# ON THE WEAK INFINITE ZERO STRUCTURES AND RELATIVE DEGREES FOR LINEAR TIME-DELAY SYSTEMS*

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**Abstract.** In this article the weak infinite zero structures for linear time-delay systems and related problems are investigated. First, by linking the expansion of the transfer function matrix with output derivatives, a structural algorithm to determine the weak infinite zero structures is established, which is instrumental to derive the Smith-McMillan form at infinity of the transfer function matrix through a novel construction of biproper transformation matrices in the weak sense. The presented algorithm is inspired by Silverman's algorithm and relies on switching between the frequency-domain and time-domain representations and interpretations. Secondly, the relationship between the weak infinite zero structures and the leading incomplete relative degrees (LIRD) is revealed by analyzing and comparing the algorithms for their computation. Finally, as an application of these results, the transformed outputs and the weak infinite zero structures are used to design delayed output feedback laws, which naturally generalize so-called proportional-derivative controllers.

**Key words.** linear time-delay systems, structural algorithm, weak infinite zero structures, Smith-McMillan canonical form, relative degree, nonlinear eigenvalue problems

**MSC codes.** 15A03, 15A21, 93B10, 93B52, 93C05, 93D15

## 1. Introduction.

The infinite zero structure is a sequence of positive integers in systems theory. Since the infinite zero structure remains invariant under algebraic equivalence transformations (state, input and output transformation), a state feedback transformation and an output injection transformation, it is known as one of the structural invariants of linear systems. The notion of infinite zero structure plays an instrumental role in solving problems such as model matching [21], the decoupling problem [18], and system inversion [29].

Recall the linear delay-free system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1.1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{m\times m}$. The infinite zero structure of system (1.1) can be defined in two ways, via the transfer function and via the Morse normal form, which were proven to be consistent [4]. The first definition is easy to

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understand. Denote the transfer function matrix of system (1.1) by

$$G(s) = C(sI_n - A)^{-1}B + D = \sum_{i=0}^{\infty} CA^iBs^{-(i+1)} + D, \ s \in \mathbf{C},$$

which is rational and proper. Note that $G(s)$ is strictly proper if and only if $D = 0$. A complex-valued scalar rational function $f(s)$ is called proper/strictly proper, if $\lim_{|s|\to\infty} f(s)$ is finite/0. For a matrix $M(s)$ whose elements are all complex-valued rational functions, $M(s)$ is called proper/strictly proper if each element of this matrix is proper/strictly proper. $M(s)$ is called biproper, if $\lim_{|s|\to\infty} M(s)$ is invertible [28]. $G(s)$ has an infinite zero if [36, p.239]

$$\operatorname{rank}(G(\infty)) = \operatorname{rank}(\lim_{z\to\infty} G(z)) < \operatorname{nrank}(G(s)),$$

where $\operatorname{nrank}(G(s)) = \max_{s\in\mathbf{C}}\{\operatorname{rank}(G(s))\}$ stands for the normal rank of $G(s)$. $G(s)$ possesses an infinite zero of order $k$ when $G(1/z)$ has a finite zero of precisely that order at $z = 0$ [4]. Also, based on the definition of the Smith-McMillan form at infinity of a rational matrix [33, 35], the infinite zero structure of system (1.1) can be characterized by the Smith-McMillan form at infinity of the transfer function matrix, see [28] and [32]. The difference lies in the method to construct biproper transformation matrices, where the former uses elementary matrix operations and the latter uses Silverman's algorithm [31]. In what follows, we assume $D = 0$.

LEMMA 1.1. *[28, 32] Assume that the transfer function matrix $G(s)$ of system (1.1) satisfies $r = \operatorname{nrank}(G(s))$. Then there exist (non-unique) biproper transformation matrices $Q_1(s)$ and $Q_2(s)$ such that*

$$Q_1(s)G(s)Q_2(s) = \begin{bmatrix} \Pi(s) & 0 \\ 0 & 0 \end{bmatrix},$$

*where $\Pi(s) = s^{-\nu_1} \oplus s^{-\nu_2} \oplus \cdots \oplus s^{-\nu_r}$, $1 \le \nu_1 \le \nu_2 \le \cdots \le \nu_r$. The set of integers $\{\nu_1, \nu_2, \ldots, \nu_r\}$ is called the infinite zero structure, which is unique.*

Now we revisit the second definition. The Morse normal form [25] is obtained by applying a certain set of algebraic equivalence transformations, a state feedback transformation and an output injection transformation, described by $(\check{T}, \check{T}_u, \check{T}_y, \check{K}, \check{L})$, to system (1.1) under suitable assumptions, and consists of four subsystems [36, Theorem 4.7.2, p.234]. One of the subsystems, denoted by $(A_3, B_3, C_3)$, has the following structure:

$$\begin{aligned}
A_3 &= \left[\begin{array}{c|c} 0 & I_{\nu_1-1} \\ \hline 0 & 0 \end{array}\right] \oplus \left[\begin{array}{c|c} 0 & I_{\nu_2-1} \\ \hline 0 & 0 \end{array}\right] \oplus \cdots \oplus \left[\begin{array}{c|c} 0 & I_{\nu_{k_3}-1} \\ \hline 0 & 0 \end{array}\right], \\
B_3 &= \left[\begin{array}{c} 0_{(\nu_1-1)\times 1} \\ \hline 1 \end{array}\right] \oplus \left[\begin{array}{c} 0_{(\nu_2-1)\times 1} \\ \hline 1 \end{array}\right] \oplus \cdots \oplus \left[\begin{array}{c} 0_{(\nu_{k_3}-1)\times 1} \\ \hline 1 \end{array}\right], \\
C_3 &= \left[\begin{array}{c|c} 1 & 0_{1\times(\nu_1-1)} \end{array}\right] \oplus \left[\begin{array}{c|c} 1 & 0_{1\times(\nu_2-1)} \end{array}\right] \oplus \cdots \oplus \left[\begin{array}{c|c} 1 & 0_{1\times(\nu_{k_3}-1)} \end{array}\right],
\end{aligned}$$

and the infinite zero structure is defined as $\{\nu_1, \nu_2, \ldots, \nu_{k_3}\}$ ($k_3 = r$). Here, the symbol "$\oplus$" stands for the direct sum of matrices. Clearly, system (1.1) possesses infinite zeros if and only if $k_3 \ge 1$ [36, Lemma 4.7.1, p.239]. Only when system (1.1) is completely transformed into its Morse normal form can the infinite zero structure be determined. However, the construction of the transformation matrix cluster of the Morse normal form is complex. By contrast, the transfer function method is easier.

In this paper, we are concerned with the infinite zero structure of the following linear time-delay system:

$$\begin{cases} \dot{x} = A_0x(t) + A_1x(t-\tau) + Bu(t), \\ y = Cx(t), \end{cases} \tag{1.2}$$

where $A_0, A_1 \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{m\times n}$, $\tau$ stands for the time delay. For simplicity, we assume $\tau = 1$. Then, by a transformation of time, the results of this paper can be extended to the case of any $\tau \in \mathbf{R}_+$. Denote the transfer function matrix by

$$G(s, \mathrm{e}^s) = C(sI_n - A_0 - A_1\mathrm{e}^{-s})^{-1}B, \ s \in \mathbf{C},$$

which is not rational and not analytic at infinity ($\infty$ is an essential singularity). Thus, "proper" cannot be defined as for $G(s)$. Limiting the way in which $s$ tends to $\infty$ results in different notions of "proper". For example, "$s \in \mathbf{R}$ and $s \to \infty$"/"$s \in \mathbf{C}$ and $\Re e(s) \to \infty$", known as "the weak sense"/"the strong sense", leads to "weak proper"/"strong proper". A matrix is weak proper if each of its element is weak proper. We generalize the definition in [28] as follows.

DEFINITION 1.2. *A complex-valued scalar function $f(s)$ is said to be weak proper /strictly weak proper, if $\lim_{s\in\mathbf{R},s\to\infty} f(s)$ is finite/0. For a matrix $M(s)$ whose elements are complex-valued functions, $M(s)$ is called weak biproper if it is weak proper, and $\lim_{s\in\mathbf{R},s\to\infty} M(s)$ is invertible. $M(s)$ is said to be of weak full row rank (or of full row rank in the weak sense), if it is weak proper, and $\lim_{s\in\mathbf{R},s\to\infty} M(s)$ is of full row rank.*

By constructing weak/strong biproper transformation matrices through elementary matrix operations, $G(s, \mathrm{e}^s)$ can be transformed into the Smith-McMillan form at infinity, resulting in the weak/strong infinite zero structure [22, 26]. The weak structure can be defined for all linear time-delay systems and can handle control problems such as row-by-row decoupling [28], disturbance decoupling [27], model matching and disturbance rejection [22]. So far, the strong structure has only been well defined for a certain class of delay systems. A general graphical characterization of the weak infinite zero structure of system (1.2) was presented in [30]. General infinite zero structures for common time-delay systems were defined in [34] and [19]. The behavior at infinity of the transfer function matrix of linear time-delay systems was studied in [3] and [9]. Motivated by the fact that a structural algorithm to compute the weak infinite zero structure of system (1.2) is lacking in the literature, the primary goal of this paper is to extend the algorithm and the results in [32] to time-delay systems.

Another important sequence of positive integers in systems theory is the relative degree. The relative degree is an important property to solve problems such as the construction of the normal form [14, 36, section 4.3, p.179-202], decoupling [5, 8], output tracking [7, 13], input-output linearization [15, section 5.2, p.227-241], and system inversion [6].

DEFINITION 1.3. *[37] The relative degree of system* (1.1) *is defined by the ordered set of integers $\{r_1, r_2, \ldots, r_m\}$ if*

$$c_iA^kB = 0, \ k = 0, 1, \ldots, r_i - 2, \quad c_iA^{r_i-1}B \neq 0, \tag{1.3}$$

*where $c_i$ represents the ith row of $C$. The relative degree $\{r_1, r_2, \ldots, r_m\}$ is called*

*regular if*

$$\det(\Gamma) \neq 0,\ \Gamma = \begin{bmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \\ \vdots \\ c_m A^{r_m-1} B \end{bmatrix} \in \mathbf{R}^{m\times m}, \tag{1.4}$$

*where $\Gamma$ is known as the coupling matrix. The relative degree is called irregular if (1.4) is not satisfied.*

Clearly, matrix $\Gamma$ is not necessarily nonsingular. Based on the fact that the relative degree and the non-singularity of $\Gamma$ can be changed by output transformations, in [37] necessary and sufficient conditions have been derived for the existence of an output transformation such that the transformed system has a regular relative degree. The concept of relative degree was extended in [17]:

DEFINITION 1.4. *[17] A set of positive integers $\{\rho_1, \rho_2, \ldots, \rho_m\}$ is called the vector of leading incomplete relative degrees (LIRD) of system (1.1) if it satisfies (1.3) and, for any pairwise different indices $i_1, i_2, \ldots, i_l$ that satisfy $\rho_{i_1} = \rho_{i_2} = \cdots = \rho_{i_l}$, the rows $\{c_{i_j} A^{\rho_{i_j}-1} B\}_{j=1}^{l}$ are linearly independent, where $c_{i_j}$ denotes the $i_j$th row of $C$.*

The definition of the LIRD does not require all rows in the coupling matrix to be linearly independent, but only requires those rows that correspond to the output components with the same relative degree to be linearly independent. Based on the theory of normal forms, the relationship between the infinite zero structure and the relative degree of system (1.1) was tentatively explored in [36, p.245]. In Subsection 4.1, we will continue to study their relationship in light of a structural algorithm.

The relative degree of linear time-delay systems was defined in a ring framework in [10, Definition 1], which, similar to Definition 1.3, can also be interpreted as the number of times that the output must be differentiated in order to get a direct access to the inputs. Considering the sensitivity of the relative degree to small delay perturbations, the notions of the strong relative degree and the leading vector of strong relative degrees (LSRD) were introduced in [24] for linear non-commensurate delay systems. Besides, a structural algorithm was designed to get an output transformation such that the LSRD of the transformed system always exists. For system (1.2), the relative degree and the strong relative degree are equivalent, so are the LIRD and the LSRD. For this reason, we give equivalent definitions of the relative degree and the LIRD of system (1.2) in Subsection 2.2. Inspired by the fact that the relationship between the weak infinite zero structure and the LIRD of system (1.2) is unclear until now, the second objective of this paper is to discuss their relationship through the algorithms for their computation, where [24, Algorithm 1] is used to reveal the LIRD.

Because of its ease of implementation, the proportional-integral-derivative (PID) output feedback controller, described by

$$u(t) = K_{\mathrm{p}} y(t) + K_{\mathrm{d}} \dot{y}(t) + K_{\mathrm{i}} \int_0^t y(s)\mathrm{d}s,$$

where $K_{\mathrm{p}}, K_{\mathrm{d}}, K_{\mathrm{i}}$ are real-valued gain matrices, has been widely applied to linear systems [1, 20]. For the special case where $K_i = 0$, the controller is known as a PD controller. When the relative degree is greater than 1, higher-order output derivatives are allowed for feedback, resulting in an extended PD controller. Note that in general, to avoid the ill-posedness of the closed-loop system caused by small delay deviations,

a scheme using output derivatives for feedback should be designed based on the strong relative degrees rather than the relative degrees. Such a scheme was proposed in [24]. As an application of the theoretical results, the final goal of this paper is to use the weak infinite zero structure to further extend the scope of a PD output feedback controller.

Compared with the existing results, the contributions of this paper are as follows.

- Based on Silverman's algorithm, a structured algorithm is designed to compute the weak infinite zero structure of system (1.2). As a by-product, the Smith-McMillan form at infinity of the transfer function matrix is obtained with the help of this algorithm.
- For systems (1.1) and (1.2), the relationships among the infinite zero structure/weak infinite zero structure, the LIRD and the relative degree are discussed.
- An extended PD output feedback controller is designed to stabilize system (1.2), where the output derivatives used for feedback are determined by the weak infinite zero structure.

The remainder of this paper is arranged as follows. The notations and preliminaries are given in Section 2. The computation of the weak infinite zero structure of system (1.2) is presented in Section 3. Section 4 studies the relationship between the infinite zero structure/weak infinite zero structure and the relative degree. An extended PD output feedback controller is designed in Section 5. The conclusions are drawn in Section 6.

