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Positroid Stratification, Tropicalization and Toric Degenerations of Grassmannians

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Abstract

Grassmannians are algebraic varieties parametrizing the family of all subspaces of given dimension of a vector space. Flag varieties generalize Grassmannians by parametrizing sequences of nested subspaces of a vector space. They can be realized as projective algebraic varieties and appear in many areas of mathematics and science.

The first aspect of Grassmannians we study is their stratification via positroids. This stratification lies in between two other well-studied stratifications of Grassmannians, labeled by Schubert cells and matroids. Positroids are a family of matroids, introduced by Postnikov in the study of non-negative Grassmannians, with a rich combinatorial structure. In particular, given a matrix M representing a point in the Grassmannian, positroids encode and are uniquely determined by the rank of each submatrix obtained from M by selecting cyclically consecutive columns. We study and characterize the minimal family of rank conditions on cyclic intervals that uniquely defines a positroid. To this end, we introduce essential sets of positroids, which generalize Fulton's essential sets for permutations to the class of bounded affine permutations. We show that the minimal rank conditions defining a positroid on n elements are less than n and give an algorithm to recover the positroid from such rank conditions.

The second tool we use in the study of Grassmannians and flag varieties are toric degenerations. Toric varieties are well studied algebraic varieties that can be combinatorially described by polytopes and cones. This creates a dictionary between toric varieties and combinatorial objects that can be extended to more general algebraic varieties that admit toric degenerations. A classical way of constructing toric degenerations is via Gröbner degenerations with respect to points in maximal prime cones in the tropicalization. However, obtaining an explicit description of the tropicalization of a variety is computationally hard. To avoid this obstacle, we use matching fields to directly construct a large family of points in maximal prime cones of the tropicalization of Grassmannians

and partial flag varieties. A matching field encodes the combinatorial data of a polytope as well as of a monomial degeneration of Plücker coordinates. Our result relies on the fact that the polytopes of the considered matching fields are combinatorial mutation equivalent. These polytopes form a class of Newton–Okounkov bodies for particular full-rank valuations for partial flag varieties.

Another class of Newton–Okounkov bodies for full flag varieties is given by marked chain-order polytopes associated to square Young diagrams. We explore the relations between chain-order polytopes and prove that, for a fixed Young diagram, all chain-order polytopes are connected through a sequence of combinatorial mutations. Moreover, we give a way of restricting these polytopes via hyperplane intersections which gives rise to a large class of rational polytopes presenting period collapse.

The third and last aspect of Grassmannians we are interested in studying is their tropicalization. In particular, we study toric degenerations arising from prime maximal cones in the tropicalization labeled by matching fields. Computationally, we show that for small Grassmannians and flag varieties this description recovers many of the cones we are interested in.

Beknopte samenvatting

Grassmann-variëteiten zijn algebraïsche variëteiten die de familie van alle deelruimten van een gegeven dimensie van een vectorruimte parametriseren. Vlagvariëteiten veralgemenen Grassmann-variëteiten door reeksen van geneste deelruimten van een vectorruimte te parametriseren. Ze kunnen gerealiseerd worden als projectieve algebraïsche variëteiten en komen voor in veel gebieden van de wiskunde en wetenschap.

Het eerste aspect van Grassmann-variëteiten dat we bestuderen, is hun stratificatie via positroïden. Deze stratificatie ligt tussen twee andere goed bestudeerde stratificaties van Grassmann-variëteiten in, aangeduid door Schubert-cellen en matroïden. positroïden zijn een familie van matroïden, geïntroduceerd door Postnikov bij de studie van niet-negatieve Grassmann-variëteiten, met een rijke combinatorische structuur. In het bijzonder, gegeven een matrix M die een punt in de Grassmann-variëteit vertegenwoordigt, coderen positroïden de rang van elke deelmatrix verkregen uit M door cyclisch opeenvolgende kolommen te selecteren, en ze worden hier volledig door bepaald. We bestuderen en karakteriseren de minimale familie van rangvoorwaarden op cyclische intervallen die een positroïde uniek definiëren. Daartoe introduceren we essentiële verzamelingen van positroïden, die Fultons essentiële verzamelingen voor permutaties generaliseren naar de klasse van begrensde affiene permutaties. We tonen aan dat de minimale rangvoorwaarden die een positroïde op n elementen definiëren minder dan n zijn en geven een algoritme om de positroïde uit dergelijke rangvoorwaarden te reconstrueren.

Het tweede hulpmiddel dat we gebruiken bij de studie van Grassmann-variëteiten en vlagvariëteiten zijn torische degeneraties. Torische variëteiten zijn goed bestudeerde algebraïsche variëteiten die combinatorisch kunnen beschreven worden door polytopen en kegels. Dit creëert een woordenboek tussen torische variëteiten en combinatorische objecten dat kan worden uitgebreid naar meer algemene algebraïsche variëteiten die torische degeneraties toelaten. Een klassieke manier om torische degeneraties te construeren is via Gröbner-

degeneraties met betrekking tot punten in maximale priemkegels in de tropicalisatie. Echter, het verkrijgen van een expliciete beschrijving van de tropicalisatie van een variëteit is computationeel moeilijk. Om dit obstakel te vermijden, gebruiken we matching fields om direct een grote familie van punten in maximale priemkegels van de tropicalisatie van Grassmann-variëteiten en gedeeltelijke vlagvariëteiten te construeren. Een matching field codeert de combinatorische data van een polytoop en van een monomiale degeneratie van Plücker-coördinaten. Ons resultaat is gebaseerd op het feit dat de polytopen van de beschouwde matching fields combinatorisch mutatie-equivalent zijn. Deze polytopen vormen een klasse van Newton–Okounkov-lichamen voor pecifieke volle-rang waarderingen voor gedeeltelijke vlagvariëteiten.

Een andere klasse van Newton–Okounkov-lichamen voor volledige vlagvariëteiten wordt gegeven door gemarkeerde keten-orde-polytopen geassocieerd met vierkante Young-diagrammen. We verkennen de relaties tussen keten-orde-polytopen en bewijzen dat, voor een vast Young-diagram, alle keten-orde-polytopen verbonden zijn door een reeks combinatorische mutaties. Bovendien geven we een manier om deze polytopen te beperken met behulp van doorsnedes van hyperoppervlakken, wat aanleiding geeft tot een grote klasse van rationale polytopen die periode-collaps presenteren.

Het derde en laatste aspect van Grassmann-variëteiten dat we willen bestuderen is hun tropicalisatie. In het bijzonder bestuderen we torische degeneraties die voortkomen uit priem maximale kegels in de tropicalisatie, gelabeld door matchingsvelden. Computationeel tonen we aan dat voor kleine Grassmann-variëteiten en vlagvariëteiten deze beschrijving veel van de kegels herstelt waarin we geïnteresseerd zijn.

List of Symbols

$[n]$	The finite set $\{1, \ldots, n\}$
$\mathbb{N}$	The non-negative integers $\mathbb{Z}_{\geq 0}$
$\binom{E}{k}$	The family of cardinality k subsets of a finite set E
$<_i$	Cyclic order on $[n]$ where i is the smallest element
$\mathrm{Gr}(k, n)$	The Grassmannian of k-subspaces of a n-dimensional vector space
$\mathcal{F}\ell(K; n)$	The partial flag variety of subspace of a n-dimensional vector space, whose dimension lies in K
p_I	Plücker coordinate of $\mathrm{Gr}(k, n)$ labeled by $I \in \binom{[n]}{k}$
$\mathcal{M}$	A matroid
$\mathcal{P}$	A positroid
$\mathcal{B}$	The set of bases of a matroid
$\mathcal{C}$	The set of circuits of a matroid
$\mathcal{F}$	The set of flats of a matroid
$\mathcal{E}$	The set of essential sets of a matroid
$\mathcal{E}^r$	The ranked essential family of a matroid
$\mathcal{U}_{k,n}$	The uniform matroid of rank k on n elements
$\Pi_{\mathcal{P}}^{\circ}$	The open positroid variety of $\mathcal{P}$
$\Pi_{\mathcal{P}}$	The positroid variety of $\mathcal{P}$
$\mathcal{S}_{\mathcal{P}}$	The positroid cell of $\mathcal{P}$ in the totally non-negative Grassmannian
$\mathrm{in}_w(f)$	The initial form of f with respect to w

$\mathrm{in}_w(I)$	The initial ideal of I with respect to w
Y_P	The toric variety defined by a normal, full-dimensional, lattice polytope P
E_P	The Ehrhart series of a polytope P
L_P	The Ehrhart quasi-polynomial of a rational polytope P
ν	A valuation
ν_M	The (quasi)-valuation defined by the weight matrix M
(A, ν)	A graded algebra A equipped with a valuation ν
$C(A, \nu)$	The Newton–Okounkov cone of (A, ν)
$\Delta(A, \nu)$	The Newton–Okounkov body of (A, ν)
Λ	A matching field
B^σ	The matching field corresponding to the permutation σ
Δ_d	The d-dimensional simplex
P_σ^K	The matching field polytope for the partial flag variety $\mathcal{F}\ell(K; n)$ defined by B^σ
$\mathcal{O}(\lambda)$	The order polytope of the Young diagram λ
$\mathcal{C}(\lambda)$	The chain polytope of the Young diagram λ
$\mathcal{O}_C(\lambda)$	The chain-order polytope of the Young diagram λ with respect to the up-set C

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Chapter 1

Introduction

The Grassmannian (k, n) over the field $\mathbb{K}$, $\mathrm{Gr}_{\mathbb{K}}(k, n)$, is the space of k-dimensional vector subspaces of $\mathbb{K}^n$. For $k = 1$, the Grassmannian is equal to the projective space $\mathbb{P}^{n-1}_{\mathbb{K}}$. In general, it is a smooth projective algebraic variety. Grassmannians, named in honor of polymath Hermann Grassmann, have been object of extensive study both for their combinatorial properties and for their applications in problems coming from different fields, such as particle physics [8, 7], systems of partial differential equations [23, 71], computer vision [113], and statistics [38].

A generalization of Grassmannians is given by flag varieties, whose elements are increasing sequences of subspaces in $\mathbb{K}^n$ called flags. The complete flag variety in $\mathbb{K}^n$ is the space of flags of vectors subspaces of $\mathbb{K}^n$ of each dimension, that is, the space of families $\{V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_{n-1} \subsetneq \mathbb{K}^n \mid \dim_{\mathbb{K}}(V_i) = i\}$, where each V_i is a vector subspace of $\mathbb{K}^n$. Flag varieties embed into products of Grassmannians, $\mathrm{Gr}_{\mathbb{K}}(1, n) \times \mathrm{Gr}_{\mathbb{K}}(2, n) \times \cdots \times \mathrm{Gr}_{\mathbb{K}}(n-1, n)$, and they are smooth projective algebraic varieties as well. When only vector subspaces of specific dimensions are considered, we obtain partial flag varieties.

In this thesis, we study Grassmannians and flag varieties using tools from combinatorial algebraic geometry. In particular, we study positroids, objects defining one of the stratifications of Grassmannians, their toric degenerations and their tropicalizations.

1.1 Positroid Stratification of Grassmannians

Points in the Grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,n)$ can be represented by $k \times n$ matrices with entries in $\mathbb{K}$, up to row operations. Given a point in the Grassmannian, the linear independence behavior of the columns of all representative matrices is identical. This information is encoded in the corresponding matroid.

Matroids, introduced in 1935 by Whitney [116], are objects that generalize the concept of linear independence in linear algebra and similar notions in graph theory, algebraic field extensions, etc. They can be axiomatically defined from many points of view, such as independent sets, bases, circuits and flats. In particular, it is possible to define the bases of a matroid as a non-empty family of sets satisfying the so-called basis exchange property.

Representable matroids. The matroid of a point V in $\mathrm{Gr}_{\mathbb{K}}(k,n)$ is the family of subsets of the columns of a matrix representing V that form a basis of $\mathbb{K}^k$. However, not every matroid as in the definition by Whitney can be obtained in this way from a matrix. In this case we say that the matroid is not representable. In fact, almost all matroids are non-representable [93]. Moreover, it is not possible to extend the axioms of matroids to only include representable matroids, [114, 82].

Totally non-negative Grassmannian and positroids. It is however possible to restrict to a subfamily of representable matroids, by adding a positivity constraint. The concept of positivity was introduced by Lusztig [78, 79, 80] in the more general setting of flag varieties. In a preprint from 2006 [102], Postnikov specialized the definition to Grassmannians and introduced totally non-negative Grassmannians and positroids together with many combinatorial objects that describe them. The totally non-negative Grassmannian is the set of points in the real Grassmannian that can be represented by a matrix with non-negative maximal minors, or equivalently, whose Plücker coordinates all have the same sign. Positroids are matroids that can be represented by points in the totally non-negative Grassmannian.

Combinatorial properties of positroids. There are many combinatorial objects in bijection with positroids. Examples include planar bipartite graphs, Grassmann necklaces and bounded affine permutations. Furthermore, positroids are uniquely defined by the values of their rank function on cyclic intervals. It is therefore natural to ask the following questions.

Question 1. What is the minimal set of rank conditions on cyclic intervals that uniquely defines a positroid?

Question 2. Is it possible to axiomatically define positroids through such rank conditions on cyclic intervals?

Positroid cells and positroid stratification. Positroids behave better than general matroids also from a geometric point of view. Indeed, realization spaces of matroids can have almost any singularity [86]. On the other hand, open positroid varieties, spaces of matrices satisfying the rank conditions on cyclic intervals specified by the positroid, are smooth [70]. Moreover, positroid varieties are irreducible, while a characterization of matroids whose variety is irreducible is not known. For families of matroids whose variety is irreducible, see [34, 25, 29, 75]. The open positroid varieties form a stratification of the Grassmannian, called the positroid stratification. This is a coarser stratification than the one defined by matroid strata, and a finer stratification than the one given by Schubert cells. In particular, an open positroid variety is the intersection of n cyclically shifted Schubert cells. The realization spaces of positroids inside the totally non-negative Grassmannian are homeomorphic to open balls and glue together to form a CW decomposition of the totally non-negative Grassmannian [102, 103].

Question 3. How can we compute the dimension of a positroid cell, given its rank on cyclic intervals?

The amplituhedron. One important application of positroids and positroid cells is in the study of the amplituhedron. The amplituhedron is a geometric object introduced in 2013 by Arkani-Hamed and Trnka [8] in the context of the study of scattering amplitudes in the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. This object is conjectured to encode scattering amplitudes, quantities related to probabilities of the outcome of particle collisions, in its "canonical form". From a mathematical point of view, the amplituhedron $\mathcal{A}_{n,k,m}$ is defined as the image, under an $n \times (k+m)$ matrix with positive maximal minors, of the totally non-negative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, inside $\mathrm{Gr}(k,k+m)$. The relation between the amplituhedron and positroid cells has been object of extensive study. When m is equal to 2 or 4, it is possible to "tile" the amplituhedron via images of positroid cells [101, 46, 45]. For $m = 2$, it has been proved that every such tiling via images of positroid cells has the same number of tiles [100]. Moreover, it is possible to embed the amplituhedron in a larger Grassmannian, which in this context is referred to as twistor space, and describe the face stratification of the $m = 2$ amplituhedron in terms of positroid cells in twistor space [76, 74].

Tropical positive Grassmannian. Two of the tools that are used in the study of positroid tilings of the amplituhedron are tropical geometry and cluster algebras. Tropicalization gives a way of constructing a combinatorial version of varieties and semialgebraic sets that encodes properties of the

original objects. The tropical totally positive Grassmannian, introduced by Speyer and Williams [108], is the tropicalization of the semialgebraic subset of the Grassmannian where all the Plücker coordinates are positive. Similarly, the tropical version of the amplituhedron has been recently introduced by Akhmedova and Tessler [1]. The positive tropical Grassmannian, which equals the positive Dressian [109], parametrizes regular positroid subdivisions of the hypersimplex [77], which are conjectured to be in bijection with positroid tilings of the $m = 2$ amplituhedron. This statement was proved for the class of BCFW tilings [77], a class of positroid tilings relevant to physicists.

Cluster algebras and positroids. The $m = 4$ amplituhedron can also be tiled via images of positroid cells [46, 45]. In this case, BCFW tiles of the amplituhedron are proved to be the positive part of some cluster varieties. Moreover, cluster variables describe the shared facet of two adjacent tiles. More generally, the coordinate ring of an open positroid variety is a cluster algebra [52]. Combinatorially, the cluster algebra is defined by a quiver naturally associated with the plabic graph describing the considered positroid.

Question 4. What are the minimal rank conditions on cyclic intervals defining the positroids that describe the face stratification of the $m = 2$ amplituhedron?

1.1.1 Our Contribution

We answer questions 1, 2, 3 and 4 by introducing and axiomatically defining the essential sets of a positroid. Essential sets are maximally dependent cyclic intervals that give a family of rank conditions sufficient to uniquely define the positroid. Combinatorially they can be obtained as the corners of the diagram of the bounded affine permutation associated to the positroid, generalizing Fulton's construction. Essential sets were introduced by Fulton [51] as objects associated to permutations, to study the degeneracy loci of maps of flagged vector bundles. Their combinatorial characterization and properties were further investigated in [43, 42]. The introduction of essential sets within the context of positroids is due to Knutson [68, 69], who used them in Fulton's original definition, for the subclass of interval rank positroids. The latter are defined by rank conditions on intervals rather than cyclic intervals.

An axiomatic description of essential sets of positroids is given in Theorem 3.2.3, answering Question 2. Furthermore, we prove that the positroid is uniquely defined by providing an explicit formula to compute the rank of any cyclic interval given the rank of essential sets in Theorem 3.1.9.

Computational results show that for a bounded affine permutation on n elements, the average size of the family of its essential sets is of approximately $n^2/30$.

This is because not all the rank conditions given by essential sets are necessary. The minimal rank conditions on cyclic intervals uniquely defining a positroid are contained in the *core*, Corollary 3.1.25. These are essential sets with positive "excess" and there are at most n such conditions. This provides a complete answer to Question 1. We prove that the rank conditions contained in the core are necessary and sufficient by giving an algorithm that takes the core rank conditions as input and outputs the bounded affine permutation of the positroid they define, Algorithm 2.

From a geometric point of view, essential sets define the positroid variety as a scheme. We also give an explicit formula to compute the codimension of a positroid cell in the Grassmannian via its core, answering Question 3.

Finally, we study the behavior of essential sets after taking the dual positroids and the relation with Grassmann necklaces other combinatorial objects in bijection with positroids. This allows us to define twistor positroids from the point of view of essential sets. Since these are the positroids involved in the face stratification of the $m = 2$ amplituhedron [74], this answers Question 4.

1.2 Toric Degenerations of Grassmannians and Flag Varieties

A toric degeneration of an algebraic variety X is a flat family of varieties over $\mathbb{A}^1$ such that the generic fiber is isomorphic to X and the fiber over 0 is a toric variety. By abuse of language, we will refer to the fiber over 0 as the toric degeneration of X. The toric variety shares many geometric invariants with the original variety X, such as the Hilbert series. Since toric varieties can be studied using combinatorial tools [37], this give rise to easier ways of studying invariants of the original variety. For example, the Hilbert series of a projective normal toric variety coincides with the Ehrhart series of the corresponding polytope.

Besides the above-mentioned applications from algebraic geometry, toric degenerations have found applications in different areas of mathematics. Some examples come from representation theory [72, 47], mirror symmetry [57, 10, 9, 94], cluster algebras [105, 19] and algebraic statistics [13]. Moreover, many combinatorial methods have been used to give rise to toric degenerations of Grassmannians, such as matching fields [90, 30, 27] and planar bipartite graphs corresponding to uniform positroids [105, 18].

Gröbner degenerations and tropicalization. A classical way of constructing toric degenerations is via Gröbner degenerations. Given an ideal I and a weight w, if the initial ideal with respect to w is toric, i.e. prime and binomial, there

is a flat family whose generic fiber is isomorphic to $\mathcal{V}(I)$ and whose fiber over 0 is the toric variety defined by $\mathrm{in}_w(I)$. The points in the maximal cones of the tropicalization of X, with respect to the trivial valuation, are good candidates for the weight vector w with respect to which the initial ideal becomes toric [81]. Indeed, as a set, the tropicalization of an ideal in a Laurent polynomial ring can be seen as the set of weights with respect to which the initial ideal is proper, i.e. does not contain monomials.

Tropical Grassmannian. Tropical Grassmannians have been extensively studied in the literature. The cones in the tropicalization can be explicitly computed for certain rather small classes of Grassmannians. Unfortunately, for k larger than 3 or n larger than 8 it is not feasible to explicitly compute all the cones in the tropicalization. For example, $\mathrm{trop}(\mathrm{Gr}(3,7))$ has 125 maximal cones in its fan up to symmetry [61] and $\mathrm{trop}(\mathrm{Gr}(3,8))$ has 14763 maximal cones up to symmetry [12]. Therefore, it is not possible to rely on explicit computations of the tropical variety to find the candidate weights.

Matching fields. Matching fields offer a combinatorial tool to find points in prime maximal cones in the tropicalization of Grassmannians, without explicit computations of all the cones. Indeed, they encode the data of a weight vector, as well as of a monomial map from the polynomial ring of Plücker coordinates. The kernel of such a monomial map is a toric variety, and it coincides with the initial ideal with respect to the corresponding weight whenever the latter is prime. Matching fields were first introduced by Sturmfels and Zelevinsky [112] in the study of Newton polytopes of products of maximal minors. Toric degenerations arising from matching fields have been introduced and studied for various kinds of homogeneous spaces, such as Grassmannians [26, 27], Schubert varieties [31] and full flag varieties [32]. The monomial ideals arising from the class of block diagonal matching fields can be realized as monomial Gröbner degenerations of determinantal ideals [33, 87]. These ideals, along with all of their powers, possess linear resolutions. Their minimal free resolutions are supported on a regular CW complex [39].

Question 5. How can we construct matching fields that give rise to toric degenerations of Grassmannians and partial flag varieties?

Newton–Okounkov bodies. A second approach to the construction of toric degenerations of a variety is through valuations and Newton–Okounkov bodies. Newton–Okounkov bodies generalize the Newton polytope of a polynomial and the polytope associated to a toric variety. In general, they are convex bodies that are not necessarily rational or even polyhedral. However, if a Newton–Okounkov body of a variety with respect to some valuation is a rational polytope, it is possible to find a toric degeneration of the original variety to a (not necessarily

normal) toric variety whose normalization is the toric variety associated to the Newton–Okounkov polytope [4].

The study of the generalization of Newton polytopes was started by Okounkov [96, 97] and formalized by Kaveh–Khovanskii [66]. The introduction of Newton–Okounkov bodies in the context of toric degenerations is due to Anderson [4] and Kaveh–Manon [67]. The latter proved that it is possible to construct weight valuations from prime cones in the tropicalization of the variety whose Newton–Okounkov body is a rational polytope. The problem of finding points in maximal prime cones in the tropicalization can therefore be rephrased as follows.

Question 6. How can we construct Newton–Okounkov polytopes of Grassmannians and partial flag varieties?

Wall-crossing in the tropicalization. Given a variety X, toric degenerations of X arising from different prime cones in the tropicalization of X are, in general, not isomorphic. However, they do share the same geometric invariants as X. The relation between Newton–Okounkov bodies arising from maximal prime cones in the tropicalization of X has been studied by Escobar–Harada [44]. In particular, Newton–Okounkov bodies arising from adjacent maximal prime cones in the tropicalization differ by a piecewise linear map. This operation is referred to as wall-crossing. In an appendix by Ilten in [44] it is shown that these maps are combinatorial mutations.

Combinatorial mutations. Combinatorial mutations are piecewise linear maps between polytopes that were first introduced in the study of the classification of Fano varieties [2]. Like toric degenerations, they preserve the Ehrhart series of a polytope. When the combinatorial mutation divides the space into two regions of linearity, they can be seen as Fock–Goncharov tropicalizations [60, Definition 1.22] of cluster mutations [21, Example 4.65].

Cluster algebras and mutations. Cluster mutations are the basic operations in the construction of cluster algebras. There are a class of commutative ring introduced by Fomin and Zelevinsky [48, 49]. The coordinate ring of Grassmannians provide a classical example of cluster algebras [107]. In particular, for $\mathrm{Gr}_{\mathbb{C}}(2,n)$, every "cluster" describes a triangulation of the n-gon. Generalizing this idea, mutations of triangulation of the punctured ∞-gon were studied by the author in joint work with Fatemeh Mohammadi and Job Daisie Rock [89]. These give rise to infinite cluster categories of type $\mathbb{D}$. The previous work arises as a generalization of work from Schiffler [106] for cluster categories of type $\mathbb{D}_n$, and work by Holm and Jørgensen [64] on the cluster category of the ∞-gon.

Question 7. How can we use the relation between toric degenerations via combinatorial mutations to study the tropicalization of Grassmannians and flag varieties?

Non-prime cones. In order to get a complete understanding of the tropicalization of a variety, it is also necessary to understand the relations and the combinatorics of maximal non-prime cones. Note that all the constructions considered above require us to consider prime cones in the tropicalization. However, it is still possible to construct a Newton–Okounkov body from non-prime cones. Non-prime cones have recently been proved to be of interest in the context of scattering amplitudes [17]. Moreover, their understanding would be fundamental in order to solve the universal SAGBI basis conjecture for Grassmannians [22].

Question 8. How are the Newton–Okounkov bodies of two adjacent maximal cones in the tropicalization related when one of them is not prime?

1.2.1 Our Contribution

We provide an answer to questions 5, 6, 7 and 8 by considering a large family of matching fields labeled by permutations avoiding specific patterns. We prove that the associated polytopes are normal and combinatorial mutation equivalent. As a corollary, Corollary 5.1.5, we obtain a large family of matching fields that give rise to toric degenerations of partial flag varieties, answering Question 5 and Question 7.

Computationally we show that, by considering a slightly larger class of matching fields, we obtain all toric degenerations of $\mathrm{Gr}(3,6)$ arising from matching fields. The tropicalization of $\mathrm{Gr}(3,7)$ contains 69 maximal prime cones, 40 of which can be described via the considered matching fields.

The polytopes of the considered matching fields are examples of Newton–Okounkov bodies, giving an answer to Question 6. Another large family of Newton–Okounkov bodies for the full flag varieties is given by marked chain-order polytopes. We show that all chain-order polytopes of a given poset are combinatorial mutation equivalent. Moreover, we show that it is possible to intersect chain-order polytopes with hyperplanes and the resulting polytopes are still combinatorial mutation equivalent. These restricted chain-order polytopes for rectangular Young diagrams have the period collapse property, i.e. they are rational polytopes whose Ehrhart function is a polynomial.

Finally, we describe one example of combinatorial mutations through matching field polytopes that crosses a non-prime cone. Question 8 was answered for this

example in [104], by embedding the Grassmannian $\mathrm{Gr}(3,6)$ in a larger projective space, following ideas from [20].

1.3 Structure of the Thesis

The thesis is largely based on the following three collaborative works, where each author had equal contribution.

- "Combinatorics of Essential Sets for Positroids", joint work with Fatemeh Mohammadi, published in the *Journal of Algebra* [92].

  I was involved in the main aspects of the research: formulating the statements and developing the proofs presented in the paper.

  The paper is contained in Chapter 3, Sections 3.1 to 3.4.

- "Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes", joint work with Oliver Clarke and Fatemeh Mohammadi, published in the *Journal of Algebra* [35].

  My main contribution has been performing the computations in *Macaulay2* and *Polymake*.

  This paper has been separated in different chapters, in order to make the text of the thesis more fluent. It is contained in Section 2.3, Section 4.2 and Chapter 5.

- "Restricted Chain-Order Polytopes via Combinatorial Mutations", joint work with Oliver Clarke and Akihiro Higashitani [28].

  In this paper, I identified a sequence of combinatorial mutations between the restricted polytopes that would respect the restriction. Additionally, I worked on the description of the intermediate polytopes obtained from the aforementioned sequence.

  The paper is contained in Section 2.3.2, and in Sections 4.1 and 4.3.

Chapter 2

Preliminaries

In this chapter we will introduce the notions and tools that will be needed to state the results in the second part of the thesis. In Section 2.1 we will introduce the main object of study of this thesis, the Grassmannian. We will then continue by introducing the main ideas needed from matroid theory, in Section 2.2, toric geometry, in Section 2.3, and tropical geometry, in Section 2.4.

This chapter is unpublished work, except for Section 2.3, which is contained in joint work with Oliver Clarke and Fatemeh Mohammadi [35], and Section 2.3.2, which is contained in joint work with Oliver Clarke and Akihiro Higashitani [28].

Notation 2.0.1. Throughout the chapter, fix k, n to be positive integers with $k \leq n$. We denote by $[n]$ the set of n elements $\{1, \dots, n\}$. The family of cardinality k subsets of $[n]$ is denoted by $\binom{[n]}{k}$. To avoid heavy notation, when the context is clear, we will write ijk for the set $\{i, j, k\}$ and similarly for larger or smaller sets.

2.1 Grassmannians and Flag Varieties

Let $\mathbb{K}$ be a field. The *Grassmannian* $\mathrm{Gr}_{\mathbb{K}}(k, n)$ is the space of k-dimensional linear subspaces of $\mathbb{K}^n$. Whenever the field is clear from the context, we will simply write $\mathrm{Gr}(k, n)$. An element $V \in \mathrm{Gr}_{\mathbb{K}}(k, n)$ can be represented by a matrix M such that $V = \mathrm{rowspan}(M)$. A $k \times n$ matrix with entries in $\mathbb{K}$ represents the same element in $\mathrm{Gr}_{\mathbb{K}}(k, n)$ up to row operations. We will denote the element in $\mathrm{Gr}_{\mathbb{K}}(k, n)$ and a matrix representing it with the same notation.

The Grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,n)$ can be embedded in projective space $\mathbb{P}_{\mathbb{K}}^{\binom{n}{k}-1}$ via the *Plücker embedding*, defined as follows:

$$\begin{aligned} \varphi : \mathrm{Gr}_{\mathbb{K}}(k,n) &\to \mathbb{P}_{\mathbb{K}}^{\binom{n}{k}-1} \\ V &\mapsto \left(\det(V[I])\right)_{I\in\binom{[n]}{k}} \end{aligned}$$

where $V[I]$ denotes the square matrix obtained from V by selecting the columns labeled by indices in I. Since the image of any matrix representing a k-dimensional vector space V is the same up to a constant and the vector is non-zero, since the matrix has full-rank, the map is well-defined. The elements in the vector $\left(\det(V[I])\right)_{I\in\binom{[n]}{k}}$ are called *Plücker coordinates* of the space V and are denoted by p_I.

The homogeneous coordinates of a point in $\mathbb{P}_{\mathbb{K}}^{\binom{n}{k}-1}$ are the Plücker coordinates of some element in $\mathrm{Gr}_{\mathbb{K}}(k,n)$ if and only if they satisfy the *Plücker equations*, given, for every $\{i_1 < i_2 < \cdots < i_{k-1}\}$ and $\{j_1 < j_2 < \cdots < j_{k+1}\}$, as follows:

$$\sum_{\ell=1}^{k+1} (-1)^{\ell} p_{i_1,\ldots,i_{k-1},j_\ell} p_{j_1,\ldots,\hat{j_\ell},\ldots,j_{k+1}} = 0,$$

where the notation $\hat{j_\ell}$ means that we are removing the element j_ℓ from the set and $p_{i_1,\ldots,i_{k-1},j_\ell}$ is the Plücker coordinate obtained by taking the determinant of the corresponding matrix with columns in the order $i_1,\ldots,i_{k-1},j_\ell$.

The Grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,n)$ is the subvariety of $\mathbb{P}_{\mathbb{K}}^{\binom{n}{k}-1}$ cut out by the Plücker equations.

Example 2.1.1. Consider the Grassmannian of planes in $\mathbb{C}^n$. Each element can be represented by a $2\times n$ matrix with entries in $\mathbb{C}$:

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ y_1 & y_2 & \cdots & y_{n-1} & y_n \end{pmatrix}.$$

In order for $\mathrm{rowspan}(X) \in \mathrm{Gr}_{\mathbb{C}}(2,n)$, we assume that $\mathrm{rk}(X) = 2$. The Plücker coordinates of X are given by $p_{i,j} = x_i y_j - x_j y_i$ for each $1 \le i < j \le n$. The Plücker equations are given by:

$$p_{i,j}p_{k,h} + p_{i,h}p_{j,k} = p_{i,k}p_{j,h}$$

for each $1 \le i < j < k < h \le n$.

A classic generalization of Grassmannians is given by (partial) flag varieties. Instead of labeling vector subspaces of a given dimension, flag varieties contain

flags of vector subspaces nested into one another. Formally they are defined as follows. Let $K \subseteq [n]$ and assume $K = \{i_1 < i_2 < \cdots < i_r\}$. The *partial flag variety* $\mathcal{F}\ell_{\mathbb{K}}(K;n)$ is the family of flags of vector spaces $\{V_1 \subset V_2 \subset \cdots \subset V_r\}$ where for each $j \in [r]$, $\dim(V_j) = i_j$. If $K = [n]$, then we denote it by $\mathcal{F}\ell_{\mathbb{K}}(n)$ and call it *(complete) flag variety*. If the field is clear from the context, we will avoid the subscript $\mathbb{K}$.

Note that since $\mathcal{F}\ell_{\mathbb{K}}(K;n) \subseteq \mathrm{Gr}_{\mathbb{K}}(i_1,n) \times \mathrm{Gr}_{\mathbb{K}}(i_2,n) \times \cdots \times \mathrm{Gr}_{\mathbb{K}}(i_r,n)$, the flag variety $\mathcal{F}\ell_{\mathbb{K}}(K;n)$ can be embedded in a product of projective spaces.

2.2 Matroid Theory

Matroids are combinatorial objects introduced to generalize the concept of linear independence. A comprehensive introduction to the topic can be found in [98]. In the next few examples, we will see how to construct a matroid from some geometric objects.

Matroid of Vectors

Let $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{K}^k$ be a set of vectors spanning $\mathbb{K}^k$, and let $M \in \mathrm{Mat}_{\mathbb{K}}(k,n)$ be the matrix whose columns are given by vectors v_i. Note that M is a full-rank matrix, hence it describes a point in $\mathrm{Gr}(k,n)$. For $I \in \binom{[n]}{k}$, the Plücker coordinate $p_I \neq 0$ if and only if the vectors $\{v_i \mid i \in I\}$ are linearly independent, i.e., they form a basis of $\mathbb{K}^k$. In linear algebra, the family of such sets is called bases of $\mathbb{K}^k$:

$$\mathcal{B}_V = \left\{ I \in \binom{[n]}{k} \mid p_I \neq 0 \right\}.$$

More generally, it is possible to study linear independence of sets of vectors of any cardinality. Such sets are called (linearly) independent sets:

$$\mathcal{L}_V = \{I \subseteq [n] \mid \dim(\mathrm{span}(v_i \mid i \in I)) = |I|\}.$$

On the other hand, it is also possible to look at linearly dependent sets of vectors. In particular, we can consider the family of minimally dependent vectors:

$$\mathcal{C}_V = \{I \subseteq [n] \mid I \notin \mathcal{L}_V \text{ and for each } i \in I, I \setminus \{i\} \in \mathcal{L}_V\}.$$

From this point of view, it is also possible to consider all the points lying on a linear subspace of $\mathbb{K}^k$. Such sets are usually called closed sets:

$$\mathcal{F}_V = \{I \subseteq [n] \mid \mathrm{span}(I \cup \{v_j\}) \supsetneq \mathrm{span}(I) \text{ for each } j \notin I\},$$

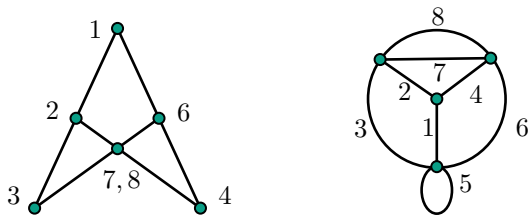


Figure 2.1: On the left, the point-line configuration of the set of vectors from Example 2.2.1. On the right, the graph considered in Example 2.2.2. They both define the same matroid.

where we denote by $\mathrm{span}(I)$ the linear span of the vectors labeled by elements in I.

Example 2.2.1. Consider the set of vectors $V = \{v_1, \ldots, v_8\} \subset \mathbb{R}^3$ given by the columns of the following matrix:

$$M = \begin{pmatrix} 1 & 2 & \pi & 1 & 0 & 0 & \pi & 1 \\ 0 & 1 & 3 & 1 & 0 & 1 & \pi - 3 & 1 - \frac{\pi}{3} \\ 0 & 1 & 3 & 0 & 0 & 0 & 3 & \frac{3}{\pi} \end{pmatrix}.$$

We can visualize the respective position of 3-dimensional vectors by removing the zero vectors and taking the intersection of the lines spanned by the remaining vectors with a plane in general position. Whenever three of these intersections lie on a line, we add a line through them. See Figure 2.1. Such a representation is called the *point-line configuration* of the set of vectors.

In this example, any vector is independent if and only if it is the non-zero vector. Any two non-zero vectors are independent if they do not label the same point in the point-line configuration. Finally, any three non-zero, pairwise independent vectors form a basis if and only if they do not lie on a line in the point-line configuration. Hence, we can easily see that the family of linearly independent sets is given by:

$$\mathcal{L}_V = \left\{ \begin{array}{c} 124, 126, 127, 128, 134, 136, 137, 138, 147, 148, 167, 168, \\ 234, 236, 237, 238, 246, 267, 268, 346, 347, 348, 467, 468, \\ 12, 13, 14, 16, 17, 18, 23, 24, 26, 27, 28, 34, 36, 37, 38, 46, 47, 48, 67, 68, \\ 1, 2, 3, 4, 6, 7, 8 \end{array} \right\}$$

The bases correspond to the first two rows of sets in $\mathcal{L}$.

Matroid of a Graph

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. A *spanning tree* T of G is a tree whose edges form a subset of the edge set covering all the vertices of G, i.e., $\bigcup_{v \in E \in T} \{v\} = V$. Denote the set of edges of spanning trees of a graph by $\mathcal{B}_G$.

A *cycle* of a graph G is a connected subgraph C of G such that every vertex in C has degree 2, i.e., it is contained in exactly two edges. Note that a tree is a graph that does not contain any cycle. The family of subsets of edges E that do not contain any cycle of the graph G is denoted by $\mathcal{L}_G$. On the other hand, the set of cycles of G is denoted by $\mathcal{C}_G$.

Example 2.2.2. Consider the graph G with 4 vertices and 8 edges from Figure 2.1. The family of subsets of edges $\{1, \ldots, 8\}$ that form spanning trees of the graph is:

$$\mathcal{B}_G = \begin{Bmatrix} 124, 126, 127, 128, 134, 136, 137, 138, 147, 148, 167, 168, \\ 234, 236, 237, 238, 246, 267, 268, 346, 347, 348, 467, 468, \end{Bmatrix}$$

Note that the sets of independent vectors, maximally independent vectors and minimally dependent vectors coming from Example 2.2.1 and Example 2.2.2 are equal: $\mathcal{L}_V = \mathcal{L}_G$, $\mathcal{B}_V = \mathcal{B}_G$ and $\mathcal{C}_V = \mathcal{C}_G$. This is because they are both representations of the same underlying matroid. In particular, the families $\mathcal{B}$, $\mathcal{L}$ and $\mathcal{C}$ defined for vectors and for graphs share common properties which are used to axiomatically define a matroid.

2.2.1 Definition and Properties

We now axiomatically introduce the families of independent sets, bases, circuits and flats, seen before for specific examples.

Definition of Matroid

The family of independent vectors or of subsets of edges not containing a cycle can be axiomatized as follows.

Definition 2.2.3 (Independent sets)**.** A family of *independent sets* is a collection $\mathcal{L}$ of subsets of a finite set E, satisfying:

(I1) $\mathcal{L} \neq \emptyset$;

(I2) For every $I_1 \in \mathcal{L}$ and $I_2 \subseteq I_1$, $I_2 \in \mathcal{L}$;

(I3) For every $I_1, I_2 \in \mathcal{L}$, if $|I_1| > |I_2|$, there exists $x \in I_1 \setminus I_2$ such that $I_2 \cup \{x\} \in \mathcal{L}$ *(independence augmentation axiom).*

The maximal sets, with respect to inclusion, in a given family of independent sets $\mathcal{L}$, are called *bases*. In the previous examples, they correspond to sets of maximally independent vectors or spanning trees. They can be axiomatized as follows.

Definition 2.2.4 (Bases)**.** A family of *bases* is a collection $\mathcal{B}$ of subsets of a finite set E, satisfying:

(B1) $\mathcal{B} \neq \emptyset$;

(B2) For every $A, B \in \mathcal{B}$ and $a \in A \setminus B$, there exists an element $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ *(basis exchange axiom).*

The size of any $B \in \mathcal{B}$ is called *rank*.

Given a set of bases, a minimally dependent set is any set $C \subseteq E$ such that C is not contained in any basis and every proper subset of C is contained in some basis. These sets are called *circuits*, and they can be axiomatized as follows.

Definition 2.2.5 (Circuits)**.** A family of *circuits* is a collection $\mathcal{C}$ of subsets of a finite set E, satisfying:

(C1) $\emptyset \notin \mathcal{C}$;

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;

(C3) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $x \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $(C_1 \cup C_2) \setminus \{x\} \supseteq C_3$ *(circuit elimination axiom).*

Given a set of circuits, we can construct the family of closed sets as the family of subsets $F \subseteq E$ such that if $|F \cap C| \geq |C| - 1$, then we have that $C \subseteq F$. These sets are also called *flats* and their family can be axiomatized as follows.

Definition 2.2.6 (Flats)**.** A family of *flats* is a collection $\mathcal{F}$ of subsets of a finite set E, satisfying:

(F1) $E \in \mathcal{F}$;

(F2) If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$;

(F3) If $X \in \mathcal{F}$ and $x \in E$, there exists a unique $Y \in \mathcal{F}$ such that $X \cup \{x\} \subseteq Y$ and there is no $Z \in \mathcal{F}$ such that $X \cup \{x\} \subsetneq Z \subsetneq Y$.

We will denote by $\langle S \rangle$ the minimal flat containing a subset $S \subseteq E$.

The information contained in any of the family above, through the relations described, is equivalent. Therefore, we obtain the following result, see [98] for more details.

Theorem/Definition 2.2.7. A *matroid* is a pair consisting of a finite set E, called *ground set*, and the information contained in any one of the following families of subsets of E.

1. Family of independent sets $\mathcal{L}$;
2. Family of bases $\mathcal{B}$;
3. Family of circuits $\mathcal{C}$;
4. Family of flats $\mathcal{F}$.

The notion of matroid is well-defined, i.e. any of the families of sets above determines the other.

Given a family of vectors V in $\mathbb{K}^k$ and a connected graph G, it is easy to see that the families $\mathcal{B}_V$, $\mathcal{L}_V$, $\mathcal{C}_V$ and $\mathcal{F}_V$ constructed from V and the families $\mathcal{B}_G$, $\mathcal{L}_G$, $\mathcal{C}_G$ and $\mathcal{F}_G$ constructed from G satisfy the axioms of, respectively, families of bases, independent sets, circuits and flats. Moreover, they satisfy the relations introduced above, hence they uniquely define a matroid $\mathcal{M}_V$ associated to the set of vectors V and a matroid $\mathcal{M}_G$ associated to the graph G.

A matroid coming from a set of vectors over $\mathbb{K}$ is called a *representable matroid over* $\mathbb{K}$. A matroid constructed from a graph is called a *graphic matroid*. Every graphic matroid is representable over any field by its incidence matrix [98, Proposition 5.1.2 and Lemma 5.1.3]. However, the converse does not hold, i.e. if a matroid is representable over any fields, then it does not imply that it is graphic.

Let V be a set of vectors in a vector space $\mathbb{K}^k$. In the following we will assume that the vectors in V span the entire vector space. For each subset $S \subseteq V$ it is possible to compute the dimension of the subspace they generate. We saw that the dimension k of the spaces spanned by all the vectors in V is called rank of the matroid $\mathcal{M}_V$. More generally, the information about the dimension of the subspace spanned by any subset of V is contained in the rank function of the matroid.

Definition 2.2.8 (Rank function)**.** Let $\mathcal{M}$ be a matroid with ground set E and bases $\mathcal{B}$. Let 2^E be the family of subsets of E. The *rank function* of $\mathcal{M}$ is the function $\mathrm{rk}_{\mathcal{M}} : 2^E \to \mathbb{Z}$ defined by:

$$\mathrm{rk}(A) = \max\{|A \cap B| \mid B \in \mathcal{B}\}, \qquad \text{for every } A \subseteq E.$$

The rank function of a matroid uniquely defines it, and it is possible to give an axiomatic characterization of the rank function of a matroid.

Proposition 2.2.9. *Let $\mathcal{M}$ be a matroid with ground set E and let $\mathrm{rk}_{\mathcal{M}} : 2^E \to \mathbb{Z}$ be its rank function. Then $\mathrm{rk}_{\mathcal{M}}$ satisfies the following properties:*

(R1) For any $A \subseteq E$, $0 \leq \mathrm{rk}_{\mathcal{M}}(A) \leq |A|$;

(R2) For any $A, B \subseteq E$, $\mathrm{rk}_{\mathcal{M}}(A) + \mathrm{rk}_{\mathcal{M}}(B) \geq \mathrm{rk}_{\mathcal{M}}(A \cup B) + \mathrm{rk}_{\mathcal{M}}(A \cap B)$ (submodularity)*;*

(R3) For any $A \subseteq E$ and $x \in E$, $\mathrm{rk}_{\mathcal{M}}(A) \leq \mathrm{rk}_{\mathcal{M}}(A \cup \{x\}) \leq \mathrm{rk}_{\mathcal{M}}(A) + 1$ (monotonicity)*.*

Moreover, any function $r : 2^E \to \mathbb{Z}$ satisfying (R1), (R2) and (R3) is the rank function of some matroid $\mathcal{M}$ with ground set E.

Other Characterizations of Matroids

As seen in the previous part, there are many equivalent definitions of matroid. We now introduce two more ways to characterize the set of bases of a matroid. These will be useful in the study of the special class of positroids. Since we always assume the ground set to be finite, we can assume without loss of generality that the ground set E is equal to $[n]$.

First, we introduce the characterization in terms of partial orderings on the set $\binom{[n]}{k}$. This characterization is due to Gale, [53], and was then generalized by Gel'fand and Serganova, [56]. It is based on the fact that the order in which we label the elements in the ground set should not be relevant.

Fix the ground set $[n]$. The group S_n of permutations on n elements naturally acts on $[n]$. Given a permutation $\omega \in S_n$, we define the order $\leq^\omega$ on $[n]$ given by $i \leq^\omega j$ if and only if $\omega^{-1}(i) \leq \omega^{-1}(j)$. If we write ω in one-line notation as $(i_1\ i_2\ \cdots\ i_n)$, then $\leq^\omega$ is the ordering on $[n]$ given by $i_1 <^\omega i_2 <^\omega \cdots <^\omega i_n$.

Definition 2.2.10 (Gale ordering)**.** Given any ordering $\preceq$ on $[n]$, we define a partial ordering $\preceq$ on $\binom{[n]}{k}$, where for each $I, J \in \binom{[n]}{k}$, with $I = \{i_1 \prec i_2 \prec$

$\cdots \prec i_k\}$ and $J = \{j_1 \prec j_2 \prec \cdots \prec j_k\}$, we have

$$I \preceq J \text{ if and only if } i_h \preceq j_h \text{ for every } h \in [k].$$

If $\preceq = \leq^\omega$ for some $\omega \in S_n$, then the induced ordering $\leq^\omega$ on $\binom{[n]}{k}$ is called *Gale ordering* induced by ω on $\binom{[n]}{k}$.

Theorem 2.2.11 ([53])**.** *Let $\mathcal{B} \subseteq \binom{[n]}{k}$. Then $\mathcal{B}$ is the set of bases of a matroid if and only if it satisfies the following* maximality property *for every $\omega \in S_n$:*

$$\textit{there exists a unique } A \in \mathcal{B} \textit{ such that } B \leq^\omega A \textit{ for every } B \in \mathcal{B}.$$

A is called the ω-maximal basis in $\mathcal{B}$.

This characterization of matroids allows the generalization to the case where S_n in substituted by a Coxeter group W acting on W/P for some parabolic subgroup P of W. This leads to the more general concept of Coxeter matroids. See [15] for more details.

The second characterization involves the construction of a polytope and describes its structure.

Definition 2.2.12 (Matroid polytope)**.** Let $\mathcal{M}$ be a matroid with ground set $[n]$ and let $\mathcal{B}$ be the set of its bases. The *matroid polytope* of $\mathcal{M}$ is given by:

$$P_{\mathcal{M}} = \text{conv}(e_B \mid B \in \mathcal{B}) \subset \mathbb{R}^n$$

where e_B is the indicator vector of the basis B, i.e., $e_B = \sum_{i \in B} e_i$, with $\{e_i\}$ the canonical basis of $\mathbb{R}^n$.

A classification of matroid polytopes is given by the following result, originally appearing in [55, Theorem 4.1].

Theorem 2.2.13. *A polytope $P \subseteq \mathbb{R}^n$ is the matroid polytope of some matroid $\mathcal{M}$ with ground set $[n]$ if and only if the vertices of P are contained in the set $\{e_I \mid I \in \binom{[n]}{k}\}$ and the edges are parallel to vectors of the form $e_i - e_j$ for $i, j \in [n]$ distinct.*

Constructing Matroids

We now describe some elementary operations between matroids.

The matroid on the ground set $[n]$ whose bases are given by $\mathcal{B} = \binom{[n]}{k}$ is called *uniform matroid* on $[n]$ of rank k and will be denoted by $\mathcal{U}_{k,n}$.

Given a matroid $\mathcal{M}$ with ground set $[n]$ and bases $\mathcal{B}$, define $\mathcal{B}^*$ to be the set $\{[n] \setminus B \mid B \in \mathcal{B}\}$. The set $\mathcal{B}^*$ satisfies the properties (B1) and (B2) in Definition 2.2.4, hence it is the family of bases of a matroid $\mathcal{M}^*$ on the ground set $[n]$. The matroid $\mathcal{M}^*$ is called *dual matroid* of $\mathcal{M}$.

Remark 2.2.14. Note that if $\mathcal{M}$ has rank k, then $\mathcal{M}^*$ has rank $n-k$. In general, let $\mathrm{rk}^* : 2^{[n]} \to \mathbb{Z}$ be the rank function associated to the dual matroid $\mathcal{M}^*$ and let $\mathrm{rk} : 2^{[n]} \to \mathbb{Z}$ be the rank function of $\mathcal{M}$. Then for every $A \subseteq [n]$:

$$\mathrm{rk}^*(A) = |A| - (k - \mathrm{rk}([n] \setminus A)).$$

2.2.2 Positroids

A class of representable matroids over $\mathbb{R}$ is given by positroids. First, we introduce some notation.

Notation 2.2.15. Fix k, n to be positive integers with $k \leq n$. For any $i, j \in \mathbb{Z}$, we define $[i, j]$ as the cyclic interval in $[n]$ from $i \mod n$, to $j \mod n$. For instance, if $n = 8$, then $[7, 10] = [7, 2] = \{7, 8, 1, 2\}$. We use $<$ to denote the normal ordering on $\mathbb{Z}$ and $<_i$ to denote the ordering on $[n]$ where i is the smallest element, i.e., $i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1$. We extend this order to a partial order on $\mathbb{Z}$ by defining, for $j, k \in \mathbb{Z}$, $j <_i k$ if and only if $j \mod n <_i k \mod n$.

A positroid is a representable matroid over $\mathbb{R}$ with a positivity condition. On Grassmannians, the positivity condition is imposed on the Plücker coordinates.

