A NOTE ON THE RELATION BETWEEN TWO CONVERGENCE ACCELERATION METHODS FOR ORDINARY CONTINUED FRACTIONS

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Abstract

In this note we relate two methods of convergence acceleration for ordinary continued fractions, the first one is due to Lorentzen and Waadeland [1, 2], the second one to Waadeland [7, 8, 9, 10, 11].

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1 Introduction and notations

In this note we discuss the relationship between the method of Lorentzen and Waadeland [1, 2] and the method of Waadeland [7, 8, 9, 10, 11] for accelerating the convergence of ordinary continued fractions. We consider the continued fraction

$$K\left(\frac{a_n}{b_n}\right) = \frac{a_1}{\left| b_1 \right.} + \frac{a_2}{\left| b_2 \right.} + \ldots + \frac{a_k}{\left| b_k \right.} + \ldots \qquad a_k,\ b_k \in \mathbb{C},\ a_k \neq 0 \tag{1.1}$$

and we assume that this continued fraction converges in $\mathbb{C}$.
If we denote by S_n the linear fractional transformation

$$S_n(w) = \frac{a_1}{\left| b_1 \right.} + \frac{a_2}{\left| b_2 \right.} + \ldots + \frac{a_n}{\left| b_n + w \right.}$$

then it is easy to prove that

$$S_n(w) = \frac{A_n + wA_{n-1}}{B_n + wB_{n-1}}$$

where A_n and B_n are defined by

$$\begin{aligned} A_{-1} &= 1,\ A_0 = 0, & B_{-1} &= 0,\ B_0 = 1, \\ A_n &= b_nA_{n-1} + a_nA_{n-2}\ , & B_n &= b_nB_{n-1} + a_nB_{n-2},\quad n \geq 1 \end{aligned}$$

A_n and B_n are called the n-th numerator and n-th denominator of (1.1), respectively.
The n-th approximant f_n (in the classical sense) of the continued fraction (1.1) is defined by

$f_n = S_n(0) = A_n/B_n$.
The n-th tail $f^{(n)}$ is given by

$$f^{(n)} = \frac{a_{n+1}}{\left| b_{n+1}\right.} + \frac{a_{n+2}}{\left| b_{n+2}\right.} + \dots$$

The value $f = f^{(0)}$ of the continued fraction (1.1) is defined by

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} S_n(0) \tag{1.2}$$

The classical way to evaluate the continued fraction (1.1) is by calculating $S_n(0)$ for a sufficiently large n. Over a number of years various methods were developed to accelerate the convergence of the limit-process (1.2). In most of these methods a new sequence f_n^* is produced that converges faster to f than the sequence of classical approximants $S_n(0)$:

$$\lim_{n\to\infty} \frac{f - f_n^*}{f - f_n} = 0 \tag{1.3}$$

In most cases $S_n(0)$ is replaced by $S_n(w^{(n)})$ where the sequence $\{w^{(n)}\}$ is suitably chosen. The sequence $S_n(w^{(n)})$ is called a modification of the continued fraction (1.1). In view of the relationship

$$f = S_n(f^{(n)})$$

the most obvious choice for $w^{(n)}$ is

$$w^{(n)} = \lim_{k\to\infty} f^{(k)} = w$$

if this limit exists (see [5],[6]). $S_n(w)$ will converge faster than $S_n(0)$. Other choices lead to even faster convergence. Assume

$$f^{(n)} = w + \epsilon_n$$

then it is a matter of estimating ϵ_n by some approximation ϵ_n' and under certain conditions $S_n(w + \epsilon_n')$ will converge faster than $S_n(w)$. This is the approach taken in [1, 2]. We have discussed this method and shown equivalence with a method by Thron and Waadeland in [4].

In this note we consider two other methods which are based on the following idea.

Let us assume that we know another continued fraction $K(\tilde{a}_n, \tilde{b}_n)$ with $\tilde{a}_n \sim a_n$, $\tilde{b}_n \sim b_n$ and $\tilde{f}^{(n)} \sim f^{(n)}$ for $n \to \infty$. ($p_n \sim q_n$ for $n \to \infty$ if $p_n/q_n \to 1$ as $n \to \infty$). Then we can write

$$f^{(n)} = \tilde{f}^{(n)} + \epsilon_n \ .$$

So the problem is to find estimates of ϵ_n.

Lorentzen and Waadeland [1, 2] show that for limit-periodic continued fractions ϵ_n is the minimal solution of a nonlinear recurrence relation. We show that with a linearization of the recurrence relation, we find an approximant $\tilde{\epsilon}_n$ for which $S_n(\tilde{f}^{(n)} + \tilde{\epsilon}_n)$ converges faster than $S_n(\tilde{f}^{(n)})$.

Another method by Waadeland [7, 8, 9, 10, 11] uses the following approach. The continued fraction (1.1) is interpreted as a function of the variables $a_1, b_1, a_2, b_2, \dots$. This function is then expanded into a Taylor series and the partial sums of the series are used as approximations to $f^{(n)}$. If we only use the linear terms the resulting modification is called the Taylor-modification of order 1.

We show that $\tilde{\epsilon}_n$ corresponds to this Taylor-modification of order 1. Furthermore it is shown that a similar linearization can also lead to Taylor-modifications of order 2.

This is discussed in the next paragraph. In the third paragraph we look at an application of this result.

2 Comparison of the two methods

2.1 The method of Lorentzen and Waadeland [1, 2]

First of all we look in some more detail at the method developed by Lorentzen and Waadeland [1, 2].

We want to compute the value of the convergent continued fraction $K(a_n/b_n)$, with tail values $f^{(n)}$. Let us assume that we know a continued fraction $K(\tilde{a}_n/\tilde{b}_n)$ and all its tail values $\tilde{f}^{(n)}$, with

$$a_n \sim \tilde{a}_n \ , \quad b_n \sim \tilde{b}_n \quad \text{as} \quad n \to \infty$$

Let us assume that this implies: $f^{(n)} \sim \tilde{f}^{(n)}$ as $n \to \infty$. Then we can write:

$$f^{(n)} = \tilde{f}^{(n)} + \epsilon_n \quad \text{with} \quad \lim_{n\to\infty} \frac{\epsilon_n}{\tilde{f}^{(n)}} = 0$$

Using this together with the recurrence relation satisfied by the tails:

$$f^{(n-1)} = \frac{a_n}{b_n + f^{(n)}}$$

we deduce that $y_n = \epsilon_n$ is a solution of the following non-linear recurrence relation:

$$\tilde{f}^{(n-1)} y_n + (b_n + \tilde{f}^{(n)} + y_n) y_{n-1} = \delta_n \tag{2.1}$$

with $\delta_n = a_n - \tilde{f}^{(n-1)}(b_n + \tilde{f}^{(n)})$.

In what follows we assume that the continued fraction we start with is limit-periodic:

$$\lim_{n\to\infty} a_n = a \in \mathbb{C}_0 \ , \quad \lim_{n\to\infty} b_n = b \in \mathbb{C}$$

and that the roots w_1 and w_2 of the auxiliary equation $x^2 + bx - a = 0$ satisfy $|w_1| < |w_2|$. In this case the continued fraction converges, $\lim_{n\to\infty} \epsilon_n = 0$, and if we choose

$$\tilde{f}^{(n)} = w_1$$

then it is possible to prove that (1.3) holds with $f_n^* = S_n(\tilde{f}^{(n)})$.
We shall continue however to write things down in the more general setting.

