Estimation of basins of attraction for controlled systems with input saturation and time-delays: extended paper

J. J. Benjamin Biemond
Wim Michiels

Report TW 644, March 31, 2014



KU Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Estimation of basins of attraction for controlled systems with input saturation and time-delays: extended paper* †

J. J. Benjamin Biemond
Wim Michiels

Report TW 644, March 31, 2014

Department of Computer Science, KU Leuven

Abstract

Basins of attraction are instrumental to study the effect of input saturation in control systems, as these sets characterise the initial conditions for which the control strategy induces attraction to the desired equilibrium. In this paper, we describe these sets when the open-loop system is exponentially unstable and the system is controlled by a single actuator with both constant time-delays and saturation. Estimates of the basin of attraction are provided and the allowable time-delay in the control loop is determined with a novel piecewise quadratic Lyapunov-Krasovskii functional that exploits the piecewise affine nature of the system. As this approach leads to sufficient, but not to necessary conditions for attractivity, we present simulations of an exemplary system to show the applicability of the results.

Keywords : Time delay, Basins of attraction, Saturation, Nonlinear control systems, Control system analysis, Lyapunov-Krasovskii functionals.

*This work has been supported by the Programme of Interuniversity Attraction Poles of the Belgian Federal Science Policy Office (IAP P6- DYSCO), by OPTEC, the Optimization in Engineering Center of the KU Leuven, and the project G.0712.11N of the Research Foundation - Flanders (FWO)

†The results in this report will appear in: J.J.B. Biemond and Wim Michiels, Estimation of basins of attraction for controlled systems with input saturation and time-delays, Proceedings of the 19th IFAC World Congress, Cape Town, August 24-29, 2014.

Estimation of basins of attraction for controlled systems with input saturation and time-delays: extended paper ⋆

J.J. Benjamin Biemond * **Wim Michiels** *

* *Department of Computer Science, KU Leuven, Celestijnenlaan 200A, Belgium (e-mail: benjamin.biemond@cs.kuleuven.be).*

Abstract: Basins of attraction are instrumental to study the effect of input saturation in control systems, as these sets characterise the initial conditions for which the control strategy induces attraction to the desired equilibrium. In this paper, we describe these sets when the open-loop system is exponentially unstable and the system is controlled by a single actuator with both constant time-delays and saturation. Estimates of the basin of attraction are provided and the allowable time-delay in the control loop is determined with a novel piecewise quadratic Lyapunov-Krasovskii functional that exploits the piecewise affine nature of the system. As this approach leads to sufficient, but not to necessary conditions for attractivity, we present simulations of an exemplary system to show the applicability of the results.

Keywords: Time delay, Basins of attraction, Saturation, Nonlinear control systems, Control system analysis, Lyapunov-Krasovskii functionals

1. INTRODUCTION

Input saturations and delays occur in virtually all control systems in mechanical, chemical and electric engineering. However, in the control design process, the nonlinear effect of saturations are often ignored, and most studies including time-delays in their analysis consider linear systems. In the present paper, we will consider the effect of both input saturations and constant time-delays on the closed-loop dynamics.

We focus on linear systems controlled by a single actuator with saturation and delays in the control implementation. Restricting our attention to static controllers, "windup"-type problems, as addressed in Grimm et al. (2003) for the delay-free case, are excluded. We present a method to estimate the basin of attraction for closed-loop systems with input saturation and delays. This is the set of initial conditions for which the controller achieves convergence to the origin, despite of the saturation and time lag. Consequently, the basin of attraction is instrumental in accessing the effect of the saturation and delays.

In the literature, basins of attraction for smooth (closed-loop) systems without time-delays are well-understood, and, under some technical conditions, the geometry of these basins of attraction can be approximated arbitrarily closely with the sublevelsets of polynomial Lyapunov functions, cf. Giesl (2008). For control systems with saturation and without delays, in Hu et al. (2006), both performance of the controlled system and its basins of attraction is described. In Johansson and Rantzer (1998), piecewise quadratic Lyapunov functions are presented and in Johansson (2002); Dai et al. (2009) these are applied to delay-free systems with saturation. However, when delays occur in the control implementation, the closed-loop dynamics should be modelled as *retarded delay differential equations*, which, due to the nonsmooth effect of saturation, will have a *nonsmooth* right-hand side. While smooth, and in particular linear, retarded delay differential equations are relatively well-understood, cf. Michiels and Niculescu (2007); Insperger and Stépán (2011); Kharitonov (2013), few of these results are applicable to nonsmooth retarded delay equations, and the nonsmooth nature of these equations necessitates more versatile analysis tools.

In Seuret et al. (2009); Tarbouriech et al. (2011), the nonsmooth character of the saturation is analysed using a polytopic overapproximation based on the observation that, given $H > 1$, the scalar saturation function $\mathrm{sat}(y) = \mathrm{sign}(y)\min(|y|, 1)$ satisfies $\mathrm{sat}(y) \in [H^{-1}, 1]y$ when $|y| < H$, or generalisations of this approach, cf. Tarbouriech et al. (2011). Hence, in the domain where $|y| < H$, the right-hand side of the nonsmooth retarded delay differential equation can be overapproximated by a set of linear functions, resulting in a linear retarded delay differential inclusion. As the stability and convergence properties of this delay inclusion is governed by the properties of the generating vertices, stability and convergence of the saturated delay system is guaranteed when a finite number of linear delay differential equations satisfy the decrease condition for a common quadratic Lyapunov-Krasovskii functional. Focussing on linear time-delay systems controlled by saturating *non-delayed* actuators, polytopic overapproximations of the functions $\mathrm{sat}(y)$ or $y - \mathrm{sat}(y)$ have been used in Gomes da Silva Jr et al. (2011); Fridman et al. (2003); Tarbouriech and Gomes da

⋆ This work has been supported by the Programme of Interuniversity Attraction Poles of the Belgian Federal Science Policy Office (IAP P6- DYSCO), by OPTEC, the Optimization in Engineering Center of the KU Leuven, and the project G.0712.11N of the Research Foundation - Flanders (FWO)

Silva Jr (2000); Cao et al. (2002), leading to controller synthesis and H_∞ performance results, where both time-varying delays and neutral systems can be considered. In these papers, quadratic Lyapunov-Krasovskii functionals are used, such that only ellipsoidal basin of attraction estimates have been attained in these references.

In this study, we follow a different approach, and will not make an overapproximation of the saturation function. Instead, we will exploit the observation that the saturation function induces a piecewise affine nature of the retarded differential equation, and analyse this dynamics with a piecewise quadratic Lyapunov-Krasovskii functional. For this purpose, firstly, we analyse the delay-free system with a piecewise quadratic Lyapunov function which is appropriate to identify the basin of attraction of the delay-free system. Addition of a functional term makes this Lyapunov function into a Lyapunov-Krasovskii functional, such that the retarded differential equation can be analysed. An overapproximation of the difference $\mathrm{sat}(Kx(t-\tau)) - \mathrm{sat}(Kx(t))$ is used to evaluate this functional along solutions.

The contributions of this paper are threefold. Firstly, a piecewise polynomial Lyapunov function is introduced which can be used to estimate the basin of attraction for linear systems controlled by a delay-free saturating control input by exploiting the piecewise affine nature of the closed-loop system. Secondly, this estimate is computed from a quadratic optimisation problem with quadratic constraints, for which an explicit solution in terms of the roots of a polynomial equation is provided. Thirdly, given the delay-free basin of attraction estimate, Lyapunov-Krasovskii techniques are used to find delay-dependent conditions for the allowed constant time-delays. As these contributions involve merely sufficient conditions for attraction and stability, and no necessary conditions are attained, we also present simulations of an exemplary system to assess the conservatism of the presented sufficient conditions.

The outline of the remainder of this paper is as follows. In the following section, we present the dynamical model and necessary notation. In Section 3, the basin of attraction is estimated for the delay-free system, and in Section 4, we analyse the allowed time-delay for which this basin of attraction estimate is accurate. An example is presented in Section 5, and conclusions are given in Section 6.

2. MODELLING AND NOTATION

Consider the linear system with a single actuator:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $u \in \mathbb{R}$ the actuation input. This input experiences delay and is given by the saturated version of a linear control action, such that $u(t) = \mathrm{sat}(Kx(t-\tau))$, where $K \in \mathbb{R}^{1\times n}$ and $\mathrm{sat}(y) := \mathrm{sign}(y)\min(|y|, 1)$. Hence, the closed-loop system is given by the nonsmooth retarded differential equation:

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}(Kx(t-\tau)). \tag{2}$$

Since (2) is a retarded differential equation, solutions should be considered in the state space of absolutely continuous functions. To describe these functions, we introduce $x_\tau(t) : [-\tau, 0] \to \mathbb{R}^n$, such that $x_\tau(t)(s) = x(t+s)$, $s \in [-\tau, 0]$. Let $AC([-\tau, 0], \mathbb{R})$ denote the set of absolutely continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^n$.

Given $P \in \mathbb{R}^{n\times n}$, let $P \succ 0$ denote that P is symmetric and positive definite, $\|x\|_P^2$, with $x \in \mathbb{R}^n$, denotes $x^T P x$, $|x|$ the Euclidean norm of x, and x_i, $i = 1, \dots n$ denotes the i-th element of x. Given a function $v : \mathbb{R}^n \to \mathbb{R}$, and set $w \subset \mathbb{R}$, $v^{-1}w$ denotes $\{x \in \mathbb{R}^n |\ v(x) \in w\}$. For $k_1, k_2 \in \mathbb{R}$, $k_1 \perp k_2$ denotes $k_1 k_2 = 0$.

3. ESTIMATING THE BASIN OF ATTRACTION FOR THE DELAY-FREE SYSTEM

We will now study the non-delayed system given by

$$\dot{x} = F_{nd}(x) := Ax + B\mathrm{sat}(Kx), \tag{3}$$

and present a Lyapunov function that can be used to estimate the basin of attraction. The basin of attraction for this ordinary differential equation can be estimated by the sublevelsets of a continuous Lyapunov function V, such that in this sublevelset, $\dot{V} < 0$ holds. As the vector field F_{nd} is piecewise affine, we propose a piecewise polynomial Lyapunov function with the same partitioning. In particular, we study the following Lyapunov function candidate:

$$V_{nd}(x) = (x - s(x))^T P(x)(x - s(x)), \tag{4}$$

with $s(x)$ and $P(x)$ the piecewise constant functions:

$$s(x) = \begin{cases} -s_s & Kx < -1 \\ 0 & |Kx| \le 1 \\ s_s & Kx > 1 \end{cases} \qquad P(x) = \begin{cases} P_s & Kx < -1 \\ P_0 & |Kx| \le 1 \\ P_s & Kx > 1, \end{cases} \tag{5}$$

where $P_0, P_s \succ 0$ and s_s satisfies the natural requirement $Ks_s < 1$, such that $V_{nd}(x) > 0$ for $x \neq 0$. Compared to the quadratic Lyapunov function $x^T P_0 x$, more design freedom for the Lyapunov function is allowed in the domain where saturation occurs. The continuity requirement implies that P_0 and s_s can be considered as the free design parameters of the Lyapunov function from which P_s follows, as stated in the following lemma.

Lemma 1. Given $P_0 \in \mathbb{R}^{n\times n}, P_0 \succ 0, K \in \mathbb{R}^{1\times n}$ and $s_s \in \mathbb{R}^n$, with $Ks_s \le 1$, there exists a unique symmetric matrix $P_s \in \mathbb{R}^{n\times n}$ such that (4) is continuous. This matrix is given by

$$\begin{aligned} P_s = &\begin{pmatrix} K \\ K_T \end{pmatrix}^T \begin{pmatrix} \frac{1}{1-Ks_s} & 0 \\ 0 & I \end{pmatrix} \\ &\begin{pmatrix} \left\| \frac{K^T}{|K|^2} + K_T^T K_T s_s \right\|_{P_0}^2 & (\frac{K}{|K|^2} + s_s^T K_T^T K_T) P_0 K_T^T \\ K_T P_0 (\frac{K^T}{|K|^2} + K_T^T K_T s_s) & K_T P_0 K_T^T \end{pmatrix} \\ &\begin{pmatrix} \frac{1}{1-Ks_s} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} K \\ K_T \end{pmatrix}, \end{aligned} \tag{6}$$

where K_T is such that $\frac{1}{|K|}\begin{pmatrix} K^T & K_T^T \end{pmatrix}$ is orthonormal. In addition, $P_s \succ 0$.

Proof. We will first prove that V_{nd} in (4) is continuous with P_s given in (6). As V_{nd} is symmetric with respect to the involution $x \to -x$, continuity can be proven by showing that $x^T P_0 x = (x - s_s)^T P_s (x - s_s)$, $\forall x \in \{x \in \mathbb{R}^n |\ Kx = 1\} = \{x = \frac{K^T}{|K|^2} + \frac{K_T^T}{|K|^2} y,\ y \in \mathbb{R}^{n-1}\}$.

To evaluate $(x-s_s)^T P_s (x-s_s)$, we first substitute $x = \frac{K^T+K_T^T y}{|K|^2}$ and observe that $\begin{pmatrix} \frac{1}{1-Ks_s} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} K \\ K_T \end{pmatrix} (x-s_s) = \begin{pmatrix} 1 \\ \frac{1}{|K|^2} y - K_T s_s \end{pmatrix}$. Hence,

$$\begin{aligned}
&(x-s_s)^T P_s (x-s_s) \\
&= \left\| \tfrac{K^T}{|K|^2} + K_T^T K_T s_s \right\|_{P_0}^2 + 2(\tfrac{K}{|K|^2} + s_s^T K_T^T K_T) P_0 K_T^T (\tfrac{1}{|K|^2} y \\
&\quad - K_T s_s) + (\tfrac{1}{|K|^2} y - K_T s_s)^T K_T P_0 K_T^T (\tfrac{1}{|K|^2} y - K_T s_s) \\
&= \tfrac{1}{|K|^4} (K^T + K_T^T y)^T P_0 (K^T + K_T^T y), \qquad (7)
\end{aligned}$$

which proves continuity of V_{nd} at the surface $\{x \in \mathbb{R}^n |\ x = \frac{K^T+K_T^T y}{|K|^2},\ y \in \mathbb{R}^{n-1}\}$.

To prove uniqueness, for the sake of contradiction, assume that there exist two unequal symmetric matrices P_s^1, P_s^2 such that V_{nd} is continuous. This implies that there exists a $x \in \mathbb{R}^n$, such that $u^T P_s^1 u - u^T P_s^2 u \neq 0$, with $u = x - s_s$. If $Ku \neq 0$, then this expression also implies that $(\frac{1-Ks_s}{Ku} u)^T P_s^1 (\frac{1-Ks_s}{Ku} u) - (\frac{1-Ks_s}{Ku} u)^T P_s^2 (\frac{1-Ks_s}{Ku} u) \neq 0$, as both terms are quadratic and $\frac{1-Ks_s}{Ku} \neq 0$. However, with substitution of $\bar{x} = \frac{1-Ks_s}{Ku} u + s_s$, we attain $K\bar{x} = 1$, such that a contradiction is attained with the required continuity of V_{nd}. Hence,

$$u^T P_s^1 u - u^T P_s^2 u = 0, \qquad (8)$$

for all u such that $Ku \neq 0$. From continuity of (8), we infer that this relation holds for all u, attaining a contradiction. Hence, the matrix P_s in (6) is unique.

To prove that P_s is positive definite, first observe that $P_s \succ 0$ if and only if

$$\begin{pmatrix} \left\| \tfrac{K^T}{|K|^2} + K_T^T K_T s_s \right\|_{P_0}^2 & (\tfrac{K}{|K|^2} + s_s^T K_T^T K_T) P_0 K_T^T \\ K_T P_0 (\tfrac{K^T}{|K|^2} + K_T^T K_T s_s) & K_T P_0 K_T^T \end{pmatrix} \succ 0, \qquad (9)$$

which, using the Schur complement, holds if firstly, $K_T P_0 K_T^T \succ 0$, which holds as $P_0 \succ 0$, and secondly

$$\tfrac{1}{|K|^4} K P_0 K^T - \tfrac{K}{|K|^2} P_0 K_T^T \left(K_T P_0 K_T^T\right)^{-1} K_T P_0 \tfrac{K^T}{|K|^2} \succ 0, \qquad (10)$$

which holds true as the left-hand side is a Schur complement of $\begin{pmatrix} \frac{1}{|K|^2} K \\ K_T \end{pmatrix} P_0 \begin{pmatrix} \frac{1}{|K|^2} K \\ K_T \end{pmatrix}^T$, that is positive definite for $P_0 \succ 0$. Positive definiteness of P_s is attained, completing the proof. □

We will now use V_{nd} to estimate the basin of attraction of the delay-free system

$$\dot{x}(t) = F_{nd}(x(t)) = Ax(t) + B\text{sat}(Kx(t)). \qquad (11)$$

Solutions of this system pass the surface $\{x \in \mathbb{R}^n |\ Kx = \pm 1\}$ instantly, such that $\nabla V_{nd}(x(t))$ is defined almost everywhere and given by $\nabla V_{nd} = 2(x(t) - s(x(t)))P(x(t))$. Hence, $\frac{dV_{nd}}{dt} = \dot{V}_{nd}(x(t))$ almost everywhere, where $\dot{V}_{nd} : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\dot{V}_{nd}(x) := \nabla V_{nd}(x) \left(Ax + B\text{sat}(Kx)\right). \qquad (12)$$

Observe that an estimate of the basin of attraction of the origin for system (11) is given by

$$\{x \in \mathbb{R}^n \mid V_{nd}(x) \leq \gamma\}, \qquad (13)$$

with $\gamma = \sup\{\bar{\gamma} |\ V_{nd}(x) \leq \bar{\gamma} \Rightarrow \dot{V}_{nd}(x) \leq 0\}$.

However, at the boundary of the set (13), no robustness to perturbations, such as the difference between (11) and (2), can be guaranteed. In order to attain such robustness properties, we define γ_ϵ as the solution of the following optimisation problem:

$$\gamma_\epsilon := \sup\{\gamma | V_{nd}(x) \leq \gamma \Rightarrow \dot{V}_{nd}(x) \leq -\epsilon V_{nd}(x)\}, \qquad (14)$$

with $\epsilon > 0$, and observe that the sublevelset $\{x \in \mathbb{R}^n \mid V_{nd}(x) \leq \gamma_\epsilon\}$ contains points that will be attracted to the origin, even if a small disturbance is applied to system (11).

We now introduce the following assumption to compute γ_ϵ in (14).

Assumption 1. Let $A + BK$ be Hurwitz, and $P_0 \succ 0$ be such that $P_0(A+BK) + (A+BK)^T P_0 \prec -\epsilon P_0$, with $\epsilon > 0$. In addition, one of the eigenvalues of A has a positive real part.

Note that the requirement on $A+BK$ implies that one can always design a P_0 such that the matrix inequality holds for sufficiently small $\epsilon > 0$. The last part of the assumption is included to ensure that the basin of attraction of (11) is bounded. Namely, sufficiently far away from the origin, the magnitude of the term $\text{sat}(Kx(t)) \in [-1, 1]$ will not suffice to compensate the unstable term $Ax(t)$.

In the following lemma, the optimisation problem (14) is rewritten as a *quadratically constrained quadratic problem.*

Lemma 2. Given Assumption 1, let $s_s \in \mathbb{R}^n$ be such that $Ks_s < 1$, P_s in (6), V_{nd} in (4) and $\dot{V}_{nd}$ in (12). Then γ_ϵ in (14) is equal to

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \ & (x-s_s)^T P_s (x-s_s) \\
\text{s.t.} \ & Kx - 1 \geq 0 \\
& (x-s_s)^T L (x-s_s) + 2(x-s_s)^T P_s (As_s + B) \geq 0,
\end{aligned} \qquad (15)$$

with

$$L := P_s A + A^T P_s + \epsilon P_s. \qquad (16)$$

Proof. Using $(V_{nd}(x) \leq \gamma \Rightarrow \dot{V}_{nd}(x) \leq -\varepsilon V_{nd}(x))$ if and only if $(\dot{V}_{nd}(x) > -\varepsilon V_{nd}(x) \Rightarrow V_{nd}(x) > \gamma)$, we observe that the optimisation problem (14) is equal to:

$$\gamma_\epsilon = \inf\{V_{nd}(x) |\ \dot{V}_{nd}(x) > -\epsilon V_{nd}(x)\}. \qquad (17)$$

From (4) and (12), we conclude that $\dot{V}_{nd}(x) = 2x^T P(A + BK)x$ if $Kx \leq 1$. Consequently, the matrix inequality in Assumption 1 implies that $\dot{V}_{nd}(x) \leq -\epsilon V_{nd}$ for all x where $|Kx| \leq 1$. Consequently, we find that the constraint set of the optimisation problem, i.e. $\{x \in \mathbb{R}^n |\ \dot{V}_{nd} > -\epsilon V_{nd}(x)\}$, is equal to $\{x \in \mathbb{R}^n |\ |Kx| > 1 \wedge 2(x-s_s)^T P_s (Ax + B) > -\epsilon V_{nd}(x)\}$. As the constraints and objective functions are invariant to the involution $x \to -x$, we may replace the first constraint with $Kx > 1$.

We will now show that the optimiser of (17) is bounded, such that γ_ϵ is bounded. For the sake of contradiction, suppose that no bounded optimiser of (17) exists. Then $\dot{V}_{nd}(x) \leq -\epsilon V_{nd}(x)$ for all bounded x. However, as A has an eigenvalue in the open right-half plane, for all $P_s \succ 0$ there exists a vector x such that $2x^T P_s A x > 0$. Since this term cannot be compensated by the remaining terms in

(12), which, as $\text{sat}(\cdot) \in [-1,1]$, can be bounded by linear terms in x, for sufficiently large x, we attain $\dot{V}_{nd}(x) > 0$ and find a contradiction. Hence, the optimiser of (17) is bounded. Consequently, we may replace the infimum operation by a minimum by closing the constraint, such that the minimum is taken over the set $\{x \in \mathbb{R}^n | \; Kx \geq 1 \wedge 2(x-s_s)^T P_s(Ax+B) \geq -\epsilon V_{nd}(x)\}$. With (4), problem (15) is attained. □

Since A has unstable eigenvalues and P_s is positive definite, we observe that L cannot be negative definite. Hence, the problem (15) has a non-convex constraint and multiple minima of this problem can exist.

The following technical lemma provides necessary and sufficient conditions for the design of s_s and P_s such that the Lyapunov function design in (4) results in a basin of attraction estimate (14) that contains points x where $|Kx| > 1$.

Lemma 3. Consider system (11), V_{nd} given in (4), s_s such that $Ks_s < 1$ and P_s in (6), let Assumption 1 hold and let γ_ϵ be given in (14). The strict inequality

$$\gamma_\epsilon > \min_{x \in \mathbb{R}^n, \; Kx \geq 1} V_{nd}(x) \tag{18}$$

holds if and only if

$$\tfrac{Ks_s - 1}{KP_s^{-1}K^T} KP_s^{-1}LP_s^{-1}K^T + K(As_s + B) < 0. \tag{19}$$

Proof. From Lemma 2, we conclude that γ_ϵ is equal to (15). We prove the present lemma by showing that (19) holds if and only if

$$\min_{x \in \mathbb{R}^n, \; Kx-1 \geq 0} (x - s_s)^T P_s (x - s_s) \tag{20}$$

is not equal to γ_ϵ.

Observe that $P_s \succ 0$ and $Ks_s < 1$, such that the optimiser x^* of (20) satisfies $Kx^* = 1$. With the Karush-Kuhn-Tucker conditions, we find $x^* = s_s + cP_s^{-1}K^T$, with $c = \frac{1 - Ks_s}{KP_s^{-1}K^T}$. We complete this proof by evaluating (15) at x^*:

$$\begin{aligned} &2(x^* - s_s)^T L(x^* - s_s) + 2(x^* - s_s)^T P_s(As_s + B), \\ &= 2c^2 KP_s^{-1}LP_s^{-1}K^T + 2cK(As_s + B), \end{aligned} \tag{21}$$

which, as $2c > 0$, is nonnegative if and only if (19) does not hold. Violation of (19) implies that x^* satisfies both constraints of (15), and, hence, $\gamma_\epsilon = V_{nd}(x^*) = \min_{x \in \mathbb{R}^n, \; Kx-1 \geq 0} V_{nd}(x)$. □

We will now find solutions to the optimisation problem (14) provided that (19) holds. The freedom to design P_s unequal to P_0 will be exploited to attain a larger basin of attraction estimate.

As the second constraint of problem (15) is non-convex, multiple minima of this problem can exist. All these minima have to satisfy the Karush-Kuhn-Tucker conditions for optimality, which, for the problem (15), imply that for each minimum x there has to exist multipliers μ, λ such that:

$$2(x - s_s)^T P_s = \mu K + 2\lambda((x - s_s)^T L + (As_s + B)^T P_s), \tag{22a}$$

$$0 \leq \mu \perp Kx - 1 \geq 0 \tag{22b}$$

$$0 \leq \lambda \perp (x-s_s)^T L(x-s_s) + 2(x-s_s)^T P_s(As_s+B) \geq 0. \tag{22c}$$

We will now provide an explicit solution to these conditions.

Proposition 4. Consider V_{nd} in (4) and $\dot{V}_{nd}$ in (12), let Assumption 1 be satisfied, let s_s be such that $Ks_s < 1$, let P_s be given in (6) and let (19) hold. In addition, we assume that there does not exists a tuple (λ, μ, v), $\lambda \geq 0, \mu \geq 0$ and $v \in \mathbb{R}^n$, such that both $(P_s - \lambda L)v = 0$ and $v^T(\frac{\mu}{2}K^T + P_s(As_s + B)) = 0$.

Let $W, U, \Sigma \in \mathbb{R}^{n \times n}$ be such that $P_s = WW^T$, $UU^T = I$, Σ diagonal with diagonal elements σ_i and $U\Sigma U^T = W^{-1}LW^{-T}$. Let K_T be such that $\frac{1}{|K|}(K^T \; K_T^T)$ is orthogonal, let $\bar{W}, \bar{U}, \bar{\Sigma} \in \mathbb{R}^{(n-1)\times(n-1)}$ be such that $K_T P_s K_T^T = \bar{W}\bar{W}^T$, $\bar{U}\bar{U}^T = I$, $\bar{\Sigma}$ diagonal with diagonal elements $\bar{\sigma}_i$ and $\bar{U}\bar{\Sigma}\bar{U}^T = \bar{W}^{-1}K_T L K_T^T \bar{W}^{-T}$.

The optimiser $\bar{x}$ of the optimisation problem (15) always exists and it satisfies one of the following conditions. In addition, $\gamma_\epsilon = V_{nd}(\bar{x})$ holds, with γ_ϵ in (14).

(i) there exist a $\lambda > 0$ such that $P_s - \lambda L$ is invertible and $\bar{x}$ is given by

$$\bar{x} = s_s + \lambda(P_s - \lambda L)^{-1} P_s(As_s + b), \tag{23}$$

$K\bar{x} > 1$ holds and $\lambda > 0$ is a real and positive solution to the polynomial expression

$$0 = \sum_{i=1}^{n} \beta_i^2 (2 - \lambda\sigma_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - \lambda\sigma_j)^2, \tag{24}$$

with $\beta = U^T W^T (As_s + B)$.

(ii) there exists a $\lambda > 0$ such that $\bar{P}_s - \lambda\bar{L}$ invertible and $\bar{x}$ is given by

$$\begin{aligned} \bar{x} =& cP_s^{-1}K^T + s_s + \lambda K_T^T \left(\bar{P}_s - \lambda\bar{L}\right)^{-1} K_T (cLP_s^{-1}K^T \\ &+ P_s(As_s + B)), \end{aligned} \tag{25}$$

with $c = \frac{1 - Ks_s}{KP_s^{-1}K^T}$, and

$$\begin{aligned} 0 =& \lambda^2 \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \Delta_{ij}\delta_i^0\delta_j^0 \prod_{\substack{k=1 \\ k\neq i}}^{n-1}\prod_{\substack{l=1 \\ l\neq j}}^{n-1}(1 - \lambda\bar{\sigma}_k)(1 - \lambda\bar{\sigma}_l) \\ &+ 2\lambda \sum_{i=1}^{n-1} (\delta_i^0)^2 \prod_{\substack{k=1 \\ k\neq i}}^{n-1}\prod_{l=1}^{n-1}(1 - \lambda\bar{\sigma}_k)(1 - \lambda\bar{\sigma}_l) \\ &+ \delta^1 \prod_{i=1}^{n-1}(1 - \lambda\bar{\sigma}_i)^2, \end{aligned} \tag{26}$$

holds with $\delta^0 = \bar{U}^T\bar{W}^{-1}K_T(P_s(As_s+B)+cLP_s^{-1}K^T)$ $\delta^1 = \left(cKP_s^{-1}L + 2(As_s + B)^T P_s\right) cP_s^{-1}K^T$ and $\Delta = \bar{U}^T\bar{W}^{-1}\bar{L}\bar{W}^{-T}\bar{U}$.

Furthermore, there exist only a finite number of $\bar{x}$ satisfying these conditions.

Proof. From Lemma 2, we conclude that the solution to the problem (14) is given by the quadratically constraint quadratic problem (15). The set in $\mathbb{R}^n$ where the constraints of this problem are feasible is nonempty, such that an optimiser always exists. Given the optimiser $\bar{x}$ of (15) there should exist multipliers λ, μ such that the Karush-Kuhn-Tucker conditions (22) hold for $(\bar{x}, \lambda, \nu)$, cf. Boyd and Vandenberghe (2004).

Now, from the assumption on tuples (λ, μ, v) stated in the proposition, we can exclude the case where $P_s - \lambda L$ is singular. Namely, when λ renders $P_s - \lambda L$ singular, multiplication of (22a) with an eigenvector v_1 such that $(P_s - \lambda L)v_1 = 0$ results in $(\frac{\mu}{2}K + (As_s + B)^T P_s)v_1 = 0$, which contradicts the assumption. Hence, $P_s - \lambda L$ is invertible at the minimisers of (15).

Since (19) is satisfied, from Lemma 3, we observe that γ_ϵ differs from the solution of the optimisation problem in (18), where the second constraint of (15) is omitted. Consequently, we observe that the second constraint in (15) is active, such that $\lambda > 0$.

Hence, (22) implies that the optimisers of (15) can be divided into two cases, $\mu = 0$ or $\mu > 0$. These two cases lead to the two options in the Proposition. We will now first focus on the option $\mu = 0$, that implies $K\bar{x} - 1 \geq 0$ and corresponds to item *(i)* of the lemma.

With $\mu = 0$, (22a) leads to $P_s(\bar{x} - s_s) = \lambda(L(\bar{x} - s_s) + P_s(As_s + B))$, which implies (23) as $P_s - \lambda L$ is invertible. Substituting (23) in (22c), we find:

$$\begin{aligned} 0 &= \lambda(As_s + B)^T P_s\big((P_s - \lambda L)^{-1}\lambda L(P_s - \lambda L)^{-1} \\ &\quad + 2(P_s - \lambda L)^{-1}\big)P_s(As_s + B), \qquad (27) \\ &= \lambda(As_s + B)^T P_s(P_s - \lambda L)^{-1}(2P_s - \lambda L) \\ &\quad (P_s - \lambda L)^{-1}P_s(As_s + B) \qquad (28) \end{aligned}$$

which, with the definitions of β, U, W, Σ in the proposition and the observations that $(P_s - \lambda L)^{-1} = W^{-T}U(I - \lambda\Sigma)^{-1}U^T W^{-1}$ and $2P_s - \lambda L = WU(2I - \lambda\Sigma)U^T W^T$, leads to:

$$= \lambda\beta^T(I - \lambda\Sigma)^{-1}(2I - \lambda\Sigma)(I - \lambda\Sigma)^{-1}\beta. \qquad (29)$$

Multiplication of this expression with $\frac{1}{\lambda}\det^2(P_s - \lambda L) = \frac{1}{\lambda}\prod_{i=1}^{n}(1 - \lambda\sigma_i)^2$, which preserves the solutions where $\lambda > 0$, results in (24).

Now, we study the second case where $\mu > 0$, which leads to conditions *(ii)* of the proposition. We observe that $\mu \geq 0$ implies $Kx - 1 = 0$, such that we may write $x = cP_s^{-1}K^T + s_s + K_T^T y$, with $y \in \mathbb{R}^{n-1}$. Substituting this expression in (22a) and premultiplying with $\frac{1}{2}K_T$ yields:

$$y = \lambda\left(\bar{P}_s - \lambda\bar{L}\right)^{-1}K_T\left(cLP_s^{-1}K^T + P_s(As_s + B)\right). \qquad (30)$$

With the definition of δ^0 in the proposition, and the observation that $\left(\bar{P}_s - \lambda\bar{L}\right)^{-1} = \bar{W}^{-T}\bar{U}(I - \lambda\bar{\Sigma})^{-1}\bar{U}^T\bar{W}^{-1}$, this leads to:

$$y = \lambda\bar{W}^{-T}\bar{U}(I - \lambda\bar{\Sigma})^{-1}\delta^0. \qquad (31)$$

Substituting this expression in (22c) and using the definitions of δ^0, δ^1 and Δ as in the proposition, we find:

$$\begin{aligned} 0 &= \big((y^T K_T + cKP_s^{-1})L + 2(As_s + B)^T P_s\big) \\ &\qquad \big(K_T^T y + cP_s^{-1}K^T\big) \qquad (32) \\ &= \lambda^2\delta^{0\,T}(I - \lambda\bar{\Sigma})^{-1}\Delta(I - \lambda\bar{\Sigma})^{-1}\delta^0 \\ &\quad + 2\lambda\delta^{0\,T}(I - \lambda\bar{\Sigma})^{-1}\delta^0 + \delta^1. \qquad (33) \end{aligned}$$

Multiplying this expression by $\det(1 - \lambda\Sigma)^2$, we attain (26), which completes the proof of the first part of the proposition.

To prove the last statement, it suffices to observe that both polynomial equations (24) and (26) have a finite number of solutions λ, which, using (23) and (25), respectively, allow for at most finitely many optimiser $\bar{x}$. □

3.1 Procedure to find γ_ϵ

Proposition 4 allows the following procedure to find γ_ϵ. First, a finite number of solutions of the polynomial expression (24) is found, for which $\bar{x}$ is computed in (23). For each $\bar{x}$ with $\bar{x} \geq 1$, we compute $V_{nd}(\bar{x})$, which is stored in a list of possible candidates for γ_ϵ.

Subsequently, a finite number of solutions for (26) can be found, which, with (25), leads to finitely many possible $\bar{x}$. Computing $V_{nd}(\bar{x})$, the other candidates for γ_ϵ are attained, that are stored in the mentioned list. Now, γ_ϵ is given by the minimum over this list.

The parameters P_0, s_s, P_s have to satisfy $Ks_s < 1$ and the conditions in Assumption 1, (6) and (19). Still, there is considerable design freedom in P_0 and s_s. How to design these parameters to attain a large basin of attraction estimate is not in the scope of this manuscript. For the non-delayed case, however, design procedures have been presented in Johansson (2002); Dai et al. (2009) for related piecewise quadratic Lyapunov functions.

4. DELAY-DEPENDENT CONDITIONS FOR THE BASIN OF ATTRACTION

In this section, we will analyse the delayed system (2) and provide an estimate for the basin of attraction, such that the initial conditions in this set always lead to trajectories converging to the origin.

System (2) can be rewritten as:

$$\dot{x}(t) = F_{nd}(x) + B\omega(x_\tau(t)) \qquad (34)$$

with

$$\omega(x_\tau(t)) := \mathrm{sat}(Kx(t - \tau)) - \mathrm{sat}(Kx(t)). \qquad (35)$$

We note that while we explicitly use the non-delayed system (11), the term ω is not considered as a random disturbance, as is common in robust control approaches. Namely, if $|x| \to 0$ for $t \to \infty$, then $\omega \to 0$ as well. With the result presented below we exploit this property, such that we can prove converge towards the origin, while robust control approaches, using only an upper bound on $|\omega|$, will only guarantee converge to a set near the origin.

We employ a Lyapunov-Krasovskii functional of the form:

$$V(x_\tau(t)) = V_{nd}(x(t)) + w(x_\tau(t)) + W(x_\tau(t)). \qquad (36)$$

Here, V_{nd} is as defined above, and the nonnegative functional w is designed as:

$$w(x_\tau(t)) = \int_{t-\tau}^{t} \dot{x}(s)^T Q\dot{x}(s)ds, \qquad (37)$$

with $Q = Q^T \succ 0$, such that $V_{nd}(x(t)) + w(x_\tau(t)) = 0$ if and only if $x_\tau(t) \equiv 0$, as x_τ is absolutely continuous. The nonnegative term W will be designed below in order to compensate two terms in the time-derivatives $\frac{dV_{nd}}{dt}$ and $\frac{dw}{dt}$.

We will now attain upper bounds for $\frac{dV_{nd}}{dt}$ and $\frac{dw}{dt}$.

Evaluating the time-derivative $\frac{dV_{nd}}{dt}$ of the function V_{nd} along the trajectories of (34), we observe that

$$\frac{dV_{nd}(x(t))}{dt} = \nabla V_{nd}(x(t))F_{nd}(x(t))+\nabla V_{nd}(x(t))B\omega(x_\tau(t)) \tag{38}$$

for almost all t. Since $V(x_\tau) \leq \gamma_\epsilon$ implies $V_{nd}(x(t)) \leq \gamma_\epsilon$, we apply (14) to conclude $\nabla V_{nd}(x(t))F_{nd} \leq -\epsilon V_{nd}(x(t))$.

A conservative estimate of the second term in (38) is:

$$\begin{aligned}
&\nabla V_{nd}(x(t))B\omega(x_\tau(t)) = \\
&= 2(x(t)-s(x(t)))^T P(x(t))B\omega(x_\tau(t)) \\
&\leq \max\left(0, 2(x(t)-s(x(t)))^T P(x(t))BK\int_{t-\tau}^{t}\dot{x}(s)ds\right) \\
&\leq \max\left(0, \int_{t-\tau}^{t} 2(x(t)-s(x(t)))^T P(x(t))BK\dot{x}(s)ds\right)
\end{aligned}$$

and, similar to Fridman (2002), we introduce $R_1 \in \mathbb{R}^{n\times n}$, with $R_1 \succ 0$, and from $\|R_1^{-1}u - v\|^2_{R_1} \geq 0$ for all $u, v, \in \mathbb{R}^n$, we conclude $2uv \leq u^T R_1^{-1} u + v^T R_1 v$. Hence, we attain:

$$\begin{aligned}
&\leq \tau(x(t)-s(x(t)))^T P(x(t))R_1^{-1}P(x(t))(x(t)-s(x(t))) \\
&+ \int_{t-\tau}^{t}\dot{x}(s)^T K^T B^T R_1 BK\dot{x}(s)ds \quad a.e.
\end{aligned} \tag{39}$$

Designing R_1 such that $P_i R_1^{-1} P_i \prec \delta_1 P_i$, $i \in \{0, s\}$, with some $\delta_1 > 0$, this function can be overapproximated with

$$\begin{aligned}
\nabla V_{nd}(x(t))B\omega(x_\tau(t)) \leq& \tau\delta_1 V_{nd}(x(t)) \\
&+ \int_{t-\tau}^{t}\dot{x}(s)^T K^T B^T R_1 BK\dot{x}(s)ds.
\end{aligned}$$

Directly evaluating $\frac{dw}{dt}$, we attain:

$$\frac{dw}{dt} = \dot{x}(t)^T Q\dot{x}(t) - \dot{x}(t-\tau)^T Q\dot{x}(t-\tau). \tag{40}$$

In order to prove that the time derivative of V is non-positive, we will add a term $\dot{x}(t)^T P_3(-\dot{x}(t) + Ax(t) + B\text{sat}(Kx(t-\tau)))$, with $P_3 \in \mathbb{R}^{n\times n}$ that vanishes, cf. (2). Hence,

$$\begin{aligned}
\frac{dw}{dt} =& \dot{x}(t)^T(Q-P_3)\dot{x}(t) - \dot{x}(t-\tau)^T Q\dot{x}(t-\tau) \\
&+ \dot{x}(t)^T P_3 Ax(t) + \dot{x}(t)^T P_3 B\text{sat}(Kx(t-\tau)).
\end{aligned} \tag{41}$$

A polytopic overapproximation of the last term results in

$$\begin{aligned}
\frac{dw}{dt} \leq& \dot{x}(t)^T(Q-P_3)\dot{x}(t) - \dot{x}(t-\tau)^T Q\dot{x}(t-\tau) \\
&+ \max_{\alpha\in[0,1]}\Big(\dot{x}(t)^T P_3(A+\alpha BK)x(t) \\
&- \alpha\dot{x}(t)^T P_3 BK\int_{t-\tau}^{t}\dot{x}(s)ds\Big),
\end{aligned} \tag{42}$$

which, with $R_2 \succ 0$, implies

$$\begin{aligned}
\frac{dw}{dt} \leq& \dot{x}(t)^T(Q-P_3+\tfrac{\tau}{2}P_3^T R_2^{-1}P_3)\dot{x}(t) - \dot{x}(t-\tau)^T Q\dot{x}(t-\tau) \\
&+ \frac{1}{2}\int_{t-\tau}^{t}\dot{x}(s)^T K^T B^T R_2 BK\dot{x}(s)ds \\
&+ \max_{\alpha\in[0,1]}\dot{x}(t)^T P_3(A+\alpha BK)x(t)
\end{aligned} \tag{43}$$

We now design the functional W in order to compensate the integral terms in (39) and (43). For this purpose, we design:

$$W := \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}(\bar{s})^T K^T B^T(R_1+\tfrac{1}{2}R_2)BK\dot{x}(\bar{s})d\bar{s}d\theta \tag{44}$$

which is nonnegative as we recall that R_1 and R_2 are positive definite. The derivative of this functional with respect to t is given by:

$$\begin{aligned}
\frac{dW}{dt} =& \tau\dot{x}(t)^T K^T B^T(R_1+\tfrac{1}{2}R_2)BK\dot{x}(t) \\
&- \int_{-\tau}^{0}\dot{x}(s)^T K^T B^T(R_1+\tfrac{1}{2}R_2)BK\dot{x}(s)ds.
\end{aligned} \tag{45}$$

Given a matrix $S \succ 0$ such that $V_{nd}(x) \geq x^T Sx$ and introducing $z(t) = \left(x^T(t)\ \dot{x}^T(t)\ \dot{x}^T(t-\tau)\right)^T$, for almost all t we can write the time derivative of V as:

$$\begin{aligned}
\frac{dV}{dt} \leq& \nabla V_{nd}F_{nd} + \tau\delta_1 V_{nd} \\
&+ \max_{\alpha\in[0,1]} z(t)^T\begin{pmatrix}0 & P_3(A+\alpha BK) & 0\\ 0 & \Psi & 0\\ 0 & 0 & -Q\end{pmatrix}z(t),
\end{aligned} \tag{46}$$

$$\leq \max_{\alpha\in[0,1]} z(t)^T\begin{pmatrix}(-\epsilon+\tau\delta_1)S & P_3(A+\alpha BK) & 0\\ 0 & \Psi & 0\\ 0 & 0 & -Q\end{pmatrix}z(t), \tag{47}$$

with $\Psi = -P_3 + \tau(\frac{1}{2}P_3 R_2^{-1}P_3 + K^T B^T(R_1+\frac{1}{2}R_2)BK)$, where, in the last step, we restricted our attention to the set of functions where $V \leq \gamma_\epsilon$ and used the definition of γ_ϵ in (14). We are now ready to formulate our main result.