Definition 2.2.16 (Totally non-negative Grassmannian)**.** The *totally non-negative Grassmannian* $\mathrm{Gr}_{\geq 0}(k, n)$ is the subset of points $V \in \mathrm{Gr}_{\mathbb{R}}(k, n)$ whose Plücker coordinates have all the same sign.

Definition 2.2.17 (Positroid)**.** A matroid $\mathcal{M}$ is said to be a *positroid* if it is a representable matroid over $\mathbb{R}$ that can be represented by a matrix M with non-negative maximal minors. In other words, a positroid is a matroid $\mathcal{M}$ that can be represented by an element of the totally non-negative Grassmannian.

Positroids of rank k on n elements can be endowed with a poset structure as follows. Given two positroids $\mathcal{P}_1$ and $\mathcal{P}_2$, we say that $\mathcal{P}_1$ is smaller than $\mathcal{P}_2$ if $\{M \in \mathrm{Gr}_{\geq 0}(k, n) \mid \mathcal{M}_M = \mathcal{P}_1\} \subseteq \mathrm{cl}(\{M \in \mathrm{Gr}_{\geq 0}(k, n) \mid \mathcal{M}_M = \mathcal{P}_2\})$. We denote this poset by $Q_{n,k}$.

Positroids were first introduced in 2006 by Postnikov, [102]. Postnikov showed that positroids are a class of matroids with a rich combinatorial structure and nice geometric properties.

We now want to describe some of the combinatorial objects that describe positroids. In particular, we will focus on Grassmann necklaces and bounded affine permutations. Postnikov introduced in the original preprint [102] Grassmann necklaces, among many other objects.

Definition 2.2.18 (Grassmann necklace)**.** A *Grassmann necklace* is an ordered collection of k-subsets of $[n]$, $\mathcal{I} = (I_1, \ldots, I_n)$ satisfying the following conditions:

- If $i \in I_i$, then $I_{i+1} = I_i \setminus \{i\} \cup \{j\}$, for some $j \in [n]$;
- If $i \notin I$, then $I_{i+1} = I_i$.

Note that the order $<_i$ defined in Notation 2.2.15, is equal to the order $\leq^\omega$ where $\omega \in S_n$ is the permutation such that $\omega(a) = a + i \mod n$ for every $a \in [n]$. By Theorem 2.2.11, there exists a unique minimal element in $\mathcal{B}$ with respect to $\leq_i$. The following result is proven in [102].

Theorem 2.2.19. *Let $\mathcal{M}$ be a matroid with ground set $[n]$ and bases $\mathcal{B}$. Let I_i be the minimal element of $\mathcal{B}$ with respect to $\leq_i$, for $i \in [n]$. Then $\mathcal{I}_\mathcal{M} = (I_1, \ldots, I_n)$ is a Grassmann necklace.*

Conversely, given a Grassmann necklace $\mathcal{I}$ on $[n]$, the set

$$\mathcal{B}_\mathcal{I} = \left\{ B \in \binom{[n]}{k} \mid B \geq_i I_i \text{ for all } i \in [n] \right\}$$

is the set of bases of a positroid $\mathcal{M}_\mathcal{I}$.

Given a matroid $\mathcal{M}$, we have that $\mathcal{M} = \mathcal{M}_{\mathcal{I}_\mathcal{M}}$ if and only if $\mathcal{M}$ is a positroid. If $\mathcal{M}$ is not a positroid, then $\mathcal{M}_{\mathcal{I}_\mathcal{M}}$ is the smallest positroid in $Q_{n,k}$ whose bases contain $\mathcal{B}$, it is called positroid envelope *of $\mathcal{M}$ and denoted by* $\mathrm{env}(\mathcal{M})$.

Bounded affine permutations were introduced by Knutson, Lam and Speyer [70].

Definition 2.2.20 (Bounded affine permutation)**.** Consider a positive integer n. A *bounded affine permutation of size n* is a bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying the following conditions for all $i, k \in \mathbb{Z}$:

$$\pi(i + kn) = \pi(i) + kn \quad \text{and} \quad i \leq \pi(i) \leq i + n.$$

Bounded affine permutations and positroids are in a one-to-one correspondence. This correspondence is constructed as follows. Given a positroid $\mathcal{P}$, we define the associated bounded affine permutation $\pi : \mathbb{Z} \to \mathbb{Z}$ by

$$\pi(i) = \begin{cases} i & \text{if } i \in \langle \emptyset \rangle \\ \min\{j > i \mid i \in \langle [i+1, j] \rangle\} & \text{otherwise.} \end{cases} \quad \text{for every } i \in \mathbb{Z}.$$

In particular, if i is a loop of the positroid, then $\pi(i) = i$, and if i is a coloop of the positroid, then $\pi(i) = i + n$. Moreover, information on the rank of a positroid on cyclic intervals can be extracted from its associated bounded affine permutation.

Proposition 2.2.21. *Let π be a bounded affine permutation of size n and let $\mathcal{P}$ be the corresponding positroid.*

1. *The rank of $\mathcal{P}$ is equal to the number of $i \in [n]$ such that $\pi(i) > n$.*
2. *The rank of a cyclic interval $[i, j] \subset [n]$ is given by the number of $\ell \in [i, j]$ such that $\pi(\ell) > j$.*

Positroids allow for a characterization in terms of the corresponding matroid polytope, as follows by combining a description of matroid polytopes due to Welsh [115, Chapter 18] with [6, Proposition 5.6].

Proposition 2.2.22. *A matroid $\mathcal{M}$ is a positroid if and only if its matroid polytope $P_{\mathcal{M}}$ can be described by the equality $x_1 + \cdots + x_n = k$ and by inequalities of the form*

$$\sum_{\ell \in [i,j]} x_\ell \leq \mathrm{rk}([i, j]) \quad \text{for } i, j \in [n].$$

2.2.3 Matroids and Grassmannians

As seen in the construction of matroids from a set of vectors, it is possible to associate to any point in the Grassmannian the matroid whose bases are given by the indices of the non-zero Plücker coordinate of that point. Note that this does not depend on the choice of the representative of the point in the Grassmannian. Hence, it is possible to categorize the points of any Grassmannian, based on the matroid they define, as introduced by Gel'fand, Goresky, Macpherson, and Serganova, [55].

Definition 2.2.23 (Matroid strata)**.** Let $\mathcal{M}$ be a matroid of rank k on the ground set $[n]$ with bases $\mathcal{B}$. The *matroid stratum* of $\mathcal{M}$ inside $\mathrm{Gr}_{\mathbb{C}}(k, n)$ is the set of points in $\mathrm{Gr}_{\mathbb{C}}(k, n)$ whose associated matroid is $\mathcal{M}$:

$$\mathcal{S}_{\mathcal{M}} = \{V \in \mathrm{Gr}_{\mathbb{C}}(k, n) \mid p_I(V) \neq 0 \iff I \in \mathcal{B}\}.$$

The collection of matroid strata forms the *matroid stratification* of the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k, n)$.

Note that a matroid $\mathcal{M}$ is representable over $\mathbb{C}$ if and only if the corresponding stratum is not empty, $\emptyset \neq \mathcal{S}_{\mathcal{M}} \subseteq \mathrm{Gr}_{\mathbb{C}}(k,n)$. However, characterizing representable matroids is essentially impossible, [114]. Moreover, a matroid stratum can have almost any singularity, [86].

Positroids provide a coarser stratification of the Grassmannian, which has nicer properties.

Definition 2.2.24 (Positroid variety)**.** Let $\mathcal{P}$ be a positroid of rank k on the ground set $[n]$. The *open positroid variety* $\Pi^{\circ}_{\mathcal{P}}$ of $\mathcal{P}$ is the set of points $V \in \mathrm{Gr}_{\mathbb{C}}(k,n)$ whose matroid $\mathcal{M}_V$ has positroid envelope equal to $\mathcal{P}$.

The *positroid variety* $\Pi_{\mathcal{P}}$ of $\mathcal{P}$ is the Zariski closure of $\Pi^{\circ}_{\mathcal{P}}$.

Note that the open positroid variety is larger than the realization space of the given positroid. This allows us to decompose the Grassmannian as disjoint union of open positroid varieties:

$$\mathrm{Gr}_{\mathbb{C}}(k,n) = \bigsqcup_{\mathcal{P}} \Pi^{\circ}_{\mathcal{P}}.$$

Theorem 2.2.25 ([70, Theorem 5.9])**.** *Let $\mathcal{P}$ be a positroid of rank k on the ground set $[n]$. The varieties $\Pi^{\circ}_{\mathcal{P}}$ and $\Pi_{\mathcal{P}}$ are irreducible. The open positroid variety $\Pi^{\circ}_{\mathcal{P}}$ is smooth.*

If we restrict to the natural realization space for positroids, i.e., the totally non-negative Grassmannian, then positroids give a CW-decomposition.

Definition 2.2.26 (Positroid cell)**.** Let $\mathcal{P}$ be a positroid of rank k on $[n]$. The *positroid cell* of $\mathcal{P}$ is the realization space of the positroid inside the totally non-negative Grassmannian:

$$\mathcal{S}_{\mathcal{P}} = \{V \in Gr_{\geq 0}(k,n) \mid \mathcal{M}_V = \mathcal{P}\}.$$

The positroid cell is equal to the intersection of the open positroid variety with the totally non-negative Grassmannian. Positroid cells give a decomposition of the totally non-negative Grassmannian as proved in [102, 103].

Theorem 2.2.27. *Each positroid cell is homeomorphic to an open ball. Moreover, positroid cells fit together to form a CW-decomposition of the totally non-negative Grassmannian*

$$\mathrm{Gr}_{\geq 0}(k,n) = \bigsqcup_{\mathcal{P}} \mathcal{S}_{\mathcal{P}}.$$

In Chapter 3 we will introduce a new labeling of positroids and describe how it gives us information about the open positroid varieties and the positroid cells.

2.3 Toric Geometry

In this section we introduce the terminology and methodology to study and to construct toric varieties. We will be interested in studying them as degenerations of the Grassmannian and of its generalizations, the (partial) flag varieties. Part of this section appears in joint work with Oliver Clarke and Fatemeh Mohammadi [35] and in joint work with Oliver Clarke and Akihiro Higashitani [28].

2.3.1 Toric Varieties

Toric varieties are a class of algebraic varieties with many combinatorial properties. In particular, projective normal toric varieties correspond to lattice polytopes. We now briefly recall this construction and some of their properties. For more details on toric varieties, see [37].

Let $M \cong \mathbb{Z}^d$ be a lattice and denote by $M_{\mathbb{R}}$ the space $M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope, i.e. its vertices $V(P) \subseteq M$ are lattice points. Recall the following definitions.

Definition 2.3.1 (Normal polytope)**.** The polytope P is said to be *normal* if for any $k \in \mathbb{N}$ and any lattice point $\alpha \in (kP \cap M)$ there exist lattice points $\alpha_1, \ldots, \alpha_k \in (P \cap M)$ such that $\alpha = \alpha_1 + \cdots + \alpha_k$.

The polytope P is said to be *very ample* if for every vertex $v \in P$, the semigroup generated by $P \cap M - v$ is saturated in M, i.e. $\mathbb{N}(P \cap M - v) = \text{cone}(P \cap M - v) \cap M$.

Remark 2.3.2. Any normal lattice polytope is very ample [37, Proposition 2.2.18].

Let $P \subseteq M_{\mathbb{R}}$ be a normal lattice polytope and let $\{p_1, \ldots, p_s\} = (P \cap M)$ be its lattice points. The toric variety $Y_P \subseteq \mathbb{P}^{s-1}$ associated to P is the Zariski closure of the image of the map

$$(\mathbb{C}^*)^d \to \mathbb{P}^{s-1} \colon t \mapsto (\chi^{p_1}(t), \ldots, \chi^{p_s}(t))$$

where $\chi^{(m_1,\ldots,m_d)}(t_1, \ldots, t_d) := t_1^{m_1} t_2^{m_2} \cdots t_d^{m_d}$ is a *character of the torus* $(\mathbb{C}^*)^d$. The characters of $(\mathbb{C}^*)^d$ naturally form a lattice with the group operation $(\chi^{m_1} \cdot \chi^{m_2})(t) := \chi^{m_1}(t)\chi^{m_2}(t) = \chi^{m_1+m_2}(t)$. So the character lattice is naturally isomorphic to M and by abuse of notation we will identify these lattices.

The homogeneous coordinate ring of Y_P is given by $\mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$ where

$$C(P) := \text{Cone}(P \times \{1\}) = \{(\lambda p, \lambda) \in M_{\mathbb{R}} \times \mathbb{R} : \lambda \geq 0,\ p \in P\} \subseteq M_{\mathbb{R}} \times \mathbb{R}.$$

2.3.2 Ehrhart Series

The combinatorics of the polytope P allows us to extract geometric invariants of the toric variety Y_P from a combinatorial point of view. As an example, it is possible to encode the number of lattice points contained in dilations of a polytope in a series.

Definition 2.3.3 (Ehrhart series)**.** The *Ehrhart series* of a polytope $P \subseteq \mathbb{R}^d$ is the generating function of the number of lattice points contained in the nth dilate of P, $|nP \cap \mathbb{Z}^d|$. We denote it by

$$E_P(t) = \sum_{n \in \mathbb{N}} |nP \cap \mathbb{Z}^d| t^n.$$

If $P \subseteq \mathbb{R}^d$ is a rational polytope of dimension D, $L_P(n) = |nP \cap \mathbb{Z}^d|$ is a quasi-polynomial in n of degree D, i.e. $L_P(t) = c_D(n)n^D + c_{D-1}(n)n^{D-1} + \cdots + c_0(n)$ with $c_i(n)$ periodic functions in n. L_P is called the *Ehrhart quasi-polynomial* of P. The least common multiple of the periods of $c_0, \ldots, c_n$ is called *period* of P. Generically, the period of P is equal to its *denominator*, which is the smallest positive integer m such that all vertices of the mth dilate of P lie in $\mathbb{Z}^d$. By a well-known theorem of Ehrhart, the period divides the denominator [11, Theorem 3.23]. In particular, if P is a lattice polytope then L_P has period one and L_P is called *Ehrhart polynomial*. The converse does not hold.

Definition 2.3.4 (Period collapse)**.** If P is such that its period differs from its denominator, we say that P exhibits *period collapse*.

It was proved in [83, Theorem 2.2] that, for all positive integers d, D, and s with $d \geq 2$ such that s divides D, there exists a d-dimensional rational polytope with denominator D and period s. In Section 4.3.4 we will study a large family of rational polytopes exhibiting period collapse.

In the study of toric varieties, we always assume P to be a lattice polytope, and the Ehrhart polynomial of P gives us the following information on the corresponding toric variety Y_P [37, Proposition 9.4.3].

Theorem 2.3.5. *If P is very ample, the Ehrhart polynomial L_P of P equals the Hilbert polynomial of the toric variety Y_P. In particular, comparing the leading coefficients of these polynomials, it follows that the normalized volume of P equals the degree of Y_P.*

2.3.3 The Total Coordinate Ring and its Grading

Let Y_P be a smooth projective toric variety associated to a normal lattice polytope $P \subseteq M_{\mathbb{R}}$, where $M \cong \mathbb{Z}^{\mathrm{rk}(M)}$ is the character lattice of the torus of Y_P.

Remark 2.3.6. The assumption that Y_P is smooth can be seen as an assumption on P. Indeed, Y_P is smooth if and only if each cone σ in the normal fan Σ_P to P is such that $\{u_\rho \mid \rho \text{ ray of } \sigma\}$ forms a $\mathbb{Z}$-basis of $M \cong \mathbb{Z}^d$ [37, Theorem 2.4.3]. Here, u_ρ is the unique generator of $\rho \cap M$ for each ray of the cone σ.

The goal of this subsection is to give a uniform description of toric subvarieties of Y_P. Recall that $Y_P = \mathrm{Proj}(\mathbb{C}[C(P) \cap (M \times \mathbb{Z})])$. Therefore, any toric subvariety of Y_P will be given by a monomial subalgebra of $\mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$. Our approach to find these subalgebras is as follows. We associate to Y_P a particularly nice polynomial ring R and obtain toric subvarieties of Y_P from toric ideals in R satisfying certain homogeneity conditions.

We start with a set of inequalities defining the polytope P. Let f be the number of facets of P. An *H-description* of P is a collection of linear inequalities that define P as follows:

$$x \in P \iff \mathbf{B} \cdot x + \rho \geq \underline{0}$$

where $\mathbf{B} \in \mathbb{Z}^{f \times \mathrm{rk}(M)}$ is a full-rank matrix whose rows are primitive inner-normal vectors to the facets of P, and $\rho \in \mathbb{R}^f$, $x \in M_{\mathbb{R}}$ are column vectors. We may think of $\mathbf{B}$ as a map from M to $\mathbb{Z}^f$. Since P is full-dimensional, the columns of $\mathbf{B}$ are linearly independent. Hence, $\mathbf{B}$ embeds the character lattice M into $\mathbb{Z}^f$. Since Y_P is smooth, we have that the only lattice points in $\mathbf{B} \cdot M_{\mathbb{R}} \cap \mathbb{Z}^f$ are of the form $\mathbf{B} \cdot x$ for some $x \in M$. See [37, Chapter 4]. So the embedding can be completed to a short exact sequence of lattices

$$0 \longrightarrow M \xrightarrow{\mathbf{B}} \mathbb{Z}^f \xrightarrow{\mathbf{D}} \mathrm{Cl}(Y_P) \longrightarrow 0. \tag{2.1}$$

The geometric interpretation of this short exact sequence is as follows: $\mathbf{B}$ is the map sending a character to its corresponding torus-invariant Weil divisor; $\mathbf{D}$ is the map that takes a torus-invariant Weil divisor to its class in the class group $\mathrm{Cl}(Y_{\mathcal{P}})$ of Y_P.

Definition 2.3.7 (Total coordinate ring)**.** The *total coordinate ring* R of a smooth projective toric variety Y_P is the polynomial ring $\mathbb{C}[z_1, \ldots, z_f]$ with variables indexed by the facets of P.

The ring R is graded by $\mathrm{Cl}(Y_{\mathcal{P}})$. For each $m \in \mathbb{Z}^f$, the monomial $x^m \in R$ has degree $\deg(x^m) = \mathbf{D} \cdot m \in \mathrm{Cl}(Y_{\mathcal{P}})$. The variety Y_P can be described purely

in terms of the total coordinate ring. We may rewrite the semigroup ring as follows:

$$\mathbb{C}[C(P) \cap (M \times \mathbb{Z})] = \mathbb{C}\left[\bigoplus_{k=0}^{\infty}(kP \cap M) \times \{k\}\right].$$

The lattice points $(kP \cap M)$ of each dilation kP can be viewed as a graded piece of R. Fix $k \in \mathbb{N}$ and let

$$\iota : M_{\mathbb{R}} \hookrightarrow \mathbb{R}^f, \quad x \mapsto \mathbf{B} \cdot x + k\rho$$

be a translation of the embedding $\mathbf{B}$. By the definition of the H-description of P, we have that a point $x \in M_{\mathbb{R}}$ lies in kP if and only if $\iota(x) \in \mathbb{R}^f_{\geq 0}$, that is $\mathbf{B} \cdot x + k\rho \geq \underline{0}$. By the short exact sequence in eq. (2.1), we have that $\iota(kP) = k(\mathbf{B} \cdot P + \rho) = \mathbf{D}^{-1}(k\underline{d}) \cap \mathbb{R}^f_{\geq 0}$, where $\underline{d} = \mathbf{D} \cdot \rho$. In particular, ι gives a bijection between lattice points in kP and $k\underline{d}$-graded monomials in R, so that we have $Y_P = \operatorname{Proj}\left(\mathbb{C}\left[\bigoplus_{k=0}^{\infty}(kP \cap M) \times \{k\}\right]\right) \cong \operatorname{Proj}\left(\bigoplus_{k=0}^{\infty}(R_{k\underline{d}})\right)$.

2.3.4 Toric Subvarieties from Monomial Maps

The description of the coordinate ring of Y_P in terms of the total coordinate ring, in the previous section, allows us to generalize the correspondence between homogeneous ideals and closed subvarieties of projective spaces. Using the notation of Subsection 2.3.3 we have the following.

Lemma 2.3.8 ([37], Proposition 5.2.4)**.** *Let R be the total coordinate ring of a projective toric variety Y_P of a full dimensional lattice polytope P. Then any $\operatorname{Cl}(Y_{\mathcal{P}})$-homogeneous ideal $\mathcal{I} \subset R$ gives rise to a closed subvariety of Y_P. Moreover, all closed subvarieties of Y_P arise in this way.*

In particular, any toric subvariety of Y_P, that is, any closed subvariety of Y_P which is a toric variety with respect to a subtorus of the torus T, arises from a toric ideal in R, i.e., the kernel of some monomial map

$$\phi : \mathbb{C}[z_1, \ldots, z_f] \to \mathbb{C}[p_1, \ldots, p_k]$$

which is concurrent with the $\operatorname{Cl}(Y_{\mathcal{P}})$-grading of R. By this we mean that if $\mathbf{z}^u - \mathbf{z}^v \in \ker(\phi)$ then $\mathbf{z}^u$ and $\mathbf{z}^v$ have the same $\operatorname{Cl}(Y_{\mathcal{P}})$-degree, i.e., $\mathbf{D} \cdot u = \mathbf{D} \cdot v$. Consider the lattice $\mathbb{Z}^k$ associated to the ring $\mathbb{C}[p_1, \ldots, p_k]$ and define

$$\hat{\phi} : \mathbb{Z}^f \to \mathbb{Z}^k \tag{2.2}$$

to be the map of lattices corresponding to ϕ, i.e., $\phi(\mathbf{z}^u) = \mathbf{p}^v$ if and only if $\hat{\phi}(u) = v$. By an abuse of notation, we will use $\hat{\phi}$ to denote the map of spaces

$\mathbb{R}^f \to \mathbb{R}^k$ that naturally extends $\hat{\phi}$. Given such a monomial map ϕ, we construct a closed toric subvariety $X_\phi \subseteq Y_P$ which is the vanishing locus in Y_P of the kernel, $\ker(\phi)$. We show that X_ϕ is the toric variety arising from the polytope $P_\phi = \hat{\phi}(P)$ viewed as a polytope with respect to the lattice $\hat{\phi}(M)$.

Proposition 2.3.9. *Let Y_P be a smooth projective toric variety with lattice polytope P and total coordinate ring $R = \mathbb{C}[z_1, \ldots, z_f]$. Fix a monomial map $\phi : \mathbb{C}[z_1, \ldots, z_f] \to \mathbb{C}[p_1, \ldots, p_k]$ which is concurrent with the $\mathrm{Cl}(Y_\mathcal{P})$-grading of R. Let $\hat{\phi}$ be the map of lattices defined above. Then X_ϕ is given by the toric variety X_{P_ϕ} constructed from the polytope $P_\phi = \hat{\phi}(P)$.*

Proof. We begin by showing that X_ϕ is given by $\operatorname{Proj}\left(\bigoplus_{k=0}^\infty (\operatorname{Im}\phi)_{k\underline{d}}\right)$. Consider the natural surjection $R \twoheadrightarrow R/\ker(\phi) \cong \operatorname{Im}(\phi)$. Since ϕ is a monomial map, we have an isomorphism $\operatorname{Im}(\phi) \cong \mathbb{C}[\mathbb{N}S]$ where $S = \{s_1, \ldots, s_f\} \subset \mathbb{R}^k$ is the image of the set of unit vectors in $\mathbb{R}^f$. By assumption, ϕ is concurrent with the $\mathrm{Cl}(Y_\mathcal{P})$-grading of R and so $\ker(\phi)$ is $\mathrm{Cl}(Y_\mathcal{P})$-homogeneous. We have that $R \twoheadrightarrow R/\ker(\phi)$ restricts to a surjection on each $\mathrm{Cl}(Y_\mathcal{P})$-graded piece. In particular, we have that

$$\bigoplus_{k=0}^\infty (R_{k\underline{d}}) \twoheadrightarrow \bigoplus_{k=0}^\infty (R_{k\underline{d}})/(\ker\phi)_{k\underline{d}} \cong \bigoplus_{k=0}^\infty (\operatorname{Im}\phi)_{k\underline{d}}.$$

By [58, Proposition 13.8], the surjection above induces a closed immersion

$$\operatorname{Proj}\left(\bigoplus_{k=0}^\infty (\operatorname{Im}\phi)_{k\underline{d}}\right) \to \operatorname{Proj}\left(\bigoplus_{k=0}^\infty (R_{k\underline{d}})\right) = Y_P.$$

Next, we show that $\operatorname{Proj}\left(\bigoplus_{k=0}^\infty (\operatorname{Im}\phi)_{k\underline{d}}\right)$ coincides with toric variety X_{P_ϕ} associated to the polytope $P_\phi := \hat{\phi}(P)$. By the construction of X_{P_ϕ}, we have

$$X_{P_\phi} = \operatorname{Proj}\left(\mathbb{C}[C(P_\phi) \cap (\mathbb{Z}S \times \mathbb{Z})]\right) = \operatorname{Proj}\left(\mathbb{C}\left[\bigoplus_{k=0}^\infty (kP_\phi) \cap \mathbb{Z}S\right]\right).$$

Since ϕ is concurrent with the $\mathrm{Cl}(Y_\mathcal{P})$-grading of R, any lattice point of kP_ϕ corresponds to the image under ϕ of a $k\underline{d}$-graded monomial. It follows that

$$\mathbb{C}\left[\bigoplus_{k=0}^\infty (kP_\phi) \cap \mathbb{Z}S\right] \cong \bigoplus_{k=0}^\infty (\operatorname{Im}\phi)_{k\underline{d}}.$$

We conclude the proof by taking Proj of the above. □

In Algorithm 1 we summarize the main discussion of this subsection. That is, for any projective toric variety Y_P associated to a polytope P, we describe the polytope associated to a toric subvariety X_ϕ of Y_P, where X_ϕ is the vanishing locus of a monomial map ϕ.

Algorithm 1: Construction of polytope corresponding to toric subvariety

Input: Y_P a smooth toric variety associated to a full-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ with f facets. A toric subvariety X_ϕ of Y_P defined by a monomial map $\phi : \mathbb{C}[\mathbb{N}^f] \to \mathbb{C}[p_1, \ldots, p_k]$.

Output: A polytope P_ϕ defining X_ϕ as a toric variety.

1. Compute the H-description of P. That is, calculate the matrix $\mathbf{B} : M \to \mathbb{Z}^f$ and vector $\rho \in \mathbb{R}^f$ so that: the rows of $\mathbf{B}$ are primitive vectors; and $\mathbf{B} \cdot x + \rho \geq \underline{0}$ if and only if $x \in P$. Identify P with $\{\mathbf{B} \cdot x + \rho \mid x \in P\} \subseteq \mathbb{R}^f$.

2. Calculate the grading matrix $\mathbf{D}$ of R by completing the short exact sequence
$$0 \longrightarrow M \xrightarrow{\mathbf{B}} \mathbb{Z}^f \xrightarrow{\mathbf{D}} \mathrm{Cl}(Y_P) \longrightarrow 0$$
Define $\underline{d} = \mathbf{D} \cdot \rho \in \mathrm{Cl}(Y_{\mathcal{P}})$ and check that $\ker(\phi)$ is homogeneous with respect to the $\mathrm{Cl}(Y_{\mathcal{P}})$-grading.

3. Let $\hat{\phi} : \mathbb{Z}^f \to \mathbb{Z}^k$ be the lattice map corresponding to ϕ. The polytope of X_ϕ is given by $P_\phi := \hat{\phi}(P)$. The polytope P_ϕ is a lattice polytope with respect to the lattice $\mathbb{Z}S$ where S is the image of the set of unit vectors in $\mathbb{R}^f$ under $\hat{\phi}$.

2.4 Tropical Geometry

In this section we introduce the main tools needed in the study of the Grassmannian from a tropical geometry point of view.

Tropical geometry gives tools to construct a "combinatorial shadow" of an algebraic variety, called its tropicalization, which can help in the study of the more complex algebraic variety. In particular, points in maximal cones of the tropicalization of a variety are good candidates to give rise to toric degenerations. We will be interested in constructing points that define a toric degeneration

of a variety. Moreover, we will study the relations between different toric degenerations and their polytopes.

2.4.1 Gröbner Degenerations and Tropicalization

In this section, we introduce the main concepts and notation of Gröbner theory. We refer to [111, 36] for more details.

Let $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal in a polynomial ring and let $w \in \mathbb{R}^n_{\geq 0}$ be a *weight vector*. For any polynomial $f = \sum_{\boldsymbol{u} \in \mathbb{N}^n} a_{\boldsymbol{u}} \mathbf{x}^{\boldsymbol{u}}$, its *initial form* with respect to w is defined to be

$$\mathrm{in}_w(f) = \sum_{\substack{\boldsymbol{u} \in \mathbb{N}^n \\ \boldsymbol{u} \cdot w \text{ is minimal}}} a_{\boldsymbol{u}} \mathbf{x}^{\boldsymbol{u}}.$$

The *initial ideal* of I with respect to w is the ideal generated by the initial forms of every polynomial in the ideal:

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle.$$

The weights $w \in \mathbb{R}^n_{\geq 0}$ can be classified based on the initial ideal they define. Let $C_w = \{u \in \mathbb{R}^n_{\geq 0} \mid \mathrm{in}_w(I) = \mathrm{in}_u(I)\}$ be the space of weights that define the same initial ideal as w. Then C_w is the interior of a polyhedral cone and there are finitely many cones C_w for $w \in \mathbb{R}^n_{\geq 0}$ that fit together to form a fan. This is called *Gröbner fan* of I.

Intuitively, the initial forms contain the most important terms of the polynomial. It is possible to construct a flat family of varieties over $\mathbb{A}^1$ whose general fiber is isomorphic to $\mathcal{V}(I)$ and whose special fiber is isomorphic to the variety of the initial ideal. Formally, for $t \in \mathbb{A}^1$ and $w \in \mathbb{N}^n$, the generic fiber is defined by:

$$I_t = \left\langle t^{-\min_{\boldsymbol{u}}\{w \cdot \boldsymbol{u}\}} f(t^{w_1} x_1, \dots, t^{w_n} x_n) \mid f = \sum a_{\boldsymbol{u}} \mathbf{x}^{\boldsymbol{u}} \in \mathcal{I} \right\rangle.$$

Such a process is called *Gröbner* or *initial degeneration* of I. This gives a flat family over $\mathbb{A}^1$ [40, Theorem 15.17]. The flatness of the family guarantees that the special fiber shares many geometric invariants with the generic fiber. In particular, the variety associated to I and the one associated to $\mathrm{in}_w(I)$ will have the same Hilbert series, hence the same dimension and degree. See [40] for more details.

The goal is to construct initial ideals that can help in the study of the given variety. For generic choices of weights, the initial form of a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ will be a monomial. Tropical geometry focuses on weighs that

give rise to non-monomials initial ideals. In other words, in tropical geometry, we are interested in varieties inside the algebraic torus $(\mathbb{C}^*)^n$, i.e. defined inside a Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. From this point of view, in order to preserve the properties arising from the flatness of the degeneration, we need the initial ideals to be proper ideals in the Laurent polynomial ring. Formally, if $I \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is an ideal in a Laurent polynomial ring, then the *tropicalization* of I (as a set) is defined as

$$\operatorname{trop}(I) = \{w \in \mathbb{R}^n \mid \operatorname{in}_w(I) \text{ contains no monomials}\}.$$

It inherits the structure of a polyhedral fan from the *Gröbner fan* of I.

It is worth noticing that there is a more general approach to tropical geometry, via valuations. The construction outlined here coincides with choosing the trivial valuation on the Laurent polynomial ring. For more details about tropicalization of variety with non-trivial valuations see [81].

Definition 2.4.1 (Prime cone)**.** A cone C in the tropicalization $\operatorname{trop}(I)$ is said to be *prime* if $\operatorname{in}_w(I)$ is a prime ideal for $w \in C$.

We will be particularly interested in the study of cones in the tropicalization that give rise to prime and binomial ideals, i.e. toric ideals. Maximal prime cones in the tropicalization of a variety are good candidates.

2.4.2 Newton–Okounkov Bodies and Tropicalization

Prime cones in the tropicalization of a variety, give rise to valuations with finite Khovanskii bases. In particular, this gives a different construction of toric degenerations.

Valuations and Khovanskii bases

We start by recalling the definition of valuation and Khovanskii basis. For more details, see [66, 67].

Definition 2.4.2 (Valuation)**.** Let A be a $\mathbb{K}$-algebra and consider $\mathbb{Z}^r$ as an abelian group ordered by $\preceq$. A *valuation* is a map $\nu : A \to \mathbb{Z}^r$ such that:

(i) for all non-zero $f, g \in A$ such that $f+g \neq 0$, $\nu(f+g) \succeq \min_{\preceq}\{\nu(f), \nu(g)\}$;

(ii) for all non-zero $f, g \in A$, $\nu(fg) = \nu(f)\nu(g)$;

(iii) for all non-zero $f \in A$ and $c \in \mathbb{K}$, $\nu(cf) = \nu(f)$.

If the equality in (ii) is substituted by the requirement that $\nu(fg) \succeq \nu(f)\nu(g)$, then ν is said to be a *quasivaluation*.

Each valuation defines a graded structure on A as follows. Consider the $\mathbb{Z}^r$-filtration $\mathcal{F}_\nu = (F_{\nu\succeq a})_{a\in\mathbb{Z}^r}$ where for each $a \in \mathbb{Z}^r$

$$F_{\nu\succeq a} = \{f \in A \setminus \{0\} \mid \nu(f) \succeq a\} \cup \{0\}$$

and $F_{\nu\succ a}$ is defined similarly. The associated graded algebra structure is given by

$$\mathrm{gr}_\nu(A) = \bigoplus_{a\in\mathbb{Z}^r} F_{\nu\succeq a}/F_{\nu\succ a} \tag{2.3}$$

The *value semigroup* of A is the image through ν of non-zero elements of A:

$$S = S(A,\nu) = \{\nu(f) \mid f \in A \setminus \{0\}\}.$$

The *rank* of the valuation ν is the rank of the sublattice of $\mathbb{Z}^r$ generated by S.

In the following we will assume that $\Bbbk$ is an algebraically closed field, and ν is a full-rank valuation. Analogously to the case of initial ideals and Gröbner bases, $\mathrm{gr}_\nu(A)$ can be seen as a degeneration of A which, in principle, has simpler structure. In particular, we are interested in the cases where $\mathrm{gr}_\nu(A) \cong \Bbbk[S]$, which always happens if A is a domain [67, Proposition 2.4]. In this situation we can regard S as a combinatorial object to obtain information about A.

In order to continue the construction parallel to Gröbner degenerations, we need to consider generators of the algebra A which are also generators of its "degeneration".

Definition 2.4.3 (Khovanskii basis)**.** A set $\mathcal{K} \subseteq A$ is said to be a Khovanskii basis for (A,ν) if the image of $\mathcal{K}$ in $\mathrm{gr}_\nu(A)$ is a set of algebra generators for $\mathrm{gr}_\nu(A)$.

Note that in general such bases do not need to be finite. We are interested in the cases where they are.

Newton–Okounkov Bodies

We now introduce Newton–Okounkov bodies, convex bodies encoding information about the asymptotic behavior of the Hilbert function of A.

Definition 2.4.4 (Newton–Okounkov cone)**.** Let A be a domain and let $\nu : A\setminus\{0\} \to \mathbb{Z}^r$ be a valuation. The Newton–Okounkov cone $C(A,\nu)$ is the closure of the convex hull of S.

Let A be a graded domain, $A = \bigoplus_i A_i$ and let $\nu' : A \setminus \{0\} \to \mathbb{Z}^r$ be a valuation. Consider the order on $\mathbb{Z}^{r+1}$ such that for every $(m,a),(n,b) \in \mathbb{Z} \times \mathbb{Z}^r$, $(m,a) \succ (n,b)$ if and only if $m < n$ or $m = n$ and $a \succ b$. Define $\nu : A \setminus \{0\} \to \mathbb{N} \times \mathbb{Z}^r$ which refines the grading as follows. If $f \in A$ is of degree m, write as a sum of its homogeneous components, $f = \sum_{i=0}^{m} f_i$. Then

$$\nu(f) = (m, \nu'(f_m)).$$

Definition 2.4.5 (Newton–Okounkov body)**.** Let A, ν be as above. The *Newton–Okounkov body* is the intersection of the Newton–Okounkov cone with the hyperplane $\{1\} \times \mathbb{R}^r$:

$$\Delta(A,\nu) = \overline{\operatorname{conv}(S)} \cap (\{1\} \times \mathbb{R}^r)\,.$$

Newton–Okounkov bodies give rise to toric degenerations as follows [4, 67].

Theorem 2.4.6. *If (A,ν) has a finite Khovanskii basis, the corresponding Newton–Okounkov body is a rational polytope. Moreover, there is a toric degeneration of* $\operatorname{Proj}(A)$ *to a (not necessarily normal) toric variety, whose normalization is the toric variety defined by a large enough dilation of $\Delta(A,\nu)$.*

Weight Matrix Valuations and Prime Cones

The toric degenerations obtained as Gröbner degenerations from prime cones in the tropicalization can be constructed in the terms of Newton–Okounkov bodies as follows.

Fix an ordering $\succ$ on $\mathbb{Q}^k$ and a $k \times n$ matrix M with entries in $\mathbb{Q}$. M defines a valuation ν_M on $\mathbb{K}[x_1,\dots,x_n]$, defined for every $f \in \mathbb{K}[x_1,\dots,x_n]$ with $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$, by:

$$\nu_M(f) = \min_{\succ}\{M\alpha \mid c_\alpha \neq 0\}.$$

Let A be an algebra that can be written as a quotient of the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$. Let $\pi : \mathbb{K}[x_1,\dots,x_n] \to A$ denote the quotient map. Then ν_M induces a quasivaluation, which will be again denoted by ν_M, on A as follows:

$$\begin{aligned}\nu_M(f) &= \max_{\succ}\left\{\nu_M(\tilde{f}) \mid \tilde{f} \in \mathbb{K}[\mathbf{x}] \text{ s.t. } \pi(\tilde{f}) = f\right\} \qquad (2.4)\\ &= \max_{\succ}\left\{\min_{\succ}\{M\alpha \mid c_\alpha \neq 0\} \mid \tilde{f} = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbb{K}[\mathbf{x}] \text{ s.t. } \pi(\tilde{f}) = f\right\}.\end{aligned}$$

A prime cone in the tropicalization of the variety we are considering allows us to construct a matrix weight valuation as follows. Fix $I \subseteq \mathbb{K}[\mathbf{x}^{\pm 1}]$ a prime

ideal and let C be a maximal prime cone in the tropicalization $\mathrm{trop}(I)$. Let $\{u_1, \ldots, u_k\} \subset C$ and suppose that $\mathrm{span}(u_1, \ldots, u_n) = \mathrm{span}(C)$. Then the matrix M whose rows are given by u_i defines a valuation ν_M on $A = \mathbb{K}[\mathbf{x}^{\pm 1}]/I$. The corresponding value semigroup $S(A, \nu_M)$ is generated by the columns of M and therefore $C(A, \nu_M)$ is the cone generated by the columns of M. For more details, see [67, Section 4].

2.4.3 Tropical Grassmannian

One of the main goal of this thesis is the study of the tropical Grassmannian. Note that since we defined the tropicalization for ideals in a Laurent polynomial ring, we need to extend the Plücker ideal to the Laurent polynomial ring $\mathbb{C}[p_I^{\pm 1} \mid I \in \binom{[n]}{k}]$. This corresponds with intersecting the Grassmannian with the algebraic torus, which will be denoted by $\mathrm{Gr}(k,n)^\circ$. Similarly, for the partial flag varieties we will consider the open part $\mathcal{F}\ell(K;n)^\circ$ defined by taking the intersection with the algebraic torus. With this assumption, we will write $\mathrm{trop}(\mathrm{Gr}(k,n))$ for $\mathrm{trop}(\mathrm{Gr}(k,n)^\circ)$ and similarly for flag varieties.

The tropical Grassmannian has been topic of extensive study. An explicit characterization of the cones in the tropicalization is given for $\mathrm{Gr}(2,n)$ via metric trees [108]. In this case all the maximal cones are prime and binomial.

The Tropical Grassmannian (3,6)

An attempt at generalizing this to $\mathrm{Gr}(3,n)$ is via arrangement of trees [61]. However, the structure of the tropical Grassmannian gets more complicated. The tropicalization of $\mathrm{Gr}(3,6)$ contains 1035 maximal cones, which can be grouped into 7 symmetry classes. Recall from [108, Section 5] that these classes are as follows:

- (FFFGG) contains 45 maximal non-prime cones. The initial ideal with respect to a weight in these cones has degree 38;
- (EEEE) contains 30 maximal prime cones;
- (EEFF1) contains 90 maximal prime cones;
- (EEFF2) contains 90 maximal prime cones;
- (EFFG) contains 180 maximal prime cones;
- (EEEG) contains 240 maximal prime cones;

- (EEFG) contains 360 maximal prime cones.

In particular, we can see that it is no longer true that all the maximal cones are prime.

Higher Grassmannians

The size of the tropicalization of Grassmannians increases very fast as n and k increase. The tropical Grassmannian $\mathrm{trop}(\mathrm{Gr}(3,7))$ contains 252000 maximal cones. They form 125 equivalence classes up to symmetry, 69 of which correspond to prime cones [61].

The tropical Grassmannian $\mathrm{trop}(\mathrm{Gr}(3,8))$ contains 14763 maximal cones up to symmetry [12].

Chapter 3

Positroids and Essential Sets

In this chapter we will focus on positroids and the positroid stratification of the Grassmannian. In particular, we will introduce a new family of invariants associated to positroids and give an axiomatic characterization of such a family. We then proceed with proving applications of this new description in the study of the positroid stratification of the Grassmannian.

Sections 3.1 to 3.4 of this chapter are based on joint work with Fatemeh Mohammadi, [92].

3.1 Essential Sets

Essential sets have been introduced by Fulton [51], in the study of degeneracy loci of maps of flagged vector bundles. This problem involved considering some rank conditions defining an irreducible variety. His goal was to characterize when some rank conditions are compatible and how to specify them efficiently.

As seen in Section 2.2.2, positroids are uniquely defined by their rank on cyclic intervals. From this point of view, the generalization of essential sets to positroids gives a way to efficiently specify the rank condition on cyclic intervals so that the positroid is uniquely defined. The axiomatic characterization of essential sets for positroids will describe when the rank conditions contained in this family are compatible.

3.1.1 Essential Sets and Bounded Affine Permutations

Notation 3.1.1. Consider a $k \times n$ array of squares in the plane. We will always consider the rows modulo k, i.e., like a matrix vertically infinitely repeating. The square at coordinates (i, j) corresponds to the square contained in row i and column j of the array. Let R_i be the i-th row of the array and $A_{(i,j)}$ be the antidiagonal containing the square (i, j). The *sub-antidiagonal* $\Delta_{(i,j)}$ of the square (i, j) is the set of squares in the array contained in the antidiagonal starting at the square $(i + 1, j - 1)$ and ending in the first column, with rows considered modulo k. That is, $\Delta_{(i,j)}$ is the squares in the antidiagonal containing $(i + 1, j - 1)$ and the boxes obtained going down and to the left of it. Formally, it is defined as:

$$\Delta_{(i,j)} = \{(i + \ell \mod k, j - \ell) \mid 0 < \ell < j\}.$$

If $(i, j) = (i, 1)$, we have $\Delta_{(i,j)} = \emptyset$. For a square (i, j), we denote by $T_{(i,j)}$ the set of squares contained in the triangle defined by the horizontal segment on the left of (i, j) and the sub-antidiagonal $\Delta_{(i,j)}$. We denote by $P_{(i,j)}$ the set of squares to the right of the sub-antidiagonal $\Delta_{(i,j)}$ and contained between the horizontal segment starting to the right of (i, j) and the row starting at the end of $\Delta_{(i,j)}$. See Figure 3.1.

Let π be a bounded affine permutation of size n. We associate to it a dotted array as follows: consider an $n \times (n + 1)$ array of squares in the plane. For each $i \in [n]$, place a dot inside the square $(i, \pi(i) - (i - 1))$, while all other squares remain white. This is well-defined since $i \leq \pi(i) \leq i + n$. We denote by $D(\pi)$ the set of squares in the dotted array of π containing a dot, i.e., $D(\pi) = \{(i, \pi(i) - i + 1) \mid i \in [n]\}$.

Definition 3.1.2 (Diagram)**.** The *diagram of the bounded affine permutation* π is constructed from the dotted array by shading. Specifically, for every dot in the array, the squares on the same row to the strict left of the dot and in its sub-antidiagonal are shaded. The diagram of π consists of the remaining white squares.

A white square is called a *corner* if no white square shares its top-right corner. The *essential family* $\mathcal{E}(\pi)$ of π is defined as the set of cyclic intervals $[i, j]$ for each white corner $(i, j - i + 1)$ in the diagram of π. Each element in $\mathcal{E}(\pi)$ is

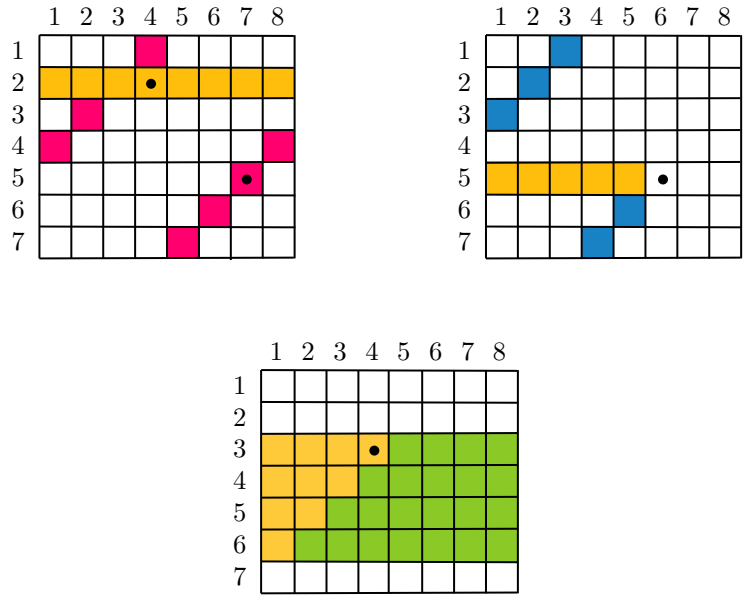


Figure 3.1: (Top-Left) Yellow represents row R_2, and pink the antidiagonal $A_{(5,7)}$ passing through $(5, 7)$.
(Top-Right) Yellow indicates squares on the same row, strictly to the left of $(5, 6)$, while blue signifies the sub-antidiagonal $\Delta_{(5,6)}$.
(Bottom) Yellow depicts $T_{(3,4)}$, and green the set of squares $P_{(3,4)}$. See Notation 3.1.1.

referred to as an *essential set*. Equivalently:

$$\mathcal{E}(\pi) = \{[i, j] \mid (i, j - i + 1) \text{ is a corner of the diagram of } \pi\}$$
$$= \left\{ [i, j] \mid i \in [n], i \le j \le i + n, \quad \begin{array}{c} \pi(i) \le j,\ \pi(i-1) > j \\ \pi^{-1}(j) \le_i j,\ j+1 <_i \pi^{-1}(j+1) \end{array} \right\}. \tag{3.1}$$

A similar construction of essential sets was described in [69] for partial permutations, in the study of interval positroid varieties.

Example 3.1.3. Consider the bounded affine permutation of size 8, defined by $\pi = (3\ 4\ 8\ 7\ 6\ 9\ 10\ 13)$. The diagram of the permutation is depicted in Figure 3.2 (left). The corners of the diagram are $(1, 4), (4, 4), (5, 2)$. Hence, the essential family is given by

$$\mathcal{E}(\pi) = \{[1, 4], [4, 7], [5, 6]\}.$$

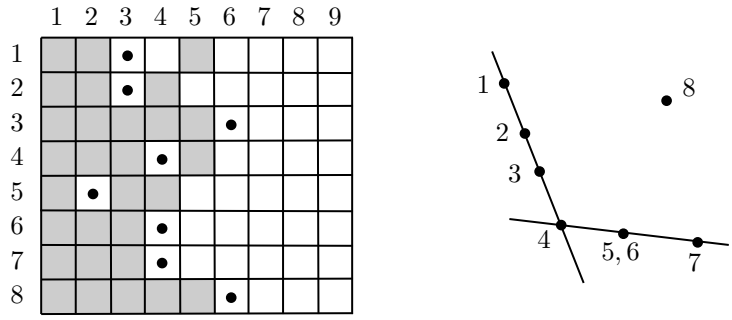


Figure 3.2: The diagram and the point-line configuration of the positroid corresponding to π from Example 3.1.3

The positroid corresponding to π has rank 3 and can be represented by the point-line configuration in Figure 3.2 (right). Note that the essential family captures the sets lying on a line and the parallel elements. However, to distinguish between them, we need to know their rank.

The properties of positroids that can be read through their associated bounded affine permutations can also be observed in their diagrams.

Proposition 3.1.4. *Let π be a bounded affine permutation of size n, and $\mathcal{P}$ its associated positroid. Then:*

1. *The rank of $\mathcal{P}$ is equal to the number of dots in $P_{(1,n)}$.*
2. *The rank of a cyclic interval $[i,j] \subseteq [n]$ is given by the number of dots in $P_{(i,j-i+1)}$.*

Proof. From Proposition 2.2.21, the rank of $[i,j]$ is the number of $\ell \in [i,j]$ such that $\pi(\ell) > j$. This is exactly the number of dots in $P_{(i,j)}$. □

Definition 3.1.5 (Ranked essential family)**.** Let π be a bounded affine permutation of order n. We define the *ranked essential family* $\mathcal{E}^{\mathrm{r}}(\pi)$ of π to be the family

$$\mathcal{E}^{\mathrm{r}}(\pi) = \left\{(r,[i,j]) \mid [i,j] \in \mathcal{E}(\pi) \cup \{[1,n]\},\ r = \text{ number of dots in } P_{(i,j-i+1)}\right\}.$$

Example 3.1.6. Consider the permutation from Example 3.1.3. Its ranked essential family is

$$\mathcal{E}^{\mathrm{r}}(\pi) = \{(1,[5,6]),(2,[1,4]),(2,[4,7]),(3,[1,8])\}.$$

Example 3.1.7. Let $\mathcal{U}_{k,n}$ be the uniform matroid of rank k on n elements. The associated bounded affine permutation is defined by $\pi_{k,n}(i) = i + k$, for every $i \in \mathbb{Z}$. It is easy to see that the set of corners of its diagram is empty, hence $\mathcal{E}^{\mathrm{r}}(\pi_{k,n}) = \{(k, [1, n])\}$.

3.1.2 Essential Sets and Rank Function

The ranked essential family can be interpreted as the rank information of "closed" dependent cyclic interval. Indeed, the essential sets can be constructed from the rank function of the positroid as follows.

Theorem 3.1.8. *Let $\mathcal{P}$ be a positroid on $[n]$ of rank k and $\mathcal{R}$ the set of pairs $(r, [i, j])$, such that*

- *if $i \neq j$, then $r = \mathrm{rk}([i, j]) = \mathrm{rk}([i+1, j]) = \mathrm{rk}([i, j-1]) = \mathrm{rk}([i-1, j]) - 1 = \mathrm{rk}([i, j+1]) - 1$.*
- *If $i = j$, then $0 = r = \mathrm{rk}([i, i]) = \mathrm{rk}([i-1, i]) - 1 = \mathrm{rk}([i, i+1]) - 1$.*

Then $\mathcal{E}^{\mathrm{r}} = \mathcal{R} \cup \{(k, [1, n])\}$.

Proof. Let π be the bounded affine permutation associated to the positroid $\mathcal{P}$. Let $(r, [i, j]) \in \mathcal{R}$. Since $\mathrm{rk}([i, j]) = \mathrm{rk}([i + 1, j])$, we have $\pi(i) \leq j$. Similarly, since $\mathrm{rk}([i - 1, j]) = \mathrm{rk}([i, j]) + 1$ and $i - 1 \notin \langle i, \ldots, j \rangle$, we have $\pi(i - 1) > j$. It remains to prove that $\pi^{-1}(j) \leq_i j$ and $\pi^{-1}(j + 1) >_i j + 1$. The condition $\mathrm{rk}([i, j]) = \mathrm{rk}([i, j - 1])$ is equivalent to the fact that the number of dots in $P_{(i,j-i+1)}$ and in $P_{(i,j-i)}$ is the same. Then either $\pi(j) = j$ or $\pi(j) > j$. In the latter case, $P_{(i,j-i+1)}$ contains a dot in the last row, which is not part of $P_{(i,j-i)}$. Hence, there must be a dot in $P_{(i,j-i)} \setminus P_{(i,j-i+1)}$, i.e. $\pi^{-1}(j) <_i j$. It follows that $\pi^{-1}(j) \leq_i j$. Finally, $\mathrm{rk}([i, j]) = \mathrm{rk}([i, j + 1]) - 1$ implies that the number of dots in $P_{(i,j-i+2)}$ is one more than the number of dots in $P_{(i,j-i+1)}$. Note that $j + 1$ is not a loop, hence $\pi(j + 1) > j + 1$, and $P_{(i,j-i+2)}$ contains a dot in the row $j + 1$, which is not in $P_{(i,j-i+1)}$. For the equality to hold there must be no dot in $P_{(i,j-i+1)} \setminus P_{(i,j-i+2)}$, i.e., $\pi^{-1}(j + 1) >_i j + 1$. It follows that $[i, j]$ is an essential set of π.

Conversely, let $(r, [i, j]) \in \mathcal{E}^{\mathrm{r}} \setminus \{(k, [1, n])\}$. By definition, $r = \mathrm{rk}([i, j])$ is the number of dots in $P_{(i,j-i+1)}$. Moreover, $\mathrm{rk}([i + 1, j])$ is equal to the number of dots in $P_{(i+1,j-i)}$, which is equal to the number of dots in $P_{(i,j-i+1)}$ since there is no dot in the first row of this region by definition of essential set. Therefore, $\mathrm{rk}([i + 1, j]) = r$. In the same way, since $\pi(i - 1) > j$, the number

of dots in $P_{(i-1,j-i+2)}$ is equal to one more than the dots in $P_{(i,j-i+1)}$, hence $\mathrm{rk}([i-1,j]) = r+1$. Since $\pi^{-1}(j) \in [i,j]$, if $\pi^{-1}(j) = j$, then j is a loop and $\mathrm{rk}([i,j-1]) = \mathrm{rk}([i,j])$. If $\pi^{-1}(j) \in [i,j-1]$, then there is a dot in $P_{(i,j-i)} \setminus P_{(i,j-i+1)}$ as well as one dot in $P_{(i,j-i+1)} \setminus P_{(i,j-i)}$, hence $\mathrm{rk}([i,j-1]) = \mathrm{rk}([i,j])$. Finally, since $\pi^{-1}(j+1) < i$, the squares in the left-most antidiagonal intersecting $P_{(i,j-i+1)}$ do not contain a dot. Hence, $\mathrm{rk}([i,j+1])$ is given by the number of dots in $P_{(i,j-i+1)}$ plus the number of dots contained in the last row of $P_{(i,j-i+2)}$, i.e. $\mathrm{rk}([i,j+1]) = r+1$. It follows that $[i,j]$ satisfies the conditions in eq. (3.1), i.e., $(r,[i,j]) \in \mathcal{E}^{\mathrm{r}}$. □

The rank of a positroid on any cyclic interval can be computed via its ranked essential family as follows.

Theorem 3.1.9. *Let $\mathcal{P}$ be a positroid and $\mathcal{E}^{\mathrm{r}}$ its ranked essential family. For any $[i,j] \subseteq [n]$, we have:*

$$\mathrm{rk}([i,j]) = \min\{|[i,j]|, r + |[i,j] \setminus I| \mid (r,I) \in \mathcal{E}^{\mathrm{r}}\}.$$

Proof. Denote by $m_{[i,j]}$ the quantity on the right-hand side of the equality. Note that $\mathrm{rk}([i,j]) \leq m_{[i,j]}$.

Suppose $m_{[i,j]} = |[i,j]|$ and suppose by contradiction that $\mathrm{rk}([i,j]) < m_{i,j} = |[i,j]|$. Then there exists a circuit $C \subseteq [i,j]$ (a minimally dependent set). Let $a = \min_{<_i} C$ and $b = \max_{<_i} C$ and note that $[a,b]$ is such that $\mathrm{rk}([a,b]) = \mathrm{rk}([a+1,b]) = \mathrm{rk}([a,b-1])$. Then there exists an essential set (r,I) with $r = \mathrm{rk}([a,b]) \leq |[a,b]| - 1$ and $[a,b] \subseteq I$. Note that $r + |[i,j] \setminus I| \leq |[a,b]| - 1 + |[i,j] \setminus [a,b]| < |[i,j] = m_{[i,j]}$. Hence, we get a contradiction and $\mathrm{rk}([i,j]) = m_{[i,j]} = |[i,j]|$.

Suppose that $m_{[i,j]} = r + |[i,j] \setminus I|$ for some essential set (r,I). Again, suppose by contradiction that $\mathrm{rk}([i,j]) < m_{i,j}$. Suppose $[i,j] \setminus I \neq \emptyset$. Then, there exists a circuit $C \subseteq [i,j] \cup I$, with $C \not\subseteq I$. Let $a = \min_{<_i} C$ and $b = \max_{<_i} C$ and note that $[a,b]$ is such that $\mathrm{rk}([a,b]) = \mathrm{rk}([a+1,b]) = \mathrm{rk}([a,b-1])$ and at least one index between a and b is not in I. Then, there exists an essential set (r',J) with $[a,b] \cup I \subseteq J$ and $r' \leq r + |(J \cap [i,j]) \setminus I| - 1$. Moreover, we have that:

$$r' + |[i,j] \setminus J| \leq r + |(J \cap [i,j]) \setminus I| - 1 + |[i,j] \setminus J| = r + |[i,j] \setminus I| - 1 < m_{[i,j]}.$$

Finally, suppose $[i,j] \subseteq I$ and that $\mathrm{rk}([i,j]) < m_{i,j} = r$. In particular, $r \leq |[i,j]|$. If $r = |[i,j]|$, then the same reasoning as in the first case works. Hence, assume that $r < |[i,j]|$. Let F be the union of circuits contained in $[i,j]$, then $F \neq \emptyset$, since $\mathrm{rk}([i,j]) < |[i,j]|$. Let $[a,b]$ be the smallest cyclic interval containing F. Since $\mathrm{rk}([a,b]) = \mathrm{rk}([a+1,b]) = \mathrm{rk}([a,b-1])$, there exists an essential set (r',J) with $r' = \mathrm{rk}([a,b]) < |[a,b]|$ and $J \supseteq [a,b]$. Then

$\mathrm{rk}([i,j]) = r' + |[i,j] \setminus J| < r = m_{[i,j]}$, contradicting the hypothesis of minimality on (r, I). □

Since a positroid is uniquely defined by its rank on every cyclic interval, it follows that the ranked essential family of a positroid uniquely determines the positroid.

Definition 3.1.10 (Connected essential set)**.** An essential set $(r, [i,j]) \in \mathcal{E}^{\mathrm{r}}$ is said to be *connected* if there exist no pairwise disjoint sets

$$(r_1, [i_1, j_1]), \ldots, (r_m, [i_m, j_m]) \in \mathcal{E}^{\mathrm{r}}$$

such that

$$r = r_1 + \cdots + r_m + \left| [i,j] \setminus \bigcup_{a=1}^{m} [i_a, j_a] \right| \quad \text{and} \quad [i,j] \supseteq \bigcup_{a=1}^{m} [i_a, j_a].$$

We denote the set of connected ranked essential sets by $\mathcal{CE}^{\mathrm{r}}$.

Remark 3.1.11. From the definition of connected essential sets and Theorem 3.1.9, it follows immediately that for every cyclic interval $[i,j]$ we have

$$\mathrm{rk}([i,j]) = \min \left\{ \sum_{a=1}^{m} r_a + \left| [i,j] \setminus \bigcup_{a=1}^{m} [i_a, j_a] \right| \mid (r_a, [i_a, j_a]) \in \mathcal{CE}^{\mathrm{r}} \cup \{(0, \emptyset)\} \right\}.$$

In particular, the rank conditions contained in connected essential intervals are sufficient to reconstruct the positroid. We will show in §3.1.4 that we can use an even smaller family of minimal rank conditions.

The rank conditions given by connected essential sets define the facets of the corresponding positroid polytope.

Corollary 3.1.12. *Let $\mathcal{P}$ be a positroid and let $\mathcal{E}^{\mathrm{r}}$ be its ranked essential family. Then the positroid polytope of $\mathcal{P}$ is given by the facets:*

$$Q_{\mathcal{P}} = \left\{ x \in \mathbb{R}^n \mid \begin{array}{c} 0 \le x_i \le 1 \, \forall i \in [n], \, \sum_{i \in [n]} x_i = k, \\ \sum_{\ell \in [i,j]} x_\ell \le r \; \forall (r, [i,j]) \in \mathcal{CE}^{\mathrm{r}} \end{array} \right\}.$$

Proof. Let $P_{\mathcal{P}}$ be the associated positroid polytope. Note that $x \in P_{\mathcal{P}}$ satisfies all the inequalities defining $Q_{\mathcal{P}}$, hence $P_{\mathcal{P}} \subseteq Q_{\mathcal{P}}$. Let $x \in Q_{\mathcal{P}}$ and $[i,j] \subseteq [n]$. By Remark 3.1.11, we have that $\mathrm{rk}([i,j]) = \sum_{a=1}^{m} r_a + \left| [i,j] \setminus \bigcup_{a=1}^{m} [i_a, j_a] \right|$ for

some pairwise disjoint intervals $(r_1, [i_1, j_1]), \ldots, (r_m, [i_m, j_m]) \in \mathcal{E}^{\mathrm{r}}$. Then

$$\sum_{\ell \in [i,j]} x_\ell = \sum_{a=1}^{m} \sum_{\ell \in [i,j] \cap [i_a, j_a]} x_\ell + \sum_{\ell \in [i,j] \setminus \bigcup_{a=1}^{m} [i_a, j_a]} x_\ell$$

$$\leq \sum_{a=1}^{m} r_a + \left| [i,j] \setminus \bigcup_{a=1}^{m} [i_a, j_a] \right| = \mathrm{rk}([i,j]).$$

Hence, $x \in P_{\mathcal{P}}$, as desired. □

3.1.3 Properties of Essential Sets

The ranked essential family contains the non-trivial rank information of a positroid. The elements of this set satisfy compatibility conditions as follows.

Theorem 3.1.13. *Let $\mathcal{E}^{\mathrm{r}}$ be the ranked essential family of a positroid $\mathcal{P}$ of rank k on the ground set $[n]$. Then, the following holds.*

(E1) *$(k, [1, n]) \in \mathcal{E}^{\mathrm{r}}$ for a unique $0 < k < n$ and for each $(r, [i, j]) \in \mathcal{E}^{\mathrm{r}}$, we have*

$$|[i,j]| > r \geq 0 \quad \textit{and} \quad |[n] \setminus [i,j]| \geq k - r > 0.$$

(E2) *For every district pair $(r_1, [i_1, j_1]), (r_2, [i_2, j_2]) \in \mathcal{E}^{\mathrm{r}} \setminus \{(k, [1, n])\}$ such that $[i_1, j_1] \subsetneq [i_2, j_2]$, we have*

$$0 < r_2 - r_1 < |[i_2, j_2] \setminus [i_1, j_1]|.$$

(E3) *Let $(r_1, [i_1, j_1]), (r_2, [i_2, j_2]) \in \mathcal{E}^{\mathrm{r}} \setminus \{(k, [1, n])\}$ such that $i_2 <_{i_1} j_2$ and $j_1 <_{i_1} j_2$.*

   (1) *If $[i_1, j_1] \cap [i_2, j_2] = \emptyset$, i.e. $j_1 <_{i_1} i_2$ and $j_2 <_{i_2} i_1$, let $(r_3, [i_3, j_3])$ be a minimal essential set containing $[i_1, j_2]$ and $(r_4, [i_4, j_4])$ a maximal essential set contained in $[j_1+1, i_2-1]$. If there is no such $(r_4, [i_4, j_4])$, then let it be $(0, \emptyset)$. Then*

$$r_1 + r_2 \geq r_3 - r_4 - |[j_1 + 1, i_2 - 1] \setminus [i_4, j_4]|.$$

   (2) *If $[i_1, j_1] \cap [i_2, j_2]$ contains $[i_2, j_1]$, i.e. $i_2 <_{i_1} j_1$, let $(r_3, [i_3, j_3])$ be a minimal essential set containing $[i_1, j_1] \cup [i_2, j_2]$ and $(r_4, [i_4, j_4])$ a maximal essential set contained in $[i_2, j_1]$. If there is no such $(r_4, [i_4, j_4])$, then let it be $(0, \emptyset)$. Then*

$$r_1 + r_2 \geq r_3 + r_4 + |[i_2, j_1] \setminus [i_4, j_4]|.$$

Proof. By construction of the essential family we have that $(k, [1, n]) \in \mathcal{E}^{\mathrm{r}}$. Moreover, if $(r, [i, j]) \in \mathcal{E}^{\mathrm{r}} \setminus \{(k, [1, n])\}$, then $0 \leq r = \mathrm{rk}([i, j]) = \mathrm{rk}([i+1, j]) \leq |[i, j]| - 1$.

Let $(r_1, [i_1, j_1]), (r_2, [i_2, j_2]) \in \mathcal{E}^{\mathrm{r}}$ be distinct ranked essential sets such that $[i_1, j_1] \subsetneq [i_2, j_2]$. Suppose $i_1 - 1 \in [i_2, j_2]$ (similar argument works if $j_1 + 1 \in [i_2, j_2]$), then $r_2 = \mathrm{rk}([i_2, j_2]) \geq \mathrm{rk}([i_1 - 1, j_1]) = \mathrm{rk}([i_1, j_1]) + 1 > r_1$, hence $r_2 - r_1 > 0$. Moreover, the following relation holds.

$$r_2 - r_1 \leq |\{\text{dots in } P_{(i_2, j_2 - i_2 + 1)} \setminus P_{(i_1, j_1 - i_1 + 1)}\}| \leq |[i_2, j_2] \setminus [i_1, j_1]|. \tag{3.2}$$

Suppose $[i_2, j_2] \neq [1, n]$ is an essential set, then the dot in the antidiagonal through $(i_2, j_2 - i_2 + 1)$ will be in row h for $i_2 <_{i_2} h <_{i_2} j_2$. If $h \in [i_1, j_1]$, then the number of dots in $P_{(i_2, j_2 - i_2 + 1)} \setminus P_{(i_1, j_1 - i_1 + 1)}$ is strictly larger than $r_2 - r_1$. If $h \in [i_2, j_2] \setminus [i_1, j_1]$, then the number of dots in $P_{(i_2, j_2 - i_2 + 1)} \setminus P_{(i_1, j_1 - i_1 + 1)}$ is strictly smaller than $|[i_2, j_2] \setminus [i_1, j_1]|$. Hence, one of the two inequalities from eq. (3.2) is strict, i.e., $r_2 - r_1 < |[i_2, j_2] \setminus [i_1, j_1]|$.

Let $(r_1, [i_1, j_1]), (r_2, [i_2, j_2])$ be as in the assumption of (E3). Let us prove some facts about the rank of the cyclic intervals involved in the statement.

Claim 1. There exists an essential set $(r, [a, b])$ containing $[i_1, j_1] \cup [i_2, j_2]$ of rank $\mathrm{rk}([i_1, j_2])$. Indeed, if $[i_1, j_2] = [1, n]$, then it is the essential set $(k, [1, n])$. Otherwise, $\mathrm{rk}([i_1, j_2]) = \mathrm{rk}([i_1 + 1, j_2]) = \mathrm{rk}([i_1, j_2 - 1])$ is either equal to k, hence $[a, b] = [1, n]$, or is strictly smaller than k and so $[a, b]$, the maximal set containing $[i_1, j_2]$ with the same rank, will be an essential set.

Claim 2. Suppose $i_2 <_{i_1} j_1$, that is, $[i_1, j_1]$ and $[i_2, j_2]$ intersect in $[i_2, j_1]$. Let $(r, [a, b])$ be the maximal essential set contained in $[i_2, j_1]$, then $\mathrm{rk}([i_2, j_1]) = r + |[i_2, j_1] \setminus [a, b]|$. First note that by Claim 1 there is indeed a unique maximal essential set contained in $[i_2, j_1]$. Moreover, $r + |[i_2, j_1] \setminus [a, b]| \leq |[i_2, j_1]|$. Suppose there exists an essential set $(r', [c, d])$ such that $r' + |[i_2, j_1] \setminus [c, d]| < r + |[i_2, j_1] \setminus [a, b]|$. Then, by Claim 1 and property (E2), $[c, d] \not\subseteq [i_2, j_1]$. Moreover, $[i_2, j_1] \not\subseteq [c, d]$ since otherwise $\mathrm{rk}([i_2, j_1]) = r'$, and by Theorem 3.1.8, $[c, d]$ would be contained in both $[i_1, j_1]$ and $[i_2, j_2]$, hence $[i_2, j_1] = [c, d] = [a, b]$. Then suppose, without loss of generality, that $[i_2, j_1] \cap [c, d] = [c, j_1]$. Since $r' + |[i_2, j_1] \setminus [c, d]| < |[i, j]|$, we have that $|[c, j_1]| > r'$ and $[c, j_1]$ does not contain an essential set. Then $\mathrm{rk}([c, j_1]) = r'$ and $[c, d]$ is contained in $[i_1, j_1] \cap [i_2, j_2]$. Hence, $\mathrm{rk}([i_2, j_1]) = r + |[i_2, j_1] \setminus [a, b]|$.

Let us now consider the two cases separately.

(1) Suppose $[i_1, j_1] \cap [i_2, j_2] = \emptyset$. By Claim 1, $(r_3, [i_3, j_3])$ has the same rank as $[i_1, j_2]$. Moreover, by Theorem 3.1.9, $\mathrm{rk}([j_1 + 1, i_2 - 1]) \leq r_4 + |[j_1 +$

$1, i_2 - 1] \setminus [i_4, j_4]|$. Hence,

$$r_1 + r_2 = \mathrm{rk}([i_1, j_1]) + \mathrm{rk}([i_2, j_2]) \geq \mathrm{rk}([i_1, j_2]) - \mathrm{rk}([j_1 + 1, i_2 - 1])$$
$$\geq r_3 - r_4 - |[j_1 + 1, i_2 - 1] \setminus [i_4, j_4]|.$$

(2) Suppose $[i_1, j_1]$ and $[i_2, j_2]$ intersect in $[i_2, j_1]$. Then, by Claim 1 and Claim 2, it follows that

$$r_1 + r_2 = \mathrm{rk}([i_1, j_1]) + \mathrm{rk}([i_2, j_2]) \geq \mathrm{rk}([i_1, j_1] \cup [i_2, j_2]) + \mathrm{rk}([i_2, j_1])$$
$$= r_3 + r_4 + |[i_2, j_1] \setminus [i_4, j_4]|$$

for $(r_3, [i_3, j_3]), (r_4, [i_4, j_4])$ as in the statement.

□

Remark 3.1.14. The properties from Theorem 3.1.13, can be interpreted as follows.

(E1) If (r, I) is a ranked essential set, then the rank is non-negative and the vectors in I are dependent.

(E2) A ranked essential set (r, I) contains all the vectors in the closure of I in the same cyclic interval as I. Hence, if there is a ranked essential set (r_2, I_2) containing I, its rank will be strictly larger and I_2 contains more dependency conditions than the ones in (r, I).

(E3) This condition corresponds to the sub-modular inequality for the rank function:

$$\mathrm{rk}(A) + \mathrm{rk}(B) \geq \mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B),$$

in terms of ranked essential sets.

Note that if two essential sets $(r_1, [i_1, j_1])$ and $(r_2, [i_2, j_2])$ intersect in two cyclic intervals, that is, $[i_1, j_1] \cap [i_2, j_2] = [i_1, j_2] \cup [i_2, j_1]$, by Theorem 25 of [84], the rank $\mathrm{rk}([i_1, j_2] \cup [i_2, j_1])$ of the intersection is given by

$$\min\{r_1 + r_2 - k, r_3 + |[i_1, j_2] \setminus [i_3, j_3]| + r_4 + |[i_2, j_1] \setminus [i_4, j_4]|\},$$

where $(r_3, [i_3, j_3])$ is the maximal essential set contained in $[i_1, j_2]$ and $(r_4, [i_4, j_4])$ is the maximal essential set contained in $[i_2, j_1]$. Hence, no additional condition is required for this case.

In Theorem 3.2.3, we will show that these properties are sufficient for a family of pairs consisting of a non-negative integer and a cyclic interval to form the ranked essential family of a positroid.

3.1.4 Minimal Rank Conditions

The rank conditions within the ranked essential family of a positroid are not minimal. For instance, consider the sets $(r_1, [i, j])$ and $(r_2, [j+1, h])$ of a rank k positroid, with $r_1 + r_2 < k$. If the smallest essential set containing $[i, h]$ is exactly $[i, h]$ and its rank is $r_1 + r_2$, then the information contained in $(r_1 + r_2, [i, h])$ is not necessary.

To restrict the ranked essential family to minimal conditions, we introduce a constant associated with each essential set. These constants intuitively count the number of additional dependent points in each essential set. We define them inductively as follows.

Definition 3.1.15 (Excess)**.** Let $\mathcal{E}^{\mathrm{r}}$ be a ranked essential family. For each ranked essential set $(r, I) \in \mathcal{E}^{\mathrm{r}}$, let

$$e_I = |I| - r - \sum_{J \in \mathcal{E}, J \subsetneq I} e_J.$$

The quantity e_I is called *excess* of the essential set I.

If the excess of a set vanishes, it indicates that there is no new information on the dependencies. Hence, we give the following definition.

Definition 3.1.16 (Core)**.** Let $\mathcal{E}^{\mathrm{r}}$ be a ranked essential family. The *core* $\mathcal{C}^{\mathrm{r}}$ of $\mathcal{E}^{\mathrm{r}}$ is the subset given by essential sets with positive excess:

$$\mathcal{C}^{\mathrm{r}} = \{(r, I) \mid e_I > 0\}.$$

Example 3.1.17. Consider the positroid from [14, Example 1]. $\mathcal{P}$ is defined to be the parallel connection of four copies of $\mathcal{U}_{2,3}$ on $123, 456, 789, 369$. The bounded affine permutation associated to $\mathcal{P}$ is $\pi = (3\ 10\ 8\ 6\ 13\ 11\ 9\ 16\ 14)$ and the ranked essential family is

$$\mathcal{E}^{\mathrm{r}} = \{(2, [1, 3]), (2, [4, 6]), (2, [7, 9]), (4, [3, 9]), (4, [6, 3]), (4, [9, 6]), (5, [1, 9])\}.$$

This example illustrates an essential family where the smallest essential set containing the union of two essential intervals is strictly larger than the union. Specifically, the smallest essential set containing $[1, 3] \cup [4, 6]$ is $[9, 6] \supsetneq [1, 6]$. In this example, all rank conditions imposed by essential sets are necessary to define $\mathcal{P}$. Moreover, the excess of each essential set is equal to 1 and $\mathcal{E}^{\mathrm{r}} = \mathcal{C}^{\mathrm{r}}$. From the rank constraints on the essential sets, we can recover $\mathrm{rk}(\{3, 6, 9\}) = 2$ by applying [84, Theorem 25].

Remark 3.1.18. Let $\mathcal{E}^{\mathrm{r}}$ be a ranked essential family. If an essential set $(r, [i, j]) \in \mathcal{E}^{\mathrm{r}}$ is disconnected, i.e., there exists pairwise disjoint essential

sets $(r_1, [i_1, j_1]), \ldots, (r_m, [i_m, j_m]) \in \mathcal{E}^{\mathrm{r}}$ such that $r = r_1 + \cdots + r_m + |[i,j] \setminus \bigcup_{a\in[m]}[i_a, j_a]|$ and $[i,j] \supseteq \bigcup_{a\in[m]}[i_a, j_a]$, then $(r, [i,j])$ is not in the core. Indeed,

$$e_{[i,j]} \leq |[i,j]| - r - \sum_{a\in[m]} (|[i_a, j_a]| - r_a) = 0.$$

However, in general the core $\mathcal{C}^{\mathrm{r}}$ is strictly smaller than the family of ranked connected essential sets $\mathcal{CE}^{\mathrm{r}}$. Consider for example the positroid with ranked essential family

$$\mathcal{E}^{\mathrm{r}} = \{(2, [1,3]), (2, [3,5]), (3, [1,5]), (4, [1,6])\}.$$

Then every essential set is connected, but the core is

$$\mathcal{C}^{\mathrm{r}} = \{(2, [1,3]), (2, [3,5]), (4, [1,6])\}.$$

Lemma 3.1.19. *Let f be a bounded affine permutation and let $\mathcal{E}^{\mathrm{r}}$ be the associated ranked essential family. For every $[i,j]$ cyclic interval, $e_{[i,j]} \geq 0$.*

Proof. By definition of excess, for any cyclic interval $[i,j]$, $e_{[i,j]}$ counts the number of dots in $T_{(i,j-i+1)}$ which are not contained in $T_{(k,\ell-k+1)}$ for any $\mathcal{E} \ni [k,\ell] \subsetneq [i,j]$. □

3.1.5 The Retrieval Algorithm

Here, we provide a robust algorithm to recover the positroid defined by certain rank conditions. The algorithm outputs a bounded affine permutation whenever the input contains rank conditions on cyclic intervals which are realizable by a positroid.

Definition 3.1.20 (Proper dotting)**.** Consider an $n \times (n+1)$ array of squares in the plane. A subset D of squares in the array is said to be a *proper dotting* if D contains at most one square in each row of the array and at most one square in each antidiagonal of the array. A proper dotting is called *maximal* if it contains exactly one square in each row of the array. The *rank* and the *excess* of a square (i,j) with respect to a proper dotting D are defined as follows:

$$r_D(i,j) = |D \cap P_{(i,j)}| \quad \text{and} \quad e_D(i,j) = |D \cap T_{(i,j)}|.$$

Remark 3.1.21. A maximal proper dotting of a $n \times (n+1)$ array corresponds uniquely to a bounded affine permutation π of size n, given by $\pi(i) = j + i - 1$ for $i \in [n]$, where $j \in [n+1]$ is such that $(i,j) \in D$. Hence, a maximal proper dotting defines a positroid.

Description of Algorithm 2. The algorithm takes as input E^{r}, a subset of the array with non-negative integer labels, containing the required rank conditions. Its objective is to identify, if feasible, such a positroid satisfying such rank conditions. In the process, $\mathcal{S}$ represents the shaded squares and D contains the position of the dots added to the array at any point during the algorithm. Additionally, the algorithm initializes r to 0.

Consider a labeled square $(r,(i,j)) \in E^{\mathrm{r}}$. Define $a_D(i,j) = j - e_D(i,j) - r$. Let $h \in [i,j]$ be the smallest row, according to the order $<_i$, not containing a dot. We consider two cases:

1. If there exists $\ell \in [h+1,j]$ such that h is dependent on $h+1,\dots,\ell$, then add a dot in the smallest such ℓ. Repeat this process until no further addition is possible.

2. If no such ℓ exists, then proceed to the next labeled square $(r,(i',j'))$ or increase the label by one. This case occurs if the number of elements in the interval corresponding to the square (i,j) minus the dots already contained in $T_{(i,j)}$ is smaller or equal to r.

Once we have applied the first part to each labeled square, we proceed to fill the remaining rows. Let k be the maximum label, which is usually appearing as the label of $(1,n)$. For each row h not containing a dot, add a dot in the minimal square with rank k, i.e., the minimal ℓ such that ℓ minus the number of dots contained in $T_{(h,\ell)}$ equals k. The algorithm is formalized in Algorithm 2.

Remark 3.1.22. A retrieval algorithm for permutations and their essential sets is given in [42]. However, the slightly different definition of essential sets introduced here, requires us to have a completely different approach in the retrieval algorithm. In particular, we use the bijection between positroids and bounded affine permutations to guide our approach.

Implementation. An implementation of the algorithm can be found at

github.com/frazaffa/essential-sets-positroid

A visual implementation of the algorithm can be found at the following website[1]. Here the required input is a list of triples, where each triple consists of the rank followed by the row and the column of the square in the array corresponding to the cyclic interval whose rank we are assigning.

[1] https://sites.google.com/view/francescazaffalon/home/positroid

Algorithm 2: Construction of positroid from rank conditions on cyclic intervals

Input: E^{r} a set of pair of integer number and labels for a square in an $n \times (n+1)$ array.

Output: A dotting D of the array.

$D \leftarrow \emptyset$, $\mathcal{S} \leftarrow \emptyset$, $r \leftarrow 0$, $k \leftarrow \max\{r \mid (r,(i,j)) \in E^{\mathrm{r}}\}$;
$E^{\mathrm{r}} \leftarrow \mathsf{Sorted}(E^{\mathrm{r}})$; ◁Sort E^{r} by label r in ascending order
foreach $(r,(i,j)) \in E^{\mathrm{r}}$ **do**
  $a \leftarrow j - r - e_D(i,j)$;
  while $a > 0$ **do**
    $h \leftarrow \min_{<_i}\{\alpha \mid R_\alpha \not\subseteq \mathcal{S}\}$;
    $\ell \leftarrow \min\{\beta \mid \beta - 1 - e_D(h,\beta) = r\}$;
    $D \leftarrow D \cup \{(h,\ell)\}$;
    $\mathcal{S} \leftarrow \mathcal{S} \cup R_h \cup A_{(h,\ell)}$;
    $b \leftarrow j - r - e_D(i,j)$;
    if $a = b$ **then**
      return E^{r} is not valid
    else
      $a \leftarrow b$;
    end
  end
end
foreach *h such that* $R_h \not\subseteq \mathcal{S}$ **do**
  $\ell \leftarrow \min\{\beta \mid \beta - 1 - e_D(h,\beta) = k\}$;
  if $\ell > n+1$ **then**
    return E^{r} is not valid
  else
    $D \leftarrow D \cup \{(h,\ell)\}$;
    $\mathcal{S} \leftarrow \mathcal{S} \cup R_h \cup A_{(h,\ell)}$;
  end
end
if $\mathcal{S} \neq [n] \times [n+1]$ **then**
  return E^{r} is not valid
end
foreach $(r,(i,j)) \in E^{\mathrm{r}}$ **do**
  if $r_D(i,j) \neq r$ **then**
    return E^{r} is not valid
  end
end
return D;

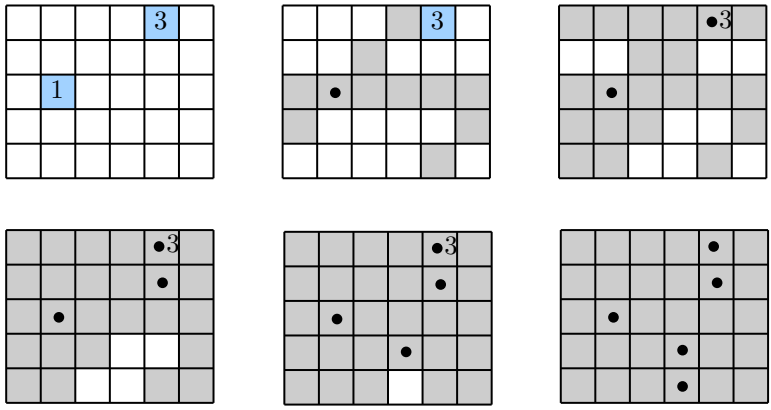


Figure 3.3: Algorithm 2 applied to the labeled squares in blue in the first array.

Example 3.1.23. Consider the set of squares labeled by

$$E^{\mathrm{r}} = \{(1,(3,2)),(3,(1,5))\}.$$

The steps of the algorithm applied to E^{r} can be seen in Figure 3.3. Hence, the corresponding decorated permutation is $\pi = (5\ 6\ 4\ 7\ 8)$.

The algorithm terminates if provided with compatible rank conditions and returns the dotting corresponding to a positroid. This positroid has a rank equal to the maximal label in the given squares and maximal among those satisfying the provided rank conditions.

Theorem 3.1.24. *Let E^{r} be a set of labeled squares from an $n \times (n+1)$ array and suppose we have a labeling of the square $(1,n)$. Then Algorithm 2 terminates and outputs a proper dotting if and only if there exists a positroid $\mathcal{P}$ satisfying the rank conditions corresponding to the input, i.e., for each $(r,(i,j)) \in E^{\mathrm{r}}$, $\mathrm{rk}([i,j+i-1]) = r$ by $\mathcal{P}$ and such that the core of $\mathcal{P}$ is contained in the rank information encoded in E^{r}.*

Proof. Let k denote the label of $(1,n)$. Note that since k denotes the rank of the positroid, if k is not the maximal label, then E^{r} is not valid.

Note that the algorithm always terminates, either with a proper dotting or an error. Indeed, the errors in line 12 and 21 avoid entering an infinite loop and the error in line 28 appears if the produced dotting is not a maximal proper dotting.

($\Leftarrow$) Suppose the rank conditions in E^{r} can be satisfied by a positroid as in the statement. We will prove that the output of the algorithm is a proper dotting. Let $\mathcal{D}$ be a maximal proper dotting such that $r_{\mathcal{D}}(i,j) = r$ for every $(r,(i,j)) \in E^{\mathrm{r}}$ and $\mathcal{D}$ is maximal with respect to the rank as in the statement. We will prove that at every step of the algorithm, the dotting D produced is a subset of $\mathcal{D}$.

In the first loop, for each $(r,(i,j)) \in E^{\mathrm{r}}$ considered in increasing rank order, we add dots inside the triangle $T_{(i,j)}$, to restrict the rank of $[i, j+i-1]$ to be at maximum r. Let D be the dotting at this step and suppose D is a subset of $\mathcal{D}$. We enter the loop if $j - e_D(i,j) > r$, that is whenever the number of dots in $T_{(i,j)}$ is not high enough. Since the quantity $j - r - d_D(i,j)$ is the excess of $[i, j+i-1]$, this is equivalent to the requirement that (i,j) corresponds to a cyclic interval in the core of $\mathcal{D}$. Let h be the first empty row starting from i. Since D can be extended to $\mathcal{D}$ and $r_{\mathcal{D}}(i,j) = r$, then $h \in [i, j+i-1]$. Let ℓ be such that $(h,\ell) \in \mathcal{D}$. Note that $r_{\mathcal{D}}(h, j+i-h) = r$. Indeed, the dots that have been added so far D in rows i to $i+h-2$ lie inside $T_{(i,j)}$. Suppose not, let the first dot in rows i to $i+h-2$ lying outside $T_{(i,j)}$ be in square (h_2,ℓ_2), added while considering some square in E^{r} labeled by $r_2 \leq r$. Then $i + j - h - e_D(h_2, i+j-h) < r_2$, hence $e_D(h_2, i+j-h) > i+j-h-r_2 \geq i+j-h-r$. Since all the rows between i and h_2 have been filled by assumption with dots inside $T_{(i,j)}$, this would imply that $e_D(i,j) > h - i + i + j - h - r = j - r$, against the assumption that $a = j - r - e_D(i,j) > 0$. It follows that $r_{\mathcal{D}}(h,\ell) = r$ and $\ell \in T_{(i,j)}$. Moreover, $\ell' = \min\{\beta \mid \beta - 1 - e_D(h,\beta) = r\}$ is such that $\ell = \ell'$ by rank maximality of $\mathcal{D}$. Hence, the algorithm does not produce an error and $D \cup \{(h,\ell')\} \subseteq \mathcal{D}$.

In the second for-loop, we add the dots outside each $T_{(i,j)}$ considered in the first loop. Suppose D is the dotting constructed so far, and it is a subset of $\mathcal{D}$ and let h be the row we are considering. Let $\ell' = \min\{\beta \mid \beta - 1 - e_D(h,\beta) = k\}$, where k is the label of the square $(1,n)$. Then, since for every $(r,(i,j)) \in E^{\mathrm{r}}$, $e_D(i,j) \geq j - r$, ℓ' will lie outside every triangle $T_{(i,j)}$. This guarantees that the rank of the square (i,j) with respect to the dotting we are constructing is actually equal to r for each $(r,(i,j)) \in E^{\mathrm{r}}$. Let ℓ be such that $(h,\ell) \in \mathcal{D}$. By rank maximality of $\mathcal{D}$, $r_{\mathcal{D}}(h,\ell) = k$ and $\ell \geq \ell'$. Moreover, suppose by contradiction that $\ell > \ell'$. Then since $r_{\mathcal{D}}(h,\ell) = r_{\mathcal{D}}(h,\ell') = k$, $\mathcal{D}$ contains a dot in the antidiagonal containing (h,ℓ') in the rows between h and $h+\ell'-1$. This is not possible by definition of ℓ'. Hence, $\ell = \ell'$, the dotting produced is still a subset of $\mathcal{D}$ and the algorithm does not produce any error at this step.

Since we assume $\mathcal{D}$ to be a maximal proper dotting satisfying the rank conditions in E^{r}, and we proved that the dotting D constructed by the algorithm is equal to $\mathcal{D}$, the last two checks will be satisfied by D, hence the algorithm will terminate and output D.

($\Rightarrow$) Suppose now that the algorithm terminates and outputs a dotting, denoted as D. Note that D must satisfy the rank conditions specified by E^{r}, as the last step in the algorithm checks for this. Moreover, at every step, we choose a row h that does not already contain a dot. As a consequence, at the end of the algorithm, all the boxes will be shaded if and only if no two dots lie in the same antidiagonal. Hence, D is a proper dotting satisfying the rank conditions given in the input; that is, there exists a positroid satisfying $\mathrm{rk}([i, j+i-1]) = r$ for every $(r, (i, j)) \in E^{\mathrm{r}}$. □

In particular, it follows that the rank information contained in the core is enough to uniquely recover the positroid.

Corollary 3.1.25. *The rank conditions contained in the core of a positroid $\mathcal{P}$ are minimal rank conditions defining $\mathcal{P}$.*

Proof. When applying Algorithm 2 to the subset of labeled squares corresponding to the ranked essential family, the quantity a in line 4 of the first for-loop is exactly equal to the excess of the essential set. Then the rank condition on the essential set is used in the algorithm if and only if the excess does not vanish, that is, if the ranked essential set is in the core. □

3.2 Combinatorial Characterization of Essential Sets

The goal of this section is to provide a combinatorial description of essential sets. We aim to introduce combinatorial conditions that guarantee a given set of labeled cyclic intervals can be viewed as the corners of the diagram of a bounded affine permutation.

The following result is a generalization of Theorem 4.1 from [43] to the setting of bounded affine permutations and our definition of the rank of an essential set. We retain the same name, which stands for CHaracterization of ESsential Sets. The use of cyclic intervals and bounded affine permutations significantly alters the third condition in the theorem. The rank interpretation of essential sets allows for an elegant proof of the result. We divide its proof into Lemmas 3.2.1 and 3.2.2. Lemma 3.2.1 demonstrates that we can use a family $\mathcal{E}^{\mathrm{r}}$ satisfying properties (E1), (E2), and (E3) to construct a function on cyclic intervals which shares many properties with the rank function of a positroid.

Lemma 3.2.1. *Let $\mathcal{E}^{\mathrm{r}}$ be a set of cyclic intervals labeled by a non-negative integer satisfying the properties (E1), (E2), and (E3) from Theorem 3.1.13. Let*

CI_n denote the family of cyclic intervals of $[n]$ and $r : \mathrm{CI}_n \to \mathbb{Z}$ be the function defined by:

$$r([i,j]) = \min\{|[i,j]|, r + |[i,j] \setminus I| \mid (r,I) \in \mathcal{E}^{\mathrm{r}}\}.$$

The function r satisfies the following properties:

(r1) *$r(I) = r$ for every $(r,I) \in \mathcal{E}^{\mathrm{r}}$ and the minimum is uniquely achieved by (r,I);*

(r2) *For every $[i,j] \in \mathrm{CI}_n$, $0 \leq r([i,j]) \leq |[i,j]|$;*

(r3) *For every $[i,j] \in \mathrm{CI}_n \setminus [n]$, $r([i,j]) \leq r([i-1,j]) \leq r([i,j]) + 1$ and $r([i,j]) \leq r([i,j+1]) \leq r([i,j]) + 1$.*

(r4) *$[i,j] \in \mathrm{CI}_n$ is such that $r([i,j]) = r([i+1,j]) = r([i,j-1]) = r([i-1,j]) - 1 = r([i,j+1]) - 1$ if and only if $(r([i,j]),[i,j]) \in \mathcal{E}^{\mathrm{r}}$.*

Proof. **(r1):** Let $(r,I) \in \mathcal{E}^{\mathrm{r}}$. By property (E1), we have $r < |I|$. Suppose by contradiction that there exists $(r_2,I_2) \in \mathcal{E}^{\mathrm{r}}$ such that $r_2 + |I_2 \setminus I| \leq r$. By (E2), I_2 neither contains nor is contained by I. Hence, let (r_3,I_3) be a minimal set with the smallest label in $\mathcal{E}^{\mathrm{r}}$ containing $I \cup I_2$, and let (r_4,I_4) be a maximal set with the largest label contained in $I \cap I_2$. Suppose I and I_2 intersect in a cyclic interval. By (E3), $r + r_2 \geq r_3 + r_4 + |(I \cap I_2) \setminus I_4|$. Moreover, $r_4 + |(I \cap I_2) \setminus I_4| + |I \setminus I_2| = r_4 + |I \setminus I_4| \geq r_2 + |I \setminus I_2|$, where the last equality holds by assuming that the minimum of the function r is achieved by (r_2,I_2). Then we can write $r + r_2 + |I \setminus I_2| \geq r_3 + r_2 + |I \setminus I_2|$, hence $r \geq r_3$. Since $I \subsetneq I_3$, by (E2), this is not possible. Finally, suppose I and I_2 intersect in two cyclic intervals. Then $r_2 + |I \setminus I_2| = r_2 + |[1,n] \setminus I_2| \geq r_2 + k - r_2 + k > r$, where we used (E2). It follows that for every $(r_2,I_2) \in \mathcal{E}^{\mathrm{r}} \setminus (r,I)$, $r_2 + |I \setminus I_2| > r$, hence (r1) holds.

(r2): Property (r2) holds by definition of the function r and condition (E1) on the family $\mathcal{E}^{\mathrm{r}}$.

(r3): Let $[i,j]$ be a cyclic interval. We will prove (r3) by considering some cases.

- If $r([i,j])$ and $r([i-1,j])$ are given by the cardinality of the sets, (r3) holds trivially.
- Suppose $r([i,j]) = |[i,j]|$ and $r([i-1,j]) = r + |[i-1,j] \setminus I|$ for some $(r,I) \in \mathcal{E}^{\mathrm{r}}$. Then $i - 1 \in I$ and $r + |[i,j] \setminus I| \geq |[i,j]|$, hence $r \geq |I \cap [i,j]|$. It follows that $r([i-1,j]) = r + |[i-1,j] \setminus I| \geq |I \cap [i,j]| + |[i-1,j] \setminus I| = |[i,j]| = r([i,j])$. Moreover, $r([i-1,j]) \leq |[i-1,j]| = r([i,j]) + 1$.

- Finally, suppose $r([i,j]) = r + |[i,j] \setminus I|$ for some essential set (r, I). Then clearly $r([i-1,j]) \leq r([i,j]) + 1$. If $r([i-1,j]) = |[i-1,j]|$ then $r([i-1,j]) \geq r([i,j]) + 1$. If $r([i-1,j]) = r' + |[i-1,j] \setminus I'|$ for some essential set (r', I') then $r + |[i,j]| \setminus I| \leq r' + |[i,j] \setminus I'| \leq r([i-1,j])$.

In an analogous way it is possible to show that $r([i,j]) \leq r([i,j+1]) \leq r([i,j])+1$, hence (r3) holds.

(r4): Suppose $[i,j] \subsetneq [n]$ is such that $r([i,j]) = r([i+1,j]) = r([i,j-1]) = r([i-1,j])-1 = r([i,j+1])-1$ holds. Let $(r, I) \in \mathcal{E}^{\mathrm{r}}$ such that $r([i,j]) = r+|[i,j] \setminus I|$. Note that by assumption, $i-1, j+1 \notin I$, hence $I \subseteq [i,j]$. Suppose $I \neq [i,j]$ and assume that $i \notin I$. Then $r([i+1,j]) \leq r+|[i+1,j] \setminus I| < r+|[i,j] \setminus I| = r([i,j])$, a contradiction. In the same way we need to have that $j \in I$. Since I is a cyclic interval, it follows that $I = [i,j]$. By (r1), $r([i,j]) = r$, hence $(r([i,j]), [i,j]) \in \mathcal{E}^{\mathrm{r}}$.

Suppose now $(r, [i,j]) \in \mathcal{E}^{\mathrm{r}}$. By (r1), $r([i,j]) = r$ and the minimum is uniquely achieved by (r, I). Hence, for every $(r_2, I_2) \in \mathcal{E}^{\mathrm{r}}$, $r_2 + |[i,j] \setminus I_2| \geq r+1$ and $r([i+1,j]) \geq r$, $r([i,j-1]) \geq r$. Since they are both contained in $[i,j]$, it follows that $r([i,j]) = r([i+1,j]) = r([i,j-1]) = r$. Moreover, $r_2+|[i-1,j] \setminus I_2| \geq r+1$ and $r_2+|[i,j+1] \setminus I_2| \geq r+1$, and $r+|[i-1,j] \setminus [i,j]| = r+1 = r+|[i,j+1] \setminus [i,j]|$, hence we can conclude that (r4) holds. □

The following result shows that we can use the function defined in Lemma 3.2.1 to construct a bounded affine permutation, which uniquely corresponds to a positroid.

Lemma 3.2.2. *Let $\mathcal{E}^{\mathrm{r}}$ be a family of labeled cyclic intervals satisfying (E1), (E2), and (E3) from Theorem 3.1.13. Let $r : \mathrm{CI}_n \to \mathbb{Z}$ be the function defined in Lemma 3.2.1 and let $\pi : \mathbb{Z} \to \mathbb{Z}$ be defined as*

$$\pi(i) = \min\{j \geq i \mid r([i,j]) = r([i+1,j])\},$$

where $[i,j] = \{i \mod n, \ldots, j \mod n\}$ for every $i \leq j$. Then π is a bounded affine permutation.

Proof. The function π satisfies the properties such that for every $i \in \mathbb{Z}$, $i \leq \pi(i) \leq i+n$ and $\pi(i+kn) = \pi(i)+kn$ for every $k \in \mathbb{Z}$ by construction. Hence, it remains to prove that π is a bijection.

Suppose $i, j \in \mathbb{Z}$, with $i < j$, are such that $\pi(i) = \pi(j) = \ell$. Then $r([j,\ell]) = r([j+1,\ell]) > r([j+1,\ell-1])$. By property (r3) of the function r, $r([j+1,\ell-1]) = r([j,\ell])-1$. We have that $r([j,\ell]) \geq r([j,\ell-1]) \geq r([j+1,\ell-1]) = r([j,\ell])-1$.

Suppose the first inequality is strict, then $r([j,\ell-1]) = r([j+1,\ell-1])$, contradicting the minimality assumption on ℓ. Hence, $r([j,\ell]) = r([j,\ell-1])$. In the same way, $r([i,\ell]) = r([i,\ell-1])$. Hence, there are $(r_i, I), (r_j, J) \in \mathcal{E}^{\mathrm{r}}$ such that $[i,\ell] \subseteq I$, $[j,\ell] \subseteq J$ and $r_i = r([i,\ell])$, $r_j = r([j,\ell])$. By definition of $\pi(i) = \ell$, it follows that $r([i+1,\ell-1]) = r_i - 1$. Suppose $r([i+1,\ell-1]) = r_2 + |[i+1,\ell-1] \setminus I_2|$ for some $(r_2, I_2) \in \mathcal{E}^{\mathrm{r}}$, then I_2 is contained in $[i+1,\ell-1]$ since if I add i or ℓ, the value of r strictly increases. Then, $r_2 + |[i+1,\ell-1] \setminus I_2| \geq r_2 + |I \setminus I_2| \geq r$ by property (E2), contradicting the assumption. Finally, note that $|[i+1,\ell-1]| \geq |[i+1,\ell-1]| + r' - |[j,\ell-1]| \geq r' + |[i+1,\ell-1] \setminus J| = r' + |[i+1,\ell] \setminus J| \geq r$. Hence, we get a contradiction, and the function π is injective. Moreover, since π is bounded and periodic, it follows that π is a bijection. □

Finally, by combining the construction from Lemmas 3.2.1 and 3.2.2, we obtain the characterization of essential sets of a positroid. This provides an axiomatic description of positroids from this perspective.

Theorem 3.2.3 (Chess theorem)**.** *Let $\mathcal{E}^{\mathrm{r}}$ be a set of cyclic intervals labeled by a non-negative integer. Then $\mathcal{E}^{\mathrm{r}}$ is the ranked essential family of a positroid if and only if $\mathcal{E}^{\mathrm{r}}$ satisfies the properties (E1), (E2), and (E3) from Theorem 3.1.13.*

Proof. If $\mathcal{E}^{\mathrm{r}}$ is the ranked essential family of a positroid, then (E1), (E2), and (E3) hold as shown in Theorem 3.1.13.

Suppose now $\mathcal{E}^{\mathrm{r}}$ is family of labeled cyclic intervals satisfying (E1), (E2), and (E3). Let $r : \mathrm{CI}_n \to \mathbb{Z}$ be the function defined in Lemma 3.2.1 and let $\pi : \mathbb{Z} \to \mathbb{Z}$ be the bounded affine permutation constructed in Lemma 3.2.2. We will prove that the positroid $\mathcal{P}$ associated to π is such that $\mathrm{rk}([i,j]) = r([i,j])$ for each cyclic interval $[i,j]$. Then, by Theorem 3.1.8 and property (r4) from Lemma 3.2.1, it follows that $\mathcal{E}^{\mathrm{r}}$ is the ranked essential family of the positroid $\mathcal{P}$. We will prove by induction on the size of the cyclic interval that the rank 'rk' and 'r' agree on each cyclic interval. For $i \in [n]$, we have

$$\mathrm{rk}([i,i]) = \begin{cases} 0 & \text{if } \pi(i) = i \quad \iff \quad r([i,i]) = 0 \\ 1 & \text{if } \pi(i) > i \quad \iff \quad r([i,i]) = 1. \end{cases}$$

Suppose now that rk and r agree on each cyclic interval on size smaller than m and suppose by contradiction $[i,j]$ is such that $|[i,j]| = m$ and $r([i,j]) \neq \mathrm{rk}([i,j])$. We will consider the following cases:

Case 1. If $r([i,j]) < \mathrm{rk}([i,j])$, since both r and rk are satisfying the property (r3) from Lemma 3.2.1, $r([i,j]) = r([i+1,j])$, hence $\pi(i) \leq j$. Moreover, $\mathrm{rk}([i,j]) = \mathrm{rk}([i+1,j]) + 1$, i.e., i is independent of $i = 1, \ldots, j$, hence $\pi(i) > j$ and we get a contradiction.

Case 2. If $r([i,j]) > \mathrm{rk}([i,j])$, since $\mathrm{rk}([i,j]) = \mathrm{rk}([i+1,j])$, i is dependent on $i+1,\ldots,j$, hence $\pi(i) < j$. Again, consider the two distinct cases. Suppose then that $r([i,j]) > \mathrm{rk}([i,j])$. Note that since $\mathrm{rk}([i,j]) = \mathrm{rk}([i+1,j])$, i is dependent on $i+1,\ldots,j$, hence $\pi(i) < j$.

Case 2.1. Suppose $\pi(j) = j$, then $r([j,j]) = 0$ hence there is $(0,I) \in \mathcal{E}^{\mathrm{r}}$ such that $j \in I$. Suppose $r([i,j-1]) = r' + |[i,j-1] \setminus I'|$ for some $(r',I') \in \mathcal{E}^{\mathrm{r}} \cup \{(0,\emptyset)\}$. First assume that $I \cap I' \neq \emptyset$. Let (r_3,I_3) be the minimal element in $\mathcal{E}^{\mathrm{r}}$ containing $I \cup I'$ and let (r_4,I_4) be the maximal element from $\mathcal{E}^{\mathrm{r}}$ contained in $I \cap I'$. Then by (E3.2) $0 + r' \geq r_3 + r_4 + |(I \cap I') \setminus I_4|$. Then $r' + |[i,j-1] \setminus I'| \geq r_3 + r_4 + |(I \cap I') \setminus I_4| + |[i,j-1] \setminus I'| \geq r_3 + |[i,j-1] \setminus I_3|$. Hence, $r([i,j-1]) = r_3 + |[i,j-1] \setminus I_3| = r([i,j])$, where the last equality holds since $j \in I_3$. Assume now that $I \cap I' \neq \emptyset$. Similarly, let (r_3,I_3) be the minimal element in $\mathcal{E}^{\mathrm{r}}$ containing $I \cup I'$ and let (r_4,I_4) be the maximal element from $\mathcal{E}^{\mathrm{r}}$ contained in the minimal cyclic interval containing $I \cup I'$ in $[i,j]$ C, minus $I \cup I'$. Then by (E3.1), $0 + r' \geq r_3 - r_4 - |C \setminus I_4|$. Then $r' + |[i,j-1] \setminus I'| \geq r_3 - r_4 - |C \setminus I_4| + |[i,j-1] \setminus I'| \geq r_3 + |[i,j-1] \setminus I_3|$ and as before we get that $r([i,j]) = r([i,j-1])$, and we get that r and rk agree on $[i,j]$.

Case 2.2. Suppose $\pi(j) > j$. Since $\mathrm{rk}([i,j-1]) = \mathrm{rk}([i,j])$, by Proposition 2.2.21, there is some $h \in [i+1,j-1]$ such that $\pi(h) = j$, i.e., $r([h,j]) = r([h+1,j]) = r([h,j-1])$. By (r4), there exists some (r,I) such that $[h,j] \subseteq I$ and $r = r([h,j]) = \mathrm{rk}([h,j])$. Let $(r',I') \in \mathcal{E}^{\mathrm{r}} \cup \{(0,\emptyset)\}$ be such that $r([i,j-1]) = r' + |[i,j-1] \setminus I'|$. Suppose $I \cap I' \neq \emptyset$, the other case will be equivalent. Let (r_3,I_3) be the minimal element in $\mathcal{E}^{\mathrm{r}}$ containing $I \cup I'$ and let (r_4,I_4) be the maximal element from $\mathcal{E}^{\mathrm{r}}$ contained in $I \cap I'$. Then by (E3), $r' + |[i,j-1] \setminus I'| = r' + |[i,j-1] \setminus (I \cup I')| + |I \setminus I'| \geq r' + |[i,j-1] \setminus I_3| + |I \setminus I'| \geq r + r' - |I \cap I'| + |[i,j-1] \setminus I_3| \geq r_3 + |[i,j-1] \setminus I_3| + r_4 + |(I \cap I') \setminus I_4| - |I \cap I'| \geq r_3 + |[i,j-1] \setminus I_3|$. Hence, $r([i,j-1]) = r_3 + |[i,j-1] \setminus I_3| = r_3 + |[i,j] \setminus I_3| = r([i,j])$, since $j \in I_3$. □

Remark 3.2.4. If $\mathcal{E}^{\mathrm{r}}$ is a family of labeled cyclic interval satisfying (E1), (E2), and (E3) we can recover the bounded affine permutation of the positroid defined by $\mathcal{E}^{\mathrm{r}}$ by applying Algorithm 2 to $E^{\mathrm{r}} = \{(r,(i,j)) \mid (r,[i,j+i-1]) \in \mathcal{E}^{\mathrm{r}}\}$.

3.3 Realization Spaces of Positroids

We now turn our attention to the realization space of a positroid in the totally non-negative Grassmannian and the Grassmannian.

3.3.1 Positroid Cells

Recall that given a positroid $\mathcal{P}$ of rank k on the ground set $[n]$, its positroid cell $\mathcal{S}_{\mathcal{P}}$ is the realization spaces of the positroid inside the totally non-negative Grassmannian.

We are now interested in computing the dimension of the cell corresponding to a given positroid in terms of essential sets. Combinatorial objects in bijection with positroids offer a way to compute the dimension of such cells. Specifically, it is possible to compute the dimension of a positroid cell using bounded affine permutations, as outlined in the following result.

Theorem 3.3.1 ([70, Theorem 3.16])**.** *Let $\mathcal{P}$ be a rank k positroid on $[n]$ and let π be the associated bounded affine permutation. The positroid cell $\mathcal{S}_{\mathcal{P}}$ has codimension $\ell(\pi)$ inside $\mathrm{Gr}_{\geq 0}(k,n)$, where*

$$\ell(\pi) = |\{(i,j) \mid i \in [n], i < j \leq i+n \text{ and } \pi(i) > \pi(j)\}|,$$

is called length *of π, and it is equal to the number of inversions in π.*

The codimension of a positroid cell can be determined from its ranked essential family or its core.

Proposition 3.3.2. *Let $\mathcal{E}^{\mathrm{r}}$ be the set of ranked essential sets of a rank k positroid $\mathcal{P}$. Then*

$$\operatorname{codim}(\mathcal{S}_{\mathcal{P}}) = \sum_{(r,[i,j])\in\mathcal{E}^{\mathrm{r}}} e_{[i,j]}(k-r) = \sum_{(r,[i,j])\in\mathcal{C}^{\mathrm{r}}} e_{[i,j]}(k-r).$$

Proof. Let π be the bounded affine permutation corresponding to the positroid $\mathcal{P}$. By Theorem 3.3.1, we need to prove that $\ell(\pi) = \sum_{(r,[i,j])\in\mathcal{E}^{\mathrm{r}}} e_{[i,j]}(k-r)$. Consider the dotted array of π and note that the length of π can be computed by looking at the array as:

$$\ell(\pi) = \sum_{(i,j)\in D(\pi)} |D(\pi) \cap T_{(i,j)}|.$$

Indeed, every time a dot $(i, \pi(i) - i + 1)$ is counted in the previous formula, there exists a $j < i$ such that $\pi(j) > \pi(i)$, indicating one inversion.

Note that, if the dot in row i lies in $D(\pi) \cap T_{(j,\pi(j)-j+1)}$, then $(j, \pi(j) - j + 1)$ defines a bounded section of the diagram of π, which is non-empty since it contains the dot $(i, \pi(i) - i + 1)$. Hence, there is a corner, i.e., an essential set inside $P_{(j,\pi(j)-j+1)}$. Let $(r, E) \in \mathcal{E}^{\mathrm{r}}$ be such an essential set. The number of

dots in the same connected component in the diagram as E is exactly the excess of E, e_E. We have to count the dot in the same connected component as the essential set $[i, j]$ as many times as the number o f inversions it is contained in, i.e., as many times as the number of h such that $(i, j - i + 1) \in T_{(h,\pi(h)-h+1)}$. Consider the antidiagonal through the point $(i, j - i + 1)$. On the right side of such a diagonal there are r dots, equal to the rank of $[1, n]$. Among these points, those that satisfy the property described above are the dots not in $P_{(i,j-i+1)}$. It follows that, if the essential set E has rank r, there are $k - r$ such points. Hence, we can write:

$$\ell(\pi) = \sum_{(i,j)\in D(\pi)} |D(\pi) \cap T_{(i,j)}| = \sum_{i\in[n]} \sum_{\mathcal{E}\ni E\subseteq[i,\pi(i)]} e_E = \sum_{(r,E)\in\mathcal{E}^{\mathrm{r}}} e_E(k - r).$$

The last equality in the statement holds because $e_E > 0$ if and only if $(r, E) \in \mathcal{C}^{\mathrm{r}}$, i.e., it is in the core of the ranked essential family. □

Example 3.3.3.

- The uniform matroid $\mathcal{U}_{k,n}$ of rank k on n elements has, as shown in Example 3.1.7, essential family given by $\{(k, [n])\}$. Hence, the codimension of the corresponding cell is 0. This is indeed the top-dimensional positroid cell in the totally non-negative Grassmannian.

- Let $\mathcal{P}$ be the positroid from Example 3.1.3. Its ranked essential family is

$$\mathcal{E}^{\mathrm{r}} = \{(1, [5, 6]), (2, [1, 4]), (2, [4, 7]), (3, [1, 8])\}.$$

  The codimension of the corresponding positroid cell is: $\operatorname{codim}(\mathcal{S}_{\mathcal{P}}) = (3 - 1) + 2(3 - 2) + (3 - 2) = 5$.

3.3.2 Positroid Varieties

We are now interested in focusing our attention on the positroid variety of $\mathcal{P}$.

The following result generalizes [68, Proposition 2.3], which states an equivalent result for interval positroid varieties, i.e., positroids defined by rank conditions on intervals.

Proposition 3.3.4. *The positroid variety Π_π is defined as a scheme by the rank conditions $\mathrm{rk}(I) \leq r$ for every $(r, I) \in \mathcal{CE}^{\mathrm{r}}_\pi$, the connected elements of the ranked essential family of $\mathcal{P}_\pi$.*

Proof. By Remark 3.1.11 and Theorem 3.1.9, the rank of each cyclic interval $[i, j]$ can be computed in terms of the rank of the connected essential sets. Hence, the rank condition for $[i, j]$ is implied by those in $\mathcal{CE}^{\mathrm{r}}$. □

3.4 Small Rank Positroids

For positroids of small rank, this construction can be explicitly related to other families associated to positroids. In particular, we are interested in studying the relation between essential sets and the family of circuit closures:

$$\widetilde{\mathcal{F}}^{\mathrm{r}} = \{(r, F) \mid F \in \mathcal{F}, F = \langle C \rangle \text{ for some } C \in \mathcal{C}\} \subseteq \mathbb{Z} \times \mathcal{F}.$$

3.4.1 Rank 2 Positroids

Let us start by considering rank 2 positroids. We will show that in this case the ranked essential family almost coincides with the family of circuit closures $\widetilde{\mathcal{F}}^{\mathrm{r}}$.

Fix a rank 2 positroid $\mathcal{P}$. The circuits of $\mathcal{P}$ can only be loops or parallel elements. The closure of a loop will be the set of all loops of $\mathcal{P}$. The closure of the circuit containing two parallel elements will be the set of all other elements parallel to the given ones. Since the positivity condition on rank 2 positroids can be seen as the requirement for the vectors giving the representation of the positroid to be arranged in a counterclockwise order, the set of all vectors parallel to a given one is a cyclic interval. Hence, it is easy to see that the relation between circuit closures and the ranked essential family is given as follows.

Proposition 3.4.1. *If $\mathcal{P}$ is a rank 2 positroid with ranked essential family $\mathcal{E}^{\mathrm{r}}$, then its set of circuit closures will be given by*

$$\widetilde{\mathcal{F}}^{\mathrm{r}} = \left\{ (0, L) \mid L = \bigcup_{(0,[i,j]) \in \mathcal{E}^{\mathrm{r}}} [i, j] \right\} \cup \{(r, [i, j] \cup L) \mid (r, [i, j]) \in \mathcal{E}^{\mathrm{r}},\ r > 0\},$$

where the first term only appears if the positroid contains loops.

In particular, we can easily characterize positroids in terms of the family of circuit closures.

Corollary 3.4.2. *If $\mathcal{M}$ is a matroid of rank 2, $\mathcal{M}$ is a positroid if and only if the terms of rank 1 in $\widetilde{\mathcal{F}}^{\mathrm{r}}$ consist of cyclic intervals.*

See [91] for more details on rank 2 positroids.

Remark 3.4.3. The equivalence between essential sets and ranked circuit closures does not hold for positroids of higher rank. For instance, consider the rank 3 positroid with ranked essential family

$$\mathcal{E}^{\mathrm{r}} = \{(1, [1, 2]), (2, [1, 5]), (2, [5, 2]), (3, [1, 7])\}.$$

Then $\{1,2,5\}$ is a circuit of rank 1 which does not appear in $\mathcal{E}^{\mathrm{r}}$. Moreover, $\{1,2\}$ is not a flat of the positroid, as its closure is given exactly by $\{1,2,5\}$.

3.5 Twistor Positroids

We now want to consider a construction on matroids, introduced by Lam [74]. This construction is relevant in the study of the amplituhedron, a geometric object encoding quantities of interest in theoretical physics.

In order to study this construction, we need to study the ranked essential family of the dual of a given positroid. This will be achieved by using the Grassmann necklace description of a positroid.

3.5.1 Essential Sets and Grassmann Necklaces

We now describe the relation between essentials sets and other combinatorial objects in bijection with positroids. In particular, we will discuss how to construct the ranked essential family from the Grassmann necklace and viceversa.

Recall that given a matroid $\mathcal{M}$ on $[n]$ with bases $\mathcal{B}$, its Grassmann necklace $\mathcal{I}_{\mathcal{M}}$ was defined as the n-tuple of minimal bases with respect to the orders $<_i$ for $i \in [n]$. We will introduce a dual concept of Grassmann necklace which would correspond to taking the maximal basis with respect to $<_i$. These were firstly introduced by Lam, [73] and studied in [65].

Definition 3.5.1 (Dual Grassmann necklace)**.** A *dual Grassmann necklace* is an ordered collection of subsets of $[n]$, $\mathcal{J} = (J_1, \ldots, J_n)$ satisfying the following conditions:

- If $i \in J_{i+1}$, then $J_i = (J_{i+1} \cup \{j\}) \setminus \{i\}$ for some $j \in [n]$;
- If $i \notin J_{i+1}$, then $J_{i+1} = J_i$.

Analogously to Grassmann necklaces, dual Grassmann necklaces can be constructing by looking at maximal elements in the bases of a matroid, as follows.

Proposition 3.5.2. *Let $\mathcal{M}$ be a matroid with ground set $[n]$ and bases $\mathcal{B}$. Let J_i be the maximal element of $\mathcal{B}$ with respect to $\leq_i$. Then $\mathcal{J}_{\mathcal{M}} = (J_1, \ldots, J_n)$ is a dual Grassmann necklace.*

Conversely, given a Grassmann necklace $\mathcal{J}$ on $[n]$, the set

$$\mathcal{B}_{\mathcal{J}} = \left\{ B \in \binom{[n]}{k} \mid B \leq_i J_i \text{ for all } i \in [n] \right\}$$

is the set of bases of a positroid $\mathcal{M}_{\mathcal{J}}$.

Moreover, given a matroid $\mathcal{M}$, we have that $\mathcal{M}_{(\mathcal{J}_{\mathcal{M}})}$ is the positroid envelope of the matroid $\mathcal{M}$.

In order to reconstruct the ranked essential family of a positroid, we need the information contained both in the Grassmann necklace and in the dual Grassmann necklace.

Proposition 3.5.3. *Let $\mathcal{P}$ be a positroid on $[n]$ with Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$ and dual Grassmann necklace $\mathcal{J} = (J_1, \ldots, J_n)$. Then the essential sets of $\mathcal{P}$ are given by*

$$\mathcal{E} = \{[i,j] \mid j \notin I_i, i \notin J_{j+1}, (j+1 \mod n) \in I_i, (i-1 \mod n) \in J_{j+1}\}$$

and, for $[i,j] \in \mathcal{E}$, the rank of $[i,j]$ is $|[i,j] \cap I_i|$.

Proof. Note that $\mathrm{rk}([i,j]) = \mathrm{rk}([i+1,j])$ if and only if i is not independent of the elements in $[i+1,j]$, i.e., $i \notin J_{j+1}$. Similarly, $\mathrm{rk}([i,j]) = \mathrm{rk}([i,j-1])$ if and only if $j \notin I_i$.

On the other hand $\mathrm{rk}([i,j]) = \mathrm{rk}([i-1,j]) - 1$ if and only if $i-1 \mod n$ is independent of the elements in $[i,j]$, i.e., $(i-1 \mod n) \in J_{j+1}$. Similarly, $\mathrm{rk}([i,j]) = \mathrm{rk}([i,j+1]) - 1$ if and only if $(j+1 \mod n) \in I_i$.

By Theorem 3.1.8, the two sets are equal.

In [84, Lemma 14] it is proven that for an interval $[i,j]$, the maximum cardinality of its intersection with any basis is achieved by I_i. Hence, the statement follows. □

Note that the Grassmann and dual Grassmann necklaces of a matroid with bases $\mathcal{B}$, define the positroid envelope of the matroid, i.e., the smallest positroid containing $\mathcal{B}$ in its bases. From the previous result it follows starting from a matroid $\mathcal{M}$, it is possible to define the ranked essential family of $\mathcal{M}$ as above, or equivalently as the maximally dependent cyclic interval in Theorem 3.1.8. This family $\mathcal{E}^{\mathrm{r}}$ is the ranked essential family of the positroid envelope of $\mathcal{M}$.

3.5.2 Dual Essential Sets

We are now interested in studying the ranked essential family of the dual of a given positroid. Recall that given a positroid $\mathcal{P}$ with bases $\mathcal{B}$, the dual positroid $\mathcal{P}^*$ was defined to be the positroid whose bases are $\mathcal{B}^* = \{[n] \setminus B \mid B \in \mathcal{B}\}$.

The following result on the Grassmann necklace of the dual of a positroid holds.

Proposition 3.5.4. *Let $\mathcal{P}$ be a positroid on $[n]$ with Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ and dual Grassmann necklace $\mathcal{J} = (J_1, \dots, J_n)$.*

Then $\mathcal{P}^*$ has Grassmann necklace $\mathcal{I}^* = ([n]\setminus J_1, \dots, [n]\setminus J_n)$ and dual Grassmann necklace $\mathcal{J}^* = ([n] \setminus I_1, \dots, [n] \setminus I_n)$.

Proof. It follows directly from definition of dual positroid and Grassmann and dual Grassmann necklace. □

We can now explicitly construct the ranked essential family of the dual of a positroid.

Proposition 3.5.5. *Let $\mathcal{P}$ be a positroid of rank k on the ground set $[n]$ and let $\mathcal{E}^{\mathrm{r}}$ be its ranked essential family. Then the ranked essential family of the dual positroid $\mathcal{P}^*$ is given by*

$$(\mathcal{E}^{\mathrm{r}})^* = \{(n-k-|I|+r, [n] \setminus I) \mid (r, I) \in \mathcal{E}^{\mathrm{r}} \setminus \{(k, [n])\}\} \cup \{(n-k, [n])\}\,.$$

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be respectively the Grassmann and the dual Grassmann necklace of the positroid $\mathcal{P}$. By Proposition 3.5.3 and Proposition 3.5.4 we have that $[i, j]$ is an essential set of $\mathcal{P}^*$ if and only if $j \in I_i, i \in J_{j+1}, (j+1 \mod n) \notin I_i, (i-1 \mod n) \notin J_{j+1}$. This happens if and only if $[j+1, i-1] \in \mathcal{E}$.

For $(r, I) \in \mathcal{E}^{\mathrm{r}}$, the rank of $[n] \setminus I$ in $\mathcal{P}^*$ can be computed using the formula in Remark 2.2.14 as

$$\mathrm{rk}^*([n] \setminus I) = |[n] \setminus I| - (k - \mathrm{rk}(I)) = n - k - |I| + r,$$

hence the statement holds. □

3.5.3 Twistor Matroids

The m-twistor of a matroid is defined as follows.

Definition 3.5.6 (Twistor matroid)**.** Let $\mathcal{M}$ be a matroid of rank k on $[n]$ and with bases $\mathcal{B}$. Fix $m \in \mathbb{Z}_{>0}$ such that $k+m \leq n$. The *m-twistor matroid* of $\mathcal{M}$ is the matroid $\mathcal{M}^{\downarrow m}$ whose bases are given by

$$\mathcal{B}^{\downarrow m} = \left\{ I \in \binom{[n]}{m} \mid I \subseteq J \text{ for some } J \in \mathcal{B}^* \right\},$$

where $\mathcal{B}^*$ denotes the bases of the dual matroid $\mathcal{M}^*$.

It is easy to check that this construction does indeed give the bases of a matroid, i.e., that $\mathcal{B}^{\downarrow m}$ satisfies axioms (B1), (B2) and (B3) in Definition 2.2.4.

Recall that $Q_{n,k}$ is the poset of rank k positroids on n elements. The following properties hold, [74].

Proposition 3.5.7. *The map $\mathcal{M} \to \mathrm{env}(\mathcal{M}^{\downarrow m})$ is an order-preserving map from $Q_{n,k}$ to $Q_{n,m}$. Moreover, $\mathrm{env}(\mathcal{M}^{\downarrow m}) = \mathrm{env}((\mathrm{env}(\mathcal{M}))^{\downarrow m})$.*

We are interested in studying the poset of envelopes of twistor matroids.

Definition 3.5.8 (Twistor positroid)**.** Let n, k, m be positive integers with $k+m \leq n$. We denote by $P_{n,k,m} \subseteq Q_{n,m}$ the subposet of positroids ordered by inclusion of the form $\mathrm{env}(\mathcal{P}^{\downarrow m})$ for $\mathcal{P} \in Q_{n,k}$. A *twistor positroid* is an element of $P_{n,k,m}$.

Note that we need to take the positroid envelope of the m-twistor positroid as this is, in general, not a positroid.

Example 3.5.9. Consider the rank 3 positroid $\mathcal{P}$ on $[6]$ whose ranked essential family is given by

$$\mathcal{E}^{\mathrm{r}} = \{(1,[2,3]), (1,[5,6]), (3,[1,6])\}.$$

The 2-twistor matroid of $\mathcal{P}$ has bases $\mathcal{B}^{\downarrow 2} = \binom{[6]}{2} \setminus \{\{1,4\}\}$, since 1 and 4 are parallel in $\mathcal{P}^*$. However, this is not a positroid. The positroid envelope of $\mathcal{P}^{\downarrow 2}$ is the uniform matroid of rank 2 on 6 elements, $\mathrm{env}(\mathcal{P}^{\downarrow 2}) = \mathcal{U}_{2,6}$.

From the definition of twistor matroid we get the following construction for the ranked essential family of the envelope of a twistor matroid.

Proposition 3.5.10. *Let $\mathcal{P}$ be a positroid on $[n]$ of rank k with ranked essential family given by $\mathcal{E}^{\mathrm{r}}$. Fix $m \in \mathbb{Z}_{>0}$ such that $k+m \leq n$.*

The ranked essential family $(\mathcal{E}^{\mathrm{r}})^{\downarrow m}$ of $\mathrm{env}(\mathcal{P}^{\downarrow m})$ is given by:

$$\{(r^*, [n] \setminus I) \mid (r, I) \in \mathcal{E}^{\mathrm{r}} \setminus \{(k, [n])\} \text{ and } r^* = n - k - |I| + r < m\} \cup \{(m, [n])\}$$

Lam studied the poset $P_{n,k,m}$ for the particular case of $m = 2$, [74]. His result can be restated in terms of essential sets as follows.

Theorem 3.5.11 ([74, Theorem 13])**.** *Let n, k be integers with $k+2 \leq n$. Then*

$$P_{n,k,2} = \left\{ \mathcal{M} \in Q_{n,2} \,\middle|\, \sum_{(r,I) \in \mathcal{C}^{\mathrm{r}}_{\mathcal{M}}, I \neq [n]} e_I \leq k \right\}.$$

Moreover, $P_{n,k,2}$ is an upper order ideal of $Q_{n,2}$ with $\binom{n}{2}$ minimal elements given by positroids with k loops. $P_{n,k,2}$ is graded of rank $2k$ and corank function given by the codimension of the corresponding positroid cell.

Twistor matroids where introduced in the study of the amplituhedron. This is object [8] is defined as follows.

Definition 3.5.12 (Amplituhedron)**.** Let k, m, n be positive integers such that $k+m \leq n$. Let Z be a $n \times (k+m)$ matrix with positive maximal minors. The *amplituhedron* (in twistor space) $\mathcal{A}_{n,k,m} = \mathcal{A}_{n,k,m}(Z)$ is the image of $\mathrm{Gr}_{\geq 0}(k, n)$ under the twistor map $\varphi : \mathrm{Gr}(k, n) \to \mathrm{Gr}(m, n)$ such that, for $V \in \mathrm{Gr}(k, n)$, the Plücker coordinates of $\varphi(V)$ are given by

$$\left(\sum_{K \in \binom{[n]}{k}} p_K(V) p_{K \cup J}(Z) \right)_{J \in \binom{[n]}{m}}.$$

Confirming predictions by Łukowski [76], Lam showed [74, Theorem 21] that, for a generic choice of Z, the semialgebraic faces of $\mathcal{A}_{n,k,2}$ agree with the stratification defined by intersecting $\mathcal{A}_{n,k,2}$ with the positroid stratification of $\mathrm{Gr}(2, n)$. Moreover, a positroid stratum has nonempty intersection with the amplituhedron if and only if it lies in $P_{n,k,2}$.

Chapter 4

Newton–Okounkov Bodies and Combinatorial Mutations

In this chapter we will study two different families of polytopes, related to Newton–Okounkov bodies for Grassmannians and flag varieties, and show that they are equivalent up to combinatorial mutations. These are piecewise linear maps between polytopes that preserve the Erhart series of a polytope.

In Section 4.1 we will recall the definition and properties of combinatorial mutations. In Section 4.2 we will introduce the definition of matching field and a particular class of such objects, labeled by permutations. We will then give a family of mutations between the polytopes they define. This is contained in joint work with Oliver Clarke and Fatemeh Mohammadi [35]. In Section 4.3 we will introduce the class of chain-order polytopes and show how to construct mutations between them. As an application, we will intersect these polytopes with a hyperplane and show that the obtained polytope presents period collapse. This section is contained in joint work with Oliver Clarke and Akihiro Higashitani [28].

4.1 Combinatorial Mutations

We start by recalling the definition of combinatorial mutation of polytopes. This is the operation we will use to study the relations between different Newton–Okounkov bodies. Combinatorial mutations were introduced in the study of mirror symmetry and classification of Fano varieties [2]. In this context, classes of Fano polytopes up to combinatorial mutations are conjectured to be in bijective correspondence with classes of Fano varieties up to $\mathbb{Q}$-Gorenstein deformations. Here, we are interested in combinatorial mutations as they preserve some geometric invariants which are also preserved by toric degenerations, such as the volume and the Erhart polynomial of polytopes.

In the study of tropical varieties, Escobar and Harada showed how to move between different prime adjacent cones in the tropical fan via piecewise linear map. In an appendix, Ilten proved that these piecewise linear maps are indeed combinatorial mutations [44]. This is the approach we will take in the study of toric degenerations and the tropical Grassmannian.

Combinatorial mutations have two definitions that correspond to dual versions of the same map. In the original definition combinatorial mutations are operations combinatorially defined via Minkowski sums, see [2, Definition 3.3] for more details. The dual version of this operation is a piecewise linear map between the dual polytopes. The latter is the definition we are interested in and that we now recall. Fix a natural number N. For each x and y in $\mathbb{R}^N$, denote by $x \cdot y$ the usual dot product of x and y. Let $w \in \mathbb{Z}^N$ be a primitive lattice point and $F \subset w^\perp \subset \mathbb{R}^N$ a lattice polytope, where $w^\perp = \{x \in \mathbb{R}^N \mid x \cdot w = 0\}$. We define the *tropical map*

$$\varphi_{w,F} : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto x - x_{\min} w,$$

where $x_{\min} = \min\{x \cdot f \mid f \in F\}$. Each vertex $v \in F$ defines a *region of linearity* of $\varphi_{w,F}$ given by $U = \{x \in \mathbb{R}^N \mid x_{\min} = x \cdot v\}$. The restriction $\varphi_{w,F}|_U : x \mapsto x - (x \cdot v)w$ is a unimodular map given by a shear.

Let $P \subset \mathbb{R}^N$ be a polytope. If $\varphi_{w,F}(P)$ is convex, we say that $\varphi_{w,F}(P)$ is a *combinatorial mutation* of P. See Figure 4.1.

Two polytopes P and Q in $\mathbb{R}^N$ are said to be *combinatorial-mutation equivalent*, or simply *mutation equivalent*, if there exists a sequence of combinatorial mutations

$$P_1 = \varphi_{w_1,F_1}(P_0),\ P_2 = \varphi_{w_2,F_2}(P_1), \ldots, P_k = \varphi_{w_i,F_i}(P_{k-1})$$

such that $P_0 = P$ and $P_k = Q$. We refer to the polytopes $P_2, P_3, \ldots, P_{k-1}$ as the *intermediate polytopes* of the sequence of combinatorial mutations.

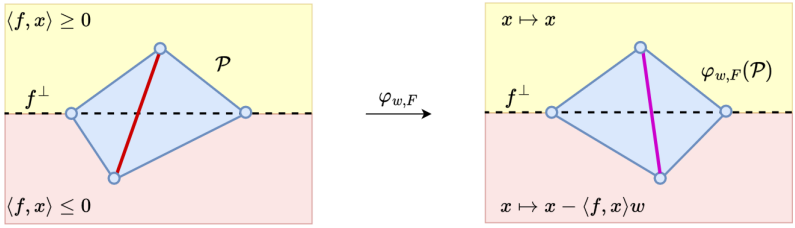


Figure 4.1: A diagram of a combinatorial mutation. In order to prove that the image of $\varphi_{w,F}$ is convex, we show that each vertex that has been moved, i.e. lying in the lower open half-space of the right-hand side picture, lies in the plane spanned by three other vertices of the polytope that were not moved, i.e. lying in the upper closed half-space of the image.

The Erhart series, see Definition 2.3.3, is equal for mutation equivalent polytopes [2, Proposition 4]. Note that here we do not assume that the dual polytopes are lattice polytopes. For clarity, we provide our own proof of this result.

Proposition 4.1.1 ([2, Proposition 4])**.** *Mutation equivalent polytopes have the same Ehrhart series.*

Proof. Suppose that $\varphi_{w,F}(P)$ is a combinatorial mutation of P. The regions of linearity of $\varphi_{w,F}$ are the maximal cones of a polyhedral fan Σ. Let $\Sigma^\circ = \{\sigma^\circ \mid \sigma \in \Sigma\}$ be the collection of relative interiors of cones in Σ. Note that Σ is a complete fan, so $\mathbb{R}^N = \bigcup_{\sigma^\circ \in \Sigma} \sigma^\circ$. Since $\varphi_{w,F}$ is piece-wise unimodular, for each $\sigma^\circ \in \Sigma^\circ$, we have that $\varphi_{w,F}(P \cap \sigma^\circ)$ and $P \cap \sigma^\circ$ have the same Ehrhart series. So, we have

$$E_{\varphi_{w,F}(P)}(t) = \sum_{\sigma^\circ \in \Sigma^\circ} E_{\varphi_{w,F}(P \cap \sigma^\circ)}(t) = \sum_{\sigma^\circ \in \Sigma^\circ} E_{P \cap \sigma^\circ}(t) = E_P(t).$$

□

Remark 4.1.2. In Section 4.2.2 we will be interested in combinatorial mutations of polytopes that do not generate new vertices. That is, we restrict to combinatorial mutations $\varphi : P \to Q$ such that $\varphi : V(P) \to V(Q)$ is a bijection. To prove that the combinatorial mutation does not create new vertices, it is enough to show that every vertex that will be moved after the combinatorial mutation, i.e. lying in the lower open half-space of the left-hand side of Figure 4.1, lies in the plane spanned by three other vertices of the polytope that will not move, i.e. lying in the upper closed half-space of the image.

4.2 Matching Fields Polytopes from Permutations

In this section we construct a class of polytopes defined by matching fields. Matching fields, introduced by Sturmfels and Zelevinsky [112], are combinatorial objects that give an ordering to each cardinality k subset of $[n]$.

4.2.1 Matching Fields

We start this section by recalling the definition of matching fields and their properties.

Definition 4.2.1 (Matching field)**.** A *matching field for* $\mathrm{Gr}(k,n)$ is a map $\Lambda : \binom{[n]}{k} \to S_k$. A *matching field for* $\mathcal{F}\ell(K;n)$ is a map

$$\Lambda : \{I \mid \varnothing \subsetneq I \subsetneq [n], |I| \in K\} \to \bigsqcup_{k\in K} S_k$$

such that for each $k \in [n-1]$ the restriction of Λ to $\binom{[n]}{k}$ is a matching field for $\mathrm{Gr}(k,n)$.

In particular, we will be interested in matching fields that can be encoded via a matrix. Formally, we have the following definition.

Definition 4.2.2 (Coherent matching field)**.** A matching field Λ is *coherent* if there exists a matrix $M \in \mathbb{R}^{(n-1)\times n}$ such that for each subset I in the domain of Λ with $I = \{i_1 < \cdots < i_k\}$ we have

$$\Lambda(I) = \mathrm{argmin}_{\sigma\in S_k} \sum_{a=1}^{k} M_{a,i_{\sigma(a)}}$$

and the minimum is attained at a unique $\sigma \in S_k$.

Each coherent matching field Λ for $\mathcal{F}\ell(K,n)$ gives rise to a monomial map. Let Q^K be defined as the product of simplices $\Delta_{\binom{n}{k_1}-1} \times \cdots \times \Delta_{\binom{n}{k_\ell}-1}$ and recall, from Section 2.3.3, that R_{Q^K} is the total coordinate ring of the toric variety defined by Q^K. Then the monomial map is defined by Λ as follows

$$\phi_\Lambda : \quad R_{\mathcal{P}} \longrightarrow \mathbb{C}[x_{i,j} \mid i \in [n-1], j \in [n]], \tag{4.1}$$

$$p_{\{j_1,\ldots,j_k\}} \mapsto \mathrm{sgn}(\Lambda(I)) \prod_{i=1}^{k} x_{i,j_{\Lambda(I)(i)}}.$$

This monomial map will be the one of interest when constructing toric varieties associated to a matching field Λ.

4.2.2 Matching Fields from Permutations

Here we are interested in studying a family of matching fields B^σ, which give rise to coherent matching fields for all partial flag varieties $\mathcal{F}\ell(K;n)$. Each such matching field is indexed by a permutation in $\sigma \in S_n$, generalizing families of coherent matching fields considered in [90] and [26].

For the Grassmannian $\mathrm{Gr}(k,n)$ this family is defined as follows.

Definition 4.2.3 (Matching field B^σ)**.** Let $k > 0$ and $\sigma \in S_n$. Given $I = \{p < q < i_3 < \cdots < i_k\} \in \binom{[n]}{k}$, the matching field $B^\sigma : \binom{[n]}{k} \to S_k$ is defined by

$$B^\sigma(I) = \begin{cases} id & \text{if } \sigma(p) > \sigma(q) \text{ or } |I| = 1, \\ (12) & \text{otherwise.} \end{cases}$$

Intuitively, B^σ is the matching field that swaps the first two entries of a subset of $[n]$ whenever σ preserves the respective order.

Fix a partial flag variety $\mathcal{F}\ell(K;n)$ with $K \subseteq [n]$. The matching field B^σ is defined simultaneously for $\mathrm{Gr}(k,n)$ for all $k \in K$. Therefore, B^σ is a matching field for the partial flag variety $\mathcal{F}\ell(K;n)$.

Lemma 4.2.4. *For all $\sigma \in S_n$, the matching fields B^σ for $\mathrm{Gr}(k,n)$ and for $\mathcal{F}\ell(K;n)$ are coherent. In particular, B^σ is induced by the weight matrix*

$$M^\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \\ Nn & N(n-1) & \cdots & N \\ N^2 n & N^2(n-1) & \cdots & N^2 \\ \vdots & \vdots & \vdots & \vdots \\ N^{n-2}n & N^{n-2}(n-1) & \cdots & N^{n-2} \end{pmatrix}$$

for any $N \geq n+1$.

Proof. Note that it is sufficient to prove that B^σ as a matching field for $\mathrm{Gr}(k,n)$ is coherent. Indeed, since we defined a unique matrix for each $k \in [n]$, it will follow that B^σ is a coherent matching field for all partial flag varieties $\mathcal{F}\ell(K;n)$.

Let $k > 0$. For $k = 1$, there is nothing to show. If $k = 2$, then let $I = \{p < q\} \in \binom{[n]}{2}$ be any 2-subset. Let Λ be the matching field induced by the matrix M^σ.

We have the following:

$$\Lambda(I) = \begin{cases} id & \text{if } M^\sigma_{1,p} + M^\sigma_{2,q} < M^\sigma_{2,p} + M^\sigma_{1,q}, \\ (12) & \text{if } M^\sigma_{2,p} + M^\sigma_{1,q} < M^\sigma_{1,p} + M^\sigma_{2,q} \end{cases}$$

$$= \begin{cases} id & \text{if } \sigma(q) < \sigma(p), \\ (12) & \text{if } \sigma(p) < \sigma(q) \end{cases} = B^\sigma(I).$$

And so the induced matching field coincides with B^σ.

For $k > 2$, let Λ be the matching field induced by M^σ and $I \in \binom{[n]}{k}$. We will show that $\Lambda(I) = B^\sigma(I)$ by induction on k, having proved the base case for $k = 1, 2$. Let $k \geq 3$. Let $I = \{i_1 < i_2 < i_3 < \cdots < i_k\}$ and let $\tau \in S_k$ be a permutation achieving the minimum of $\sum_{a=1}^k M^\sigma_{a,i_{\tau(a)}}$. We need to prove that τ is unique and $\tau(k) = k$.

Let $\pi \in S_k$ by another permutation. Then

$$\left| \sum_{a=1}^{k-1} M^\sigma_{a,i_{\pi(a)}} - \sum_{a=1}^{k-1} M^\sigma_{a,i_{\tau(a)}} \right| \leq \sum_{a=1}^{k-1} \left| M^\sigma_{a,i_{\pi(a)}} - M^\sigma_{a,i_{\tau(a)}} \right|$$

$$\leq \sum_{a=2}^{k-1} nN^{a-2} < N^{k-2}.$$

and $|M^\sigma_{k,i_{\pi(k)}} - M^\sigma_{k,i_{\tau(k)}}| \geq N^{k-2}$. Hence, $M^\sigma_{k,i_{\tau(k)}}$ has to be as small as possible. Since the rows of M^σ contain entries in descending order, we must have $\tau(k) = k$. By the induction hypothesis, we have that $\tau(j) = j$ or $3 \leq j \leq k$, that is, the behavior of the matching field only considers the first two entries of I. It follows that $\Lambda(I) = B^\sigma(I)$ for each $I \in \binom{[n]}{k}$. □

4.2.3 Toric Varieties from Matching Fields

In this section, we apply Algorithm 1 to find the polytope of the toric variety defined as the kernel of the monomial map ϕ_Λ from eq. (4.1).

We begin by fixing the notation for the toric varieties defining the ambient spaces. For simplicity of notation we will work with full flag varieties $\mathcal{F}\ell_n$. However, the following procedure can be applied to all partial flag varieties $\mathcal{F}\ell(K; n)$.

For each $d \in \mathbb{N}$, we let $\Delta_d \subseteq \mathbb{R}^d$ be the d-dimensional simplex given by the convex hull of the standard basis vectors of $\mathbb{R}^d$ together with the origin. Let Q

be the product of $(\binom{n}{k}-1)$ dimensional simplices,

$$Q = \Delta_{\binom{n}{1}-1} \times \cdots \times \Delta_{\binom{n}{n-1}-1}$$

and let Y_Q be the toric variety defined by Q. For the partial flag variety $\mathcal{F}\ell(K;n)$ with $K = \{k_1 < \cdots < k_\ell\}$, we would consider instead the polytope $Q^K = \Delta_{\binom{n}{k_1}-1} \times \cdots \times \Delta_{\binom{n}{k_\ell}-1}$.

Remark 4.2.5. Given two polytopes P and Q with s and t facets respectively, their product $P \times Q$ has $s+t$ facets. Explicitly, if $F_1, \ldots, F_s$ are the facets of P and $G_1, \ldots, G_t$ are the facets of Q, then the facets of $P \times Q$ are given by $F_i \times Q$ along with $P \times G_j$ where $i \in [s]$ and $j \in [t]$. So, each facet of P and Q gives rise to a facet of $P \times Q$.

Given a polytope P, recall from Definition 2.3.7 that the total coordinate ring of Y_P, which we write as R_P, is a polynomial ring that has one variable for each facet of P. For each $k \in [n-1]$ the simplex $\Delta_{\binom{n}{k}-1}$ has $\binom{n}{k}$ facets, so we may write

$$R_{\Delta_{\binom{n}{k}-1}} = \mathbb{C}\left[p_I \mid I \subseteq [n],\ |I| = k\right] \text{ and}$$

$$R_Q = R_{\Delta_{\binom{n}{1}-1}} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} R_{\Delta_{\binom{n}{n-1}-1}} = \mathbb{C}\left[p_I \mid \varnothing \subsetneq I \subsetneq [n]\right].$$

Each polynomial ring $R_{\Delta_{\binom{n}{k}-1}}$ is the homogeneous coordinate ring of $\mathbb{P}^{\binom{n}{k}-1}$. Recall that the Plücker embedding $\mathcal{F}\ell_n \hookrightarrow \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ is defined by the ring map

$$\phi : R_Q \to \mathbb{C}[x_{i,j} |\ i \in [n-1], j \in [n]], \quad p_I \mapsto \det(X_I) \tag{4.2}$$

where $X = (x_{i,j})$ is an $(n-1) \times n$ matrix of variables and for each subset $I \subset [n]$ we write X_I for the submatrix of X whose columns are those indexed by I and whose rows are indexed by $1, \ldots, |I|$. A matching field can be thought of as a modification of ϕ to a monomial map which sends each Plücker variable p_I to a single term of $\det(X_I)$. The kernel of this monomial map, as in eq. (4.1), gives rise to a toric variety which, under certain conditions, is a toric degeneration of the flag variety.

We proceed to apply Algorithm 1 to the toric variety Y_Q with total coordinate ring R_Q together with the monomial map ϕ_Λ in eq. (4.1) for some coherent matching field Λ.

Step 1. Let us compute the H-description of Q. For each $k \in \mathbb{N}$ we write $\mathbf{I}_k$ for the $k \times k$ identity matrix and $\mathbf{1}_k$ for the column vector of all ones. We define the $(k+1) \times k$ matrix $\mathbf{J}_k = [\mathbf{I}_k \mid -\mathbf{1}_k]^t$ given by the identity matrix with a row of -1's below it. We define the column vector $\rho_{k+1} = (0, \ldots, 0, 1)^t \in \mathbb{R}^{k+1}$. The simplex Δ_k is defined to be the convex hull of the standard basis vectors in $\mathbb{R}^k$ together with the origin. Equivalently, we have that

$$\Delta_k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \colon x_i \geq 0 \text{ for all } i \text{ and } x_1 + \cdots + x_k \leq 1\}.$$

Therefore, the H-description of Δ_k is given by $\Delta_k = \{x \in \mathbb{R}^k \colon \mathbf{J}_k \cdot x \geq -\rho_{k+1}\}$. Since Q is the direct product of the simplices $\Delta_{\binom{n}{1}-1}, \ldots, \Delta_{\binom{n}{n-1}-1}$, we define the $\left(\binom{n}{1} + \cdots + \binom{n}{n-1}\right) \times \left(\binom{n}{1} - 1 + \cdots + \binom{n}{n-1} - 1\right)$ matrix $\mathbf{B}$ to be the direct sum of $\mathbf{J}_{\binom{n}{1}-1}, \ldots, \mathbf{J}_{\binom{n}{n-1}-1}$ which is given by

$$\mathbf{B} = \begin{pmatrix} \mathbf{J}_{\binom{n}{1}-1} & 0 & \cdots & 0 \\ 0 & \mathbf{J}_{\binom{n}{2}-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}_{\binom{n}{n-1}-1} \end{pmatrix}.$$

Similarly, we define the column vector $\rho = \left(\rho_{\binom{n}{1}} | \ldots | \rho_{\binom{n}{n-1}}\right)^t \in \mathbb{Z}^{\binom{n}{1}+\cdots+\binom{n}{n-1}}$. Hence, by construction we have that the H-description of Q is

$$Q = \{x \in \mathbb{R}^{\binom{n}{1}-1+\cdots+\binom{n}{n-1}-1} \colon \mathbf{B} \cdot x \geq -\rho\}.$$

Step 2. Let us compute the grading matrix $\mathbf{D}$, given by the cokernel of $\mathbf{B}$. Observe that for each $k \in \mathbb{N}$, the cokernel of $\mathbf{J}_k$ is $\mathbb{Z}^{k+1}/\operatorname{Im}(\mathbf{J}_k)$ which we identify with $\mathbb{Z}$ by sending $(x_1, \ldots, x_{k+1}) \in \mathbb{Z}^{k+1}$ to $x_1 + \cdots + x_{k+1} \in \mathbb{Z}$. We obtain a short exact sequence given by $\mathbb{Z}^k \to \mathbb{Z}^{k+1} \to \mathbb{Z}$ where the left-hand map is given by $\mathbf{J}_k$ and the right-hand map is the all-ones matrix $\mathbf{1}_{k+1}^t$. So, we define

$$\mathbf{D} = \begin{pmatrix} \mathbf{1}^t_{\binom{n}{1}} & 0 & \cdots & 0 \\ 0 & \mathbf{1}^t_{\binom{n}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^t_{\binom{n}{n-1}} \end{pmatrix} \in \mathbb{Z}^{(n-1)\times\left(\binom{n}{1}+\cdots+\binom{n}{n-1}\right)}.$$

Hence, we obtain a short exact sequence

$$\mathbb{Z}^{\binom{n}{1}-1+\cdots+\binom{n}{n-1}-1} \xrightarrow{\mathbf{B}} \mathbb{Z}^{\binom{n}{1}+\cdots+\binom{n}{n-1}} \xrightarrow{\mathbf{D}} \mathbb{Z}^{n-1}.$$

We also note that $\underline{d} := \mathbf{D} \cdot \rho = \mathbf{1}_{n-1} \in \mathbb{Z}^{n-1} =: \mathrm{Cl}(Y_Q)$.

Explicitly, the grading induced by $\mathbf{D}$ on the total coordinate ring $R_Q = \mathbb{C}[p_I \mid \varnothing \subsetneq I \subsetneq [n]]$ is given by $\deg(p_I) = e_{|I|} \in \mathbb{Z}^{n-1}$ for each subset I, where $e_1, \ldots, e_{n-1}$ are the standard unit vectors in $\mathbb{Z}^{n-1}$. We also define the $\mathrm{Cl}(Y_Q)$-grading on the ring $\mathbb{C}[x_{i,j}]$ by $\deg(x_{i,j}) = e_i - e_{i-1}$ if $i > 1$ and $\deg(x_{1,j}) = e_1$, for all $j \in [n]$. We now show that the matching field ideal $\ker(\phi_\Lambda)$ is homogeneous with respect to the grading given by $\mathbf{D}$.

Lemma 4.2.6. *The monomial map ϕ_Λ is concurrent with respect to the $\mathrm{Cl}(Y_Q)$-grading of R_Q.*

Proof. The result follows immediately from the fact that the grading on the total coordinate ring R_Q is induced by the $\mathrm{Cl}(Y_Q)$-grading on $\mathbb{C}[x_{i,j}]$. Explicitly, if $I \subseteq [n]$ is a k-subset such that $\phi_\Lambda(p_I) = c x_{1,i_1} x_{2,i_2} \ldots x_{k,i_k}$ for some $i_j \in [n]$ and $c \in \{+1, -1\}$, then we have that

$$\deg(\phi_\Lambda(p_I)) = \deg(x_{k,i_k}) + \cdots + \deg(x_{2,i_2}) + \deg(x_{1,i_1}) =$$

$$= (e_k - e_{k-1}) + \cdots + (e_2 - e_1) + (e_1) = e_k = \deg(p_I). \qquad \square$$

Step 3. For the final step of Algorithm 1, we identify Q with its image under the map $x \mapsto \mathbf{B} \cdot x + \rho$ which lives in $\mathbb{R}^{\binom{n}{1} + \cdots + \binom{n}{n-1}}$. Consider the lattice $\mathbb{Z}^{n(n-1)}$ associated to the polynomial ring $\mathbb{C}[x_{i,j}]$. The monomial map ϕ_Λ gives rise to a lattice map $\hat{\phi}_\Lambda : \mathbb{Z}^{\binom{n}{1} + \cdots + \binom{n}{n-1}} \to \mathbb{Z}^{n(n-1)}$ which sends Q to the polytope Q_Λ of the toric variety associated to the monomial map ϕ_Λ.

Definition 4.2.7 (Matching field polytope)**.** The *matching field polytope* associated to the matching field Λ is $Q_\Lambda = (\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(Q)$.

In order to distinguish between matching field polytopes for different varieties we write Q_Λ^K for the matching field polytope associated to the partial flag variety $\mathcal{F}\ell(K; n)$. In particular Q_Λ^k denotes the matching field polytope associated to $\mathrm{Gr}(k, n)$ and $Q_\Lambda^{[n]}$ the matching field polytope associated to $\mathcal{F}\ell_n$.

By applying the previous procedure to a coherent matching field for the Grassmannian $\mathrm{Gr}(k, n)$, we get the following explicit description of the vertices of the matching field polytope.

Corollary 4.2.8. *Fix Λ a coherent matching field for $\mathrm{Gr}(k, n)$. The matching field polytope of Λ is given by*

$$Q_\Lambda^k = \mathrm{conv}\left(\sum_{a=1}^{k} e_{a, i_{\Lambda(I)(a)}} \mid I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k} \right) \subseteq \mathbb{R}^{k \times n},$$

where $e_{i,j}$ is the standard basis vector of $\mathbb{R}^{k\times n}$ such that the only non-zero entry is in row i and column j.

Minkowski Sum Property

We now use the description given in Corollary 4.2.8 to give an explicit description for any matching field polytope Q^K_Λ. In particular, we will show that a matching field polytope for $\mathcal{F}\ell(K;n)$ can be constructed as a Minkowski sum of matching field polytopes for the Grassmannians $\mathrm{Gr}(k,n)$ for $k \in K$.

Notation 4.2.9. For each subset $K = \{k_1, \dots, k_t\} \subseteq [n-1]$ we write ϕ^K for the restriction of the map ϕ from Equation (4.2) to the subring

$$R_{Q^K} = R_{\Delta_{\binom{n}{k_1}-1}} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} R_{\Delta_{\binom{n}{k_t}-1}} = \mathbb{C}\left[p_I \mid I \subseteq [n],\ |I| \in K\right].$$

The ring map ϕ^K defines the Plücker embedding of the partial flag variety $\mathcal{F}\ell(K;n)$ into $\mathbb{P}^{\binom{n}{k_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{k_t}-1}$. Similarly, for each matching field Λ for $\mathcal{F}\ell_n$, we define the monomial map ϕ^K_Λ to be the restriction of ϕ_Λ from Equation (4.1) to R_{Q^K}. