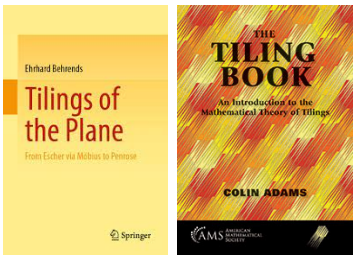
Tilings of the plane. From Escher via Möbius to Penrose Ehrhard Berendt. Springer Nature, 2022 (294 p.), isbn: 978-3-030-42652-1.
The tiling book Colin Adams, AMS/MAA, 2022 (298 p.), isbn 978-1-4704-6897-2

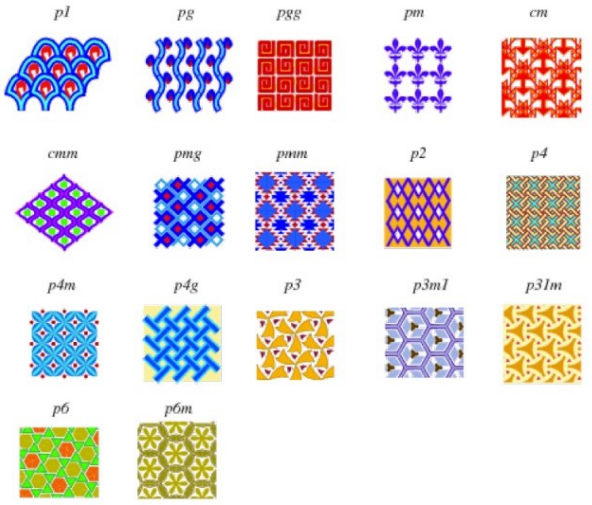



Ehrhard Behrends

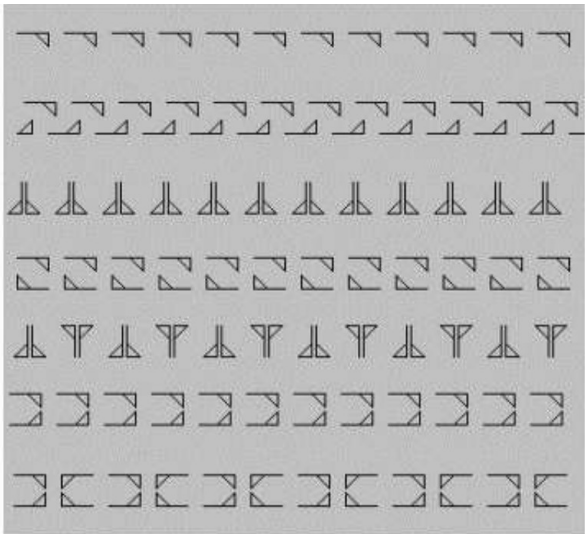

Colin Adams

These are two marvelous books on tilings. The subject is very visual so that you see in the layout of the tessellations what you have to prove, and conversely when you have some symmetry rules for the tiles and or for the whole tiling, then trying to satisfy these rules leads to a number of different solutions that may not have been obvious when you formulated the rules. A genuine mathematician will then try to list all the possible tilings that satisfy these conditions. Of course the random tilings are not very interesting, it are imposed symmetries that give the nice results and that are naturally conceived as pleasing to look at. Symmetries of course means mathematically that there are groups involved. But in general the mathematics are rather simple. However, there are still some open problems that are easy to formulate, yet still unsolved for the moment.

The authors of these books are taking their own approach to the subject. Behrends starts with a general setting where each point in $\mathbb{R}^2$ is associated with a colour. All kind of transformations can be applied which can be settled down to a group of symmetries (translations, rotations, reflections, glide reflections) such that the colored plane before and after the transformation looks the same. The groups need to be discrete and must have a fundamental domain, i.e., a *single tile* which, by applying all the transformations, will tile the whole plane. There are some natural restrictions to make this happen and by investigating all the possibilities it leads to the 7 frieze groups (only 1 direction of translation) and the 17 wallpaper or crystallographic groups (2 independent translations).