## 2. Notations and preliminaries.

**Notations:** We denote the inverse/transpose/rank/determinant of matrix $A$ by $A^{-1}/A^{\mathrm{T}}/\mathrm{rank}(A)/\det(A)$. Denote the direct sum of matrices $A$ and $B$ by $A\oplus B$. Let $\Re e(s)/\Im m(s)$ denote the real part/imaginary part of complex number $s$. $\mathbf{Z}/\mathbf{R}/\mathbf{R}_+/\mathbf{C}$ denotes the set of integers/real numbers/positive real numbers/complex numbers. $\mathbf{R}^{p\times q}/\mathbf{R}_m^{p\times q}$ denotes the set of $p\times q$ real-valued matrices/$p\times q$ real-valued matrices with rank $m$. $I_p\in\mathbf{R}^{p\times p}$ represents an identity matrix. Denote by $\mathrm{row}\{a_i: i=1,2,\ldots,k\}$ the matrix generated by arranging the matrices $a_i\in\mathbf{R}^{p\times q}$, $i=1,2,\ldots,k$ column by column, i.e., $\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_k \end{array}\right]\in\mathbf{R}^{p\times kq}$. $\mathbf{Z}^n/\mathbf{C}^n$ denotes the set of $n$-dimensional integers/complex numbers. Denote the $j$th derivative of an univariate function $y$ at time $t$ by $y^{(j)}(t)$, in particular, $y^{(1)}(t)=\dot{y}(t)$. Let $s/\mathrm{e}^s/\mathrm{e}^{-s}$ denote the Laplace variable/the unitary shift operator/the unitary delay operator. $\mathbf{R}^{p\times q}[z]/\mathbf{R}^{p\times q}(z)$ denotes the set of $p\times q$ polynomial/rational function matrices with real coefficients in variable $z$. Let $\mathbf{R}(s,z)$ denote the set of rational functions with real coefficients in two variables $s$ and $z$. Then for any $f(s,z)\in\mathbf{R}(s,z)$, $f(s,z)$ is rational with respect to variable $s$ or $z$. $\mathbf{R}^{p\times q}(s,z)$ denotes the set of $p\times q$ rational function matrices with real coefficients in two variables $s$ and $z$. Let $\mathbf{R}^{p\times q}(s)[z]$ denote the set of matrix polynomials of $z$ with $p\times q$ real rational function matrices in variable $s$ as the coefficients.

### 2.1. The weak infinite zero structure of system (1.2).

Define a recursive sequence $\{Q(i,j)\}_{i,j\in\mathbf{Z}}$ as follows

$$Q(i,j)=Q(i-1,j)A_0+Q(i-1,j-1)A_1,$$

where

$$\begin{aligned} &Q(0,0)=I_n,\quad Q(i,0)=A_0^i,\quad Q(i,i)=A_1^i,\\ &Q(i,j)=0_{n\times n},\ \text{if } i<j;\quad Q(i,j)=0_{n\times n},\ \text{if } i<0 \text{ or } j<0. \end{aligned}$$

The transfer function matrix $G(s,\mathrm{e}^s)$ can be expanded as

$$G(s,\mathrm{e}^s)=\sum_{j=0}^{\infty}G_j(s)\mathrm{e}^{-js}=G_0(s)+G_1(s)\mathrm{e}^{-s}+G_2(s)\mathrm{e}^{-2s}+\cdots, \tag{2.1}$$

where, by using the identity $(sI_n-A_0)^{-1}=\sum_{i=0}^{\infty}A_0^i s^{-(i+1)}$, the coefficients are described by

$$G_j(s)=C(sI_n-A_0)^{-1}(A_1(sI_n-A_0)^{-1})^jB=\sum_{i=0}^{\infty}CQ(i,j)Bs^{-(i+1)}.$$

LEMMA 2.1. *[28] Let $G(s,\mathrm{e}^s)$ be the transfer function matrix of system (1.2). There exist (non-unique) weak biproper rational matrices $\bar{B}_1(s,\mathrm{e}^s)\in\mathbf{R}^{m\times m}(s,\mathrm{e}^s)$ and $\bar{B}_2(s,\mathrm{e}^s)\in\mathbf{R}^{m\times m}(s,\mathrm{e}^s)$ such that $G(s,\mathrm{e}^s)$ is transformed into the Smith-McMillan form at infinity:*

$$\bar{B}_1(s,\mathrm{e}^s)G(s,\mathrm{e}^s)\bar{B}_2(s,\mathrm{e}^s)=\left[\begin{array}{cccc|c}\Pi_0(s)&0&\cdots&0&0\\0&\Pi_1(s)\mathrm{e}^{-s}&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&\Pi_q(s)\mathrm{e}^{-qs}&0\\\hline0&0&\cdots&0&0\end{array}\right], \tag{2.2}$$

*where $\Pi_i(s)=s^{-n_{i,1}}\oplus s^{-n_{i,2}}\oplus\cdots\oplus s^{-n_{i,p_i}}$, $0\le n_{i,1}\le n_{i,2}\le\cdots\le n_{i,p_i}$, $i=1,2,\ldots,q$. The list of integer*

$$\{n_{0,1},\ldots,n_{0,p_0};n_{1,1},\ldots,n_{1,p_1};\ldots;n_{q,1},\ldots,n_{q,p_q}\}, \tag{2.3}$$

*is said to be the weak infinite zero structure of system (1.2), which is unique up to the order.*

*Remark* 2.2. Assume that $r=\mathrm{nrank}(G(s,\mathrm{e}^s))=\max_{s\in\mathbf{C}}\{\mathrm{rank}(G(s,\mathrm{e}^s))\}$. Then $r=\sum_{i=0}^{q}p_i$, namely, the total number of weak infinite zeros is equal to the normal rank $r$. The normal rank $r$ can be obtained by means of the determinant divisors of $G(s,\mathrm{e}^s)$. In fact, according to the expansion (2.1), $G(s,\mathrm{e}^s)$ can be regarded as a matrix polynomial of $\mathrm{e}^{-s}$ of "sufficiently large degree". Denote the $i$th order determinant divisor of $G(s,\mathrm{e}^s)$ by $d_i(s,\mathrm{e}^s)$, which is the greatest common divisor of all the $i$th order minors of $G(s,\mathrm{e}^s)$. The property that $d_k(s,\mathrm{e}^s)=0$ for all $k>r$ can be used to compute the normal rank $r$.

The proof of Lemma 2.1 was derived based on Definition 1.2 and the expansion (2.1). By using elementary matrix operations, a recursive procedure was designed to construct step by step weak biproper transformation matrices, starting from transforming $G_0(s)$ into its Smith-McMillan form at infinity. However, as each step in this design is dependent on the previous one, $\bar{B}_1(s,\mathrm{e}^s)$ and $\bar{B}_2(s,\mathrm{e}^s)$ cannot be expressed in explicit forms. Let us explain this procedure with the following example.

EXAMPLE 2.3. *Consider the system (1.2) with coefficient matrices*

$$A_0=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ A_1=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix},\ C=\begin{bmatrix}0&1\\0&0\\0&0\\1&0\end{bmatrix}^{\mathrm{T}}.$$

*The transfer function matrix is*

$$G(s,\mathrm{e}^s)=\begin{bmatrix} 0 & s^{-1} \\ s^{-3} & (s^{-2}+s^{-4})\mathrm{e}^{-s} \end{bmatrix},$$

*and satisfies* $\operatorname{nrank}(G(s,\mathrm{e}^s))=2$. *By means of (2.1), one obtains an expansion of the form* $G(s,\mathrm{e}^s)=G_0(s)+G_1(s)\mathrm{e}^{-s}$, *where* $G_0(s)=\left[\begin{smallmatrix} 0 & s^{-1} \\ s^{-3} & 0 \end{smallmatrix}\right]$ *and* $G_1(s)=\left[\begin{smallmatrix} 0 & 0 \\ 0 & s^{-2}+s^{-4} \end{smallmatrix}\right]$.

*Firstly, let* $\bar{B}_1^a=I_2$ *and* $\bar{B}_2^a=\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $G_0(s)$ *is transformed into its Smith-McMillan form at infinity:* $\mathit{\Pi}_0(s)=\bar{B}_1^aG_0(s)\bar{B}_2^a=\left[\begin{smallmatrix} s^{-1} & 0 \\ 0 & s^{-3} \end{smallmatrix}\right]$. *Then* $\bar{B}_1^aG(s,\mathrm{e}^s)\bar{B}_2^a=\left[\begin{smallmatrix} s^{-1} & 0 \\ (s^{-2}+s^{-4})\mathrm{e}^{-s} & s^{-3} \end{smallmatrix}\right]$. *Next, let* $\bar{B}_1^b=I_2$ *and* $\bar{B}_2^b=\left[\begin{smallmatrix} 1 & 0 \\ -(s+s^{-1})\mathrm{e}^{-s} & 1 \end{smallmatrix}\right]$, $G(s,\mathrm{e}^s)$ *can be transformed into the Smith-McMillan form at infinity, i.e., (2.2),*

$$\bar{B}_1^b\bar{B}_1^aG(s,\mathrm{e}^s)\bar{B}_2^a\bar{B}_2^b=\bar{B}_1^b\begin{bmatrix} s^{-1} & 0 \\ (s^{-2}+s^{-4})\mathrm{e}^{-s} & s^{-3} \end{bmatrix}\bar{B}_2^b=\begin{bmatrix} s^{-1} & 0 \\ 0 & s^{-3} \end{bmatrix},$$

*where the transformation matrices*

$$\bar{B}_1(s,\mathrm{e}^s)=\bar{B}_1^b\bar{B}_1^a=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \bar{B}_2(s,\mathrm{e}^s)=\bar{B}_2^a\bar{B}_2^b=\begin{bmatrix} -(s+s^{-1})\mathrm{e}^{-s} & 1 \\ 1 & 0 \end{bmatrix},$$

*are weak biproper and, the weak infinite zero structure is* $\{1,3\}$.

This paper considers another equivalent expansion of $G(s,\mathrm{e}^s)$:

$$G(s,\mathrm{e}^s)=\sum_{i=0}^{\infty}P_i(\mathrm{e}^{-s})s^{-(i+1)}=P_0(\mathrm{e}^{-s})s^{-1}+P_1(\mathrm{e}^{-s})s^{-2}+P_2(\mathrm{e}^{-s})s^{-3}+\cdots, \tag{2.4}$$

where $P_i(\mathrm{e}^{-s})=\sum_{j=0}^{i}CQ(i,j)B\mathrm{e}^{-js}$ are polynomial matrices of $\mathrm{e}^{-s}$. For instance,

$$\begin{aligned} P_0(\mathrm{e}^{-s}) =& CQ(0,0)B=CB,\\ P_1(\mathrm{e}^{-s}) =& CQ(1,0)B+CQ(1,1)B\mathrm{e}^{-s}=CA_0B+CA_1B\mathrm{e}^{-s},\\ P_2(\mathrm{e}^{-s}) =& CQ(2,0)B+CQ(2,1)B\mathrm{e}^{-s}+CQ(2,2)B\mathrm{e}^{-2s}\\ =& CA_0^2B+C\big(A_0A_1+A_1A_0\big)B\mathrm{e}^{-s}+CA_1^2B\mathrm{e}^{-2s}. \end{aligned}$$

The expansion (2.4) establishes connections between the transfer function matrix and output derivatives of system (1.2). For example, we can compute

$$\begin{aligned} y^{(1)}(t) =& CA_0x(t)+CA_1x(t-1)+CBu(t),\\ y^{(2)}(t) =& CA_0^2x(t)+C(A_0A_1+A_1A_0)x(t-1)\\ &+CA_1^2x(t-2)+CA_0Bx(t)+CA_1Bx(t-1),\\ y^{(3)}(t) =& CA_0^3x(t)+C\big(A_0^2A_1+A_1A_0^2+A_0A_1A_0\big)x(t-1)\\ &+C\big(A_1^2A_0+A_0A_1^2+A_1A_0A_1\big)x(t-2)+CA_1^3x(t-3)+CA_0^2Bu(t)\\ &+C(A_0A_1+A_1A_0)Bu(t-1)+CA_1^2Bu(t-2), \end{aligned}$$

whose Laplace transformations are

$$\begin{aligned} y(s) =& s^{-1}(CA_0+CA_1\mathrm{e}^{-s})x(s)+s^{-1}P_0(\mathrm{e}^{-s})u(s),\\ y(s) =& s^{-2}\big(CA_0^2+C(A_0A_1+A_1A_0)\mathrm{e}^{-s}+CA_1^2\mathrm{e}^{-2s}\big)x(s)+s^{-2}P_1(\mathrm{e}^{-s})u(s),\\ y(s) =& s^{-3}\big[CA_0^3+C\big(A_0^2A_1+A_1A_0^2+A_0A_1A_0\big)\mathrm{e}^{-s}\\ &+C\big(A_1^2A_0+A_0A_1^2+A_1A_0A_1\big)\mathrm{e}^{-2s}+CA_1^3\mathrm{e}^{-3s}\big]x(s)+s^{-3}P_2(\mathrm{e}^{-s})u(s). \end{aligned}$$

Hence, the coefficients of $u(s)$ correspond exactly to the terms in expansion (2.4).

**2.2. The relative degree of system (1.2).** According to [24, Theorem 3.12], we obtain equivalent definitions of the relative degree and the LIRD of system (1.2):

DEFINITION 2.4. *[24] The relative degree of system* (1.2) *is defined by the ordered set of positive integers* $\{r_1, r_2, \ldots, r_m\}$ *if and only if*

$$c_k Q(i,j) B = 0, \quad \forall\, (i,j) \in \mathbf{Z}^2 :\ 0 \le i < r_k - 1,\ 0 \le j \le i, \tag{2.5}$$

*and there exists a pair* $(i,j) \in \{(l_1,l_2) \in \mathbf{Z}^2 : l_1 = r_k - 1,\ 0 \le l_2 \le l_1\}$ *such that* $c_k Q(i,j) B \neq 0$, *where* $c_k$ *represents the* $k$*th row of* $C$.

DEFINITION 2.5. *[24] A set of positive integers* $\{\rho_1, \rho_2, \ldots, \rho_m\}$ *is called the LIRD of system* (1.2) *if it satisfies Definition 2.4 and, for any pairwise different indices* $i_1, i_2, \ldots, i_l$ *that satisfies* $\rho_{i_1} = \rho_{i_2} = \cdots = \rho_{i_l}$, *the rows* $\{c_{i_j} H_{\rho_{i_j}-1}\}_{j=1}^{l}$ *are linearly independent, where* $c_{i_j}$ *represents the* $i_j$*th row of* $C$, $H_k = \operatorname{row}\big\{Q(i,j)B : (i,j) \in \mathbf{Z}^2,\ i = k,\ 0 \le j \le i\big\}$. *The matrices in* $H_k$ *are arranged in the lexicographical order by treating* $Q(i,j)$ *as a polynomials in* $A_0$ *and* $A_1$.

Readers unfamiliar with the terminology are referred to the following enumeration:

$$H_0 = \operatorname{row}\big\{Q(0,0)B\big\} = CB,$$
$$H_1 = \operatorname{row}\big\{Q(1,0)B, Q(1,1)B\big\} = \begin{bmatrix} A_0B & A_1B \end{bmatrix},$$
$$H_2 = \operatorname{row}\big\{Q(2,0)B, Q(2,1)B, Q(2,2)B\big\} = \begin{bmatrix} A_0^2B & (A_0A_1 + A_1A_0)B & A_1^2B \end{bmatrix}.$$

**3. Main results.** This section begins with a new structural algorithm. Then we construct the Smith-McMillan form at infinity of system (1.2) by means of the outputs of this algorithm, in order to obtain the weak infinite zero structure. Based on the connection between the transfer function matrix and the output derivatives discussed in Subsection 2.1, we design an algorithm switching between the time domain and the Laplace domain, see Algorithm 3.1.

Since $\operatorname{rank}(\lim_{s\in\mathbf{R}, s\to\infty} G(s, \mathrm{e}^s)) = 0$, this algorithm relies on differential transformations, and nonsingular transformations in Laplace domain, such that when $s \in \mathbf{R}$ and $s \to \infty$, the rank of the transformed transfer function matrix is equal to $\operatorname{nrank}(G(s, \mathrm{e}^s))$. Algorithm 3.1 extends the algorithm in [32] and Silverman's algorithm [31]. Silverman's algorithm was designed for constructing an inverse of system (1.1). It relies on repeatedly executing nonsingular transformations and differential transformations on block matrices of outputs, in order to obtain relations that allow to uniquely determine the input from the output derivatives and the state. Interested readers are referred to [2] for an extension of Silverman's algorithm to dynamical systems over a ring, where a submodule is constructed that can be used to characterize the zero dynamics of time-delay systems.

