt $\langle\langle f, g\rangle\rangle_{\ell,\mu}$

(left) Hilbert module

- $\langle\langle af + bg, h\rangle\rangle_{\ell,\mu} = a\langle\langle f, h\rangle\rangle_{\ell,\mu} + b\langle\langle g, h\rangle\rangle_{\ell,\mu}$
  $\langle\langle f, g\rangle\rangle_{\ell,\mu} = [\langle\langle g, f\rangle\rangle_{\ell,\mu}]^H$
  $\langle\langle f, f\rangle\rangle_{\ell,\mu} \geq 0$ and $\operatorname{tr}\langle\langle f, f\rangle\rangle_{\ell,\mu} = 0 \Leftrightarrow f = 0$
- Pre-Hilbert space. Need some topology. Scalar inner product and scalar norm.
- $\langle f, g\rangle_{\ell,\mu} = \operatorname{tr}\langle\langle f, g\rangle\rangle_{\ell,\mu} = \int \operatorname{tr}(f\mu'_\tau h^H)d\tau$
  Norm: $|f|_{\ell,\mu} = [\langle f, f\rangle_{\ell,\mu}]^{1/2}$.
- $\mathbb{L}_2(\mu)$ Hilbert space, with $\langle\cdot,\cdot\rangle_{\ell,\mu}$, complete wrt $|f|_{\ell,\mu}$, Hilbert module wrt $\langle\langle f, g\rangle\rangle_{\ell,\mu}$

$\mathbb{L}_p$ and $\mathbb{H}_p$ for $d\mu = d\lambda$

- Euclidean norm: $|f|_E = [\mathrm{tr}\,(f^H f)]^{1/2} = [\mathrm{tr}\,(f f^H)]^{1/2}$
- $\mathbb{L}_p$: $f \in \mathbb{L}_p \Leftrightarrow f_{ij} \in L_p \Leftrightarrow |f|_E \in L_p$,
  $|f|_p = [\int |f|_E^p d\lambda]^{1/p}$, $|f|_\infty = \sup_{\mathbb{D}} |f|_E$
- $\mathbb{H}_p$: analytic in $\mathbb{D}$ and in $\mathbb{L}_p$
- $\Omega_\mu(z) = \int D(t,z) d\mu(t) = \underbrace{c_0}_{=I} + 2\sum_{k\geq 1} c_k z^k$,

  $c_k = \int t^{-k} d\mu(t)$,
  $\mathrm{Re}\Omega_\mu(z) = \int P(t,z) d\mu(t)$, $\rightarrow \mu'(t) = \frac{d\mu}{d\lambda}(t)$
- $\log\det\mu' \in L_1$: $\mu' = \sigma^L \sigma_*^L = \sigma_*^R \sigma^R$
  σ^L, σ^R outer in $\mathbb{H}_2$: i.e. the determinant is outer in $H_{2/N}$
  Unique up to right/left unitary constant factor.

$\mathbb{L}_p$ and $\mathbb{H}_p$ for $d\mu = d\lambda$

- Euclidean norm: $|f|_E = [\mathrm{tr}\,(f^H f)]^{1/2} = [\mathrm{tr}\,(f f^H)]^{1/2}$
- $\mathbb{L}_p$: $f \in \mathbb{L}_p \Leftrightarrow f_{ij} \in L_p \Leftrightarrow |f|_E \in L_p$,
  $|f|_p = [\int |f|_E^p d\lambda]^{1/p}$, $|f|_\infty = \sup_{\mathbb{D}} |f|_E$
- $\mathbb{H}_p$: analytic in $\mathbb{D}$ and in $\mathbb{L}_p$
- $\Omega_\mu(z) = \int D(t,z) d\mu(t) = \underbrace{c_0}_{=I} + 2\sum_{k\geq 1} c_k z^k$,

  $c_k = \int t^{-k} d\mu(t)$,
  $\mathrm{Re}\Omega_\mu(z) = \int P(t,z) d\mu(t)$, $\rightarrow \mu'(t) = \frac{d\mu}{d\lambda}(t)$
- $\log\det \mu' \in L_1$: $\mu' = \sigma^L \sigma^L_* = \sigma^R_* \sigma^R$
  σ^L, σ^R outer in $\mathbb{H}_2$: i.e. the determinant is outer in $H_{2/N}$
  Unique up to right/left unitary constant factor.

$\mathbb{L}_p$ and $\mathbb{H}_p$ for $d\mu = d\lambda$

- Euclidean norm: $|f|_E = [\mathrm{tr}\,(f^H f)]^{1/2} = [\mathrm{tr}\,(f f^H)]^{1/2}$
- $\mathbb{L}_p$: $f \in \mathbb{L}_p \Leftrightarrow f_{ij} \in L_p \Leftrightarrow |f|_E \in L_p$,
  $|f|_p = [\int |f|_E^p d\lambda]^{1/p}$, $|f|_\infty = \sup_{\mathbb{D}} |f|_E$
- $\mathbb{H}_p$: analytic in $\mathbb{D}$ and in $\mathbb{L}_p$
- $\Omega_\mu(z) = \int D(t,z) d\mu(t) = \underbrace{c_0}_{=I} + 2\sum_{k\geq 1} c_k z^k$,

  $c_k = \int t^{-k} d\mu(t)$,
  $\mathrm{Re}\,\Omega_\mu(z) = \int P(t,z) d\mu(t)$, $\to \mu'(t) = \frac{d\mu}{d\lambda}(t)$
- $\log\det \mu' \in L_1$: $\mu' = \sigma^L \sigma^L_* = \sigma^R_* \sigma^R$
  σ^L, σ^R outer in $\mathbb{H}_2$: i.e. the determinant is outer in $H_{2/N}$
  Unique up to right/left unitary constant factor.

$\mathbb{L}_p$ and $\mathbb{H}_p$ for $d\mu = d\lambda$

- Euclidean norm: $|f|_E = [\mathrm{tr}\,(f^H f)]^{1/2} = [\mathrm{tr}\,(f f^H)]^{1/2}$
- $\mathbb{L}_p$: $f \in \mathbb{L}_p \Leftrightarrow f_{ij} \in L_p \Leftrightarrow |f|_E \in L_p$,
  $|f|_p = [\int |f|_E^p d\lambda]^{1/p}$, $|f|_\infty = \sup_{\mathbb{D}} |f|_E$
- $\mathbb{H}_p$: analytic in $\mathbb{D}$ and in $\mathbb{L}_p$
- $\Omega_\mu(z) = \int D(t,z) d\mu(t) = \underbrace{c_0}_{=I} + 2\sum_{k\geq 1} c_k z^k$,

  $c_k = \int t^{-k} d\mu(t)$,
  $\mathrm{Re}\,\Omega_\mu(z) = \int P(t,z) d\mu(t)$, $\rightarrow \mu'(t) = \frac{d\mu}{d\lambda}(t)$
- $\log\det \mu' \in L_1$: $\mu' = \sigma^L \sigma^L_* = \sigma^R_* \sigma^R$
  σ^L, σ^R outer in $\mathbb{H}_2$: i.e. the determinant is outer in $H_{2/N}$
  Unique up to right/left unitary constant factor.

$\mathbb{L}_p$ and $\mathbb{H}_p$ for $d\mu = d\lambda$

- Euclidean norm: $|f|_E = [\mathrm{tr}\,(f^H f)]^{1/2} = [\mathrm{tr}\,(f f^H)]^{1/2}$
- $\mathbb{L}_p$: $f \in \mathbb{L}_p \Leftrightarrow f_{ij} \in L_p \Leftrightarrow |f|_E \in L_p$,
  $|f|_p = [\int |f|_E^p d\lambda]^{1/p}$, $|f|_\infty = \sup_{\mathbb{D}} |f|_E$
- $\mathbb{H}_p$: analytic in $\mathbb{D}$ and in $\mathbb{L}_p$
- $\Omega_\mu(z) = \int D(t,z) d\mu(t) = \underbrace{c_0}_{=I} + 2\sum_{k\geq 1} c_k z^k$,

  $c_k = \int t^{-k} d\mu(t)$,
  $\mathrm{Re}\,\Omega_\mu(z) = \int P(t,z) d\mu(t)$, $\rightarrow \mu'(t) = \frac{d\mu}{d\lambda}(t)$
- $\log\det \mu' \in L_1$: $\mu' = \sigma^L \sigma^L_* = \sigma^R_* \sigma^R$
  σ^L, σ^R outer in $\mathbb{H}_2$: i.e. the determinant is outer in $H_{2/N}$
  Unique up to right/left unitary constant factor.

ϕ_n^L and ϕ_n^R

- GS on $\{B_k I\}$: $\tilde{\phi}_n^L = B_n I - \sum_{k=0}^{n-1} \langle\langle B_n I, \phi_k^L \rangle\rangle_{\ell,\mu} \phi_k^L$ and $\phi_n^L = m_n^{-1} \tilde{\phi}_n^L$, m_n left square root of $\left\langle\left\langle \tilde{\phi}_n^L, \tilde{\phi}_n^L \right\rangle\right\rangle_{\ell,\mu}$
- If $\log\det \mu' \in L_1$ then m_n is invertible if $\tilde{\phi}_n^L \neq 0$
- ϕ_n^L unique up to constant unitary factor from the left. Fix it to make leading coefficient $[\phi_n^{L*}(\alpha_n)]^H = \kappa_n^L > 0$.
  Note: $\langle\langle f, g \rangle\rangle_{\ell,\mu} = \langle\langle f_*, g_* \rangle\rangle_{r,\mu} = \langle\langle f^*, g^* \rangle\rangle_{r,\mu}$

ϕ_n^L and ϕ_n^R

- GS on $\{B_k I\}$: $\tilde{\phi}_n^L = B_n I - \sum_{k=0}^{n-1} \langle\langle B_n I, \phi_k^L \rangle\rangle_{\ell,\mu} \phi_k^L$ and $\phi_n^L = m_n^{-1} \tilde{\phi}_n^L$, m_n left square root of $\left\langle\left\langle \tilde{\phi}_n^L, \tilde{\phi}_n^L \right\rangle\right\rangle_{\ell,\mu}$
- If $\log\det \mu' \in L_1$ then m_n is invertible if $\tilde{\phi}_n^L \neq 0$
- ϕ_n^L unique up to constant unitary factor from the left. Fix it to make leading coefficient $[\phi_n^{L*}(\alpha_n)]^H = \kappa_n^L > 0$. Note: $\langle\langle f, g \rangle\rangle_{\ell,\mu} = \langle\langle f_*, g_* \rangle\rangle_{r,\mu} = \langle\langle f^*, g^* \rangle\rangle_{r,\mu}$

ϕ_n^L and ϕ_n^R

- GS on $\{B_k I\}$: $\tilde{\phi}_n^L = B_n I - \sum_{k=0}^{n-1} \langle\langle B_n I, \phi_k^L \rangle\rangle_{\ell,\mu} \phi_k^L$ and $\phi_n^L = m_n^{-1} \tilde{\phi}_n^L$, m_n left square root of $\left\langle\left\langle \tilde{\phi}_n^L, \tilde{\phi}_n^L \right\rangle\right\rangle_{\ell,\mu}$
- If $\log\det \mu' \in L_1$ then m_n is invertible if $\tilde{\phi}_n^L \neq 0$
- ϕ_n^L unique up to constant unitary factor from the left. Fix it to make leading coefficient $[\phi_n^{L*}(\alpha_n)]^H = \kappa_n^L > 0$.
  Note: $\langle\langle f, g \rangle\rangle_{\ell,\mu} = \langle\langle f_*, g_* \rangle\rangle_{r,\mu} = \langle\langle f^*, g^* \rangle\rangle_{r,\mu}$

k_n^L and k_n^R

- $k_n^L(z,w) = \sum_{k=0}^{n} [\phi_k^L(z)]^H \phi_k^L(w)$
  $k_n^R(w,z) = \sum_{k=0}^{n} \phi_k^R(z) [\phi_k^R(w)]^H$
  $\langle\langle f, k_n^L(\cdot, w)\rangle\rangle_{\ell,\mu} = f(w) = \langle\langle k_n^R(w,\cdot), f\rangle\rangle_{r,\mu}$, $f \in \mathcal{L}_n$