Theorem 5. Consider system (2) with $\tau > 0$, let Assumption 1 hold, let P_s be given in (6) and let γ_ϵ be given in (14).

If there exist matrices $P_3, R_1, R_2, S \in \mathbb{R}^{n\times n}$, with $R_1, R_2, S \succ 0$, and scalar $\delta_1 > 0$ such that

$$\begin{pmatrix}(-\epsilon+\tau\delta_1)S & \frac{1}{2}P_3(A+\alpha BK)\\ \frac{1}{2}(A+\alpha BK)^T P_3 & \Psi\end{pmatrix} \prec 0, \tag{48}$$

$$\begin{pmatrix}-\delta_1 P_i & P_i\\ P_i & -R_1\end{pmatrix} \prec 0, \qquad S \prec P_0, \qquad S \prec P_s. \tag{49}$$

with $\Psi = -P_3 + \tau(\frac{1}{2}P_3 R_2^{-1}P_3 + K^T B^T(R_1+\frac{1}{2}R_2)BK)$, $\alpha \in \{0,1\}$ and $i \in \{0, s\}$, then there exists a $Q \succ 0$ such that all trajectories of (2) with initial condition in $\{x_\tau \in AC[-\tau, 0] |\ V(x_\tau) \leq \gamma_\epsilon\}$ are attracted towards the origin.

Proof. First, we will show that $S \prec P_0$ and $S \prec P_s$ imply $V_{nd}(x) \geq x^T Sx$. This statement is obvious for $|Kx| \leq 1$, such that we now evaluate

$$\inf_{Kx-1>0} z(x), \tag{50}$$

with $z(x) = V_{nd}(x) - x^T Sx = x^T(P_s - S)x - 2x^T P_s s_s + s_s^T S s_s$. As $P_s - S$ is positive definite, the infimum is attained at a bounded x. The Karush-Kuhn-Tucker conditions state that there exists a multiplier λ, $0 \leq \lambda \perp Kx - 1 \geq 0$ such that

$$2(x-s_s)^T P_s - 2x^T S = \lambda K$$

For $\lambda = 0$, we find $x = -(P_s - S)^{-1}P_s s_s$, such that $x - s = -(P_s - S)^{-1}(2P_s - S)s$. Then, $z(x) = s^T Ds$, where $D = (2P_s - S)(P_s - S)^{-1}P_s(P_s - S)^{-1}(2P_s - S) - S$. We find

$$\begin{aligned}
D =& (2P_s - S)(P_s - S)^{-1}(2P_s - S) \\
&+ (2P_s - S)(P_s - S)^{-1}S(P_s - S)^{-1}(2P_s - S) - S
\end{aligned}$$

and, as $(P_s - S)^{-1}(2P_s - S) = I + (P_s - S)^{-1}P_s$ and $P_s - S \succ 0$,

$$D \succ 2S(P_s - S)^{-1}P_s + P_s(P_s - S)^{-1}S(P_s - S)^{-1}P_s$$

such that $S \succ 0$ and, $2S(P_s - S)^{-1}P_s = 2S + 2S(P_s - S)^{-1}S \succ 0$ imply $D \succ 0$. Hence, $z(x) > 0$ when the optimiser of (50) satisfies $Kx > 1$. When this optimiser satisfies $Kx = 1$, then we directly observe $z(x) = x^T P_0 x - x^T S x > 0$. Hence, we have proven $V_{nd}(x) - x^T S x \geq 0$ for all x where $Kx \geq -1$. By symmetry of V_{nd}, the global result is attained.

Observe that the inequality (48) is strict, such that one can always choose a positive definite matrix $Q \in \mathbb{R}^{n\times n}$, with sufficiently small eigenvalues, such that

$$\begin{pmatrix} (-\epsilon + \tau\delta_1)S & \frac{1}{2}P_3(A + \alpha BK) & 0 \\ \frac{1}{2}(A + \alpha BK)^T P_3 & \Psi & 0 \\ 0 & 0 & -Q \end{pmatrix} \prec 0 \qquad (51)$$

holds for $\alpha = 0$ and $\alpha = 1$.

Now, let V_{nd} and $\dot{V}_{nd}$ be given in (4) and (12), with Q chosen to satisfy the condition above. Assuming the matrix inequalities in the theorem hold, we will show that V in (36) is a Lyapunov-Krasovskii functional that proves convergence of these trajectories to the origin. V is positive definite as $R_1, R_2, Q \succ 0$ imply that V is positive definite if V_{nd} is positive definite, which follows from $V_{nd}(x) \geq x^T S x$.

Restricting our attention to the sublevelset $\{x_\tau \in AC[-\tau, 0] |\ V(x_\tau) \leq \gamma_\epsilon\}$, we observe that the decrease condition $\frac{dV}{dt} \leq -\delta_1 x(t) S x(t)$ directly follows from (47) and (51).

Since, in addition, V is a positive definite functional, in the set $\{x_\tau \in AC[-\tau, 0] |\ V(x_\tau) \leq \gamma_\epsilon\}$, the Lyapunov-Krasovskii functional decreases for every time interval of length τ, which implies that V converges to zero and, consequently x converges to the origin. □

By the requirement that $x_\tau(s)$ is absolutely continuous, we directly infer that $\dot{x}(s)$ is integrable. We note that for $\tau \to 0^+$, the mentioned matrix inequalities can always be satisfied. Hence, when τ is sufficiently small, the minimal ϵ such that (49) is satisfied can be found with a line search in ϵ. This minimal ϵ then leads to a basin of attraction estimate with γ_ϵ in (14).

The conditions (48), (49) are nonlinear matrix inequality constraints. By setting $R_2 = P_3$ and $\delta_1 = (1 - \delta_2)\frac{\epsilon}{\tau}$, with fixed and small $\delta_2 > 0$, at the price of increased conservatism, one attains linear matrix inequalities from the matrix inequalities in the theorem. We note that this step might restrict the set of time delays τ for which the theorem proves attraction to the origin.

Remark 1. The basin of attraction estimate in this theorem is a sublevelset in the space of initial functions. Note that for the subset of constant initial functions, the value x_0 of these function has to lie within $\{x_0 |\ V_{nd}(x_0) \leq \gamma_\epsilon\}$ for the initial function to be contained in the basin of attraction. Alternatively, in various control applications, it is reasonable to assume that $u = 0$ for $t \in [0, \tau]$, such that $x_\tau(s) = e^{As}x_0,\ s \in [0, \tau]$ is a natural initial trajectory, which allows to link a sublevelset of V at time τ to an initial state x_0.

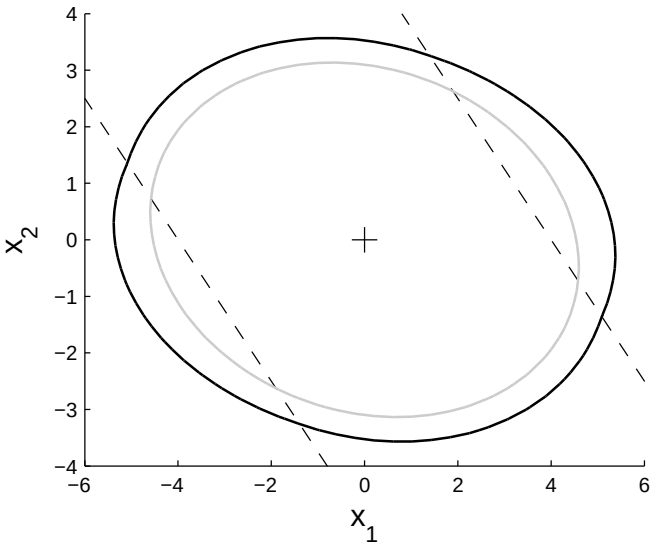

Fig. 1. Interior of black solid curve: sublevelset $V_{nd}^{-1}[0, \gamma_\epsilon]$ with $\gamma_\epsilon = 133.78$ V_{nd} in (4), and P_0, s_s and P_s as in Section 5. Interior of gray solid curve: sublevelset of the quadratic function, i.e. ${V_{nd}^q}^{-1}[0, \gamma_\epsilon^q]$ with $\gamma_\epsilon^q = 103.31$ and V_{nd} in (4). Dashed lines represent the set $\{x \in \mathbb{R}^2 |\ Kx = \pm 1\}$.

5. EXAMPLE

We will illustrate our results with a two-dimensional example. Let $A = \begin{pmatrix} 0 & 1 \\ -0.2 & 0.05 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $K = (-0.25\ \ -0.2)$. Selecting $\epsilon = 0.05$ and $P_0 = \begin{pmatrix} 5.00 & 1.11 \\ 1.11 & 10.74 \end{pmatrix}$, we observe that Assumption 1 is satisfied. For $s_s = (0.65\ \ -0.05)^T$, (6) leads to $P_s = \begin{pmatrix} 3.71 & 0.59 \\ 0.59 & 10.74 \end{pmatrix}$. Using the procedure presented in Section 3.1, we find $\gamma_\epsilon = 133.78$. Hence, if $\tau = 0$, we find the set $\{x \in \mathbb{R}^2 |\ V_{nd}(x) \leq \gamma_\epsilon\}$ as an estimate of the basin of attraction. This set is depicted with the solid black line in Figure 1. For comparison, in gray, we depict the set attained with a quadratic Lyapunov function, i.e. the set $\{x \in \mathbb{R}^n |\ V_{nd}^q(x) \leq \gamma_\epsilon^q\}$, with $\gamma_\epsilon^q = \sup\{\gamma |\ V_{nd}^q \leq \gamma \Rightarrow \dot{V}_{nd}^q(x) \leq -\epsilon V_{nd}^q(x)\} = 103.31$ and $V_{nd}^q(x) = x^T P_0 x$. Clearly, the extra design freedom introduced by the piecewise quadratic nature of V_{nd} allows to attain a larger basin of attraction estimate.

To find an estimate for the allowed time delay, we set $R_2 = P_3$ and $\delta_1 = 0.99\frac{\epsilon}{\tau}$. Fixing τ, the inequalities in Theorem 5 lead to linear matrix inequalities in the variables Q, S, R_1 and P_3. For $\tau = 0.015$, using an LMI solver, we observe that these conditions are satisfied for $S = \begin{pmatrix} 1.83 & 0.31 \\ 0.31 & 5.30 \end{pmatrix}$, $Q = 10^{-4}\begin{pmatrix} 0.32 & -0.09 \\ -0.09 & 0.91 \end{pmatrix}$, $R_1 = \begin{pmatrix} 6.64 & 0.26 \\ 0.26 & 3.29 \end{pmatrix}$, $P_3 = 10^{-2}\begin{pmatrix} 0.49 & 0.17 \\ 0.17 & 0.92 \end{pmatrix}$. Hence, from Theorem 5, we conclude that for $\tau = 0.015$, all trajectories from initial conditions in $\{x_\tau \in AC[-\tau, 0] |\ V(x_\tau) \leq \gamma_\epsilon\}$ are attracted towards the origin.

To illustrate this result, we restrict our attention to constant functions x_τ. In this case, $V(x_\tau) = V_{nd}(x_\tau(0))$. Hence, the set $\{x_\tau \in AC[-\tau, 0] |\ V(x_\tau) \leq \gamma_\epsilon\}$ contains all constant functions with $x_\tau(0)$ inside the black curve of Figure 1.

6. CONCLUSION

A method has been presented that provides an estimate of the basin of attraction of linear systems controlled by a single saturating controller with delay. A novel piecewise quadratic Lyapunov-Krasovskii functional is introduced which exploits the piecewise affine nature of the retarded delay differential equation that describes the closed-loop system. Given a fixed value of the time-delay, conditions have been presented that guarantee that trajectories from a sublevelset of the Lyapunov-Krasovskii functional converge to the origin. These results are illustrated with an exemplary system.

REFERENCES

Boyd, S. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press, Cambridge.

Cao, Y.Y., Lin, Z., and Hu, T. (2002). Stability analysis of linear time-delay systems subject to input saturation. *IEEE Transactions on Circuits and Systems I*, 49(2), 233–240.

Dai, D., Hu, T., Teel, A.R., and Zaccarian, L. (2009). Piecewise-quadratic Lyapunov functions for systems with deadzones or saturations. *Systems & Control Letters*, 58(5), 365–371.

Fridman, E. (2002). Effects of small delays on stability of singularly perturbed systems. *Automatica*, 38(5), 897–902.

Fridman, E., Pila, A., and Shaked, U. (2003). Regional stabilization and H_∞ control of time-delay systems with saturating actuators. *International Journal of Robust and Nonlinear Control*, 13, 885–907.

Giesl, P. (2008). Construction of a local and global Lyapunov function using radial basis functions. *IMA Journal of Applied Mathematics*, 73(5), 782–802.

Gomes da Silva Jr, J.M., Seuret, A., Fridman, E., and Richard, J.P. (2011). Stabilisation of neutral systems with saturating control inputs. *International Journal of Systems Science*, 42(7), 1093–1103.

Grimm, G., Hatfield, J., Postlethwaite, I., Teel, A.R., Turner, M.C., and Zaccarian, L. (2003). Antiwindup for stable linear systems with input saturation: an LMI-based approach. *IEEE Transactions on Automatic Control*, 48(9), 1509–1525.

Hu, T., Teel, A.R., and Zaccarian, L. (2006). Stability and performance for saturated systems via quadratic and nonquadratic Lyapunov functions. *IEEE Transactions on Automatic Control*, 51(11), 1770–1786.

Insperger, T. and Stépán, G. (2011). *Semi-discretization for time-delay systems*. Springer-Verlag, New York.

Johansson, M. (2002). Piecewise quadratic estimates of domains of attraction for linear control systems with saturation. In *Proceedings of the 15th Triennial World Congress, Barcelona, Spain*.

Johansson, M. and Rantzer, A. (1998). Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4), 555–559.

Kharitonov, V.L. (2013). *Time-delay systems*. Springer, New York.

Michiels, W. and Niculescu, S.I. (2007). *Stability and stabilization of time-delay systems: an eigenvalue approach*, volume 12 of *Advances in Design and Control*. SIAM.

Seuret, A., Edwards, C., Spurgeon, S.K., and Fridman, E. (2009). Static output feedback sliding mode control design via an artificial stabilizing delay. *IEEE Transactions on Automatic Control*, 54(2), 256–265.

Tarbouriech, S. and Gomes da Silva Jr, J.M. (2000). Synthesis of controllers for continuous-time delay systems with saturating controls via lmi's. *IEEE Transactions on Automatic Control*, 45(1), 105–111.

Tarbouriech, S., Garcia, G., Gomes da Silva Jr., J.M., and Queinnec, I. (2011). *Stability and stabilization of linear systems with saturating actuators*. Springer-Verlag, London.