Algorithm 3.1 repeatedly executes five operations: (i) transfer to the time domain, (ii) differentiate in the time domain, (iii) transfer to the Laplace domain, (iv) take nonsingular transformations, (v) check the rank of D-part. Some descriptions of this algorithm are as follows.

**Definitions of symbols:** Denote $r = \operatorname{nrank}(G(s, \mathrm{e}^s))$, $r \le m$, and

$$\mathcal{X}_j = \begin{bmatrix} x(t) \\ x(t-1) \\ \vdots \\ x(t-j+1) \end{bmatrix} \in \mathbf{R}^{jn}, \quad \mathcal{V}_i = \begin{bmatrix} u(t) \\ u(t-1) \\ \vdots \\ u(t-i+1) \end{bmatrix} \in \mathbf{R}^{im}. \tag{3.7}$$

**Algorithm 3.1** A structural decomposition algorithm switching between time domain and Laplace domain

**Initialization**: Denote $\tilde{y}_0(t) = y(t) = Cx(t)$. Take the time derivative of $\tilde{y}_0(t)$:

$$\dot{\tilde{y}}_0(t) = CA_0x(t) + CA_1x(t-1) + CBu(t). \tag{3.1}$$

Convert equation (3.1) to the Laplace domain:

$$L_0(s)\tilde{y}_0(s) = (CA_0 + CA_1\mathrm{e}^{-s})x(s) + CBu(s), \tag{3.2}$$

where $L_0(s) = sI_m$, $s \in \mathbf{C}$ denotes a Laplace operator matrix.

Apply a nonsingular transformation to equation (3.2): Denote $q_1 = \mathrm{rank}(CB)$. There exists a constant nonsingular matrix $S_1 \in \mathbf{R}^{m\times m}$ such that

$$\begin{bmatrix} \bar{y}_1(s) \\ \tilde{y}_1(s) \end{bmatrix} \triangleq S_1L_0(s)\tilde{y}_0(s) \triangleq \begin{bmatrix} \bar{C}_1(\mathrm{e}^{-s}) \\ \tilde{C}_1(\mathrm{e}^{-s}) \end{bmatrix} x(s) + \begin{bmatrix} \bar{D}_1 \\ 0 \end{bmatrix} u(s),$$

where $\bar{D}_1 \in \mathbf{R}^{q_1\times m}$ is of full row rank, $\bar{C}_1(\mathrm{e}^{-s}) \in \mathbf{R}^{q_1\times n}[\mathrm{e}^{-s}]$, $\tilde{C}_1(\mathrm{e}^{-s}) \in \mathbf{R}^{(m-q_1)\times n}[\mathrm{e}^{-s}]$, $\bar{y}_1(s) \in \mathbf{C}^{q_1}$, $\tilde{y}_1(s) \in \mathbf{C}^{m-q_1}$.

**if** $q_1 = r$ **then** return

**end if**

**for** $k = 1$ to $2n-1$ **do**

By the previous step, it holds that

$$\begin{bmatrix} \bar{y}_k(s) \\ \tilde{y}_k(s) \end{bmatrix} = \begin{bmatrix} \bar{C}_k(\mathrm{e}^{-s}) \\ \tilde{C}_k(\mathrm{e}^{-s}) \end{bmatrix} x(s) + \begin{bmatrix} \bar{D}_k(\mathrm{e}^{-s}) \\ 0 \end{bmatrix} u(s), \tag{3.3}$$

where $\bar{D}_k(\mathrm{e}^{-s}) \in \mathbf{R}^{q_k\times m}[\mathrm{e}^{-s}]$ is of weak full row rank, $\bar{C}_k(\mathrm{e}^{-s}) \in \mathbf{R}^{q_k\times n}[\mathrm{e}^{-s}]$, $\tilde{C}_k(\mathrm{e}^{-s}) \in \mathbf{R}^{(m-q_k)\times n}[\mathrm{e}^{-s}]$, $\bar{y}_k(s) \in \mathbf{C}^{q_k}$, $\tilde{y}_k(s) \in \mathbf{C}^{m-q_k}$.

Convert the equation $\tilde{y}_k(s) = \tilde{C}_k(\mathrm{e}^{-s})x(s)$ in (3.3) to the time domain, denoted by $\tilde{y}_k(t) = \tilde{\mathfrak{C}}_k\mathcal{X}_{k+1}$, where $\mathcal{X}_{k+1} \in \mathbf{R}^{(k+1)n}$ is defined in (3.7), and $\tilde{\mathfrak{C}}_k \in \mathbf{R}^{(m-q_k)\times(k+1)n}$.

Take the time derivative of $\tilde{y}_k(t)$, namely,

$$\dot{\tilde{y}}_k(t) = \tilde{\mathcal{C}}_k\mathcal{X}_{k+2} + \tilde{\mathcal{D}}_k\mathcal{V}_{k+1}, \tag{3.4}$$

where $\mathcal{X}_{k+2} \in \mathbf{R}^{(k+2)n}$ and $\mathcal{V}_{k+1} \in \mathbf{R}^{(k+1)m}$ are defined in (3.7), $\tilde{\mathcal{C}}_k \in \mathbf{R}^{(m-q_k)\times(k+2)n}$, $\tilde{\mathcal{D}}_k \in \mathbf{R}^{(m-q_k)\times(k+1)m}$.

Convert equation (3.4) to the Laplace domain, namely,

$$sI_{m-q_k}\tilde{y}_k(s) = \tilde{C}_k'(\mathrm{e}^{-s})x(s) + \tilde{D}_k'(\mathrm{e}^{-s})u(s), \tag{3.5}$$

where $\tilde{C}_k'(\mathrm{e}^{-s}) \in \mathbf{R}^{(m-q_k)\times n}[\mathrm{e}^{-s}]$, $\tilde{D}_k'(\mathrm{e}^{-s}) \in \mathbf{R}^{(m-q_k)\times m}[\mathrm{e}^{-s}]$. By replacing $\tilde{y}_k(s) = \tilde{C}_k(\mathrm{e}^{-s})x(s)$ in equation (3.3) with (3.5) gives

$$L_k(s)\begin{bmatrix} \bar{y}_k(s) \\ \tilde{y}_k(s) \end{bmatrix} = \begin{bmatrix} \bar{C}_k(\mathrm{e}^{-s}) \\ \tilde{C}_k'(\mathrm{e}^{-s}) \end{bmatrix} x(s) + \begin{bmatrix} \bar{D}_k(\mathrm{e}^{-s}) \\ \tilde{D}_k'(\mathrm{e}^{-s}) \end{bmatrix} u(s), \tag{3.6}$$

where $L_k(s) = I_{q_k} \oplus sI_{m-q_k}$, $s \in \mathbf{C}$ denotes a Laplace operator matrix.

Apply a nonsingular transformation to equation (3.6). Denote $q_{k+1} = \mathrm{nrank}\begin{bmatrix} \bar{D}_k(\mathrm{e}^{-s}) \\ \tilde{D}_k'(\mathrm{e}^{-s}) \end{bmatrix}$. There exists a nonsingular matrix $S_{k+1}(\mathrm{e}^s) \in \mathbf{R}^{m\times m}(\mathrm{e}^s)$ such that

$$\begin{bmatrix} \bar{y}_{k+1}(s) \\ \tilde{y}_{k+1}(s) \end{bmatrix} \triangleq S_{k+1}(\mathrm{e}^s)L_k(s)\begin{bmatrix} \bar{y}_k(s) \\ \tilde{y}_k(s) \end{bmatrix} \triangleq \begin{bmatrix} \bar{C}_{k+1}(\mathrm{e}^{-s}) \\ \tilde{C}_{k+1}(\mathrm{e}^{-s}) \end{bmatrix} x(s) + \begin{bmatrix} \bar{D}_{k+1}(\mathrm{e}^{-s}) \\ 0 \end{bmatrix} u(s),$$

where $\bar{D}_{k+1}(\mathrm{e}^{-s}) \in \mathbf{R}^{q_{k+1}\times m}[\mathrm{e}^{-s}]$ is of weak full row rank, $\bar{C}_k(\mathrm{e}^{-s}) \in \mathbf{R}^{q_{k+1}\times n}[\mathrm{e}^{-s}]$, $\tilde{C}_k(\mathrm{e}^{-s}) \in \mathbf{R}^{(m-q_{k+1})\times n}[\mathrm{e}^{-s}]$, $\bar{y}_{k+1}(s) \in \mathbf{C}^{q_{k+1}}$, $\tilde{y}_{k+1}(s) \in \mathbf{C}^{m-q_{k+1}}$.

**if** $q_{k+1} = r$ **then** return

**end if**

**end for**

**Output**: Matrix $\bar{D}_\alpha(\mathrm{e}^{-s}) \in \mathbf{R}^{r\times m}[\mathrm{e}^{-s}]$ of weak full row rank, the Laplace operator matrices $L_{k-1}(s)$, the nonsingular transformation matrices $S_k(\mathrm{e}^s)$, and rank indices $q_k$, $k = 1, 2, \ldots, \alpha$, where $\alpha = \min\{k \mid q_k = r\}$.

The notation $\triangleq$ defines new quantities describing how the vectors/matrices are partitioned by the rank. For example, $\left[\begin{smallmatrix}\bar{y}_1(s)\\ \tilde{y}_1(s)\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\bar{C}_1(\mathrm{e}^{-s})\\ \tilde{C}_1(\mathrm{e}^{-s})\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\bar{D}_1\\ 0\end{smallmatrix}\right]$ represent the partitioning of $S_1L_0(s)\tilde{y}_0(s)$, $S_1(CA_0+CA_1\mathrm{e}^{-s})$ and $S_1CB$, respectively, according to the rank $q_1$.

**Conversion to the time domain:** In line 7, $\tilde{y}_k$ is obtained from $\tilde{y}_0$ by $k$ differentiation operations and nonsingular transformations. Thus, the time-domain representation of equation $\tilde{y}_k(s)=\tilde{C}_k(\mathrm{e}^{-s})x(s)$ contains $x(t), x(t-1),\dots,x(t-k)$, which, by using (3.7), can be expressed in a compact form: $\tilde{y}_k(t)=\tilde{\mathfrak{C}}_k\mathcal{X}_{k+1}$, where $\tilde{\mathfrak{C}}_k\in\mathbf{R}^{(m-q_k)\times(k+1)n}$ is a constant matrix.

**Differentiation in the time domain:** In line 8, when differentiating equation $\tilde{y}_k(t)=\tilde{\mathfrak{C}}_k\mathcal{X}_{k+1}$, each delayed state $x(t-i)$ introduces $x(t-i-1)$ and $u(t-i)$, $i=0,1,\dots,k$. Using (3.7), we can express $\dot{\tilde{y}}_k(t)$ as (3.4).

**Conversion to the Laplace domain:** In line 9, applying the Laplace transformation to equation (3.4) yields equation (3.5). Since the dimensions of $\bar{C}_k(\mathrm{e}^{-s})$ and $\tilde{C}_k'(\mathrm{e}^{-s})$, $\bar{D}_k(\mathrm{e}^{-s})$ and $\tilde{D}_k'(\mathrm{e}^{-s})$, are compatible, we replace $\tilde{C}_k(\mathrm{e}^{-s})$ and 0 in equation (3.3) by $\tilde{C}_k'(\mathrm{e}^{-s})$ and $\tilde{D}_k'(\mathrm{e}^{-s})$ respectively.

**Nonsingular matrices:** The nonsingular matrix $S_k(\mathrm{e}^s)$ allows to gather linearly independent rows in matrix $\left[\begin{smallmatrix}\bar{D}_{k-1}(\mathrm{e}^{-s})\\ \tilde{D}_{k-1}'(\mathrm{e}^{-s})\end{smallmatrix}\right]$ on the top by using row operations such that the matrix $\bar{D}_k(\mathrm{e}^{-s})$ is of weak full row rank, see line 11. The existence of the transformation matrix $S_k(\mathrm{e}^s)$ is guaranteed by [37, Lemma 2]. To simplify the presentation, we assume that the row permutations required for each step has been completed by rearranging the order of outputs before executing the algorithm. Then the nonsingular transformation matrix $S_k(\mathrm{e}^s)$ has the form:

$$S_k(\mathrm{e}^s)=\left[\begin{array}{c|c} S_{k1}(\mathrm{e}^s) & \\ \hline S_{k3}(\mathrm{e}^s) & S_{k2}(\mathrm{e}^s)\end{array}\right]\in\mathbf{R}^{m\times m}(\mathrm{e}^s), \tag{3.8}$$

where $S_{k3}(\mathrm{e}^s)\in\mathbf{R}^{(m-q_k)\times q_k}(\mathrm{e}^s)$ is weak proper in variable $s$, the diagonal matrices

$$\begin{aligned} S_{k1}(\mathrm{e}^s)&=\mathrm{e}^{i_{k,1}s}\oplus\mathrm{e}^{i_{k,2}s}\oplus\cdots\oplus\mathrm{e}^{i_{k,q_k}s}\in\mathbf{R}^{q_k\times q_k}[\mathrm{e}^s],\\ S_{k2}(\mathrm{e}^s)&=\mathrm{e}^{j_{k,1}s}\oplus\mathrm{e}^{j_{k,2}s}\oplus\cdots\oplus\mathrm{e}^{j_{k,m-q_k}s}\in\mathbf{R}^{(m-q_k)\times(m-q_k)}[\mathrm{e}^s],\end{aligned} \tag{3.9}$$

are such that $i_{k,1}\le i_{k,2}\le\cdots\le i_{k,q_k}\le j_{k,1}\le\cdots\le j_{k,m-q_k}$. In particular, $S_{11}=I_{q_1}$, $S_{12}=I_{m-q_1}$, and $S_{13}\in\mathbf{R}^{(m-q_1)\times q_1}$.

**Termination and number of steps:** This algorithm stops at step $\alpha$, where $\alpha=\min\{k\,|\,q_k=r\}$, namely, $\alpha$ is the minimum $k$ such that $q_k=r$. By using the expansion (2.4) and the generalized Cayley-Hamilton theorem [11, Proposition 1 and Theorem 5], the algorithm runs at most $2n$ steps.

**Nonsingular operator matrices:** Denote $N_0(s,\mathrm{e}^s)=I_m$, and

$$N_k(s,\mathrm{e}^s)=\prod_{i=1}^{k}S_i(\mathrm{e}^s)L_{i-1}(s),\quad k=1,2,\dots,\alpha. \tag{3.10}$$

which is lower triangular and therefore invertible in $\mathbf{R}^{m\times m}(s,\mathrm{e}^s)$. Let $Y_k$ denote the system output at step $k$. Then $Y_k(s)\in\mathbf{C}^m$ satisfies

$$Y_k(s)=N_k(s,\mathrm{e}^s)y(s)=\left[\begin{array}{c}\bar{C}_k(\mathrm{e}^{-s})\\ \tilde{C}_k(\mathrm{e}^{-s})\end{array}\right]x(s)+\left[\begin{array}{c}\bar{D}_k(\mathrm{e}^{-s})\\ 0\end{array}\right]u(s). \tag{3.11}$$

Denote the transformed transfer function matrix by $G_\alpha(s,\mathrm{e}^s)$. Then

$$\begin{aligned} G_\alpha(s,\mathrm{e}^s) &= N_\alpha(s,\mathrm{e}^s)G(s,\mathrm{e}^s) \\ &= \begin{bmatrix} \bar{C}_\alpha(\mathrm{e}^{-s}) \\ \tilde{C}_\alpha(\mathrm{e}^{-s}) \end{bmatrix} (sI_n - A_0 - A_1\mathrm{e}^{-s})^{-1}B + \begin{bmatrix} \bar{D}_\alpha(\mathrm{e}^{-s}) \\ 0 \end{bmatrix}, \end{aligned} \tag{3.12}$$

clearly, we have $\mathrm{rank}(\lim_{s\in\mathbf{R},s\to\infty} G_\alpha(s,\mathrm{e}^s)) = \mathrm{nrank}(G(s,\mathrm{e}^s)) = r$.

Next, examples are given to illustrate the procedure of Algorithm 3.1.

EXAMPLE 3.1. *Reconsider the system in Example 2.3.*

**Initialization :** *Denote $\tilde{y}_0(t) = y(t) = Cx(t)$. Take the time derivative of $\tilde{y}_0(t)$:*

$$\dot{\tilde{y}}_0(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t). \tag{3.13}$$

*Convert equation (3.13) to the Laplace domain:*

$$L_0(s)\tilde{y}_0(s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(s),$$

*where $L_0(s) = sI_2$. Denote $q_1 = \mathrm{rank}\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] = 1$. There exists a nonsingular matrix $S_1 = I_2$ such that*

$$\left[\begin{array}{c} \bar{y}_1(s) \\ \hline \tilde{y}_1(s) \end{array}\right] \triangleq S_1L_0(s)\tilde{y}_0(s) = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \mathrm{e}^{-s} \end{array}\right] x(s) + \left[\begin{array}{cc} 0 & 1 \\ \hline 0 & 0 \end{array}\right] u(s). \tag{3.14}$$

**Step 1 :** *Convert the equation $\tilde{y}_1(s) = \begin{bmatrix} 0 & 1 & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s)$ to the time domain, namely, $\tilde{y}_1(t) = x_2(t) + x_4(t-1)$, where $x_i$ denotes the $i$th component of $x$. Take the time derivative of $\tilde{y}_1(t)$:*

$$\dot{\tilde{y}}_1(t) = \dot{x}_2(t) + \dot{x}_4(t-1) = x_3(t) + u_2(t-1), \tag{3.15}$$

*where $u_i$ denotes the $i$th component of $u$. Convert equation (3.15) to the Laplace domain:*

$$s\tilde{y}_1(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(s) + \begin{bmatrix} 0 & \mathrm{e}^{-s} \end{bmatrix} u(s). \tag{3.16}$$

*Replace the equation $\tilde{y}_1(s) = \begin{bmatrix} 0 & 1 & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s)$ in (3.14) by (3.16), to get*

$$L_1(s)\begin{bmatrix} \bar{y}_1(s) \\ \tilde{y}_1(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 0 & \mathrm{e}^{-s} \end{bmatrix} u(s),$$

*where $L_1(s) = 1 \oplus s$. Denote $q_2 = \mathrm{nrank}\left[\begin{smallmatrix} 0 & 1 \\ 0 & \mathrm{e}^{-s} \end{smallmatrix}\right] = 1$. There exists a nonsingular matrix $S_2(\mathrm{e}^s) = \left[\begin{smallmatrix} 1 & 0 \\ -\mathrm{e}^{-s} & 1 \end{smallmatrix}\right]$ such that*

$$\left[\begin{array}{c} \bar{y}_2(s) \\ \hline \tilde{y}_2(s) \end{array}\right] \triangleq S_2(\mathrm{e}^s)L_1(s)\begin{bmatrix} \bar{y}_1(s) \\ \tilde{y}_1(s) \end{bmatrix} = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] x(s) + \left[\begin{array}{cc} 0 & 1 \\ \hline 0 & 0 \end{array}\right] u(s). \tag{3.17}$$

**Step 2 :** *Convert the equation $\tilde{y}_2(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(s)$ to the time domain, namely, $\tilde{y}_2(t) = x_3(t)$. Take the time derivative of $\tilde{y}_2(t)$:*

$$\dot{\tilde{y}}_2(t) = \dot{x}_3(t) = x_4(t-1) + u_1(t). \tag{3.18}$$

*Convert equation (3.18) to the Laplace domain:*

$$s\tilde{y}_2(s) = \begin{bmatrix} 0 & 0 & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 1 & 0 \end{bmatrix} u(s). \tag{3.19}$$

*Replace the equation* $\tilde{y}_2(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(s)$ *in (3.17) by (3.19), namely,*

$$L_2(s) \begin{bmatrix} \bar{y}_2(s) \\ \tilde{y}_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(s),$$

*where* $L_2(s) = 1 \oplus s$. *Denote* $q_3 = \operatorname{nrank}\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = 2$. *There exists a nonsingular matrix* $S_3 = I_2$ *such that*

$$\bar{y}_3(s) \triangleq S_3 L_2(s) \begin{bmatrix} \bar{y}_2(s) \\ \tilde{y}_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(s).$$

*Since* $q_3 = \operatorname{nrank}(G(s, \mathrm{e}^s))$, *the algorithm is terminates with* $\alpha = 3$ *and a matrix with weak full row rank,* $\bar{D}_\alpha(\mathrm{e}^{-s}) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. *Also, the total nonsingular operator matrix has the form*

$$N_\alpha(s, \mathrm{e}^s) = \prod_{i=1}^{\alpha} S_i(\mathrm{e}^s) L_{i-1}(s) = \begin{bmatrix} s & 0 \\ -s^2\mathrm{e}^{-s} & s^3 \end{bmatrix}, \tag{3.20}$$

*which implies the following transformed transfer function matrix,*

$$G_\alpha(s, \mathrm{e}^s) = N_\alpha(s, \mathrm{e}^s) G(s, \mathrm{e}^s) = \begin{bmatrix} 0 & 1 \\ 1 & s^{-1}\mathrm{e}^{-s} \end{bmatrix}. \tag{3.21}$$

EXAMPLE 3.2. *Consider the system* (1.2) *with coefficient matrices*

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}^{\mathrm{T}}.$$

*The transfer function matrix is given by*

$$G(s, \mathrm{e}^s) = \begin{bmatrix} 0 & \frac{1}{s-2} \\ \frac{2(1+\mathrm{e}^{-s})}{s^2(s-2)} & \frac{2\mathrm{e}^{-s}(1+\mathrm{e}^{-s})+\mathrm{e}^{-s}s(s-2)}{s^2(s-2)^2} \end{bmatrix}$$

*and it satisfies* $\operatorname{nrank}(G(s, \mathrm{e}^s)) = 2$.

**Initialization :** *Denote* $\tilde{y}_0(t) = y(t) = Cx(t)$. *Take the time derivative of* $\tilde{y}_0(t)$*:*

$$\dot{\tilde{y}}_0(t) = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t). \tag{3.22}$$

*Convert equation (3.22) to the Laplace domain:*

$$L_0(s)\tilde{y}_0(s) = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1+\mathrm{e}^{-s} & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(s),$$

*where* $L_0(s) = sI_2$. *Denote* $q_1 = \operatorname{rank}\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] = 1$. *There exists a nonsingular matrix* $S_1 = I_2$ *such that*

$$\begin{bmatrix} \bar{y}_1(s) \\ \tilde{y}_1(s) \end{bmatrix} \triangleq S_1 L_0(s) \tilde{y}_0(s) = \left[\begin{array}{cccc} 0 & 0 & 0 & 2 \\ 0 & 1+\mathrm{e}^{-s} & 0 & \mathrm{e}^{-s} \end{array}\right] x(s) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(s). \tag{3.23}$$

**Step 1 :** *Convert the equation* $\tilde{y}_1(s) = \begin{bmatrix} 0 & 1+\mathrm{e}^{-s} & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s)$ *to the time domain, namely,* $\tilde{y}_1(t) = x_2(t) + x_2(t-1) + x_4(t-1)$. *Taking the time derivative of* $\tilde{y}_1(t)$ *gives*

$$\dot{\tilde{y}}_1(t) = 2x_3(t) + 2x_3(t-1) + 2x_4(t-1) + u_2(t-1). \tag{3.24}$$

*Convert equation* (3.24) *to the Laplace domain:*

$$s\tilde{y}_1(s) = \begin{bmatrix} 0 & 0 & 2+2\mathrm{e}^{-s} & 2\mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & \mathrm{e}^{-s} \end{bmatrix} u(s). \tag{3.25}$$

*Replace the equation* $\tilde{y}_1(s) = \begin{bmatrix} 0 & 1+\mathrm{e}^{-s} & 0 & \mathrm{e}^{-s} \end{bmatrix} x(s)$ *in* (3.23) *by* (3.25)*:*

$$L_1(s) \begin{bmatrix} \bar{y}_1(s) \\ \tilde{y}_1(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2+2\mathrm{e}^{-s} & 2\mathrm{e}^{-s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 0 & \mathrm{e}^{-s} \end{bmatrix} u(s),$$

*where* $L_1(s) = 1 \oplus s$. *Denote* $q_2 = \operatorname{nrank}\begin{bmatrix} 0 & 1 \\ 0 & \mathrm{e}^{-s} \end{bmatrix} = 1$. *There exists a nonsingular matrix* $S_2(\mathrm{e}^s) = \begin{bmatrix} 1 & 0 \\ -\mathrm{e}^{-s} & 1 \end{bmatrix}$ *such that*

$$\begin{aligned} \begin{bmatrix} \bar{y}_2(s) \\ \tilde{y}_2(s) \end{bmatrix} &\triangleq S_2(\mathrm{e}^s) L_1(s) \begin{bmatrix} \bar{y}_1(s) \\ \tilde{y}_1(s) \end{bmatrix} \\ &= \left[\begin{array}{cccc} 0 & 0 & 0 & 2 \\ \hline 0 & 0 & 2+2\mathrm{e}^{-s} & 0 \end{array}\right] x(s) + \left[\begin{array}{cc} 0 & 1 \\ \hline 0 & 0 \end{array}\right] u(s). \end{aligned} \tag{3.26}$$

**Step 2 :** *Convert the equation* $\tilde{y}_2(s) = \begin{bmatrix} 0 & 0 & 2+2\mathrm{e}^{-s} & 0 \end{bmatrix} x(s)$ *to the time domain, namely,* $\tilde{y}_2(t) = 2x_3(t) + 2x_3(t-1)$. *Take the time derivative of* $\tilde{y}_2(t)$*:*

$$\dot{\tilde{y}}_2(t) = 4x_3(t) + 4x_3(t-1) + 2x_4(t-1) + 2x_4(t-2) + 2u_1(t) + 2u_1(t-1). \tag{3.27}$$

*Convert equation* (3.27) *to the Laplace domain, namely,*

$$s\tilde{y}_2(s) = \begin{bmatrix} 0 & 0 & 4+4\mathrm{e}^{-s} & 2\mathrm{e}^{-s}+2\mathrm{e}^{-2s} \end{bmatrix} x(s) + \begin{bmatrix} 2+2\mathrm{e}^{-s} & 0 \end{bmatrix} u(s). \tag{3.28}$$

*Replace the equation* $\tilde{y}_2(s) = \begin{bmatrix} 0 & 0 & 2+2\mathrm{e}^{-s} & 0 \end{bmatrix} x(s)$ *in* (3.26) *by* (3.28)*, such that*

$$L_2(s) \begin{bmatrix} \bar{y}_2(s) \\ \tilde{y}_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 4+4\mathrm{e}^{-s} & 2\mathrm{e}^{-s}+2\mathrm{e}^{-2s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 2+2\mathrm{e}^{-s} & 0 \end{bmatrix} u(s),$$

*where* $L_2(s) = 1 \oplus s$. *Denote* $q_3 = \operatorname{nrank}\begin{bmatrix} 0 & 1 \\ 2+2\mathrm{e}^{-s} & 0 \end{bmatrix} = 2$. *There exists a nonsingular matrix* $S_3 = I_2$ *such that*

$$\begin{aligned} \bar{y}_3(s) &\triangleq S_3 L_2(s) \begin{bmatrix} \bar{y}_2(s) \\ \tilde{y}_2(s) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 4+4\mathrm{e}^{-s} & 2\mathrm{e}^{-s}+2\mathrm{e}^{-2s} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 1 \\ 2+2\mathrm{e}^{-s} & 0 \end{bmatrix} u(s). \end{aligned}$$

*Since* $q_3 = \operatorname{nrank}(G(s, \mathrm{e}^s))$, *the algorithm terminates with* $\alpha = 3$ *and a matrix with weak full row rank,* $\bar{D}_\alpha(\mathrm{e}^{-s}) = \begin{bmatrix} 0 & 1 \\ 2+2\mathrm{e}^{-s} & 0 \end{bmatrix}$. *The total nonsingular operator matrix possesses the form* (3.20)*, and the transformed transfer function matrix satisfies*

$$G_\alpha(s, \mathrm{e}^s) = N_\alpha(s, \mathrm{e}^s) G(s, \mathrm{e}^s) = \begin{bmatrix} 0 & \frac{s}{s-2} \\ \frac{2s(1+\mathrm{e}^{-s})}{s-2} & \frac{2s\mathrm{e}^{-s}(1+\mathrm{e}^{-s})}{(s-2)^2} \end{bmatrix}. \tag{3.29}$$

*Besides, according to* (3.11) *and* (3.12)*, the system outputs* $Y_k$ *at steps* $k = 0, 1, 2, 3$*, in the time domain, are expressed as*

$$
\begin{aligned}
&Y_0(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \ Y_1(t) = \begin{bmatrix} y_1^{(1)}(t) \\ y_2^{(1)}(t) \end{bmatrix}, \\
&Y_2(t) = \begin{bmatrix} y_1^{(1)}(t) \\ y_2^{(2)}(t) - y_1^{(1)}(t-1) \end{bmatrix}, \ Y_3(t) = \begin{bmatrix} y_1^{(1)}(t) \\ y_2^{(3)}(t) - y_1^{(2)}(t-1) \end{bmatrix},
\end{aligned} \tag{3.30}
$$

*where* $y_i$ *stands for the* $i$*th component of* $y$*.*

The following two lemmas are needed in order to obtain the Smith-McMillan form at infinity of system (1.2).

LEMMA 3.3. *Let* $V = \begin{bmatrix} V_{\rm ri} & V_{\rm ns} \end{bmatrix} \in \mathbf{R}^{m\times m}$*, where* $V_{\rm ri}$ *is the right inverse matrix of* $\bar{D}_\alpha(\mathrm{e}^{-s})$ *in the weak sense, and the columns of* $V_{\rm ns}$ *span the null space of* $\bar{D}_\alpha(\mathrm{e}^{-s})$ *in the weak sense. Then matrix* $V$ *is nonsingular.*

*Proof.* According to Algorithm 3.1, matrix $\bar{D}_\alpha(\mathrm{e}^{-s}) \in \mathbf{R}^{r\times m}[\mathrm{e}^{-s}]$ is of full row rank in the weak sense. Let

$$
\bar{D}_\alpha^* = \lim_{s\in\mathbf{R}, s\to\infty} \bar{D}_\alpha(\mathrm{e}^{-s}),
$$

then $\bar{D}_\alpha^* \in \mathbf{R}_r^{r\times m}$. By assumption, we have

$$
\begin{aligned}
&\big(\lim_{s\in\mathbf{R}, s\to\infty} \bar{D}_\alpha(\mathrm{e}^{-s})\big)V_{\rm ri} = \bar{D}_\alpha^* V_{\rm ri} = I_r, \quad V_{\rm ri} \in \mathbf{R}_r^{m\times r}, \\
&\big(\lim_{s\in\mathbf{R}, s\to\infty} \bar{D}_\alpha(\mathrm{e}^{-s})\big)V_{\rm ns} = \bar{D}_\alpha^* V_{\rm ns} = 0, \quad V_{\rm ns} \in \mathbf{R}_{m-r}^{m\times(m-r)}.
\end{aligned} \tag{3.31}
$$

Denote the left inverse matrix of $V_{\rm ns}$ by $U_{\rm li}$. Then it holds that

$$
U_{\rm li} V_{\rm ns} = I_{m-r}, \quad U_{\rm li} \in \mathbf{R}_{m-r}^{(m-r)\times m}. \tag{3.32}
$$

Construct the following matrix:

$$
U = \begin{bmatrix} \bar{D}_\alpha^* \\ -U_{\rm li} V_{\rm ri} \bar{D}_\alpha^* + U_{\rm li} \end{bmatrix} \in \mathbf{R}^{m\times m}.
$$

By using equations (3.31) and (3.32), we get $UV = I_r \oplus I_{m-r}$, that is, the matrix $V$ is nonsingular. Therefore, the proof is complete. ∎

LEMMA 3.4. *Let the matrix* $N_k(s, \mathrm{e}^s)$*,* $k = 1, 2, \ldots, \alpha$*, be defined in* (3.10)*. Then* $N_k^{-1}(s, \mathrm{e}^s) \in \mathbf{R}^{m\times m}(s, \mathrm{e}^s)$ *has the form*

$$
N_k^{-1}(s, \mathrm{e}^s) = \begin{bmatrix}
\Delta_1(\mathrm{e}^{-s})s^{-1} & 0 & \cdots & 0 & 0 \\
N_{21}(s, \mathrm{e}^s) & \Delta_2(\mathrm{e}^{-s})s^{-2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
N_{k1}(s, \mathrm{e}^s) & N_{k2}(s, \mathrm{e}^s) & \cdots & \Delta_k(\mathrm{e}^{-s})s^{-k} & 0 \\
\Psi_{k1}(s, \mathrm{e}^s) & \Psi_{k2}(s, \mathrm{e}^s) & \cdots & \Psi_{kk}(s, \mathrm{e}^s) & \Omega_k(\mathrm{e}^{-s})s^{-k}
\end{bmatrix}, \tag{3.33}
$$

*where*

(1) $\Delta_i(\mathrm{e}^{-s}) = \mathrm{e}^{-m_{i,1}s} \oplus \mathrm{e}^{-m_{i,2}s} \oplus \cdots \oplus \mathrm{e}^{-m_{i,j_i}s}$*, where* $0 \le m_{i,1} \le m_{i,2} \le \cdots \le m_{i,j_i} \le m_{i+1,1} \le \cdots \le m_{i+1,j_{i+1}}$*,* $j_i = q_i - q_{i-1}$*,* $q_0 = 0$*,* $i = 1, 2, \ldots, k$*.*

(2) $\Omega_k(e^{-s}) = \mathrm{e}^{-\mu_{k,1}s} \oplus \mathrm{e}^{-\mu_{k,2}s} \oplus \cdots \oplus \mathrm{e}^{-\mu_{k,m-q_k}s}$, *where* $m_{k,j_k} \le \mu_{k,1} \le \mu_{k,2} \le \cdots \le \mu_{k,m-q_k}$.

(3) *The matrices* $N_{hl}(s,\mathrm{e}^s)s^l\Delta_l^{-1}(\mathrm{e}^{-s})$, $h = l+1, l+2, \ldots, k$, $l = 1, 2, \ldots, k-1$ *and* $\Psi_{ki}(s,\mathrm{e}^s)s^i\Delta_i^{-1}(\mathrm{e}^{-s})$, $i = 1, 2, \ldots, k$ *are weak proper.*

*Proof.* This proof is based on induction. For brevity, we omit the variables $s, \mathrm{e}^s$ in some places. Denote

$$\Lambda_k = \Delta_1(\mathrm{e}^{-s})s^{-1} \oplus \Delta_2(\mathrm{e}^{-s})s^{-2} \oplus \cdots \oplus \Delta_k(\mathrm{e}^{-s})s^{-k} \in \mathbf{R}^{q_k\times q_k}(s,\mathrm{e}^s),$$

$$\Psi_k = \begin{bmatrix} \Psi_{k1}(s,\mathrm{e}^s) & \Psi_{k2}(s,\mathrm{e}^s) & \cdots & \Psi_{kk}(s,\mathrm{e}^s) \end{bmatrix} \in \mathbf{R}^{(m-q_k)\times q_k}(s,\mathrm{e}^s),$$

$$E_k = \begin{bmatrix} \Delta_1(\mathrm{e}^{-s})s^{-1} & 0 & \cdots & 0 \\ N_{21}(s,\mathrm{e}^s) & \Delta_2(\mathrm{e}^{-s})s^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{k1}(s,\mathrm{e}^s) & N_{k2}(s,\mathrm{e}^s) & \cdots & \Delta_k(\mathrm{e}^{-s})s^{-k} \end{bmatrix} \in \mathbf{R}^{q_k\times q_k}(s,\mathrm{e}^s).$$

Partition matrices $\Psi_k$ and $\Omega_k(\mathrm{e}^{-s}) \in \mathbf{R}^{(m-q_k)\times(m-q_k)}[\mathrm{e}^{-s}]$ as

$$\Psi_k = \begin{bmatrix} \Psi_k^{[1]} \\ \Psi_k^{[2]} \end{bmatrix}, \Psi_k^{[1]} \in \mathbf{R}^{(q_{k+1}-q_k)\times q_k}(s,\mathrm{e}^s), \Psi_k^{[2]} \in \mathbf{R}^{(m-q_{k+1})\times q_k}(s,\mathrm{e}^s),$$

$$\Omega_k = \Omega_k^{[1]} \oplus \Omega_k^{[2]}, \Omega_k^{[1]} \in \mathbf{R}^{(q_{k+1}-q_k)\times(q_{k+1}-q_k)}[\mathrm{e}^{-s}], \Omega_k^{[2]} \in \mathbf{R}^{(m-q_{k+1})\times(m-q_{k+1})}[\mathrm{e}^{-s}].$$

According to (3.10), we have

$$N_k^{-1}(s,\mathrm{e}^s) = L_0^{-1}(s)S_1^{-1}L_1^{-1}(s)\cdots S_{k-1}^{-1}(\mathrm{e}^s)L_{k-1}^{-1}(s)S_k^{-1}(\mathrm{e}^s),$$

where $L_k^{-1}(s) = I_{q_k} \oplus s^{-1}I_{m-q_k}$. Besides, it follows from (3.8) and (3.9) that

$$S_k^{-1}(\mathrm{e}^s) = \left[\begin{array}{c|c} H_{k1}(\mathrm{e}^s) & 0 \\ \hline J_k(\mathrm{e}^s) & H_{k2}(\mathrm{e}^s) \end{array}\right],$$

where

$$H_{k1} = S_{k1}^{-1} = \mathrm{e}^{-i_{k,1}s} \oplus \mathrm{e}^{-i_{k,2}s} \oplus \cdots \oplus \mathrm{e}^{-i_{k,q_k}s} \in \mathbf{R}^{q_k\times q_k}[\mathrm{e}^{-s}],$$

$$H_{k2} = S_{k2}^{-1} = \mathrm{e}^{-j_{k,1}s} \oplus \mathrm{e}^{-j_{k,2}s} \oplus \cdots \oplus \mathrm{e}^{-j_{k,m-q_k}s} \in \mathbf{R}^{(m-q_k)\times(m-q_k)}[\mathrm{e}^{-s}],$$

$$i_{k,1} \le i_{k,2} \le \cdots \le i_{k,q_k} \le j_{k,1} \le \cdots \le j_{k,m-q_k},$$

$$J_k = -S_{k2}^{-1}S_{k3}S_{k1}^{-1} = -H_{k2}S_{k3}H_{k1} \in \mathbf{R}^{(m-q_k)\times q_k}(\mathrm{e}^{-s}).$$

Since $S_{k3}$ is weak proper, we notice that both $J_k$ and $J_kH_{k1}^{-1}$ are weak proper.

Now, the induction process begins. Firstly, we prove that the lemma holds when $k = 1$. In fact, we have

$$N_1^{-1}(s,\mathrm{e}^s) = L_0^{-1}(s)S_1^{-1} = \left[\begin{array}{c|c} s^{-1}I_{q_1} & \\ \hline -s^{-1}S_{13} & s^{-1}I_{m-q_1} \end{array}\right],$$

which clearly coincides with equation (3.33). In addition, conditions (1)-(3) can be simply verified.

Assume that when $i = k$, $N_i^{-1}(s,\mathrm{e}^s)$ possesses the form (3.33), and satisfies conditions (1)-(3). Now we prove that $N_i^{-1}(s,\mathrm{e}^s)$ also has the form (3.33) and satisfies conditions (1)-(3) when $i = k+1$. Denote

$$H_{k+1,1} = H_{k+1,1}^* \oplus H_{k+1,1}^{[k+1]} \in \mathbf{R}^{q_{k+1}\times q_{k+1}}[\mathrm{e}^{-s}],$$

$$H_{k+1,1}^* = H_{k+1,1}^{[1]} \oplus H_{k+1,1}^{[2]} \oplus \cdots \oplus H_{k+1,1}^{[k]} \in \mathbf{R}^{q_k\times q_k}[\mathrm{e}^{-s}],$$

where

$$H_{k+1,1}^{[l]} = \mathrm{e}^{-i_{k+1,q_{l-1}+1}s} \oplus \cdots \oplus \mathrm{e}^{-i_{k+1,q_l}s} \in \mathbf{R}^{(q_l-q_{l-1})\times(q_l-q_{l-1})}[\mathrm{e}^{-s}].$$

Also, denote

$$J_{k+1} = \left[\begin{array}{cc} J_{k+1}^{*} & J_{k+1}^{[k+1]} \end{array}\right] \in \mathbf{R}^{(m-q_{k+1})\times q_{k+1}}(\mathrm{e}^{-s}),$$
$$J_{k+1}^{*} = \left[\begin{array}{ccccc} J_{k+1}^{[1]} & J_{k+1}^{[2]} & \cdots & J_{k+1}^{[k]} \end{array}\right] \in \mathbf{R}^{(m-q_{k+1})\times q_k}(\mathrm{e}^{-s}).$$

where $J_{k+1}^{[l]} \in \mathbf{R}^{(m-q_{k+1})\times(q_l-q_{l-1})}(\mathrm{e}^{-s})$. Using this terminology, we can express

$$\begin{aligned}
&N_{k+1}^{-1}(s,\mathrm{e}^s)\\
&= N_k^{-1}(s,\mathrm{e}^s)L_k^{-1}(s)S_{k+1}^{-1}(\mathrm{e}^s)\\
&= \left[\begin{array}{c|c} E_k & 0 \\ \hline \Psi_k & s^{-(k+1)}\Omega_k \end{array}\right]\left[\begin{array}{c|c} H_{k+1,1} & 0 \\ \hline J_{k+1} & H_{k+1,2} \end{array}\right]\\
&= \left[\begin{array}{cc|c} E_k & 0 & 0 \\ \Psi_k^{[1]} & s^{-(k+1)}\Omega_k^{[1]} & 0 \\ \hline \Psi_k^{[2]} & 0 & s^{-(k+1)}\Omega_k^{[2]} \end{array}\right]\left[\begin{array}{cc|c} H_{k+1,1}^{*} & 0 & 0 \\ 0 & H_{k+1,1}^{[k+1]} & 0 \\ \hline J_{k+1}^{*} & J_{k+1}^{[k+1]} & H_{k+1,2} \end{array}\right]\\
&= \left[\begin{array}{cc|c} E_kH_{k+1,1}^{*} & 0 & 0 \\ \Theta_{2,1} & s^{-(k+1)}\Omega_k^{[1]}H_{k+1,1}^{[k+1]} & 0 \\ \hline \Theta_{3,1} & \Theta_{3,2} & s^{-(k+1)}\Omega_k^{[2]}H_{k+1,2} \end{array}\right],
\end{aligned}$$

where

$$\begin{aligned}
\Theta_{2,1} &= \Psi_k^{[1]}H_{k+1,1}^{*},\\
\Theta_{3,1} &= \Psi_k^{[2]}H_{k+1,1}^{*} + s^{-(k+1)}\Omega_k^{[2]}J_{k+1}^{*},\\
\Theta_{3,2} &= s^{-(k+1)}\Omega_k^{[2]}J_{k+1}^{[k+1]}
\end{aligned}$$

and

$$E_kH_{k+1,1}^{*} = \left[\begin{array}{cccc} s^{-1}\Delta_1H_{k+1,1}^{[1]} & & & \\ N_{21}H_{k+1,1}^{[1]} & s^{-2}\Delta_2H_{k+1,1}^{[2]} & & \\ \vdots & \vdots & \ddots & \\ N_{k1}H_{k+1,1}^{[1]} & N_{k2}H_{k+1,1}^{[2]} & \cdots & s^{-k}\Delta_kH_{k+1,1}^{[k]} \end{array}\right].$$

By the inequality $i_{k+1,1} \le i_{k+1,2} \le \cdots \le i_{k+1,q_{k+1}} \le j_{k+1,1} \le \cdots \le j_{k+1,m-q_{k+1}}$, the diagonal matrix

$$\Lambda_kH_{k+1,1}^{*} \oplus s^{-(k+1)}\Omega_k^{[1]}H_{k+1,1}^{[k+1]} \oplus s^{-(k+1)}\Omega_k^{[2]}H_{k+1,2},$$

in $N_{k+1}^{-1}(s,\mathrm{e}^s)$ clearly satisfies conditions (1) and (2).

By assumption, matrices $\Psi_k^{[1]}H_{k+1,1}^{*}(\Lambda_kH_{k+1,1}^{*})^{-1}$, $\Psi_k^{[2]}H_{k+1,1}^{*}(\Lambda_kH_{k+1,1}^{*})^{-1}$, as well as $N_{hl}H_{k+1,1}^{[l]}(s^{-l}\Delta_lH_{k+1,1}^{[l]})^{-1}$, $h = l+1, l+2, \ldots, k$, $l = 1, 2, \ldots, k-1$, are weak proper. In addition, we note that matrices

$$s^{-(k+1)}\Omega_k^{[2]}J_{k+1}^{*}(\Lambda_kH_{k+1,1}^{*})^{-1} \text{ and } s^{-(k+1)}\Omega_k^{[2]}J_{k+1}^{[k+1]}(s^{-(k+1)}\Omega_k^{[1]}H_{k+1,1}^{[k+1]})^{-1}$$

are weak proper, which follows from the induction hypothesis and the weak properness of matrix $J_{k+1}H_{k+1,1}^{-1}$.

Based on the above analysis, we know that the matrix $N_{k+1}^{-1}(s,\mathrm{e}^s)$ has the form (3.33), and satisfies conditions (1)-(3). Thus the proof is finished. ∎

Utilizing Algorithm 3.1, as well as Lemmas 3.3 and 3.4, we get the Smith-McMillan form at infinity of the transfer function matrix, as well as a novel construction of weak biproper transformation matrices, which are summarized as follows.

THEOREM 3.5. *Let the transfer function matrix of system* (1.2) *be* $G(s,\mathrm{e}^s)$. *Then there are weak biproper matrices* $B_1(s,\mathrm{e}^s)\in\mathbf{R}^{m\times m}(s,\mathrm{e}^s)$ *and* $B_2(s,\mathrm{e}^s)\in\mathbf{R}^{m\times m}(s,\mathrm{e}^s)$ *such that*

$$G(s,\mathrm{e}^s)=B_1(s,\mathrm{e}^s)\left[\begin{array}{ccc|c} \Delta_1(\mathrm{e}^{-s})s^{-1} & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & \Delta_\alpha(\mathrm{e}^{-s})s^{-\alpha} & 0\\ \hline 0 & \cdots & 0 & 0\end{array}\right]B_2(s,\mathrm{e}^s), \tag{3.34}$$

*where* $\Delta_i(\mathrm{e}^{-s})=\mathrm{e}^{-m_{i,1}s}\oplus\mathrm{e}^{-m_{i,2}s}\oplus\cdots\oplus\mathrm{e}^{-m_{i,j_i}s}$, $0\le m_{i,1}\le m_{i,2}\le\cdots\le m_{i,j_i}\le m_{i+1,1}\le\cdots\le m_{i+1,j_{i+1}}$, $j_i=q_i-q_{i-1}$, $q_0=0$, $i=1,2,\ldots,\alpha$.

*Proof.* For simplicity, we will omit the variables $s,\mathrm{e}^s$ in some places of this proof. According to (3.12) and Lemma 3.3, we have

$$\begin{aligned} G_\alpha(s,\mathrm{e}^s)V&=\left[\begin{array}{c}\bar{C}_\alpha(\mathrm{e}^{-s})\\ \tilde{C}_\alpha(\mathrm{e}^{-s})\end{array}\right](sI_n-A_0-A_1\mathrm{e}^{-s})^{-1}BV+\left[\begin{array}{c}\bar{D}_\alpha(\mathrm{e}^{-s})\\ 0\end{array}\right]V\\ &\triangleq\left[\begin{array}{cc}T_{\alpha1}(s,\mathrm{e}^s) & T_{\alpha2}(s,\mathrm{e}^s)\\ T_{\alpha3}(s,\mathrm{e}^s) & T_{\alpha4}(s,\mathrm{e}^s)\end{array}\right],\end{aligned} \tag{3.35}$$

where

$$\begin{aligned} T_{\alpha1}&=\bar{C}_\alpha(\mathrm{e}^{-s})(sI_n-A_0-A_1\mathrm{e}^{-s})^{-1}BV_{\mathrm{ri}}+\bar{D}_\alpha(\mathrm{e}^{-s})V_{\mathrm{ri}}\in\mathbf{R}^{r\times r}(s,\mathrm{e}^s),\\ T_{\alpha2}&=\bar{C}_\alpha(\mathrm{e}^{-s})(sI_n-A_0-A_1\mathrm{e}^{-s})^{-1}BV_{\mathrm{ns}}+\bar{D}_\alpha(\mathrm{e}^{-s})V_{\mathrm{ns}}\in\mathbf{R}^{r\times(m-r)}(s,\mathrm{e}^s),\\ T_{\alpha3}&=\tilde{C}_\alpha(\mathrm{e}^{-s})(sI_n-A_0-A_1\mathrm{e}^{-s})^{-1}BV_{\mathrm{ri}}\in\mathbf{R}^{(m-r)\times r}(s,\mathrm{e}^s),\\ T_{\alpha4}&=\tilde{C}_\alpha(\mathrm{e}^{-s})(sI_n-A_0-A_1\mathrm{e}^{-s})^{-1}BV_{\mathrm{ns}}\in\mathbf{R}^{(m-r)\times(m-r)}(s,\mathrm{e}^s).\end{aligned}$$

We note that $T_{\alpha2}$, $T_{\alpha3}$ and $T_{\alpha4}$ are weak proper. Moreover, $T_{\alpha1}$ is weak biproper, which implies: (i) $\mathrm{nrank}(T_{\alpha1})=r$, (ii) $T_{\alpha1}$ is invertible in $\mathbf{R}^{r\times r}(s,\mathrm{e}^s)$, and its inverse is also weak biproper.

Since $\mathrm{nrank}(G_\alpha(s,\mathrm{e}^s)V)=\mathrm{nrank}(G_\alpha(s,\mathrm{e}^s))=\mathrm{nrank}(G(s,\mathrm{e}^s))=r$, there exists an elementary matrix $\Phi=\left[\begin{smallmatrix}I_r & \Phi_1\\ 0 & I_{m-r}\end{smallmatrix}\right]\in\mathbf{R}^{m\times m}(s,\mathrm{e}^s)$ such that the last $m-r$ columns of matrix $G_\alpha(s,\mathrm{e}^s)V$ are transformed into zero by elementary column operations. From the relations

$$G_\alpha(s,\mathrm{e}^s)V\Phi=\left[\begin{array}{cc}T_{\alpha1} & T_{\alpha2}\\ T_{\alpha3} & T_{\alpha4}\end{array}\right]\left[\begin{array}{cc}I_r & \Phi_1\\ 0 & I_{m-r}\end{array}\right]=\left[\begin{array}{cc}T_{\alpha1} & T_{\alpha1}\Phi_1+T_{\alpha2}\\ T_{\alpha3} & T_{\alpha3}\Phi_1+T_{\alpha4}\end{array}\right],$$

we derive $T_{\alpha4}=T_{\alpha3}T_{\alpha1}^{-1}T_{\alpha2}$.

Then we have

$$
\begin{aligned}
G(s,\mathrm{e}^s) &= N_\alpha^{-1}(s,\mathrm{e}^s)G_\alpha(s,\mathrm{e}^s)\\
&= N_\alpha^{-1}(s,\mathrm{e}^s)\begin{bmatrix} T_{\alpha1} & T_{\alpha2}\\ T_{\alpha3} & T_{\alpha4}\end{bmatrix}V^{-1}\\
&= N_\alpha^{-1}(s,\mathrm{e}^s)\begin{bmatrix} I_r & 0\\ T_{\alpha3}T_{\alpha1}^{-1} & T_{\alpha4}-T_{\alpha3}T_{\alpha1}^{-1}T_{\alpha2}\end{bmatrix}\begin{bmatrix} T_{\alpha1} & T_{\alpha2}\\ 0 & I_{m-r}\end{bmatrix}V^{-1}\\
&= N_\alpha^{-1}(s,\mathrm{e}^s)\begin{bmatrix} I_r & 0\\ T_{\alpha3}T_{\alpha1}^{-1} & 0\end{bmatrix}\begin{bmatrix} T_{\alpha1} & T_{\alpha2}\\ 0 & I_{m-r}\end{bmatrix}V^{-1}.
\end{aligned}
\tag{3.36}
$$

Denote

$$
B_2(s,\mathrm{e}^s)=\begin{bmatrix} T_{\alpha1} & T_{\alpha2}\\ 0 & I_{m-r}\end{bmatrix}V^{-1},
\tag{3.37}
$$

which is obviously weak biproper. On the other side, denote

$$
\begin{aligned}
F_1(s,\mathrm{e}^s) &= N_\alpha^{-1}(s,\mathrm{e}^s)\begin{bmatrix} \Lambda_\alpha & 0\\ 0 & s^{-\alpha}\Omega_\alpha\end{bmatrix}^{-1}\\
&= \left[\begin{array}{c|c} E_\alpha & 0\\ \hline \Psi_\alpha & s^{-\alpha}\Omega_\alpha\end{array}\right]\left[\begin{array}{c|c} \Lambda_\alpha^{-1} & 0\\ \hline 0 & s^{\alpha}\Omega_\alpha^{-1}\end{array}\right]\\
&= \begin{bmatrix} I_{q_1-q_0} & 0 & \cdots & 0 & 0\\ sN_{21}\Delta_1^{-1} & I_{q_2-q_1} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ sN_{\alpha1}\Delta_1^{-1} & s^2N_{\alpha2}\Delta_2^{-1} & \cdots & I_{q_\alpha-q_{\alpha-1}} & 0\\ s\Psi_{\alpha1}\Delta_1^{-1} & s^2\Psi_{\alpha2}\Delta_2^{-1} & \cdots & s^\alpha\Psi_{\alpha\alpha}\Delta_\alpha^{-1} & I_{m-q_\alpha}\end{bmatrix},
\end{aligned}
\tag{3.38}
$$

which is weak biproper, according to Lemma 3.4. Let

$$
B_1(s,\mathrm{e}^s)=F_1(s,\mathrm{e}^s)\begin{bmatrix} I_{q_\alpha} & 0\\ s^{-\alpha}\Omega_\alpha T_{\alpha3}T_{\alpha1}^{-1}\Lambda_\alpha^{-1} & I_{m-q_\alpha}\end{bmatrix},
\tag{3.39}
$$

which is clearly weak biproper. According to (3.36), we verify that the weak biproper matrices $B_1(s,\mathrm{e}^s)$ and $B_2(s,\mathrm{e}^s)$, defined in (3.39) and (3.37), respectively, satisfy the decomposition (3.34). Therefore, the proof is complete. ∎

Algorithm 3.1 and Theorem 3.5 directly derive the following result, which reveals the relationship between the weak infinite zero structure and the rank indices.

PROPOSITION 3.6. *Let $q_0=0$ and $q_i$, $i=1,2,\ldots,\alpha$ be determined by Algorithm 3.1. Then system* (1.2) *has the weak infinite zero structure*

$$
\{w_1,w_2,\ldots,w_r\}=\{\underbrace{1,1,\ldots,1}_{q_1-q_0},\underbrace{2,2,\ldots,2}_{q_2-q_1},\ldots,\underbrace{\alpha,\alpha,\ldots,\alpha}_{q_\alpha-q_{\alpha-1}}\}.
\tag{3.40}
$$

We summarize the challenges and contributions in this section. Firstly, the design of the algorithm is constructive. The operations involving delays can only be conducted in the Laplace domain, while the differential transformations need to be done in the time domain. Secondly, the delays lead to more complicated transformation matrices. Compared with the existing results, the main advantages of the results obtained in this manuscript are two-fold. On the one side, Algorithm 3.1 and

Proposition 3.6 reveal the connection between the weak infinite zero structure and the output derivatives. On the other side, unlike Lemma 2.1, Theorem 3.5 provides an explicit construction scheme of weak biproper transformation matrices, which may be useful for other purposes.

Finally, examples are given to illustrate the effectiveness of the previous results. To be specific, we construct the Smith-McMillan form at infinity of the transfer function matrix, based on Theorem 3.5, and compute the weak infinite zero structure in light of Proposition 3.6.

EXAMPLE 3.7. *We continue with Examples 3.1.*

*Recall* $\alpha = 3$ *and* $\bar{D}_\alpha(\mathrm{e}^{-s}) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. *According to Lemma 3.3, we have* $V = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. *It follows from* (3.21) *and* (3.35) *that*

$$G_\alpha(s, \mathrm{e}^s)V = T_{\alpha 1} = \begin{bmatrix} 1 & 0 \\ s^{-1}\mathrm{e}^{-s} & 1 \end{bmatrix},$$

*which, together with* (3.37), *yields*

$$B_2(s, \mathrm{e}^s) = T_{\alpha 1}V^{-1} = \begin{bmatrix} 1 & 0 \\ s^{-1}\mathrm{e}^{-s} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & s^{-1}\mathrm{e}^{-s} \end{bmatrix}.$$

*On the other hand, utilizing* (3.20), (3.38) *and* (3.39), *one obtains*

$$B_1(s, \mathrm{e}^s) = N_\alpha^{-1}(s, \mathrm{e}^s)\Lambda_\alpha^{-1} = \begin{bmatrix} s^{-1} & 0 \\ s^{-2}\mathrm{e}^{-s} & s^{-3} \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s^{-1}\mathrm{e}^{-s} & 1 \end{bmatrix}.$$

*By the decomposition* (3.34), *the Smith-McMillan form at infinity satisfies*

$$G(s, \mathrm{e}^s) = B_1(s, \mathrm{e}^s) \begin{bmatrix} s^{-1} & 0 \\ 0 & s^{-3} \end{bmatrix} B_2(s, \mathrm{e}^s).$$

*Recall the rank indices* $q_1 = q_2 = 1$, $q_3 = 2$. *In light of Proposition 3.6, we derive the weak infinite zero structure* $\{1, 3\}$, *which is consistent with the one in Example 2.3.*

EXAMPLE 3.8. *We carry on with Example 3.2.*

*Recall* $\alpha = 3$ *and* $\bar{D}_\alpha(\mathrm{e}^{-s}) = \left[\begin{smallmatrix} 0 & 1 \\ 2+2\mathrm{e}^{-s} & 0 \end{smallmatrix}\right]$. *Since* $\operatorname{nrank}(G(s, \mathrm{e}^s)) = m$, *we have* $G_\alpha(s, \mathrm{e}^s)V = T_{\alpha 1}$. *By virtue of* (3.29) *and* (3.37), *we obtain*

$$B_2(s, \mathrm{e}^s) = T_{\alpha 1}V^{-1} = G_\alpha(s, \mathrm{e}^s) = \begin{bmatrix} 0 & \frac{s}{s-2} \\ \frac{2s(1+\mathrm{e}^{-s})}{s-2} & \frac{2s\mathrm{e}^{-s}(1+\mathrm{e}^{-s})}{(s-2)^2} \end{bmatrix}.$$

*According to* (3.20), (3.38) *and* (3.39), *we get*

$$B_1(s, \mathrm{e}^s) = N_\alpha^{-1}(s, \mathrm{e}^s)\Lambda_\alpha^{-1} = \begin{bmatrix} s^{-1} & 0 \\ s^{-2}\mathrm{e}^{-s} & s^{-3} \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s^{-1}\mathrm{e}^{-s} & 1 \end{bmatrix}.$$

*It follows from the decomposition* (3.34) *that the Smith-McMillan form at infinity satisfies*

$$G(s, \mathrm{e}^s) = B_1(s, \mathrm{e}^s) \begin{bmatrix} s^{-1} & 0 \\ 0 & s^{-3} \end{bmatrix} B_2(s, \mathrm{e}^s).$$

*Recall the rank indices* $q_1 = q_2 = 1$, $q_3 = 2$. *Then we obtain the weak infinite zero structure* $\{1, 3\}$ *through Proposition 3.6.*

**4. The infinite zero structure and relative degree.** Subsection 4.1 discusses the relationship between the infinite zero structure and relative degree of system (1.1). Subsection 4.2 explores the relationship between the weak infinite zero structure and relative degree of system (1.2). In order to avoid overloading notations, for systems (1.1) and (1.2), we denote the relative degree and the LIRD by the non-descending ordered sets $\{r_1, r_2, \ldots, r_m\}$ and $\{\rho_1, \rho_2, \ldots, \rho_m\}$, respectively.

**4.1. The infinite zero structure and relative degree of system (1.1).** Assume that system (1.1) is invertible, namely, $\operatorname{nrank}(G(s)) = m$. Denote the infinite zero structure of system (1.1) by the non-descending ordered set $\{\nu_1, \nu_2, \ldots, \nu_m\}$. The main results of this subsection are as follows.

PROPOSITION 4.1. *When system* (1.1) *has a regular relative degree, or if there exists an output transformation such that the transformed system has a regular relative degree, then* $r_i = \rho_i = \nu_i$, $i = 1, 2, \ldots, m$.

PROPOSITION 4.2. *When system* (1.1) *has an irregular relative degree, and there is no output transformation such that the transformed system has a regular relative degree, then* $r_i \le \rho_i \le \nu_i$, $i = 1, 2, \ldots, m$.

Firstly, we use the theory of normal forms to prove Proposition 4.1.

*Proof.* Recall that $(\check{T}, \check{T}_y, \check{T}_u) \in (\mathbf{R}_n^{n\times n}, \mathbf{R}_m^{m\times m}, \mathbf{R}_m^{m\times m})$ is the set of algebraic transformation matrices required to transform system (1.1) into the Morse normal form. Let

$$\check{T}x = \begin{bmatrix} \check{\eta} \\ \check{\xi} \end{bmatrix}, \ \check{\eta} \in \mathbf{R}^{\check{n}_0}, \ \check{\xi} = \begin{bmatrix} \check{\xi}_1 \\ \check{\xi}_2 \\ \vdots \\ \check{\xi}_m \end{bmatrix} \in \mathbf{R}^{n-\check{n}_0}, \ \check{\xi}_i = \begin{bmatrix} \check{\xi}_{i1} \\ \check{\xi}_{i2} \\ \vdots \\ \check{\xi}_{i,\nu_i} \end{bmatrix}, \ i = 1, 2, \ldots, m.$$

$$\check{T}_y y = \check{y} \triangleq \begin{bmatrix} \check{y}_1 \\ \check{y}_2 \\ \vdots \\ \check{y}_m \end{bmatrix} \in \mathbf{R}^m, \quad T_u^{-1} u = \check{u} \triangleq \begin{bmatrix} \check{u}_1 \\ \check{u}_2 \\ \vdots \\ \check{u}_m \end{bmatrix} \in \mathbf{R}^m.$$

Then, under the transformation $(\check{T}, \check{T}_y, \check{T}_u)$, system (1.1) is transformed into [36, p.246]

$$\begin{cases} \dot{\check{\eta}} = \check{A}_0 \check{\eta} + \check{L}_0 \check{y}, \\ \dot{\check{\xi}}_{i1} = \check{\xi}_{i2} + \check{L}_{i1} \check{y}, \\ \dot{\check{\xi}}_{i2} = \check{\xi}_{i3} + \check{L}_{i2} \check{y}, \\ \quad \vdots \\ \dot{\check{\xi}}_{i,\nu_i - 1} = \check{\xi}_{i,\nu_i} + \check{L}_{i,\nu_i - 1} \check{y}, \\ \dot{\check{\xi}}_{i,\nu_i} = \check{a}_{i1} \check{\eta} + \check{a}_{i2} \check{\xi} + \check{L}_{i,\nu_i} \check{y} + \check{u}_i, \\ \check{y}_i = \check{\xi}_{i1}, \quad i = 1, 2, \ldots, m, \end{cases} \tag{4.1}$$

where $\check{A}_0, \check{L}_0, \check{a}_{i1}, \check{a}_{i2}, \check{L}_{ij}$ are real matrices of suitable dimensions.

Assume that system (1.1) has a regular relative degree $\{r_1, r_2, \ldots, r_m\}$ (if system (1.1) can be transformed into a system with regular relative degree by output trans-

formations, it can be analyzed in a simialr way). According to [37, Theorem 1], there exists a state transformation, described by

$$\tilde{T}x = \begin{bmatrix} \tilde{\eta} \\ \tilde{\xi} \end{bmatrix},\ \tilde{\eta} \in \mathbf{R}^{\check{n}_0},\ \tilde{\xi} = \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \vdots \\ \tilde{\xi}_m \end{bmatrix} \in \mathbf{R}^{n-\check{n}_0},\ \tilde{\xi}_i = \begin{bmatrix} \tilde{\xi}_{i1} \\ \tilde{\xi}_{i2} \\ \vdots \\ \tilde{\xi}_{i,r_i} \end{bmatrix},\ i = 1, 2, \ldots, m,$$

such that system (1.1) is transformed into the normal form

$$\begin{cases} \dot{\tilde{\eta}} = \tilde{A}_0\tilde{\eta} + \tilde{B}_0 y, \\ \dot{\tilde{\xi}}_{i1} = \tilde{\xi}_{i2}, \\ \dot{\tilde{\xi}}_{i2} = \tilde{\xi}_{i3}, \\ \quad \vdots \\ \dot{\tilde{\xi}}_{i,r_i-1} = \tilde{\xi}_{i,r_i}, \\ \dot{\tilde{\xi}}_{i,r_i} = \tilde{a}_{i1}\tilde{\eta} + \tilde{a}_{i2}\tilde{\xi} + c_i A^{r_i-1} B u, \\ y_i = \tilde{\xi}_{i1}, \quad i = 1, 2, \ldots, m, \end{cases} \tag{4.2}$$

where $\tilde{A}_0, \tilde{B}_0, \tilde{a}_{i1}, \tilde{a}_{i2}$ are real matrices with suitable dimensions. Here, $y_i$ represents the $i$th component of $y$, and $c_i$ denotes the $i$th row of $C$.

It can be observed that (4.1) coincides with (4.2) in form, except for the output injection terms $\check{L}_{ij}$, $j = 1, 2, \ldots, \nu_i$, $i = 1, 2, \ldots, m$. According to [36, Corollary 4.3.2, p.198], the non-singularity of coupling matrix $\Gamma$ implies the invertibility of system (1.1). Thus, when system (1.1) has a regular relative degree, we have $\check{L}_{ij} = 0$. Then (4.1) completely degenerates into (4.2), which implies $r_i = \nu_i$. Besides, it follows from [17, Lemma 2 and Corollary] that $r_i = \rho_i$ when system (1.1) has a regular relative degree. Hence, this proof is finished. □

Next, we give a concise proof of Proposition 4.2 in light of structural algorithms.

*Proof.* According to [17, Lemma 2], $r_i \le \rho_i$ holds when the relative degree $\{r_1, r_2, \ldots, r_m\}$ of system (1.1) is irregular. It remains to prove $\rho_i \le \nu_i$. We use [32, Algorithm] and [24, Algorithm 1] (for the delay-free case) to reveal the infinite zero structure and the LIRD of system (1.1), respectively. Based on differential transformations and nonsingular transformations, these two algorithms both yield a relation between the output and the input, denoted as $y_*(t) = C_* x(t) + D_* u(t)$.

First, we recall [32, Algorithm] in brief. To obtain the infinite zero structure, this algorithm requires the linear independence of all rows in the $D_*$-part. From this, the nonsingular transformation matrix is expressed as $\check{S} = \left[\begin{smallmatrix} I & 0 \\ \star_1 & I \end{smallmatrix}\right]$, where $\star_1$ stands for a nonzero matrix which plays the role of an output injection transformation matrix. Then the transformed output and the infinite zero structure satisfy

$$\begin{aligned} &\check{y}_*(t) = \check{T}_0 y(t) + \check{T}_1 y^{(1)}(t) + \cdots + \check{T}_{\check{\beta}} y^{(\check{\beta})}(t), \\ &\{\nu_1, \nu_2, \ldots, \nu_m\} = \{\underbrace{1, \ldots, 1}_{\check{d}_1}, \underbrace{2, \ldots, 2}_{\check{d}_2}, \ldots, \underbrace{\check{\beta}, \ldots, \check{\beta}}_{\check{d}_{\check{\beta}}}\}, \end{aligned}$$

where $\check{\beta} < n$ represents the number of iterations of this algorithm, $\check{T}_i$ is a real matrix, and $\check{d}_j$ is determined by this algorithm.

Second, we summarize [24, Algorithm 1]. In order to get the LIRD, this algorithm does not require the linear independence of all rows in the $D_*$-part, but requires the linear independence of those rows that correspond to the output components with the same relative degree. For this reason, the nonsingular transformation matrix is represented as $\tilde{S} = \left[\begin{smallmatrix} I & 0 \\ 0 & \star_2 \end{smallmatrix}\right]$, where $\star_2$ stands for an orthogonal matrix. Also, the transformed output and the LIRD take the form

$$\tilde{y}_*(t) = \tilde{T}_0 y(t), \tag{4.3}$$

$$\{\rho_1, \rho_2, \dots, \rho_m\} = \{\underbrace{1, \dots, 1}_{\tilde{d}_1}, \underbrace{2, \dots, 2}_{\tilde{d}_2}, \dots, \underbrace{\tilde{\beta}, \dots, \tilde{\beta}}_{\tilde{d}_{\tilde{\beta}}}\},$$

where $\tilde{\beta} < n$ represents the number of iterations of the algorithm, $\tilde{T}_0$ is a real matrix, and $\tilde{d}_i$ is determined by the algorithm.

It can be observed that the requirements in each iteration in [32, Algorithm] are more stringent, which implies $\check{d}_i \leq \tilde{d}_i$. Also, we note that

$$\check{d}_1 + \check{d}_2 + \cdots + \check{d}_{\check{\beta}} = \tilde{d}_1 + \tilde{d}_2 + \cdots + \tilde{d}_{\tilde{\beta}} = m,$$

which leads to $\tilde{\beta} \leq \check{\beta}$. In words, [32, Algorithm] may require more differentiation operations. Thus $\rho_i \leq \nu_i$ holds true. Therefore, the proof is finished. ∎

**4.2. The weak infinite zero structure and LIRD of system (1.2).** In this subsection, we assume that the system (1.2) is invertible, that is, $\operatorname{nrank}(G(s, \mathrm{e}^s)) = m$. The relationship among the weak infinite zero structure, the LIRD and the relative degree of system (1.2) is described by the following result.

PROPOSITION 4.3. *Let* $\{w_1, w_2, \dots, w_m\}$, $w_1 \leq w_2 \leq \cdots \leq w_m$ *be the weak infinite zero structure of system* (1.2). *Then* $r_i \leq \rho_i \leq w_i$, $i = 1, 2, \dots, m$.

Similarly, we prove this proposition by structural algorithms.

*Proof.* According to [24, Theorem 4.5], we have $r_i \leq \rho_i$. Next we prove $\rho_i \leq w_i$. Assume that the weak infinite zero structure and the LIRD are revealed by Algorithm 3.1 and [24, Algorithm 1] (let the number of delays be equal to one), respectively.

A summary of [24, Algorithm 1] can be found in the proof of Proposition 4.2, with the only difference being that for system (1.2), $\tilde{\beta} < 2n$ holds.

Now we give a quick review of Algorithm 3.1. The block $S_{k3}$ in the nonsingular transformation matrix $S_k$ amounts to an output injection transformation matrix, which renders that the transformed output depends on the output derivatives, up to $\alpha$th order, namely,

$$Y_\alpha(t) = T_0 y(t) + T_1 y^{(1)}(t) + \cdots + T_\alpha y^{(\alpha)}(t),$$

where every $T_i$ is a real matrix. An illustration is provided by (3.30). It follows from (3.11) that the transformed output and the input satisfy, in the Laplace domain,

$$Y_\alpha(s) = \bar{C}_\alpha(\mathrm{e}^{-s}) x(s) + \bar{D}_\alpha(\mathrm{e}^{-s}) u(s), \tag{4.4}$$

where $\bar{D}_\alpha(\mathrm{e}^{-s}) \in \mathbf{R}^{m \times m}[e^{-s}]$ is weak biproper. Since the degree of the matrix polynomial $\bar{D}_\alpha(\mathrm{e}^{-s})$ is not greater than $\alpha - 1$, without loss of generality, we assume $\bar{D}_\alpha(\mathrm{e}^{-s}) = D_0^* + D_1^* \mathrm{e}^{-s} + \cdots + D_{\alpha-1}^* \mathrm{e}^{-(\alpha-1)s}$, where $D_i^* \in \mathbf{R}^{m \times m}$. According to [28], $\bar{D}_\alpha(\mathrm{e}^{-s})$ is weak biproper if and only if $D_0^*$ is invertible. Then, we transform equation

(4.4) into an equivalent form in the time domain:

$$
\begin{aligned}
Y_\alpha(t) &= \bar{\mathcal{C}}^*_\alpha \mathcal{X}_{\alpha+1} + \bar{\mathcal{D}}^*_\alpha \mathcal{V}_\alpha \\
&= \bar{\mathcal{C}}^*_\alpha \begin{bmatrix} x(t) \\ x(t-1) \\ \vdots \\ x(t-\alpha) \end{bmatrix} + \begin{bmatrix} D^*_0 & D^*_1 & \cdots & D^*_{\alpha-1} \end{bmatrix} \begin{bmatrix} u(t) \\ u(t-1) \\ \vdots \\ u(t-\alpha+1) \end{bmatrix},
\end{aligned} \tag{4.5}
$$

where $\bar{\mathcal{C}}^*_\alpha \in \mathbf{R}^{m\times(\alpha+1)n}$ is a real matrix determined by $\bar{C}_\alpha(\mathrm{e}^{-s})$, and $\bar{\mathcal{D}}^*_\alpha \in \mathbf{R}^{m\times\alpha m}$ is of full row rank.

Compared with [24, Algorithm 1], the requirement that $\bar{\mathcal{D}}^*_\alpha$ is of full row rank in Algorithm 3.1 is obviously more stringent. Let $d_i = q_i - q_{i-1}$ in (3.40), then $d_i \le \tilde{d}_i$ holds. Since

$$
d_1 + d_2 + \cdots + d_\alpha = \tilde{d}_1 + \tilde{d}_2 + \cdots + \tilde{d}_{\tilde{\beta}} = m,
$$

we obtain $\tilde{\beta} \le \alpha$, namely, Algorithm 3.1 may require more differentiation operations, which implies that $\rho_i \le w_i$. Thus this proof is finished. ∎

**5. Extended PD Control.** This section investigates the application of the weak infinite zero structure in the design of PD output feedback controllers.

**5.1. Overview of the extended PD controllers.** This subsection provides the design approaches of some extended PD controllers for the invertible system (1.2).

When the relative degree is larger than 1, an extended PD controller is obtained by employing higher-order output derivatives for feedback. For example, assume that the output $\{y_1, y_2, y_3\}$ of system (1.2) has the relative degree $\{1, 2, 3\}$, and consider the extended PD controller

$$
u(t) = \bar{K}_1 \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + \bar{K}_2 \begin{bmatrix} y_2^{(1)}(t) \\ y_3^{(1)}(t) \end{bmatrix} + \bar{K}_3 y_3^{(2)}(t), \tag{5.1}
$$

where $\bar{K}_k$, $k = 1, 2, 3$ are gain matrices, and the output derivatives explicitly depend on states only.

Next, we introduce further extensions of the PD output feedback controllers.

(1) Let the highest-order output derivatives, used for feedback, rely on the relative degree $\{r_1, r_2, \ldots, r_m\}$. Then we can construct an extended PD controller of the form

$$
u(t) = \sum_{k=1}^{m} \sum_{l=0}^{r_k} \hat{K}_{kl} y_k^{(l)}(t - \tau_f), \tag{5.2}
$$

where $\hat{K}_{kl}$, $l = 0, 1, \ldots, r_k$, $k = 1, 2, \ldots, m$ are gain matrices, $y_k$ represents the $k$th component of $y$, and $\tau_f \ge 0$ denotes a delay in the feedback control loop. In addition to states, the control law (5.2) may also implicitly depend on the input. Thus the closed-loop system is of retarded type if $\hat{K}_{k,r_k} = 0$, $k = 1, 2, \ldots, m$, otherwise, it is of neutral type.