Christoffel-Darboux

- $k_n^L(z,w) = \frac{[\phi_{n+1}^{R*}(w)]^H \phi_{n+1}^{R*}(z) - [\phi_{n+1}^{L}(w)]^H \phi_{n+1}^{L}(z)}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

  $k_n^L(z,w) = \frac{[\phi_{n}^{R*}(w)]^H \phi_{n}^{R*}(z) - \zeta_n(z)\overline{\zeta_n(w)}[\phi_{n}^{L}(w)]^H \phi_{n}^{L}(z)}{1-\zeta_{n}(z)\overline{\zeta_{n}(w)}}$

- $k_n^R(w,z) = \frac{\phi_{n+1}^{L*}(z)[\phi_{n+1}^{L*}(w)]^H - \phi_{n+1}^{R}(z)[\phi_{n+1}^{R}(w)]^H}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

  $k_n^R(w,z) = \frac{\phi_{n}^{L*}(z)[\phi_{n}^{L*}(w)]^H - \zeta_n(z)\overline{\zeta_n(w)}\phi_{n}^{R}(z)[\phi_{n}^{R}(w)]^H}{1-\zeta_{n}(z)\overline{\zeta_{n}(w)}}$

- proof: exercise.
- Note:

  $k_n^L(z,\alpha_n) = \kappa_n^R \phi_n^{R*}(z)$, $k_n^L(\alpha_n,\alpha_n) = \kappa_n^R[\kappa_n^R]^H$

  $k_n^R(\alpha_n,z) = \phi_n^{L*}(z)\kappa_n^L$, $k_n^R(\alpha_n,\alpha_n) = [\kappa_n^L]^H \kappa_n^L$

Christoffel-Darboux

- $k_n^L(z,w) = \frac{[\phi_{n+1}^{R*}(w)]^H \phi_{n+1}^{R*}(z) - [\phi_{n+1}^{L}(w)]^H \phi_{n+1}^{L}(z)}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

  $k_n^L(z,w) = \frac{[\phi_{n}^{R*}(w)]^H \phi_{n}^{R*}(z) - \zeta_n(z)\overline{\zeta_n(w)}[\phi_{n}^{L}(w)]^H \phi_{n}^{L}(z)}{1-\zeta_{n}(z)\overline{\zeta_n(w)}}$

- $k_n^R(w,z) = \frac{\phi_{n+1}^{L*}(z)[\phi_{n+1}^{L*}(w)]^H - \phi_{n+1}^{R}(z)[\phi_{n+1}^{R}(w)]^H}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

  $k_n^R(w,z) = \frac{\phi_{n}^{L*}(z)[\phi_{n}^{L*}(w)]^H - \zeta_n(z)\overline{\zeta_n(w)}\phi_{n}^{R}(z)[\phi_{n}^{R}(w)]^H}{1-\zeta_{n}(z)\overline{\zeta_n(w)}}$

- proof: exercise.
- Note:

  $k_n^L(z,\alpha_n) = \kappa_n^R \phi_n^{R*}(z)$, $k_n^L(\alpha_n,\alpha_n) = \kappa_n^R[\kappa_n^R]^H$

  $k_n^R(\alpha_n,z) = \phi_n^{L*}(z)\kappa_n^L$, $k_n^R(\alpha_n,\alpha_n) = [\kappa_n^L]^H \kappa_n^L$

Christoffel-Darboux

- $k_n^L(z,w) = \frac{[\phi_{n+1}^{R*}(w)]^H \phi_{n+1}^{R*}(z) - [\phi_{n+1}^{L}(w)]^H \phi_{n+1}^{L}(z)}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

  $k_n^L(z,w) = \frac{[\phi_{n}^{R*}(w)]^H \phi_{n}^{R*}(z) - \zeta_n(z)\overline{\zeta_n(w)}[\phi_{n}^{L}(w)]^H \phi_{n}^{L}(z)}{1-\zeta_{n}(z)\overline{\zeta_n(w)}}$

- $k_n^R(w,z) = \frac{\phi_{n+1}^{L*}(z)[\phi_{n+1}^{L*}(w)]^H - \phi_{n+1}^{R}(z)[\phi_{n+1}^{R}(w)]^H}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$

  $k_n^R(w,z) = \frac{\phi_{n}^{L*}(z)[\phi_{n}^{L*}(w)]^H - \zeta_n(z)\overline{\zeta_n(w)}\phi_{n}^{R}(z)[\phi_{n}^{R}(w)]^H}{1-\zeta_{n}(z)\overline{\zeta_n(w)}}$

- proof: exercise.
- Note:
  $k_n^L(z,\alpha_n) = \kappa_n^R \phi_n^{R*}(z)$, $k_n^L(\alpha_n,\alpha_n) = \kappa_n^R[\kappa_n^R]^H$
  $k_n^R(\alpha_n,z) = \phi_n^{L*}(z)\kappa_n^L$, $k_n^R(\alpha_n,\alpha_n) = [\kappa_n^L]^H \kappa_n^L$

Christoffel-Darboux

- $$k_n^L(z,w) = \frac{[\phi_{n+1}^{R*}(w)]^H \phi_{n+1}^{R*}(z) - [\phi_{n+1}^{L}(w)]^H \phi_{n+1}^{L}(z)}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$$
  $$k_n^L(z,w) = \frac{[\phi_{n}^{R*}(w)]^H \phi_{n}^{R*}(z) - \zeta_n(z)\overline{\zeta_n(w)}[\phi_{n}^{L}(w)]^H \phi_{n}^{L}(z)}{1-\zeta_{n}(z)\overline{\zeta_{n}(w)}}$$
- $$k_n^R(w,z) = \frac{\phi_{n+1}^{L*}(z)[\phi_{n+1}^{L*}(w)]^H - \phi_{n+1}^{R}(z)[\phi_{n+1}^{R}(w)]^H}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(w)}}$$
  $$k_n^R(w,z) = \frac{\phi_{n}^{L*}(z)[\phi_{n}^{L*}(w)]^H - \zeta_n(z)\overline{\zeta_n(w)}\phi_{n}^{R}(z)[\phi_{n}^{R}(w)]^H}{1-\zeta_{n}(z)\overline{\zeta_{n}(w)}}$$
- proof: exercise.
- Note:
  $k_n^L(z,\alpha_n) = \kappa_n^R \phi_n^{R*}(z)$, $k_n^L(\alpha_n,\alpha_n) = \kappa_n^R[\kappa_n^R]^H$
  $k_n^R(\alpha_n,z) = \phi_n^{L*}(z)\kappa_n^L$, $k_n^R(\alpha_n,\alpha_n) = [\kappa_n^L]^H \kappa_n^L$

Recurrence for kernels

Theorem

$$\begin{bmatrix} \{I - [\rho_n^R]^H \rho_n^L\} & k_n^{R*}(w,z) \\ \{I - \rho_n^L [\rho_n^R]^H\} & k_n^L(z,w) \end{bmatrix} = t_n(z,w) \begin{bmatrix} k_{n-1}^{R*}(w,z) \\ k_{n-1}^L(z,w) \end{bmatrix}$$

$$t_n(z,w) = T_n^\rho \begin{bmatrix} \zeta_n(z) I & 0 \\ 0 & I \end{bmatrix} T_n^\gamma, \qquad T_n^\beta = \begin{bmatrix} I & [\beta_n^R]^H \\ \beta_n^L & I \end{bmatrix}$$

$$\rho_n^L = [\phi_n^L(w)]^H [\phi_n^{L*}(w)]^{-1}, \quad \gamma_n^L = -\zeta_n(w)\rho_n^L(w)$$

$$\rho_n^R = [\phi_n^{R*}(w)]^{-1} [\phi_n^R(w)]^H, \quad \gamma_n^R = -\zeta_n(w)\rho_n^R(w)$$

proof: Like scalar case. Use $k_n^L(z,w) = B_n(z)\overline{B_n(w)} k_n^R(1/\overline{z}, 1/\overline{w})$

Recurrence for kernels

Theorem

$$\begin{bmatrix} \{I - [\rho_n^R]^H \rho_n^L\} & k_n^{R*}(w,z) \\ \{I - \rho_n^L [\rho_n^R]^H\} & k_n^L(z,w) \end{bmatrix} = t_n(z,w) \begin{bmatrix} k_{n-1}^{R*}(w,z) \\ k_{n-1}^L(z,w) \end{bmatrix}$$

$$t_n(z,w) = T_n^\rho \begin{bmatrix} \zeta_n(z) I & 0 \\ 0 & I \end{bmatrix} T_n^\gamma, \qquad T_n^\beta = \begin{bmatrix} I & [\beta_n^R]^H \\ \beta_n^L & I \end{bmatrix}$$

$$\rho_n^L = [\phi_n^L(w)]^H [\phi_n^{L*}(w)]^{-1}, \quad \gamma_n^L = -\zeta_n(w)\rho_n^L(w)$$

$$\rho_n^R = [\phi_n^{R*}(w)]^{-1} [\phi_n^R(w)]^H, \quad \gamma_n^R = -\zeta_n(w)\rho_n^R(w)$$

proof: Like scalar case. Use $k_n^L(z,w) = B_n(z)\overline{B_n(w)} k_n^R(1/\overline{z}, 1/\overline{w})$

Corollaries

- $z = w \in \mathbb{D}$
  $\{I - \rho_n^L[\rho_n^R]^H\}[\phi_n^{R*}]^H\phi_n^{R*} = [\phi_n^{R*}]^H\phi_n^{R*} - [\phi_n^L]^H\phi_n^L > 0$
- $\therefore \{\cdots\}$ nonsingular, $[\phi_n^{R*}]^H\phi_n^{R*} > 0 \Rightarrow \det\phi_n^{R*} \neq 0$
- $(1 - \gamma_n^L[\gamma_n^R]^H)k_{n-1}^L(w,w) = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
  $(1 - [\gamma_n^R]^H\gamma_n^L)k_{n-1}^R(w,w) = (I - [\rho_n^R]^H\rho_n^L)k_n^R(w,w)$

Corollaries

- $z = w \in \mathbb{D}$
  $\{I - \rho_n^L[\rho_n^R]^H\}[\phi_n^{R*}]^H\phi_n^{R*} = [\phi_n^{R*}]^H\phi_n^{R*} - [\phi_n^L]^H\phi_n^L > 0$
- $\therefore \{\cdots\}$ nonsingular, $[\phi_n^{R*}]^H\phi_n^{R*} > 0 \Rightarrow \det\phi_n^{R*} \neq 0$
- $(1 - \gamma_n^L[\gamma_n^R]^H)k_{n-1}^L(w,w) = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
  $(1 - [\gamma_n^R]^H\gamma_n^L)k_{n-1}^R(w,w) = (I - [\rho_n^R]^H\rho_n^L)k_n^R(w,w)$

Corollaries

- $z = w \in \mathbb{D}$
  $\{I - \rho_n^L[\rho_n^R]^H\}[\phi_n^{R*}]^H\phi_n^{R*} = [\phi_n^{R*}]^H\phi_n^{R*} - [\phi_n^L]^H\phi_n^L > 0$
- $\therefore \{\cdots\}$ nonsingular, $[\phi_n^{R*}]^H\phi_n^{R*} > 0 \Rightarrow \det\phi_n^{R*} \neq 0$
- $(1 - \gamma_n^L[\gamma_n^R]^H)k_{n-1}^L(w,w) = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
  $(1 - [\gamma_n^R]^H\gamma_n^L)k_{n-1}^R(w,w) = (I - [\rho_n^R]^H\rho_n^L)k_n^R(w,w)$

Symmetrizing the recurrence

- $L_n(w) = [k_n^L(w,w)]^{L/2}$, $K_n^L(z,w) = [L_n(w)]^{-1} k_n^L(z,w)$
  $R_n(w) = [k_n^R(w,w)]^{R/2}$, $K_n^R(w,z) = k_n^R(w,z)[R_n(w)]^{-1}$
- CD $\Rightarrow \rho_n^L(w) k_{n-1}^R(w,w) = k_{n-1}^L(w,w)\rho_n^R(w)$
- $[L_{n-1}(w)]^{-1}\rho_n^L(w)[R_{n-1}(w)]^{-1} =$
  $[L_{n-1}(w)]^H \rho_n^R(w)[R_{n-1}(w)]^{-1}$
- symmetrized $\rho_n(w)$
- CD $\Rightarrow [\rho_n(w)]^H \rho_n(w) < I$ and $\rho_n(w)[\rho_n(w)]^H < I$ in $\mathbb{D}$
- similarly for $\gamma_n(w)$

Symmetrizing the recurrence

- $L_n(w) = [k_n^L(w,w)]^{L/2}$, $K_n^L(z,w) = [L_n(w)]^{-1} k_n^L(z,w)$
  $R_n(w) = [k_n^R(w,w)]^{R/2}$, $K_n^R(w,z) = k_n^R(w,z)[R_n(w)]^{-1}$
- CD $\Rightarrow \rho_n^L(w) k_{n-1}^R(w,w) = k_{n-1}^L(w,w)\rho_n^R(w)$
- $[L_{n-1}(w)]^{-1}\rho_n^L(w)[R_{n-1}(w)]^{-1} =$
  $[L_{n-1}(w)]^H \rho_n^R(w)[R_{n-1}(w)]^{-1}$
- symmetrized $\rho_n(w)$
- CD $\Rightarrow [\rho_n(w)]^H \rho_n(w) < I$ and $\rho_n(w)[\rho_n(w)]^H < I$ in $\mathbb{D}$
- similarly for $\gamma_n(w)$

Symmetrizing the recurrence

- $L_n(w) = [k_n^L(w,w)]^{L/2}$, $K_n^L(z,w) = [L_n(w)]^{-1} k_n^L(z,w)$
  $R_n(w) = [k_n^R(w,w)]^{R/2}$, $K_n^R(w,z) = k_n^R(w,z)[R_n(w)]^{-1}$
- CD $\Rightarrow \rho_n^L(w) k_{n-1}^R(w,w) = k_{n-1}^L(w,w)\rho_n^R(w)$
- $[L_{n-1}(w)]^{-1}\rho_n^L(w)[R_{n-1}(w)]^{-1} =$
  $[L_{n-1}(w)]^H \rho_n^R(w)[R_{n-1}(w)]^{-1}$
- symmetrized $\rho_n(w)$
- CD $\Rightarrow [\rho_n(w)]^H \rho_n(w) < I$ and $\rho_n(w)[\rho_n(w)]^H < I$ in $\mathbb{D}$
- similarly for $\gamma_n(w)$

Symmetrizing the recurrence

- $L_n(w) = [k_n^L(w,w)]^{L/2}$, $K_n^L(z,w) = [L_n(w)]^{-1} k_n^L(z,w)$
  $R_n(w) = [k_n^R(w,w)]^{R/2}$, $K_n^R(w,z) = k_n^R(w,z)[R_n(w)]^{-1}$
- CD $\Rightarrow \rho_n^L(w) k_{n-1}^R(w,w) = k_{n-1}^L(w,w)\rho_n^R(w)$
- $[L_{n-1}(w)]^{-1}\rho_n^L(w)[R_{n-1}(w)]^{-1} =$
  $[L_{n-1}(w)]^H \rho_n^R(w)[R_{n-1}(w)]^{-1}$
- symmetrized $\rho_n(w)$
- CD $\Rightarrow [\rho_n(w)]^H \rho_n(w) < I$ and $\rho_n(w)[\rho_n(w)]^H < I$ in $\mathbb{D}$
- similarly for $\gamma_n(w)$

Symmetrizing the recurrence

- $L_n(w) = [k_n^L(w,w)]^{L/2}$, $K_n^L(z,w) = [L_n(w)]^{-1} k_n^L(z,w)$
  $R_n(w) = [k_n^R(w,w)]^{R/2}$, $K_n^R(w,z) = k_n^R(w,z)[R_n(w)]^{-1}$
- CD $\Rightarrow \rho_n^L(w) k_{n-1}^R(w,w) = k_{n-1}^L(w,w)\rho_n^R(w)$
- $[L_{n-1}(w)]^{-1}\rho_n^L(w)[R_{n-1}(w)]^{-1} =$
  $[L_{n-1}(w)]^H \rho_n^R(w)[R_{n-1}(w)]^{-1}$
- symmetrized $\rho_n(w)$
- CD $\Rightarrow [\rho_n(w)]^H \rho_n(w) < I$ and $\rho_n(w)[\rho_n(w)]^H < I$ in $\mathbb{D}$
- similarly for $\gamma_n(w)$

Symmetrizing the recurrence

- $L_n(w) = [k_n^L(w,w)]^{L/2}$, $K_n^L(z,w) = [L_n(w)]^{-1} k_n^L(z,w)$
  $R_n(w) = [k_n^R(w,w)]^{R/2}$, $K_n^R(w,z) = k_n^R(w,z)[R_n(w)]^{-1}$
- CD $\Rightarrow \rho_n^L(w) k_{n-1}^R(w,w) = k_{n-1}^L(w,w)\rho_n^R(w)$
- $[L_{n-1}(w)]^{-1}\rho_n^L(w)[R_{n-1}(w)]^{-1} =$
  $[L_{n-1}(w)]^H \rho_n^R(w)[R_{n-1}(w)]^{-1}$
- symmetrized $\rho_n(w)$
- CD $\Rightarrow [\rho_n(w)]^H \rho_n(w) < I$ and $\rho_n(w)[\rho_n(w)]^H < I$ in $\mathbb{D}$
- similarly for $\gamma_n(w)$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1} L_n(w) = F(\rho_n(w))^{-1/2} F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2} G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2 \rho_n^L [\rho_n^R]^H) L_{n-1}$
- $L_{n-1} F(\gamma_n) L_{n-1}^H = (I - \rho_n^L [\rho_n^R]^H) k_n^L(w, w)$
- similarly $I - \rho_n^L [\rho_n^R]^H = L_{n-1} F(\rho_n) L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n) L_{n-1}^H L_n^{-H} = F(\rho_n) L_{n-1}^{-1} L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1}L_n(w) = F(\rho_n(w))^{-1/2}F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2}G(\gamma_n(w))^{1/2}$$

proof:

Skip proof

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)L_{n-1}$
- $L_{n-1}F(\gamma_n)L_{n-1}^H = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
- similarly $I - \rho_n^L[\rho_n^R]^H = L_{n-1}F(\rho_n)L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n)L_{n-1}^H L_n^{-H} = F(\rho_n)L_{n-1}^{-1}L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1}L_n(w) = F(\rho_n(w))^{-1/2}F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2}G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)L_{n-1}$
- $L_{n-1}F(\gamma_n)L_{n-1}^H = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
- similarly $I - \rho_n^L[\rho_n^R]^H = L_{n-1}F(\rho_n)L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n)L_{n-1}^H L_n^{-H} = F(\rho_n)L_{n-1}^{-1}L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1}L_n(w) = F(\rho_n(w))^{-1/2}F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2}G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)L_{n-1}$
- $\therefore L_{n-1}F(\gamma_n)L_{n-1}^H = (I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)k_{n-1}^L(w,w)$

  $= (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$ (previous corol.)
- similarly $I - \rho_n^L[\rho_n^R]^H = L_{n-1}F(\rho_n)L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n)L_{n-1}^H L_n^{-H} = F(\rho_n)L_{n-1}^{-1}L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1}L_n(w) = F(\rho_n(w))^{-1/2}F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2}G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)L_{n-1}$
- $L_{n-1}F(\gamma_n)L_{n-1}^H = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
- similarly $I - \rho_n^L[\rho_n^R]^H = L_{n-1}F(\rho_n)L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n)L_{n-1}^H L_n^{-H} = F(\rho_n)L_{n-1}^{-1}L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1}L_n(w) = F(\rho_n(w))^{-1/2}F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2}G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)L_{n-1}$
- $L_{n-1}F(\gamma_n)L_{n-1}^H = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
- similarly $I - \rho_n^L[\rho_n^R]^H = L_{n-1}F(\rho_n)L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n)L_{n-1}^H L_n^{-H} = F(\rho_n)L_{n-1}^{-1}L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1}L_n(w) = F(\rho_n(w))^{-1/2}F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2}G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) = L_{n-1}^{-1}(I - |\zeta_n|^2\rho_n^L[\rho_n^R]^H)L_{n-1}$
- $L_{n-1}F(\gamma_n)L_{n-1}^H = (I - \rho_n^L[\rho_n^R]^H)k_n^L(w,w)$
- similarly $I - \rho_n^L[\rho_n^R]^H = L_{n-1}F(\rho_n)L_{n-1}^{-1}$
- $\Rightarrow F(\gamma_n)L_{n-1}^H L_n^{-H} = F(\rho_n)L_{n-1}^{-1}L_n$

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1} L_n(w) = F(\rho_n(w))^{-1/2} F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2} G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) L_{n-1}^H L_n^{-H} = F(\rho_n) L_{n-1}^{-1} L_n$
- $\Rightarrow [F(\rho_n)]^{-1} F(\gamma_n) = [L_{n-1}^{-1} L_n L_n^H L_{n-1}^H] > 0$
- $Q_n = [\cdots]^{1/2} = L_{n-1}^{-1} L_n U_n$, U_n unitary
- include U_n in L_n
- QED

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1} L_n(w) = F(\rho_n(w))^{-1/2} F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2} G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) L_{n-1}^H L_n^{-H} = F(\rho_n) L_{n-1}^{-1} L_n$
- $\Rightarrow [F(\rho_n)]^{-1} F(\gamma_n) = [L_{n-1}^{-1} L_n L_n^H L_{n-1}^H] > 0$
- $Q_n = [\cdots]^{1/2} = L_{n-1}^{-1} L_n U_n$, U_n unitary
- include U_n in L_n
- QED

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1} L_n(w) = F(\rho_n(w))^{-1/2} F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2} G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) L_{n-1}^H L_n^{-H} = F(\rho_n) L_{n-1}^{-1} L_n$
- $\Rightarrow [F(\rho_n)]^{-1} F(\gamma_n) = [L_{n-1}^{-1} L_n L_n^H L_{n-1}^H] > 0$
- $Q_n = [\cdots]^{1/2} = L_{n-1}^{-1} L_n U_n$, U_n unitary
- include U_n in L_n
- QED

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1} L_n(w) = F(\rho_n(w))^{-1/2} F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2} G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) L_{n-1}^H L_n^{-H} = F(\rho_n) L_{n-1}^{-1} L_n$
- $\Rightarrow [F(\rho_n)]^{-1} F(\gamma_n) = [L_{n-1}^{-1} L_n L_n^H L_{n-1}^H] > 0$
- $Q_n = [\cdots]^{1/2} = L_{n-1}^{-1} L_n U_n$, U_n unitary
- include U_n in L_n
- QED

Symmetrizing the recurrence

Lemma

$F(\rho) := I - \rho\rho^H$, $G(\rho) := I - \rho^H\rho$

One may choose L_n and R_n such that

$$[L_{n-1}(w)]^{-1} L_n(w) = F(\rho_n(w))^{-1/2} F(\gamma_n(w))^{1/2}$$
$$R_n(w)[R_{n-1}(w)]^{-1} = G(\rho_n(w))^{-1/2} G(\gamma_n(w))^{1/2}$$

proof:

- $F(\gamma_n) L_{n-1}^H L_n^{-H} = F(\rho_n) L_{n-1}^{-1} L_n$
- $\Rightarrow [F(\rho_n)]^{-1} F(\gamma_n) = [L_{n-1}^{-1} L_n L_n^H L_{n-1}^H] > 0$
- $Q_n = [\cdots]^{1/2} = L_{n-1}^{-1} L_n U_n$, U_n unitary
- include U_n in L_n
- QED

Symmetrized recurrence

Theorem

$F(\rho) = I - \rho\rho^H$, $G(\rho) = I - \rho^H\rho$

Halmos extension:

$$H(\rho) = \begin{bmatrix} G(\rho)^{-1/2} & 0 \\ 0 & F(\rho)^{-1/2} \end{bmatrix} \begin{bmatrix} I & \rho^H \\ \rho & I \end{bmatrix} =$$

$$H(\rho) = \begin{bmatrix} I & \rho^H \\ \rho & I \end{bmatrix} \begin{bmatrix} G(\rho)^{-1/2} & 0 \\ 0 & F(\rho)^{-1/2} \end{bmatrix}$$

$$\text{Then } \begin{bmatrix} K_n^{R*}(w,z) \\ K_n^L(z,w) \end{bmatrix} = \theta_n(z,w) \begin{bmatrix} K_{n-1}^{R*}(w,z) \\ K_{n-1}^L(z,w) \end{bmatrix}$$

$$\theta_n(z,w) = H(\rho_n(w))Z_n(z)H(\gamma_n(w)), \quad Z_n(w) = \begin{bmatrix} \zeta_n(z)I & 0 \\ 0 & I \end{bmatrix}$$

Kernels of 2nd kind

- $L_0^{R*} = I = L_0^L$ and for $n \geq 1$
$$\begin{bmatrix} L_n^{R*}(w,z) \\ -L_n^L(z,w) \end{bmatrix} = \theta_n(z,w) \begin{bmatrix} L_{n-1}^{R*}(w,z) \\ -L_{n-1}^L(z,w) \end{bmatrix}$$
- $$\begin{bmatrix} K_n^{R*} & L_n^{R*} \\ K_n^L & -L_n^L \end{bmatrix} = \Theta_n \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$
$\Theta_n = \theta_n \theta_{n-1} \cdots \theta_1$
- $\Leftrightarrow \Theta_n = \frac{1}{2} \begin{bmatrix} K_n^{R*} + L_n^{R*} & K_n^{R*} - L_n^{R*} \\ K_n^L - L_n^L & K_n^L + L_n^L \end{bmatrix}$

Kernels of 2nd kind

- $L_0^{R*} = I = L_0^L$ and for $n \geq 1$
$$\begin{bmatrix} L_n^{R*}(w,z) \\ -L_n^L(z,w) \end{bmatrix} = \theta_n(z,w) \begin{bmatrix} L_{n-1}^{R*}(w,z) \\ -L_{n-1}^L(z,w) \end{bmatrix}$$
- $$\begin{bmatrix} K_n^{R*} & L_n^{R*} \\ K_n^L & -L_n^L \end{bmatrix} = \Theta_n \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$
$\Theta_n = \theta_n \theta_{n-1} \cdots \theta_1$
- $\Leftrightarrow \Theta_n = \frac{1}{2} \begin{bmatrix} K_n^{R*} + L_n^{R*} & K_n^{R*} - L_n^{R*} \\ K_n^L - L_n^L & K_n^L + L_n^L \end{bmatrix}$

Kernels of 2nd kind

- $L_0^{R*} = I = L_0^L$ and for $n \geq 1$
$$\begin{bmatrix} L_n^{R*}(w,z) \\ -L_n^L(z,w) \end{bmatrix} = \theta_n(z,w) \begin{bmatrix} L_{n-1}^{R*}(w,z) \\ -L_{n-1}^L(z,w) \end{bmatrix}$$
- $$\begin{bmatrix} K_n^{R*} & L_n^{R*} \\ K_n^L & -L_n^L \end{bmatrix} = \Theta_n \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$
$\Theta_n = \theta_n \theta_{n-1} \cdots \theta_1$
- $\Leftrightarrow \Theta_n = \frac{1}{2} \begin{bmatrix} K_n^{R*} + L_n^{R*} & K_n^{R*} - L_n^{R*} \\ K_n^L - L_n^L & K_n^L + L_n^L \end{bmatrix}$

$$\Theta_n^* J \Theta_n = \Theta_n J \Theta_n^* = J$$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is *-J-unitary

- $K_n^L(z,w)L_n^{L*}(z,w) + L_n^L(z,w)K_n^{L*}(z,w) = 2B_n(z)I$
- $K_n^{R*}(w,z)L_n^R(w,z) + L_n^{R*}(w,z)K_n^R(w,z) = 2B_n(z)I$
- $K_n^{L*}(z,w)K_n^L(z,w) = K_n^R(w,z)K_n^{R*}(w,z)$
- $K_n^{L*}(z,w)L_n^L(z,w) = L_n^R(w,z)L_n^{R*}(w,z)$

$$\Theta_n^* J \Theta_n = \Theta_n J \Theta_n^* = J$$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is *-J-unitary

- $K_n^L(z,w)L_n^{L*}(z,w) + L_n^L(z,w)K_n^{L*}(z,w) = 2B_n(z)I$
- $K_n^{R*}(w,z)L_n^R(w,z) + L_n^{R*}(w,z)K_n^R(w,z) = 2B_n(z)I$
- $K_n^{L*}(z,w)K_n^L(z,w) = K_n^R(w,z)K_n^{R*}(w,z)$
- $K_n^{L*}(z,w)L_n^L(z,w) = L_n^R(w,z)L_n^{R*}(w,z)$

$$\Theta_n^* J \Theta_n = \Theta_n J \Theta_n^* = J$$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is *-J-unitary

- $K_n^L(z,w)L_n^{L*}(z,w) + L_n^L(z,w)K_n^{L*}(z,w) = 2B_n(z)I$
- $K_n^{R*}(w,z)L_n^R(w,z) + L_n^{R*}(w,z)K_n^R(w,z) = 2B_n(z)I$
- $K_n^{L*}(z,w)K_n^L(z,w) = K_n^R(w,z)K_n^{R*}(w,z)$
- $K_n^{L*}(z,w)L_n^L(z,w) = L_n^R(w,z)L_n^{R*}(w,z)$

$$\Theta_n^* J \Theta_n = \Theta_n J \Theta_n^* = J$$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is *-J-unitary

- $K_n^L(z,w)L_n^{L*}(z,w) + L_n^L(z,w)K_n^{L*}(z,w) = 2B_n(z)I$
- $K_n^{R*}(w,z)L_n^R(w,z) + L_n^{R*}(w,z)K_n^R(w,z) = 2B_n(z)I$
- $K_n^{L*}(z,w)K_n^L(z,w) = K_n^R(w,z)K_n^{R*}(w,z)$
- $K_n^{L*}(z,w)L_n^L(z,w) = L_n^R(w,z)L_n^{R*}(w,z)$

$$\Theta_n^* J \Theta_n = \Theta_n J \Theta_n^* = J$$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is *-J-unitary

- $K_n^L(z,w)L_n^{L*}(z,w) + L_n^L(z,w)K_n^{L*}(z,w) = 2B_n(z)I$
- $K_n^{R*}(w,z)L_n^R(w,z) + L_n^{R*}(w,z)K_n^R(w,z) = 2B_n(z)I$
- $K_n^{L*}(z,w)K_n^L(z,w) = K_n^R(w,z)K_n^{R*}(w,z)$
- $K_n^{L*}(z,w)L_n^L(z,w) = L_n^R(w,z)L_n^{R*}(w,z)$

$\Theta_n J \Theta_n^H \leq J$ in $\mathbb{D}$, $\Theta_n J \Theta_n^H = J$ on $\mathbb{T}$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is J-contractive in $\mathbb{D}$ and J-unitary on $\mathbb{T}$

- $K_n^L(z,w), L_n^L(z,w), K_n^R(w,z), L_n^R(w,z)$ invertible $w \in \mathbb{D}$, $z \in \mathbb{D} \cup \mathbb{T}$. Hence also ϕ_n^{R*} and ϕ_n^{L*} invertible in $\mathbb{D} \cup \mathbb{T}$
- $[K_n^L(z,w)]^{-1} L_n^L(z,w)$ and $L_n^R(w,z)[K_n^R(w,z)]^{-1}$ are Carathéodory for $w \in \mathbb{D}$
- $[K_n^L(z,w)]^{-1}$ and $[K_n^R(w,z)]^{-1}$ are in $\mathbb{H}_2$

$\Theta_n J \Theta_n^H \leq J$ in $\mathbb{D}$, $\Theta_n J \Theta_n^H = J$ on $\mathbb{T}$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is J-contractive in $\mathbb{D}$ and J-unitary on $\mathbb{T}$

- $K_n^L(z,w), L_n^L(z,w), K_n^R(w,z), L_n^R(w,z)$ invertible $w \in \mathbb{D}$, $z \in \mathbb{D} \cup \mathbb{T}$. Hence also ϕ_n^{R*} and ϕ_n^{L*} invertible in $\mathbb{D} \cup \mathbb{T}$
- $[K_n^L(z,w)]^{-1} L_n^L(z,w)$ and $L_n^R(w,z)[K_n^R(w,z)]^{-1}$ are Carathéodory for $w \in \mathbb{D}$
- $[K_n^L(z,w)]^{-1}$ and $[K_n^R(w,z)]^{-1}$ are in $\mathbb{H}_2$

$\Theta_n J \Theta_n^H \leq J$ in $\mathbb{D}$, $\Theta_n J \Theta_n^H = J$ on $\mathbb{T}$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is J-contractive in $\mathbb{D}$ and J-unitary on $\mathbb{T}$

- $K_n^L(z,w), L_n^L(z,w), K_n^R(w,z), L_n^R(w,z)$ invertible $w \in \mathbb{D}$, $z \in \mathbb{D} \cup \mathbb{T}$. Hence also ϕ_n^{R*} and ϕ_n^{L*} invertible in $\mathbb{D} \cup \mathbb{T}$
- $[K_n^L(z,w)]^{-1} L_n^L(z,w)$ and $L_n^R(w,z)[K_n^R(w,z)]^{-1}$ are Carathéodory for $w \in \mathbb{D}$
- $[K_n^L(z,w)]^{-1}$ and $[K_n^R(w,z)]^{-1}$ are in $\mathbb{H}_2$

$\Theta_n J \Theta_n^H \leq J$ in $\mathbb{D}$, $\Theta_n J \Theta_n^H = J$ on $\mathbb{T}$

with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Θ_n is J-contractive in $\mathbb{D}$ and J-unitary on $\mathbb{T}$

- $K_n^L(z,w), L_n^L(z,w), K_n^R(w,z), L_n^R(w,z)$ invertible $w \in \mathbb{D}$, $z \in \mathbb{D} \cup \mathbb{T}$. Hence also ϕ_n^{R*} and ϕ_n^{L*} invertible in $\mathbb{D} \cup \mathbb{T}$
- $[K_n^L(z,w)]^{-1} L_n^L(z,w)$ and $L_n^R(w,z)[K_n^R(w,z)]^{-1}$ are Carathéodory for $w \in \mathbb{D}$
- $[K_n^L(z,w)]^{-1}$ and $[K_n^R(w,z)]^{-1}$ are in $\mathbb{H}_2$

Lemma

- !! temporarily ϕ_n^L and ϕ_n^R are orthogonal (not normalized) !!
- Lemma: ϕ_0^L and ϕ_0^R constant in $\mathbb{C}^{N\times N}$

  There exist C and D matrices such that

  $$f_k^L(z) = \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L(z) \qquad f_k^R(z) = \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^R(z)$$
  $$\phi_k^L(z) = D_k^L f_k^L(z) + C_k^L f_k^{R*}(z) \qquad \phi_k^R(z) = f_k^R(z) D_k^R + f_k^{L*}(z) C_k^R$$

  defines an orthogonal set of L and R MORF.

Lemma

- !! temporarily ϕ_n^L and ϕ_n^R are orthogonal (not normalized) !!
- Lemma: ϕ_0^L and ϕ_0^R constant in $\mathbb{C}^{N\times N}$

  There exist C and D matrices such that

$$f_k^L(z) = \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L(z) \qquad f_k^R(z) = \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^R(z)$$
$$\phi_k^L(z) = D_k^L f_k^L(z) + C_k^L f_k^{R*}(z) \qquad \phi_k^R(z) = f_k^R(z) D_k^R + f_k^{L*}(z) C_k^R$$

  defines an orthogonal set of L and R MORF.

Proof (induction step)

Skip proof

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}\right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k);\quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \left\langle\left\langle f_k^{R*}, \phi_0^L\right\rangle\right\rangle_{\ell,\mu} \Rightarrow \left\langle\left\langle \phi_k^L, 1\right\rangle\right\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\left\langle \left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2} \right\rangle \right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \left\langle \left\langle f_k^{R*}, \phi_0^L \right\rangle \right\rangle_{\ell,\mu} \Rightarrow \left\langle \left\langle \phi_k^L, 1 \right\rangle \right\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\langle\langle \phi_{k-1}^L, \mathcal{L}_{n-2} \rangle\rangle_{\ell,\mu} = 0 \Rightarrow \left\langle\left\langle \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{n-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{k-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \langle\langle f_k^{R*}, \phi_0^L \rangle\rangle_{\ell,\mu} \Rightarrow \langle\langle \phi_k^L, 1 \rangle\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1}z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\langle\langle \phi_{k-1}^L, \mathcal{L}_{n-2} \rangle\rangle_{\ell,\mu} = 0 \Rightarrow \left\langle\left\langle \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{n-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $\langle\langle \phi_{k-1}^R, \mathcal{L}_{k-2} \rangle\rangle_{r,\mu} = 0 \Rightarrow \langle\langle \phi_{k-1}^{R*}, \zeta_{k-1}\mathcal{L}_{n-2} \rangle\rangle_{\ell,\mu}$
  $\Rightarrow \left\langle\left\langle \frac{1-\overline{\alpha}_{k-1}z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2} \right\rangle\right\rangle_{\ell,\mu}$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k);\quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \langle\langle f_k^{R*}, \phi_0^L \rangle\rangle_{\ell,\mu} \Rightarrow \langle\langle \phi_k^L, 1 \rangle\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\langle\langle \phi_{k-1}^L, \mathcal{L}_{n-2} \rangle\rangle_{\ell,\mu} = 0 \Rightarrow \left\langle\left\langle \frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{n-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $\langle\langle \phi_{k-1}^R, \mathcal{L}_{k-2} \rangle\rangle_{r,\mu} = 0 \Rightarrow \langle\langle \phi_{k-1}^{R*}, \zeta_{k-1}\mathcal{L}_{n-2} \rangle\rangle_{\ell,\mu}$
  $\Rightarrow \left\langle\left\langle \frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{k-2} \right\rangle\right\rangle_{\ell,\mu}$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{k-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1} z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \langle\langle f_k^{R*}, \phi_0^L \rangle\rangle_{\ell,\mu} \Rightarrow \langle\langle \phi_k^L, I \rangle\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2} \right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \left\langle\left\langle f_k^{R*}, \phi_0^L \right\rangle\right\rangle_{\ell,\mu} \Rightarrow \left\langle\left\langle \phi_k^L, I \right\rangle\right\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1} z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}\right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \left\langle\left\langle f_k^{R*}, \phi_0^L\right\rangle\right\rangle_{\ell,\mu} \Rightarrow \left\langle\left\langle \phi_k^L, I\right\rangle\right\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1}z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}\right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \left\langle\left\langle f_k^{R*}, \phi_0^L\right\rangle\right\rangle_{\ell,\mu} \Rightarrow \left\langle\left\langle \phi_k^L, I\right\rangle\right\rangle_{\ell,\mu} = 0$
- QED

Proof (induction step)

- $\phi_k^L = D_k^L \overbrace{\frac{z-\alpha_{k-1}}{1-\overline{\alpha}_k z}\phi_{k-1}^L}^{f_k^L} + C_k^L \overbrace{\frac{1-\overline{\alpha}_{k-1}z}{1-\overline{\alpha}_k z}\phi_{k-1}^{R*}}^{f_k^{R*}} \in \mathcal{L}_k$
- $\left\langle\left\langle \phi_k^L, \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}\right\rangle\right\rangle_{\ell,\mu} = 0$
- $f \in \mathcal{L}_{k-1} \Rightarrow f(z) = g(z) - g(\alpha_k); \quad g \in \frac{z-\alpha_k}{1-\overline{\alpha}_{k-1}z}\mathcal{L}_{k-2}$
- $D_k^L = -C_k^L \left\langle\left\langle f_k^{R*}, \phi_0^L\right\rangle\right\rangle_{\ell,\mu} \Rightarrow \left\langle\left\langle \phi_k^L, I\right\rangle\right\rangle_{\ell,\mu} = 0$
- QED

Another lemma

$C_n^L = \frac{1-\overline{\alpha}_n\alpha_{n-1}}{1-|\alpha_{n-1}|^2}\phi_n^L(\alpha_{n-1})\underbrace{[\phi_{n-1}^{L*}(\alpha_{n-1})]^{-1}}_{\kappa_{n-1}^L}$ used for normalization.

Lemma

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}\begin{bmatrix} I & [\lambda_n^L]^H \\ \lambda_n^R & I \end{bmatrix}\begin{bmatrix} \zeta_{n-1}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \phi_{n-1}^L \\ \phi_{n-1}^{R*} \end{bmatrix}$$

$$\lambda_n^L = -\overline{z}_{n-1}[[D_n^L]^{-1}C_n^L]^H = -\overline{z}_{n-1}\left\langle\left\langle \phi_{n-1}^L, \phi_n^{R*}\right\rangle\right\rangle_{\ell,\mu}^{-1}\left\langle\left\langle \phi_{n-1}^L, \phi_n^L\right\rangle\right\rangle_{\ell,\mu}$$
$$\lambda_n^R = -\overline{z}_{n-1}[C_n^R[D_n^R]^{-1}]^H =$$
$$-\overline{z}_{n-1}\left\langle\left\langle \phi_n^R, \phi_{n-1}^R\right\rangle\right\rangle_{r,\mu}\left\langle\left\langle \phi_n^{L*}, \phi_{n-1}^R\right\rangle\right\rangle_{r,\mu}^{-1}$$

Another lemma

$C_n^L = \frac{1-\overline{\alpha}_n\alpha_{n-1}}{1-|\alpha_{n-1}|^2}\phi_n^L(\alpha_{n-1})\underbrace{[\phi_{n-1}^{L*}(\alpha_{n-1})]^{-1}}_{\kappa_{n-1}^L}$ used for normalization.

Lemma

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}\begin{bmatrix} I & [\lambda_n^L]^H \\ \lambda_n^R & I \end{bmatrix}\begin{bmatrix} \zeta_{n-1}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \phi_{n-1}^L \\ \phi_{n-1}^{R*} \end{bmatrix}$$

$\lambda_n^L = -\overline{z}_{n-1}[[D_n^L]^{-1}C_n^L]^H = -\overline{z}_{n-1}\left\langle\!\left\langle \phi_{n-1}^L, \phi_n^{R*}\right\rangle\!\right\rangle_{\ell,\mu}^{-1}\left\langle\!\left\langle \phi_{n-1}^L, \phi_n^L\right\rangle\!\right\rangle_{\ell,\mu}$
$\lambda_n^R = -\overline{z}_{n-1}[C_n^R[D_n^R]^{-1}]^H =$
$-\overline{z}_{n-1}\left\langle\!\left\langle \phi_n^R, \phi_{n-1}^R\right\rangle\!\right\rangle_{r,\mu}\left\langle\!\left\langle \phi_n^{L*}, \phi_{n-1}^R\right\rangle\!\right\rangle_{r,\mu}^{-1}$

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

Skip proof

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- def: $N_n^L = \langle\langle \phi_n^L, \phi_n^L \rangle\rangle_{\ell,\mu}^{L/2}, \quad N_n^R = \langle\langle \phi_n^L, \phi_n^L \rangle\rangle_{r,\mu}^{R/2}$
- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- def: $N_n^L = \langle\langle \phi_n^L, \phi_n^L \rangle\rangle_{\ell,\mu}^{L/2}, \quad N_n^R = \langle\langle \phi_n^L, \phi_n^L \rangle\rangle_{r,\mu}^{R/2}$
- $\phi_n^L \rightarrow [N_n^L]^{-1} \phi_n^L [N_n^R]^{-1}$ is L-orthonormal
- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- def: $N_n^L = \langle\langle \phi_n^L, \phi_n^L \rangle\rangle_{\ell,\mu}^{L/2}, \quad N_n^R = \langle\langle \phi_n^L, \phi_n^L \rangle\rangle_{r,\mu}^{R/2}$
- $\phi_n^L \rightarrow [N_n^L]^{-1} \phi_n^L [N_n^R]^{-1}$ is L-orthonormal
- L- or R-square roots s.t. $\kappa_n^L > 0$
- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \underbrace{[\phi_n^L(\alpha_{n-1})]^H [\phi_n^{L*}(\alpha_{n-1})]^{-1}}_{\rho_n^L(\alpha_{n-1})} [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H = \kappa_{n-1}^R \rho_n^L [\kappa_{n-1}^L]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H = \eta_n \rho_n(\alpha_{n-1})$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H = \kappa_{n-1}^R \rho_n^L [\kappa_{n-1}^L]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Normalized MORF

Lemma

Normalized MORF: $\Rightarrow \lambda_n^L = \lambda_n^R = \eta_n \rho_n(\alpha_{n-1})$,
$\eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$

proof:

- $\lambda_n^L = \eta_n [\kappa_{n-1}^R]^{-1} \rho_n^L(\alpha_{n-1}) [\kappa_{n-1}^L]^H = \eta_n \rho_n(\alpha_{n-1})$
- $\rho_n = [K_{n-1}^L]^{L/2} \rho_n^L [[K_{n-1}^R]^{R/2}]^H = \kappa_{n-1}^R \rho_n^L [\kappa_{n-1}^L]^H$
  $K_n^L(\alpha_n, \alpha_n) = \kappa_n^R [\kappa_n^R]^H$, $K_n^R(\alpha_n, \alpha_n) = [\kappa_n^L]^H \kappa_n^L$
- QED

Recurrence orthonormal ϕ_n

Theorem

$$\begin{bmatrix} \phi_n^L(z) \\ \phi_n^{R*}(z) \end{bmatrix} = t_n(z) \begin{bmatrix} \phi_{n-1}^L(z) \\ \phi_{n-1}^{R*}(z) \end{bmatrix}$$

$$t_n(z) = \frac{\sqrt{1-|\alpha_n|^2}}{\sqrt{1-|\alpha_{n-1}|^2}} \frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z} H(\lambda_n) Z_{n-1}(z)$$

$$\lambda_n = \eta_n \rho_n(\alpha_{n-1}), \quad \eta_n = \frac{z_n}{z_{n-1}} \frac{1-\overline{\alpha}_{n-1}\alpha_n}{1-\overline{\alpha}_n\alpha_{n-1}} \in \mathbb{T}$$

Proof

Skip proof

- Result of RHS is $\hat{\phi}_n^{L/R*}$ is orthogonal
- $\phi_n^L = N_n^{-1}\hat{\phi}_n^L$, $\phi_n^{R*} = M_n^{-1}\hat{\phi}_n^{R*}$
- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1-|\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R[\kappa_{n-1}^R]^H$$

- $$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\alpha_{n-1}|^2}{1-\bar{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H[\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$$

- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1-|\alpha_j|^2}$
- Solutions:

$$N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}, \quad M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2},$$

U_n, V_n unitary, used to make leading coef > 0

Proof

- Result of RHS is $\hat{\phi}_n^{L/R*}$ is orthogonal
- $\phi_n^L = N_n^{-1}\hat{\phi}_n^L$, $\phi_n^{R*} = M_n^{-1}\hat{\phi}_n^{R*}$
- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1-|\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R [\kappa_{n-1}^R]^H$$

- $$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\alpha_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H [\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$$

- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1-|\alpha_j|^2}$
- Solutions:
  $N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
  U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- Result of RHS is $\hat{\phi}_n^{L/R*}$ is orthogonal
- $\phi_n^L = N_n^{-1}\hat{\phi}_n^L$, $\phi_n^{R*} = M_n^{-1}\hat{\phi}_n^{R*}$
- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1-|\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R[\kappa_{n-1}^R]^H$$

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\alpha_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H[\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$$

- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1-|\alpha_j|^2}$
- Solutions:

$$N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}, \quad M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2},$$

U_n, V_n unitary, used to make leading coef > 0

- QED

Proof

- Result of RHS is $\hat{\phi}_n^{L/R*}$ is orthogonal
- $\phi_n^L = N_n^{-1}\hat{\phi}_n^L$, $\phi_n^{R*} = M_n^{-1}\hat{\phi}_n^{R*}$
- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1-|\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R[\kappa_{n-1}^R]^H$$

- $\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\overline{\alpha}_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H[\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1-|\alpha_j|^2}$
- Solutions:
$N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$

$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1 - |\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R [\kappa_{n-1}^R]^H$$

- $$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\overline{\alpha}_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H [\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$$

- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1 - |\alpha_j|^2}$
- Solutions:

$$N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}, \quad M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2},$$

U_n, V_n unitary, used to make leading coef > 0

- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$
$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1 - |\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R [\kappa_{n-1}^R]^H$$
- $\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\overline{\alpha}_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H [\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1 - |\alpha_j|^2}$
- Solutions:
$N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$
  $$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1 - |\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R [\kappa_{n-1}^R]^H$$
- $$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\overline{\alpha}_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H [\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1 - |\alpha_j|^2}$
- Solutions:
  $N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
  U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- CD: $(z = \alpha_{n-1})$
$$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix}^H J \begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = -(1 - |\zeta_n(\alpha_{n-1})|^2)\kappa_{n-1}^R [\kappa_{n-1}^R]^H$$
- $$\begin{bmatrix} \phi_n^L \\ \phi_n^{R*} \end{bmatrix} = \frac{1-|\overline{\alpha}_{n-1}|^2}{1-\overline{\alpha}_n\alpha_{n-1}} \begin{bmatrix} N_n & 0 \\ 0 & M_n \end{bmatrix} \begin{bmatrix} \lambda_n^H [\kappa_{n-1}^R]^H \\ [\kappa_{n-1}^R]^H \end{bmatrix}$$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1 - |\alpha_j|^2}$
- Solutions:
$N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1 - |\alpha_j|^2}$
- Solutions:
  $N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
  U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1-|\alpha_j|^2}$
- Solutions:
  $N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
  U_n, V_n unitary, used to make leading coef > 0
- QED

Proof

- $N_n = [\langle\langle \phi_n^L, \phi_k^L \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$, $M_n = [\langle\langle \phi_n^{R*}, \phi_k^{R*} \rangle\rangle_{\ell,\mu}^{L/2}]^{-1}$
- substitute: $\lambda_n N_n^H \lambda_n^H - M_n^H M_n = -\frac{\gamma_n}{\gamma_{n-1}} I$, $\gamma_j = \sqrt{1 - |\alpha_j|^2}$
- Solutions:
  $N_n = \frac{\gamma_n}{\gamma_{n-1}} U_n (I - \lambda_n^H \lambda_n)^{-1/2}$, $\quad M_n = \frac{\gamma_n}{\gamma_{n-1}} V_n (I - \lambda_n \lambda_n^H)^{-1/2}$,
  U_n, V_n unitary, used to make leading coef > 0
- QED

The ψ_n^L and ψ_n^R

- $\begin{bmatrix} \psi_n^L \\ -\psi_n^{R*} \end{bmatrix} = t_n \begin{bmatrix} \psi_{n-1}^L \\ -\psi_{n-1}^{R*} \end{bmatrix}$
- $n \geq 1$
  $\psi_n^L(z) = \int D(t,z)[\phi_n^L(t) - \phi_n^L(z)]d\mu(t)$
  $\psi_n^R(z) = \int D(t,z)d\mu(t)[\phi_n^R(t) - \phi_n^R(z)]$
- $T_n = t_n t_{n-1} \cdots t_1 = \frac{1}{2} \begin{bmatrix} \phi_n^L + \psi_n^L & \phi_n^L - \psi_n^L \\ \phi_n^{R*} - \psi_n^{R*} & \phi_n^{R*} + \psi_n^{R*} \end{bmatrix}$
- $(1 - \overline{\alpha}_n z) T_n(z)/\gamma_n$ is lossles and *-J-unitary. This implies:

The ψ_n^L and ψ_n^R

- $\begin{bmatrix} \psi_n^L \\ -\psi_n^{R*} \end{bmatrix} = t_n \begin{bmatrix} \psi_{n-1}^L \\ -\psi_{n-1}^{R*} \end{bmatrix}$
- $n \geq 1$
  $\psi_n^L(z) = \int D(t,z)[\phi_n^L(t) - \phi_n^L(z)]d\mu(t)$
  $\psi_n^R(z) = \int D(t,z)d\mu(t)[\phi_n^R(t) - \phi_n^R(z)]$
- $T_n = t_n t_{n-1} \cdots t_1 = \frac{1}{2} \begin{bmatrix} \phi_n^L + \psi_n^L & \phi_n^L - \psi_n^L \\ \phi_n^{R*} - \psi_n^{R*} & \phi_n^{R*} + \psi_n^{R*} \end{bmatrix}$
- $(1 - \overline{\alpha}_n z) T_n(z)/\gamma_n$ is lossles and *-J-unitary. This implies:

The ψ_n^L and ψ_n^R

- $\begin{bmatrix} \psi_n^L \\ -\psi_n^{R*} \end{bmatrix} = t_n \begin{bmatrix} \psi_{n-1}^L \\ -\psi_{n-1}^{R*} \end{bmatrix}$
- $n \geq 1$
  $\psi_n^L(z) = \int D(t,z)[\phi_n^L(t) - \phi_n^L(z)]d\mu(t)$
  $\psi_n^R(z) = \int D(t,z)d\mu(t)[\phi_n^R(t) - \phi_n^R(z)]$
- $T_n = t_n t_{n-1} \cdots t_1 = \frac{1}{2} \begin{bmatrix} \phi_n^L + \psi_n^L & \phi_n^L - \psi_n^L \\ \phi_n^{R*} - \psi_n^{R*} & \phi_n^{R*} + \psi_n^{R*} \end{bmatrix}$
- $(1 - \overline{\alpha}_n z) T_n(z)/\gamma_n$ is lossles and *-J-unitary. This implies:

The ψ_n^L and ψ_n^R

- $\begin{bmatrix} \psi_n^L \\ -\psi_n^{R*} \end{bmatrix} = t_n \begin{bmatrix} \psi_{n-1}^L \\ -\psi_{n-1}^{R*} \end{bmatrix}$
- $n \geq 1$
  $\psi_n^L(z) = \int D(t,z)[\phi_n^L(t) - \phi_n^L(z)]d\mu(t)$
  $\psi_n^R(z) = \int D(t,z)d\mu(t)[\phi_n^R(t) - \phi_n^R(z)]$
- $T_n = t_n t_{n-1} \cdots t_1 = \frac{1}{2} \begin{bmatrix} \phi_n^L + \psi_n^L & \phi_n^L - \psi_n^L \\ \phi_n^{R*} - \psi_n^{R*} & \phi_n^{R*} + \psi_n^{R*} \end{bmatrix}$
- $(1 - \overline{\alpha}_n z) T_n(z)/\gamma_n$ is lossles and *-J-unitary. This implies:

The ψ_n^L and ψ_n^R

- ϕ_n^{R*} and ψ_n^{R*} no zeros in $\mathbb{D} \cup \mathbb{T}$ (i.e. det $\neq 0$)
- $[\phi_n^{R*}]^{-1}\psi_n^{R*}$ and $\psi_n^{R*}[\phi_n^{R*}]^{-1}$: Carathéodory
- $[\phi_n^{L*}]^{-1}$ and $[\psi_n^{R*}]^{-1}$ in $\mathbb{H}_2$
- $\phi_{n*}^R\psi_n^R + \psi_{n*}^R\phi_n^R = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^L\psi_{n*}^L + \psi_n^L\phi_{n*}^L = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^R\phi_n^{R*} = \phi_n^{L*}\phi_n^L$ and $\psi_n^R\psi_n^{R*} = \psi_n^{L*}\psi_n^L$

The ψ_n^L and ψ_n^R

- ϕ_n^{R*} and ψ_n^{R*} no zeros in $\mathbb{D} \cup \mathbb{T}$ (i.e. det $\neq 0$)
- $[\phi_n^{R*}]^{-1}\psi_n^{R*}$ and $\psi_n^{R*}[\phi_n^{R*}]^{-1}$: Carathéodory
- $[\phi_n^{L*}]^{-1}$ and $[\psi_n^{R*}]^{-1}$ in $\mathbb{H}_2$
- $\phi_{n*}^R\psi_n^R + \psi_{n*}^R\phi_n^R = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^L\psi_{n*}^L + \psi_n^L\phi_{n*}^L = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^R\phi_n^{R*} = \phi_n^{L*}\phi_n^L$ and $\psi_n^R\psi_n^{R*} = \psi_n^{L*}\psi_n^L$

The ψ_n^L and ψ_n^R

- ϕ_n^{R*} and ψ_n^{R*} no zeros in $\mathbb{D} \cup \mathbb{T}$ (i.e. det $\neq 0$)
- $[\phi_n^{R*}]^{-1}\psi_n^{R*}$ and $\psi_n^{R*}[\phi_n^{R*}]^{-1}$: Carathéodory
- $[\phi_n^{L*}]^{-1}$ and $[\psi_n^{R*}]^{-1}$ in $\mathbb{H}_2$
- $\phi_{n*}^R\psi_n^R + \psi_{n*}^R\phi_n^R = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^L\psi_{n*}^L + \psi_n^L\phi_{n*}^L = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^R\phi_n^{R*} = \phi_n^{L*}\phi_n^L$ and $\psi_n^R\psi_n^{R*} = \psi_n^{L*}\psi_n^L$

The ψ_n^L and ψ_n^R

- ϕ_n^{R*} and ψ_n^{R*} no zeros in $\mathbb{D} \cup \mathbb{T}$ (i.e. det $\neq 0$)
- $[\phi_n^{R*}]^{-1}\psi_n^{R*}$ and $\psi_n^{R*}[\phi_n^{R*}]^{-1}$: Carathéodory
- $[\phi_n^{L*}]^{-1}$ and $[\psi_n^{R*}]^{-1}$ in $\mathbb{H}_2$
- $\phi_{n*}^R\psi_n^R + \psi_{n*}^R\phi_n^R = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^L\psi_{n*}^L + \psi_n^L\phi_{n*}^L = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^R\phi_n^{R*} = \phi_n^{L*}\phi_n^L$ and $\psi_n^R\psi_n^{R*} = \psi_n^{L*}\psi_n^L$

The ψ_n^L and ψ_n^R

- ϕ_n^{R*} and ψ_n^{R*} no zeros in $\mathbb{D} \cup \mathbb{T}$ (i.e. det $\neq 0$)
- $[\phi_n^{R*}]^{-1}\psi_n^{R*}$ and $\psi_n^{R*}[\phi_n^{R*}]^{-1}$: Carathéodory
- $[\phi_n^{L*}]^{-1}$ and $[\psi_n^{R*}]^{-1}$ in $\mathbb{H}_2$
- $\phi_{n*}^R\psi_n^R + \psi_{n*}^R\phi_n^R = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^L\psi_{n*}^L + \psi_n^L\phi_{n*}^L = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^R\phi_n^{R*} = \phi_n^{L*}\phi_n^L$ and $\psi_n^R\psi_n^{R*} = \psi_n^{L*}\psi_n^L$

The ψ_n^L and ψ_n^R

- ϕ_n^{R*} and ψ_n^{R*} no zeros in $\mathbb{D} \cup \mathbb{T}$ (i.e. det $\neq 0$)
- $[\phi_n^{R*}]^{-1}\psi_n^{R*}$ and $\psi_n^{R*}[\phi_n^{R*}]^{-1}$: Carathéodory
- $[\phi_n^{L*}]^{-1}$ and $[\psi_n^{R*}]^{-1}$ in $\mathbb{H}_2$
- $\phi_{n*}^R\psi_n^R + \psi_{n*}^R\phi_n^R = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^L\psi_{n*}^L + \psi_n^L\phi_{n*}^L = 2P(\cdot,\alpha_n)I$ on $\mathbb{T}$
- $\phi_n^R\phi_n^{R*} = \phi_n^{L*}\phi_n^L$ and $\psi_n^R\psi_n^{R*} = \psi_n^{L*}\psi_n^L$

Inner product

Theorem
$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\langle\langle p/\pi_n, k_n^L(\cdot, w)\rangle\rangle_{\ell,\mu_n^L} = -I_2$ (CD)
- QED

Inner product

Theorem
$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

Skip proof

- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\langle\langle p/\pi_n, k_n^L(\cdot, w)\rangle\rangle_{\ell,\mu_n^L} = -I_2$ (CD)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- $k_n^L(z, \xi_k)$, $k = 0, \ldots, n$ is basis for $\mathcal{L}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\langle\langle p/\pi_n, k_n^L(\cdot, w)\rangle\rangle_{\ell,\mu_n^L} = -l_2$ (CD)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- $k_n^L(z, \xi_k)$, $k = 0, \ldots, n$ is basis for $\mathcal{L}_n$
- Hence prove $\left\langle\left\langle f, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu} = \left\langle\left\langle f, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L}$
- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = -l_2$ (CD)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle f, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu} = \left\langle\left\langle f, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L}$
- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = -I_2$ (CD)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = -I_2$ (CD)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = I_1 - I_2$ (CD)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = I_1 - I_2$ (CD)
- $I_1 = [\int p(t)[(\phi_{nn}^{R*})_*\phi_{nn}^{R*}]^{-1}[\phi_{nn}^{R*}]_* \frac{t}{t-w} d\lambda(t)] \frac{\phi_{nn}^{R*}(w)}{\pi_n(w)} = \frac{p(w)}{\pi_n(w)}$ (Cauchy)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)} - I_2$ (CD)
- $I_1 = [\int p(t)[(\phi_{nn}^{R*})_*\phi_{nn}^{R*}]^{-1}[\phi_{nn}^{R*}]_* \frac{t}{t-w} d\lambda(t)] \frac{\phi_{nn}^{R*}(w)}{\pi_n(w)} = \frac{p(w)}{\pi_n(w)}$ (Cauchy)
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)} - I_2$ (CD)
- $I_2 = [\int p(t)[(\phi_{nn}^{R*})_*\phi_{nn}^{R*}]^{-1}[\phi_{nn}^L]_* \frac{1}{t-w} d\lambda(t)] \frac{\phi_{nn}^L(w)}{\pi_n(w)} = 0$
  Because $(\phi_{nn}^{R*})_*\phi_{nn}^{R*} = \phi_{nn}^{L*}(\phi_{nn}^{L*})_*$
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)} - I_2$ (CD)
- $I_2 = [\int p(t)[(\phi_{nn}^{R*})_*\phi_{nn}^{R*}]^{-1}[\phi_{nn}^L]_* \frac{1}{t-w} d\lambda(t)] \frac{\phi_{nn}^L(w)}{\pi_n(w)} = 0$
  Because $(\phi_{nn}^{R*})_*\phi_{nn}^{R*} = \phi_{nn}^{L*}(\phi_{nn}^{L*})_*$
  $Int = \int [\phi_{nn}^{L*}]^{-1} \frac{p^*(t)t}{1-\overline{w}t} d\lambda(t) = 0$
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$
- $I_2 = [\int p(t)[(\phi_{nn}^{R*})_*\phi_{nn}^{R*}]^{-1}[\phi_{nn}^L]_* \frac{1}{t-w} d\lambda(t)] \frac{\phi_{nn}^L(w)}{\pi_n(w)} = 0$
  Because $(\phi_{nn}^{R*})_*\phi_{nn}^{R*} = \phi_{nn}^{L*}(\phi_{nn}^{L*})_*$
  $Int = \int [\phi_{nn}^{L*}]^{-1} \frac{p^*(t)t}{1-\overline{w}t} d\lambda(t) = 0$
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\mu_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1} d\lambda(t)$.

proof:

- Hence prove $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$, $p \in \mathcal{P}_n$
- $\phi_{k,n}^R$ MOPVM $\mu/|\pi_n|^2$ then $K_n^L(z,0) = \phi_{nn}^{R*}(z)/\pi_n(z)$
- $\left\langle\left\langle p/\pi_n, k_n^L(\cdot, w)\right\rangle\right\rangle_{\ell,\mu_n^L} = \frac{p(w)}{\pi_n(w)}$
- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- QED

Skip proof

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $K_n^L(z,0)]^{-1}L_n^L(z,0)$ is Carathéodory and $\frac{1}{2}[\Omega_n^L(t) + \Omega_{n*}^L(t)] = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}$
- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\sigma_n^L(z) = [K_n^L(z,0)]^{-1}$ is outer left spectral factor in $\mathbb{H}_2$
- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\nu = \mu - \mu_n^L$
  $\frac{\Omega_\mu(z)-\Omega_n^L(z)}{B_n(z)} = \int[\frac{1}{B_n(z)} - \frac{1}{B_n(t)}]D(t,z)d\nu(t) + \int \frac{D(t,z)}{B_n(t)}d\nu(t)$
- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\nu = \mu - \mu_n^L$

$$\frac{\Omega_\mu(z)-\Omega_n^L(z)}{B_n(z)} = \int[\frac{1}{B_n(z)} - \frac{1}{B_n(t)}]D(t,z)d\nu(t) + \underbrace{\int \frac{D(t,z)}{B_n(t)}d\nu(t)}_{=0(Cauchy)}$$

- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\nu = \mu - \mu_n^L$
  $\frac{\Omega_\mu(z)-\Omega_n^L(z)}{B_n(z)} = \int[\frac{1}{B_n(z)} - \frac{1}{B_n(t)}]D(t,z)d\nu(t)$
- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\nu = \mu - \mu_n^L$

$$\frac{\Omega_\mu(z)-\Omega_n^L(z)}{B_n(z)} = \int \underbrace{[\frac{1}{B_n(z)} - \frac{1}{B_n(t)}]D(t,z)}_{\in \mathcal{L}_{n*}}\, d\nu(t)$$

- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\nu = \mu - \mu_n^L$

$$\frac{\Omega_\mu(z)-\Omega_n^L(z)}{B_n(z)} = \underbrace{\int [\frac{1}{B_n(z)} - \frac{1}{B_n(t)}]D(t,z)d\nu(t)}_{=\langle\langle I, fI\rangle\rangle_{\ell,\nu}=0 \quad f\in\mathcal{L}_n}$$

- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\Omega_\mu(0) - \Omega_n^L(0) = \int D(t,0)d\nu(t) = \int d\nu(t) = 0$
- QED

Interpolation $d\mu_n^L(t) = [K_{n*}^L(t,0)K_n^L(t,0)]^{-1}d\lambda(t)$

Theorem

$\Omega_n^L(z) := \int D(t,z)d\mu_n^L(t) = [K_n^L(z,0)]^{-1}L_n^L(z,0)$
$\sigma_n^L(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\mu_n^L)'(t)d\lambda(t)\} = [K_n^L(z,0)]^{-1}$
$\Omega_\mu(z) - \Omega_n^L(z) = zB_n(z)h(z),\ h \in H(\mathbb{D})$

proof:

- $\Omega_\mu(0) - \Omega_n^L(0) = \int D(t,0)d\nu(t) = \int d\nu(t) = 0$
- QED

Corollary

$$\Omega_n^L(z) = [K_n^L(z,0)]^{-1} L_n^L(z,0) = L_n^R(0,z)[K_n^R(0,z))]^{-1} = \Omega_n^R(z)$$

proof: $[K_{n*}^L(t,0)K_n^L(t,0)]^{-1} = [K_n^R(0,t)K_{n*}^R(0,t)]^{-1}$

Riesz-Herglotz $\Rightarrow \Omega_n^L = \Omega_n^R$ in $\mathbb{D}$

Rational $\Rightarrow$ equality in $\mathbb{C}$

QED

Corollary

$$\Omega_n^L(z) = [K_n^L(z,0)]^{-1} L_n^L(z,0) = L_n^R(0,z)[K_n^R(0,z))]^{-1} = \Omega_n^R(z)$$

proof: $[K_{n*}^L(t,0)K_n^L(t,0)]^{-1} = [K_n^R(0,t)K_{n*}^R(0,t)]^{-1}$

Riesz-Herglotz $\Rightarrow \Omega_n^L = \Omega_n^R$ in $\mathbb{D}$

Rational $\Rightarrow$ equality in $\mathbb{C}$

QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t).$

proof:

- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t).$

proof:

- QED

Skip proof

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t)$.

proof:

- Prove ϕ_n^L is L-orthonormal basis for $\hat{\mu}_n^L$
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t)$.

proof:

- Use backward recursion
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t)$.

proof:

- $\langle\langle \phi_n^L, \phi_n^L\rangle\rangle_{\ell,\hat{\mu}_n^L} = I$
- QED

Inner product

Theorem
$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t)$.

proof:

- $\left\langle\left\langle \phi_n^L, \phi_n^L \right\rangle\right\rangle_{\ell,\hat{\mu}_n^L} = I$
- $k = 0, \ldots, n-1$
  $\left\langle\left\langle \phi_n^L, \phi_n^L \right\rangle\right\rangle_{\ell,\hat{\mu}_n^L} = \int [\phi_n^{L*}(t)]^{-1} \phi_k^{L*}(t) P(t, \alpha_n) \frac{B_n(t)}{B_k(t)} d\lambda(t)$
  and this is 0 by Poisson.
- QED

Inner product

Theorem

$\langle\langle f, g\rangle\rangle_{\ell,\mu} = \langle\langle f, g\rangle\rangle_{\ell,\hat{\mu}_n^L}$, $\forall f, g \in \mathcal{L}_n$ *if*
$d\hat{\mu}_n^L(t) = P(t, \alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1} d\lambda(t)$.

proof:

- $\left\langle\left\langle \phi_n^L, \phi_n^L \right\rangle\right\rangle_{\ell,\hat{\mu}_n^L} = I$
- $k = 0, \ldots, n-1$
  $\left\langle\left\langle \phi_n^L, \phi_n^L \right\rangle\right\rangle_{\ell,\hat{\mu}_n^L} = \int [\phi_n^{L*}(t)]^{-1} \phi_k^{L*}(t) P(t, \alpha_n) \frac{B_n(t)}{B_k(t)} d\lambda(t)$
  and this is 0 by Poisson.
- QED

Interpolation $d\hat{\mu}_n^L(t) = P(t,\alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1}d\lambda(t)$

Theorem

$\hat{\Omega}_n^L(z) := \int D(t,z)d\hat{\mu}_n^L(t) = \psi_n^{L*}(z)[\phi_n^{L*}(z)]^{-1}$

$\hat{\sigma}_n^R(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\hat{\mu}_n^L)'(t)d\lambda(t)\} = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z}[\phi_n^{L*}(z)]^{-1}$

$\Omega_\mu(z) - \hat{\Omega}_n^L(z) = zB_n(z)\hat{h}(z),\ \hat{h} \in H(\mathbb{D})$

proof: Similar to the kernel case.

Corollary

$\hat{\Omega}_n^L = \hat{\Omega}_n^R$

Theorem (Favard)

ϕ_n^L, ϕ_n^R generated by recurrence $\Rightarrow$ they are L- and R- orthogonal w.r.t. a positive measure on $\mathbb{T}$

Interpolation $d\hat{\mu}_n^L(t) = P(t,\alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1}d\lambda(t)$

Theorem

$\hat{\Omega}_n^L(z) := \int D(t,z)d\hat{\mu}_n^L(t) = \psi_n^{L*}(z)[\phi_n^{L*}(z)]^{-1}$

$\hat{\sigma}_n^R(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\hat{\mu}_n^L)'(t)d\lambda(t)\} = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z}[\phi_n^{L*}(z)]^{-1}$

$\Omega_\mu(z) - \hat{\Omega}_n^L(z) = zB_n(z)\hat{h}(z),\ \hat{h} \in H(\mathbb{D})$

proof: Similar to the kernel case.

Corollary

$\hat{\Omega}_n^L = \hat{\Omega}_n^R$

Theorem (Favard)

ϕ_n^L, ϕ_n^R generated by recurrence $\Rightarrow$ they are L- and R- orthogonal w.r.t. a positive measure on $\mathbb{T}$

Interpolation $d\hat{\mu}_n^L(t) = P(t,\alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1}d\lambda(t)$

Theorem

$\hat{\Omega}_n^L(z) := \int D(t,z)d\hat{\mu}_n^L(t) = \psi_n^{L*}(z)[\phi_n^{L*}(z)]^{-1}$

$\hat{\sigma}_n^R(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\hat{\mu}_n^L)'(t)d\lambda(t)\} = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z}[\phi_n^{L*}(z)]^{-1}$

$\Omega_\mu(z) - \hat{\Omega}_n^L(z) = zB_n(z)\hat{h}(z),\ \hat{h} \in H(\mathbb{D})$

proof: Similar to the kernel case.

Corollary

$\hat{\Omega}_n^L = \hat{\Omega}_n^R$

Theorem (Favard)

ϕ_n^L, ϕ_n^R generated by recurrence $\Rightarrow$ they are L- and R- orthogonal w.r.t. a positive measure on $\mathbb{T}$

Interpolation $d\hat{\mu}_n^L(t) = P(t,\alpha_n)[\phi_{n*}^L(t)\phi_n^L(t)]^{-1}d\lambda(t)$

Theorem

$\hat{\Omega}_n^L(z) := \int D(t,z)d\hat{\mu}_n^L(t) = \psi_n^{L*}(z)[\phi_n^{L*}(z)]^{-1}$

$\hat{\sigma}_n^R(z) = \exp\{\frac{1}{2}\int D(t,z)\log(\hat{\mu}_n^L)'(t)d\lambda(t)\} = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z}[\phi_n^{L*}(z)]^{-1}$

$\Omega_\mu(z) - \hat{\Omega}_n^L(z) = zB_n(z)\hat{h}(z),\ \hat{h} \in H(\mathbb{D})$

proof: Similar to the kernel case.

Corollary

$\hat{\Omega}_n^L = \hat{\Omega}_n^R$

Theorem (Favard)

ϕ_n^L, ϕ_n^R generated by recurrence $\Rightarrow$ they are L- and R- orthogonal w.r.t. a positive measure on $\mathbb{T}$

Szegő's optimization problem

Note $\inf_s \Phi(s)$ and $\inf_s \operatorname{tr} \Phi(s)$ have the same solutions.

Theorem

$$\min_{f \in \mathcal{L}_n, f(w)=I} \langle\langle f, f \rangle\rangle_{\ell,\mu} = [k_n^L(w,w)]^{-1} \text{ obtained for}$$

$$f = [k_n^L(w,w)]^{-1} k_n^L(\cdot, w), \text{ provided that } \det k_n^L(w,w) \neq 0$$

- Left Szegő kernel: $s^L(z,w) = [(1-\overline{w}z)\sigma^L(z)(\sigma^L(w))^H]^{-1}$, $\sigma^L(0) > 0$, is L-reproducing in $\mathbb{H}_2$.
- Complete predictable if $\exp\{\int \log\det \mu'(t) d\lambda(t)\} = 0$, or $\log\det \mu' \notin L_1$.
- $A \in BD$ and $\mu \in SZ$, then $\{B_n\}_{n\geq 0}$ is complete in $\mathbb{H}_2$ (w.r.t. trace norm $|\cdot|_{\ell,\mu}$)

Szegő's optimization problem

Note $\inf_s \Phi(s)$ and $\inf_s \operatorname{tr} \Phi(s)$ have the same solutions.

Theorem

$\min_{f \in \mathcal{L}_n, f(w)=I} \langle\langle f, f \rangle\rangle_{\ell,\mu} = [k_n^L(w,w)]^{-1}$ *obtained for*
$f = [k_n^L(w,w)]^{-1} k_n^L(\cdot, w)$, *provided that* $\det k_n^L(w,w) \neq 0$

- Left Szegő kernel: $s^L(z,w) = [(1-\overline{w}z)\sigma^L(z)(\sigma^L(w))^H]^{-1}$, $\sigma^L(0) > 0$, is L-reproducing in $\mathbb{H}_2$.
- Complete predictable if $\exp\{\int \log\det \mu'(t) d\lambda(t)\} = 0$, or $\log\det \mu' \notin L_1$.
- $A \in BD$ and $\mu \in SZ$, then $\{B_n\}_{n\geq 0}$ is complete in $\mathbb{H}_2$ (w.r.t. trace norm $|\cdot|_{\ell,\mu}$)

Szegő's optimization problem

Note $\inf_s \Phi(s)$ and $\inf_s \operatorname{tr} \Phi(s)$ have the same solutions.

Theorem

$\min_{f \in \mathcal{L}_n, f(w)=I} \langle\langle f, f \rangle\rangle_{\ell,\mu} = [k_n^L(w,w)]^{-1}$ *obtained for*
$f = [k_n^L(w,w)]^{-1} k_n^L(\cdot, w)$, *provided that* $\det k_n^L(w,w) \neq 0$

- Left Szegő kernel: $s^L(z,w) = [(1-\overline{w}z)\sigma^L(z)(\sigma^L(w))^H]^{-1}$, $\sigma^L(0) > 0$, is L-reproducing in $\mathbb{H}_2$.
- Complete predictable if $\exp\{\int \log\det \mu'(t) d\lambda(t)\} = 0$, or $\log\det \mu' \notin L_1$.
- $A \in BD$ and $\mu \in SZ$, then $\{B_n\}_{n\geq 0}$ is complete in $\mathbb{H}_2$ (w.r.t. trace norm $|\cdot|_{\ell,\mu}$)

Szegő's optimization problem

Note $\inf_s \Phi(s)$ and $\inf_s \operatorname{tr} \Phi(s)$ have the same solutions.

Theorem

$\min_{f \in \mathcal{L}_n, f(w)=I} \langle\langle f, f \rangle\rangle_{\ell,\mu} = [k_n^L(w,w)]^{-1}$ *obtained for*
$f = [k_n^L(w,w)]^{-1} k_n^L(\cdot, w)$, *provided that* $\det k_n^L(w,w) \neq 0$

- Left Szegő kernel: $s^L(z,w) = [(1-\overline{w}z)\sigma^L(z)(\sigma^L(w))^H]^{-1}$, $\sigma^L(0) > 0$, is L-reproducing in $\mathbb{H}_2$.
- Complete predictable if $\exp\{\int \log\det \mu'(t) d\lambda(t)\} = 0$, or $\log\det \mu' \notin L_1$.
- $A \in BD$ and $\mu \in SZ$, then $\{B_n\}_{n\geq 0}$ is complete in $\mathbb{H}_2$ (w.r.t. trace norm $|\cdot|_{\ell,\mu}$)

Convergence in $\mathbb{L}_2(\mu)$ and $L_2(\mu)$

Theorem

$\mu \in SZ$, $s_w^L(z) = s^L(z,w)$

$\min_{f \in \mathcal{L}_n} \|f - s_w^L\|_{\ell,\mu} = s_w^L(w) - k_n^L(w,w)$, $f = k_n^L(z,w)$

If $A \in BD$: $\lim_{n\to\infty} |k_n^L(t,w) - s_w^L(t)|_{\ell,\mu} = 0.$