Let us assume that

$$\lim_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = t \in \mathbb{C}$$

(with $|t| \leq 1$). Then we also have: ([1, 2])

$$\lim_{n\to\infty} \frac{\epsilon_n}{\epsilon_{n-1}} = t$$

and, using this in (2.1):

$$\frac{\delta_n}{\epsilon_{n-1}} \sim \tilde{f}^{(n-1)} t + b + \tilde{f}^{(n)} \quad \text{as} \quad n \to \infty \ .$$

We can rewrite this in the following way:

$$\epsilon_n \sim \frac{\delta_{n+1}}{b + \tilde{f}^{(n+1)} + t\tilde{f}^{(n)}} \quad \text{as} \quad n \to \infty \ .$$

Hence if we define

$$\hat{\epsilon}_n = \frac{\delta_{n+1}}{b + \tilde{f}^{(n+1)} + t\tilde{f}^{(n)}}$$

then $\hat{\epsilon}_n$ is an approximation to ϵ_n, and in [1, 2] it is shown that

$$\lim_{n\to\infty} \frac{f - S_n(\tilde{f}^{(n)} + \hat{\epsilon}_n)}{f - S_n(\tilde{f}^{(n)})} \ = \ 0 \ .$$

This is the method of Lorentzen and Waadeland discussed in [1, 2]. Note that in [3] we use another choice for $w^{(n)}$ based on:

$$\epsilon_n \sim \check{\epsilon}_n = \frac{\delta_{n+1}}{b_{n+1} + \tilde{f}^{(n+1)} + t\tilde{f}^{(n)}} \quad \text{as} \quad n \to \infty$$

Let us look at a variation of this method. Instead of using the recurrence relation (2.1) we replace it by a linear one:

$$\tilde{f}^{(n-1)} y_n + (b_n + \tilde{f}^{(n)}) y_{n-1} = \delta_n \tag{2.2}$$

leaving out the non-linear part $y_n y_{n-1}$ which will be small compared to the other terms. It is easy to prove that this new recurrence relation has a minimal solution (see [3]) $y_n = \tilde{\epsilon}_n$:

$$\lim_{n\to\infty} \frac{\tilde{\epsilon}_n}{z_n} = 0 \tag{2.3}$$

for every solution z_n of the homogeneous equation associated with (2.2). Furthermore it also satisfies:

$$\lim_{n\to\infty} \frac{\tilde{\epsilon} n}{\tilde{\epsilon}_{n-1}} = t \ .$$

(To prove this we can use the same method as in [3].)
From (2.1) we get

$$\lim_{n\to\infty} \frac{\delta_n/\tilde{\epsilon}_{n-1}}{\delta_n/\epsilon_{n-1}} = \lim_{n\to\infty} \frac{\tilde{f}^{(n-1)}\tilde{\epsilon}_n/\tilde{\epsilon}_{n-1} + (b_n + \tilde{f}^{(n)})}{\tilde{f}^{(n-1)}\epsilon_n/\epsilon_{n-1} + (b_n + \tilde{f}^{(n)} + \epsilon_n)} = 1$$

and hence:

$$\lim_{n\to\infty} \frac{\epsilon_n}{\tilde{\epsilon}_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\hat{\epsilon}_n}{\tilde{\epsilon}_n} = 1 \ .$$

Note that we could equally well have used

$$\tilde{f}^{(n-1)} y_n + (b + \tilde{f}^{(n)}) y_{n-1} = \delta_n \tag{2.4}$$

instead of equation (2.2).
Using the methods from [1, 2], we can now prove the following result:

Theorem *Under the assumptions given above, we have:*

$$\lim_{n\to\infty} \frac{f - S_n(\tilde{f}^{(n)} + \tilde{\epsilon}_n)}{f - S_n(\tilde{f}^{(n)})} \ = \ 0 \ .$$

2.2 The method of Waadeland [7, 8, 9, 10, 11]

Since $\tilde{\epsilon}_n$ is the minimal solution of the recurrence relation (2.2), it is possible to compute it using backward recursion: if we denote by $\tilde{\epsilon}_n^{(N)}$ the solution of (2.2) with

$$\tilde{\epsilon}_{N+1}^{(N)} = 0 \ ,$$

then we have:

$$\lim_{N\to\infty} \tilde{\epsilon}_n^{(N)} = \tilde{\epsilon}_n$$

(This follows from

$$\tilde{\epsilon}_n^{(N)} = \tilde{\epsilon}_n - \frac{\tilde{\epsilon}_{N+1}}{z_{N+1}} z_n$$

where z_n is as in (2.3).)

We now calculate an explicit expression for $\tilde{\epsilon}_n$. Using (2.2) and starting from $\tilde{\epsilon}_{N+1}^{(N)} = 0$, we get

$$\tilde{\epsilon}_N^{(N)} = \frac{\delta_{N+1}}{b_{N+1} + \tilde{f}^{(N+1)}}$$

and

$$\tilde{\epsilon}_{N-1}^{(N)} = \frac{\delta_N}{b_N + \tilde{f}^{(N)}} - \frac{\tilde{f}^{(N-1)}\delta_{N+1}}{(b_N + \tilde{f}^{(N)})(b_{N+1} + \tilde{f}^{(N+1)})}$$

Continuing in this way we find:

$$\tilde{\epsilon}_n^{(N)} = \sum_{i=0}^{N-n} \frac{\tilde{f}^{(n)}}{\tilde{f}^{(i+n)}(b_{i+n+1} + \tilde{f}^{(i+n+1)})} \prod_{m=1}^{i} \left(-\frac{\tilde{f}^{(m+n)}}{b_{m+n} + \tilde{f}^{(m+n)}} \right) \delta_{i+n+1}$$

We now let $N \to \infty$:

$$\tilde{\epsilon}_n = \tilde{f}^{(n)} \sum_{i=0}^{\infty} \frac{1}{\tilde{f}^{(i+n)}(b_{i+n+1} + \tilde{f}^{(i+n+1)})} \prod_{m=1}^{i} \left(-\frac{\tilde{f}^{(m+n)}}{b_{m+n} + \tilde{f}^{(m+n)}} \right) \delta_{i+n+1}$$

For $n = 0$ this gives us:

$$\tilde{\epsilon}_0 = \tilde{f}^{(0)} \sum_{i=0}^{\infty} \frac{1}{\tilde{f}^{(i)}(b_{i+1} + \tilde{f}^{(i+1)})} \prod_{m=1}^{i} \left(-\frac{\tilde{f}^{(m)}}{b_m + \tilde{f}^{(m)}} \right) \delta_{i+1}$$

The modification $S_n(\tilde{f}^{(n)} + \tilde{\epsilon}_n)$ of the continued fraction $K(a_n/b_n)$ is exactly the Taylor-modification of the first order defined and discussed in [7, 8, 9, 10, 11] for the special case $b_n = 1$ for all n. However, the method by which it is obtained is very different from ours.

The idea behind Taylor-modifications is the following: A convergent continued fraction $K(a_n/1)$ can be seen as a function of an infinite number of variables

$$F(a_1, a_2, \ldots, a_n, \ldots) = K\left(\frac{a_n}{1}\right) \ .$$

If we assume that all its tail values $f^{(n)}$ are finite, the derivative of F with respect to one of its variables is given by

$$\frac{\partial F}{\partial a_n} = \frac{f}{a_n} \prod_{m=1}^{n-1} \left(-\frac{f^{(m)}}{1 + f^{(m)}} \right)$$

(see [8]). We now expand the function F in a Taylor series around the point $(a_1, a_2, \ldots, a_n, \ldots) = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n, \ldots)$:

$$F(a_1, a_2, \ldots, a_n, \ldots) = F(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n, \ldots) + \sum_{i=1}^{\infty} \left.\frac{\partial F}{\partial a_i}\right|_{a_j=\tilde{a}_j} \cdot (a_i - \tilde{a}_i) + \text{higher order terms}$$

$$\Rightarrow \quad F(a_1, a_2, \ldots, a_n, \ldots) = \tilde{f}^{(0)} + \tilde{f}^{(0)} \sum_{i=1}^{\infty} \frac{1}{\tilde{a}_i} \prod_{m=1}^{i-1} \left(-\frac{\tilde{f}^{(m)}}{1 + \tilde{f}^{(m)}} \right) \delta_i + \text{h. o. terms}$$

and similarly

$$F(a_{n+1}, a_{n+2}, \ldots) = \tilde{f}^{(n)} + \tilde{f}^{(n)} \sum_{i=1}^{\infty} \frac{1}{\tilde{a}_{i+n}} \prod_{m=1}^{i-1} \left(-\frac{\tilde{f}^{(m+n)}}{1 + \tilde{f}^{(m+n)}} \right) \delta_{i+n} + \text{h. o. terms} \ .$$

If we replace the tail $f^{(n)}$ of the original continued fraction by the sum of the first two terms in this expression, we call the result the Taylor-modification of order 1.

Using the same method it is possible to recuperate the Taylor-modification of second order (see [11]). If we put

$$f^{(n)} = \tilde{f}^{(n)} + \tilde{\epsilon}_n + \mu_n$$

in the recurrence relation for the tails, then we get a recurrence relation for the sequence μ_n :

$$(\tilde{f}^{(n-1)} + \tilde{\epsilon}_{n-1})\mu_n + (b_n + \tilde{f}^{(n)} + \tilde{\epsilon}_n + \mu_n)\mu_{n-1} = -\tilde{\epsilon}_n \tilde{\epsilon}_{n-1} \ .$$

We leave out the products of $\tilde{\epsilon}$'s and μ's, assuming these terms will be small compared to the others, and we define $\tilde{\mu}_n$ as the minimal solution of the recurrence relation:

$$\tilde{f}^{(n-1)} y_n + (b_n + \tilde{f}^{(n)}) y_{n-1} = -\tilde{\epsilon}_{n-1} \tilde{\epsilon}_n \ . \tag{2.5}$$

Note that the lefthand side of this equation is the same as that of equation (2.3), and so it is easy to see that

$$\tilde{\mu}_n = -\tilde{f}^{(n)} \sum_{i=0}^{\infty} \frac{1}{\tilde{f}^{(i+n)}(b_{i+n+1} + \tilde{f}^{(i+n+1)})} \prod_{m=1}^{i} \left(-\frac{\tilde{f}^{(m+n)}}{b_{m+n} + \tilde{f}^{(m+n)}} \right) \tilde{\epsilon}_{n+i} \tilde{\epsilon}_{n+i+1} \ .$$

We have checked that $S_n(\tilde{f}^{(n)} + \tilde{\epsilon}_n + \tilde{\mu}_n)$ coincides with the second-order Taylor-modification in the special case

$$b_n = 1 \ .$$

3 An application

We look at the special case

$$a_n = a + C \cdot T^n \ , \quad b_n = 1 \quad \text{with } a \in \mathbb{C} \backslash]-\infty, -\frac{1}{4}] \ , \ |T| < 1$$

which has been studied in [11]. Note that for the continued fraction $K(a/1)$ we have for all n:

$$\tilde{f}^{(n)} = \Gamma$$

with Γ the root of $x^2 + x - a$ with smallest modulus. The other root is $1 + \Gamma$. (We use the notations from [11].)

Using the results from the previous paragraph, it is now very easy to find the Taylor-modifications of the first and the second order for $K(a_n/1)$. We only have to calculate $\tilde{\epsilon}_n$ and $\tilde{\mu}_n$. Now, $\tilde{\epsilon}_n$ is the minimal solution of (2.2):

$$\Gamma y_n + (1 + \Gamma) y_{n-1} = CT^n \ . \tag{3.1}$$

This recurrence relation clearly has a solution of the form

$$y_n = A_1 \cdot T^n$$

and this will be the minimal solution, since the homogeneous equation associated with (3.1) has the solution

$$(-1)^n \left(\frac{1 + \Gamma}{\Gamma} \right)^n$$

for which we have

$$\lim_{n \to \infty} \frac{A_1 \cdot T^n}{(-1)^n \left(\frac{1+\Gamma}{\Gamma} \right)^n} = 0 \ .$$

(Note that $|\frac{1+\Gamma}{\Gamma}| > 1$.)
We can determine the coefficient A_1 using (3.1):

$$\Gamma A_1 T^n + (1 + \Gamma) A_1 T^{n-1} = CT^n \quad \Rightarrow \quad A_1 = \frac{CT}{1 + \Gamma + \Gamma T}$$

Hence:

$$\tilde{\epsilon}_n = \frac{CT^{n+1}}{1 + \Gamma + \Gamma T} \ .$$

It follows from the theorem in the previous paragraph that $S_n(\Gamma + \tilde{\epsilon}_n)$ converges faster than $S_n(\Gamma)$.

We calculate $\tilde{\mu}_n$ using the same method. $y_n = \tilde{\mu}_n$ satisfies (2.5):

$$\Gamma y_n + (1 + \Gamma) y_{n-1} = -\tilde{\epsilon}_n \tilde{\epsilon}_{n-1} \quad \Leftrightarrow \quad \Gamma y_n + (1 + \Gamma) y_{n-1} = -\frac{C^2 T^{2n+1}}{(1 + \Gamma + \Gamma T)^2} \tag{3.2}$$

This equation has a minimal solution of the form

$$y_n = A_2 T^{2n}$$

Substituting this into equation (3.2) we find:

$$A_2 = -\frac{C^2 T^3}{(1 + \Gamma + \Gamma T)^2 (1 + \Gamma + \Gamma T^2)} \quad \Rightarrow \quad \tilde{\mu}_n = -\frac{C^2 T^{2n+3}}{(1 + \Gamma + \Gamma T)^2 (1 + \Gamma + \Gamma T^2)}$$

It is easy to proving (using the method from [3]) that $S_n(\Gamma + \tilde{\epsilon}_n + \tilde{\mu}_n)$ converges faster than $S_n(\Gamma + \tilde{\epsilon}_n)$.

The two modifications discussed above may be found in [11]. We note however that we can continue in the same way to get better and better approximations to $\tilde{f}^{(n)}$, with much less effort than in [11]. For instance, if we assume that

$$f^{(n)} = \Gamma + \tilde{\epsilon}_n + \tilde{\mu}_n + \nu_n \ ,$$

then the corresponding approximation $y_n = \tilde{\nu}_n$ to ν_n can be shown to satisfy:

$$\Gamma y_n + (b_n + \Gamma) y_{n-1} = -\tilde{\epsilon}_{n-1}\tilde{\mu}_n - \tilde{\epsilon}_n \tilde{\mu}_{n-1}$$

(using the same technique used to obtain (2.5)). This is equivalent with

$$\Gamma y_n + (b_n + \Gamma) y_{n-1} = \frac{C^3(1+T)T^{3n+2}}{(1+\Gamma+\Gamma T)^3(1+\Gamma+\Gamma T^2)}$$

and solving this equation for the minimal solution we find

$$\tilde{\nu}_n = \frac{C^3(1+T)T^{3n+5}}{(1+\Gamma+\Gamma T)^3(1+\Gamma+\Gamma T^2)(1+\Gamma+\Gamma T^3)} \ .$$

$S_n(\Gamma + \tilde{\epsilon}_n + \tilde{\mu}_n + \tilde{\nu}_n)$ converges faster than $S_n(\Gamma + \tilde{\epsilon}_n + \tilde{\mu}_n)$.

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