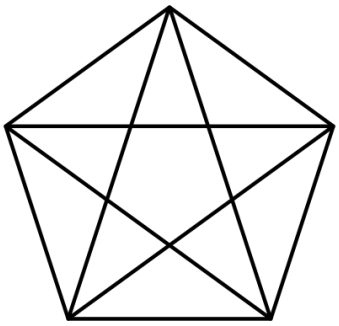
Wallpaper groups ©wolfram mathworld

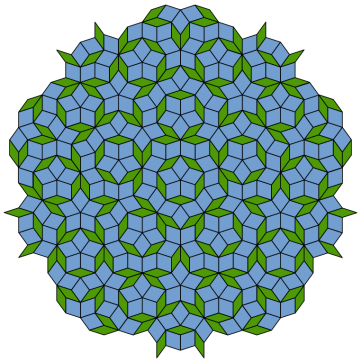

Frieze groups

Each of them is investigated in detail. The translations define the lattice where the tiles have to be placed after the transformation. However to make sure that they really cover the plane without overlap, the tiles have to be designed in an appropriate way. How to do that is described by Heinrich Heersch (1906-1995). At every vertex of the grid, the angles of the tiles meeting there should add up to 2π. That implies that it is impossible to obtain a tiling with only n-gons if

$n \geq 7$. Depending on the number and type of n-gons that share the different vertices of a tile, the tile can be designed. The straight edges of the n-gon tile can be replaced by curves that are such that they can match with edges of itself or of the other tiles. Each of the 28 Heersch constructions is discussed in detail.

Then there is a slight shift of topic. The second part of the book is devoted to Möbius transforms $z \to \frac{az+b}{cz+d}$, $ac - bd \neq 0$, with normalization $ad - bc = 1$. These are holomorphic 1-1 mappings of the Riemann sphere $\hat{\mathbb{C}}$ leaving circles invariant (straight lines are degenerate circles with center at infinity). These transforms also form a group. Aside from the identity, a Möbius transform can be parabolic (a translation), hyperbolic (a dilation), elliptic (a rotation), or loxodromic (a scaled rotation). A repeated application of these give transforms that can be considered as a dynamical system for which the orbits can accumulate in fixed points, that are attractors or repellors. Of course the discrete subgroups (Kleinian groups) are the ones that are interesting for tiling. Also here one has to define a fundamental domain (a basic tile). The most classical Möbius transforms are the ones that leave either the upper half plane or the unit circle invariant. Here the modular group $SL(2, \mathbb{Z})$, generated by $z \mapsto -1/z$ and $z \mapsto z + 1$ results in a tiling of the upper half plane, and using a Cayley transform thus also of the unit circle. Both are typical representations of hyperbolic geometry and it is only in this geometry that the transformed tiles keep shape and size. But other groups like the Schottky group generated by a transforms that are defined by how they map the outside of a given circle onto the inside of another one. Investigating the limit circles of the dynamical system (circles are mapped into smaller circles inside previous circles) giving eventually fractal curves of fixed points (like Julia sets).





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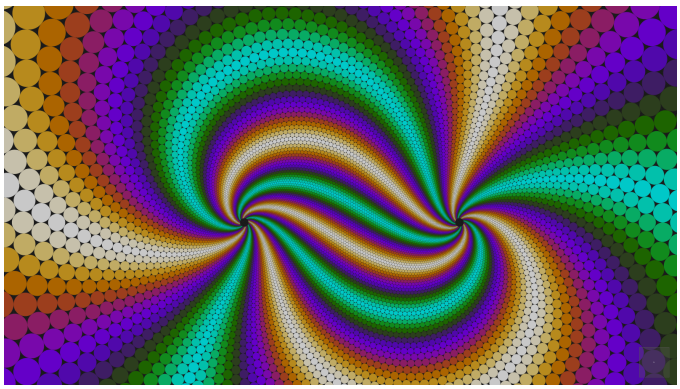
R. Penrose in Texas A&M University

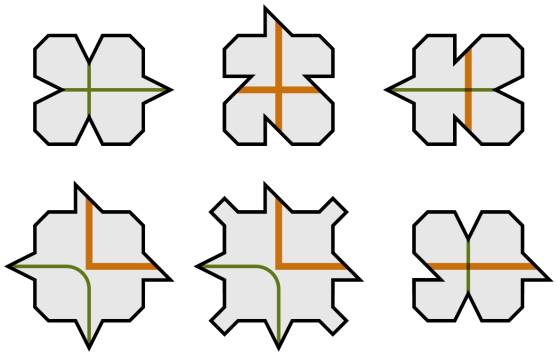
There is a third part that is devoted to nonperiodic (Penrose) tiling. That is when there is no translation that can map the tiling onto itself. The simplest example has 2 basic tiles which are the 2 types of isosceles triangles between the outer and the inner pentagon when a pentagram is drawn in a regular pentagon. On condition that these are arranged following some rules, they will form a Penrose tiling of the plane. An example is shown in the figure where 2 triangles with an acute top (green) and 2 triangles with obtuse top (blue) are glued at their base and the plane is tiled with these lozenges. The Robinson tiles are singles tiles that allow aperiodic tiling. An *einstein* (2023) is a single prototile that only allows aperiodic tilings.