(2) Let the highest-order output derivatives, used for feedback, depend on the LIRD $\{\rho_1, \rho_2, \ldots, \rho_m\}$. Note that, to reveal the LIRD, a preliminary output transformation might be necessary [24]. Then we design an extended PD controller

$$
u(t) = \sum_{k=1}^{m} \sum_{l=0}^{\rho_k} \tilde{K}_{kl} y_k^{(l)}(t - \tau_f), \tag{5.3}
$$

where $\tilde{K}_{kl}$, $l = 0, 1, \ldots, \rho_k$, $k = 1, 2, \ldots, m$ are gain matrices. If $\tilde{K}_{k,\rho_k} = 0$, $k = 1, 2, \ldots, m$, the closed-loop system is of retarded type. Otherwise, the controller (5.3) can be expressed in terms of the states and the inputs, and the closed-loop system corresponds to a delay differential algebraic equation of neutral type [23] with associated delay difference equation

$$u(t) = \sum_{k=1}^{m} \tilde{K}_{k,\rho_k} \Big\{ \sum_{i=\rho_k-1,\, 0\le j\le i} c_k Q(i,j) B u(t-j-\tau_f) \Big\}, \tag{5.4}$$

where $c_k$ represents the $k$th row of $C$.

(3) Let the highest-order output derivatives, used for feedback, be determined by the weak infinite zero structure $\{w_1, w_2, \ldots, w_m\}$. Assume that the weak infinite zero structure of system (1.2) is revealed by Algorithm 3.1. By (3.11) and (3.40), we partition the system output $Y_k(t)$ at step $k$ according to the weak infinite zero structure, and represent it as

$$Y_k(t) = \begin{bmatrix} Y_{k1}(t) \\ Y_{k2}(t) \\ \vdots \\ Y_{k\alpha}(t) \end{bmatrix} \in \mathbf{R}^m, \ k = 0, 1, \ldots, \alpha, \quad Y_{kl}(t) \in \mathbf{R}^{q_l - q_{l-1}}.$$

Then we can construct an extended PD controller of the form

$$u(t) = \sum_{k=1}^{\alpha-1} \sum_{l=k+1}^{\alpha} \check{K}_{kl} Y_{kl}(t-\tau_f) + \sum_{i=0,\alpha} \sum_{j=1}^{\alpha} \check{K}_{ij} Y_{ij}(t-\tau_f), \tag{5.5}$$

where $\check{K}_{kl}$, $l = k+1, k+2, \ldots, \alpha$, $k = 1, 2, \ldots, \alpha-1$ and $\check{K}_{ij}$, $j = 1, 2, \ldots, \alpha$, $i = 0, \alpha$ are gain matrices. We note that the controller (5.5) can also be expressed in terms of the states and the inputs, which implies that the closed-loop system is of neutral type [23]. According to (4.5), its associated delay difference equation has the form

$$u(t) = \sum_{j=1}^{\alpha} \sum_{h=0}^{\alpha-1} \check{K}_{\alpha j} D^*_{jh} u(t-h-\tau_f), \tag{5.6}$$

where $D^*_{jh} \in \mathbf{R}^{(q_j - q_{j-1}) \times m}$ is a block in matrix $D^*_h \in \mathbf{R}^{m \times m}$, which satisfies

$$D^*_h = \begin{bmatrix} D^*_{1h} \\ D^*_{2h} \\ \vdots \\ D^*_{\alpha h} \end{bmatrix}, \quad h = 0, 1, \ldots, \alpha-1.$$

Since the closed-loop system formed by the feedback interconnection of (1.2) with either control law (5.3) or (5.5) is of neutral type, its stability may be sensitive to small delay perturbations. Therefore we adopt the approach of [23] for synthesizing the controller gains, which is based on minimizing the so-called robust spectral abscissa, a stability measure that takes infinitesimal delay perturbations explicitly into account. Note that a strictly negative robust spectral abscissa implies that the closed-loop system is strongly exponentially stable (in the sense of [12]), which implies on its turn that the difference equation (5.4) or (5.6) is strongly exponentially stable.

*Remark* 5.1. If a new output is considered as a linear combination of the original outputs, then its relative degree may be sensitive to infinitesimal perturbation to the corresponding weights (causing the relative degree to become equal to the minimum of the original relative degrees). Likewise, small perturbations in the transformation of the outputs to (4.3) and (4.4) may induce the appearance of terms with small coefficient in the right-hand side of (5.4) and (5.6) that depend on (delayed) derivatives of the input, which render the closed-loop system ill-posed. However, this situation can be avoided by including a low-pass filter in the control law, even though care is needed in its design [1].

**5.2. Numerical example.** In this subsection, an example is given to show the design procedure of the extended PD controllers proposed in Subsection 5.1.

EXAMPLE 5.2. *We revisit once more the system in Examples 3.2 and 3.8. By simple computations, we derive that the LIRD exists and coincides with the relative degree, namely,* $\{\rho_1, \rho_2\} = \{r_1, r_2\} = \{1, 2\}$. *Besides, we have obtained the weak infinite zero structure* $\{w_1, w_2\} = \{1, 3\}$. *The transformed outputs in all steps are given by* (3.30). *Note that since* $q_1 = q_2$, *their partitioning is of the form* $Y_0 = \left[\begin{array}{c} Y_{0,1} \\ \hline Y_{0,2} \\ \hline Y_{0,3} \end{array}\right] = \left[\begin{array}{c} y_1 \\ \hline\hline y_2 \end{array}\right]$ *etc.*

*Assume* $\tau_f = 0$. *We consider the gain matrices as free parameters, and design the following output feedback controllers:*

$$u(t) = K_{11}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = K_{11}\begin{bmatrix} x_4(t) \\ x_1(t) \end{bmatrix}, \tag{5.7}$$

$$\begin{aligned} u(t) =& K_{21}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + K_{22}y_2^{(1)}(t) \\ =& K_{21}\begin{bmatrix} x_4(t) \\ x_1(t) \end{bmatrix} + K_{22}\big(x_2(t) + x_2(t-1) + x_4(t-1)\big), \end{aligned} \tag{5.8}$$

$$\begin{aligned} u(t) =& K_{31}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + K_{32}\begin{bmatrix} y_1^{(1)}(t) \\ y_2^{(1)}(t) \end{bmatrix} + K_{33}y_2^{(2)}(t) \\ =& K_{31}\begin{bmatrix} x_4(t) \\ x_1(t) \end{bmatrix} + K_{32}\begin{bmatrix} 2x_4(t) + u_2(t) \\ x_2(t) + x_2(t-1) + x_4(t-1) \end{bmatrix} \\ &+ K_{33}\big(2x_3(t) + 2x_3(t-1) + 2x_4(t-1) + u_2(t-1)\big), \end{aligned} \tag{5.9}$$

$$\begin{aligned} u(t) =& K_{41}\begin{bmatrix} Y_{01}(t) \\ Y_{03}(t) \end{bmatrix} + K_{42}\begin{bmatrix} Y_{11}(t) \\ Y_{13}(t) \end{bmatrix} + K_{43}Y_{23}(t) + K_{44}Y_{33}(t) \\ =& K_{41}\begin{bmatrix} x_4(t) \\ x_1(t) \end{bmatrix} + K_{42}\begin{bmatrix} 2x_4(t) + u_2(t) \\ x_2(t) + x_2(t-1) + x_4(t-1) \end{bmatrix} \\ &+ K_{43}\big(2x_3(t) + 2x_3(t-1)\big) + K_{44}\big\{4x_3(t) + 4x_3(t-1) \\ &+ 2x_4(t-1) + 2x_4(t-2) + 2u_1(t) + 2u_1(t-1)\big\}, \end{aligned} \tag{5.10}$$

*where* $K_{ij}, j = 1, 2, \ldots, i,\ i = 1, 2, 3, 4$ *are gain matrices. It should be noted that the controllers* (5.9) *and* (5.10) *are designed according to* (5.3) *and* (5.5), *respectively. Their associated delay difference equations*

$$u(t) = K_{32}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t) + K_{33}\begin{bmatrix} 0 & 1 \end{bmatrix} u(t-1),$$

TABLE 1
*The spectral abscissa of the open-loop system and the results of minimizing the (robust) spectral abscissa by controllers* (5.7)-(5.10)

| Controller | Gain matrices | (Robust) spectral abscissa |
|---|---|---|
| no | — | 2 |
| (5.7) | $K_{11} = \begin{bmatrix} -1.9657 & -2.0753 \\ -3.0058 & -1.9503 \end{bmatrix}$ | 0.6626 |
| (5.8) | $K_{21} = \begin{bmatrix} -0.6877 & -0.7793 \\ -1.7861 & -0.5741 \end{bmatrix}$<br>$K_{22} = \begin{bmatrix} -0.6503 & -1.0562 \end{bmatrix}^{\mathrm{T}}$ | 0.3993 |
| (5.9) | $K_{31} = \begin{bmatrix} 861.3112 & -693.9282 \\ -1175.8826 & -200.1317 \end{bmatrix}$<br>$K_{32} = \begin{bmatrix} -18.4544 & -15.6575 \\ 0.9699 & 1.7602 \end{bmatrix}$<br>$K_{33} = \begin{bmatrix} -63.7489 & -0.0304 \end{bmatrix}^{\mathrm{T}}$ | 0.0182 |
| (5.10) | $K_{41} = \begin{bmatrix} -3.5983 & -0.7437 \\ -5.4461 & -0.9451 \end{bmatrix}$<br>$K_{42} = \begin{bmatrix} 0.2566 & -0.3891 \\ 0.0829 & -0.3261 \end{bmatrix}$<br>$K_{43} = \begin{bmatrix} -0.4688 & 0.1011 \end{bmatrix}^{\mathrm{T}}$<br>$K_{44} = \begin{bmatrix} -0.3048 & 0.0471 \end{bmatrix}^{\mathrm{T}}$ | −0.9956 |

$$u(t) = K_{42}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t) + K_{44}\begin{bmatrix} 2 & 0 \end{bmatrix} u(t) + K_{44}\begin{bmatrix} 2 & 0 \end{bmatrix} u(t-1),$$

*are necessarily required to be strongly exponentially stable, which is an essential constraint to minimize the robust spectral abscissa. Table 1 displays the results of minimizing the (robust) spectral abscissa. Note that in the process of minimizing this function, the gain parameters* $K_{ij}, j = 2, \ldots, i,\ i = 2, 3, 4$ *were rescaled by a factor of 100 to counter large differences in sensitivity. In the case of zero input, the open-loop system is unstable due to its nonnegative spectral abscissa, see Figure 1. The spectra of the open-loop system* (1.2) *and the closed-loop systems with controllers* (5.7)-(5.10) *are shown in Figures 2 and 3 respectively.*

*It is observed that only when using the controller* (5.10)*, the closed-loop system realizes strong exponential stability. It is shown that a system that cannot be stabilized by the controller* (5.3)*, designed based on the LIRD, may still be stabilized by the controller* (5.5)*, designed based on the weak infinite zero structure.*

**6. Conclusions.** This paper is concerned with the weak infinite zero structures of linear time-delay systems. First of all, a structural algorithm switching between Laplace domain and time domain is presented, grounded in Silverman's algorithm. With the assistance of this algorithm, a novel construction of the weak biproper transformation matrices that bring the transfer function matrix to the Smith-McMillan canonical form at infinity is given, and the weak infinite zero structure is thereby obtained. Then, the relationship between the infinite zero structure and the relative degree for delay-free systems is discussed, and this discussion is generalized to linear time-delay systems. At last, the transformed outputs and the weak infinite zero structures are utilized to design an extended PD output feedback controller. Note that the computation of characteristic roots and the optimization of the spectral abscissa involve solving a nonlinear eigenvalue problem [16]. A numerical example shows the

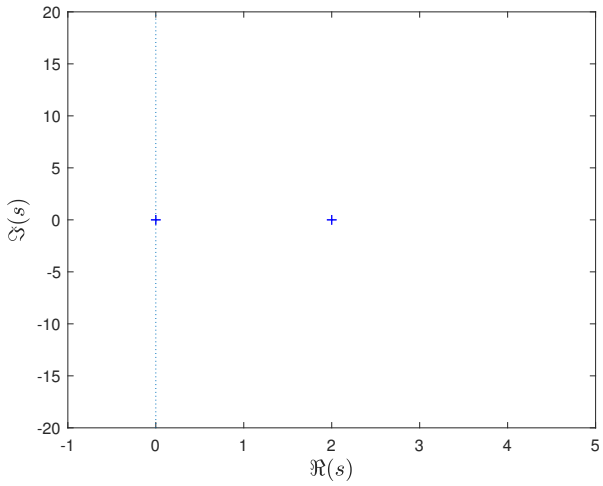



FIG. 1. *Rightmost characteristic roots of the open-loop system* (1.2).

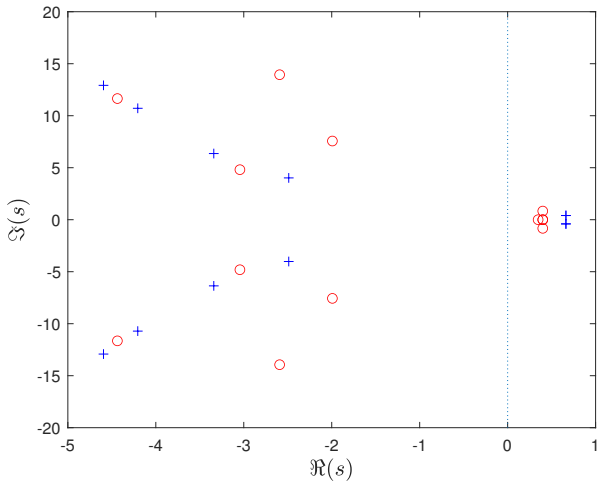



FIG. 2. *Rightmost characteristic roots of the closed-loop systems with controllers* (5.7) *('+') and* (5.8) *('o') after minimizing the spectral abscissa.*

887 effectiveness of such a controller.

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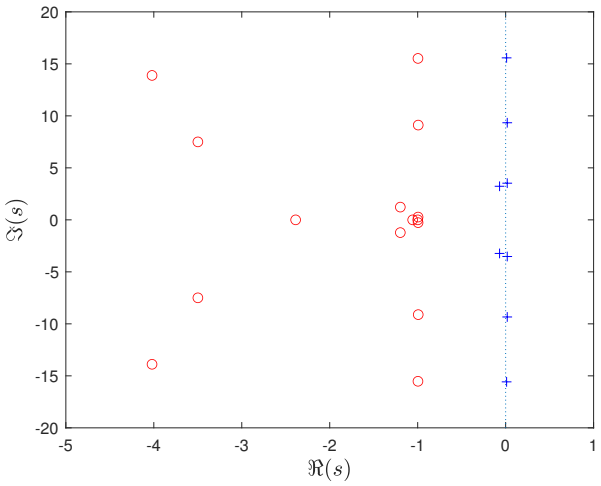



FIG. 3. *Rightmost characteristic roots of the closed-loop systems with controllers* (5.9) *('+') and* (5.10) *('o') after minimizing the robust spectral abscissa.*

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