Colin Adams takes a slightly different approach. He starts form the protoset, which is the set of basic tiles that are then transformed (translate, rotate, reflect, glide reflect) to tile the plane. Tilings are investigated by classifying the transitivity classes of tiles (the orbits of the tiles in the protoset). Unlike in Behrends' book the tiles can be much 'wilder', they are not simple modifications of the edges of a (generally convex) polygon. Tiles can be convex or not, tilings can be edge-to-edge or not, regular or semiregular, uniform, periodic, etc. All this results in

different classifications. There are for exampe 11 Laves, 7 Frieze, and 17 wallpaper tilings. A tile can be surrounded by transforms of itself to form patches, which van be used in turn to fill up the plane. Theorems can be proved about all these settings and for example find formulas for an average number of edges or vertices per tile. Of course there are the aperiodic tilings too. Here Adams introduces different methods to generate the aperiodic results. There are for example reptiles, which is recursively filling up a tile with scaled replicas of itself or with scaled tiles from a protoset. Blowing up everything means that these can fill the whole plane. Other examples are the single Robinson tiles, protosets obtained via projection from a higher dimensional lattice, and the Taylor-Socolar hexagonal tiles, which all result in aperiodic tilings.



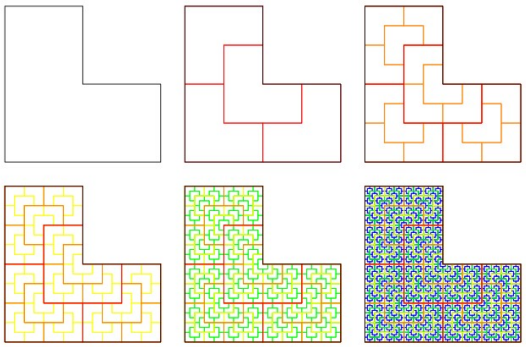
Loxodromic spiral ©wikimedia commons



Robinson tiles ©futility closet

The last chapter deals with tilings in other geometries and other dimensions. Cyclic tilings can best be represented on a cylinder or a torus, but here it concerns tilings of the sphere, the hyperbolic plane (here represented in the disk), or the 3D Euclidean space, and putting in some topology, there can be knotted tilings, jumping to a more advanced level.

Both books contain theorems and proofs and are amply and beautifully illustrated. Definitions, propositions and theorems are highlighted. The Behrends book has a standard format, while the Adams book has larger format and might suggest a (definitely wrong) impression of a picture book. The approach of Behrends is one of a (student) mathematician who wants to learn about the subject, and the part on Möbius transforms is perhaps more addressing a mathematically minded community. The Adams book on the other hand seems to be more addressing the hobbyist who is intrigued by pictures and wants to learn the mathematics behind them, which turns out to be in some aspects more advanced than Behrends. Adams adds a lot of exercises at the end of sections. No solutions are provided. To some extent, his approach is more heuristic. For example the protoset is the starting point, while Behrends finds a single fundamental set only after the required symmetry properties of the tilings are explained. Adams also includes many 'projects' which require more exploration than just solving an exercise. There are some 'open problems', questions that are still open and that need further research. I would have hoped for more discussion about available software to let the reader experiment with. It exists but is not discussed in these books.



L-shaped rep-tile ©Wolfram.com

There are some typos in Behrends's book: The tiling of the Heersch construction 12 seems to be wrong (p.124), circles with centers m_1 and m_2 and radii r_1, r_2 are disjunct if $|m_1 - m_2| > r_1 + r_2$, not $|z_1 - z_2| > r_1 + r_2$ (p. 206), and 'canstant' should be 'constant' (p.177). But let that not hold you back from reading (and studying) these marvelous books.

Adhemar Bultheel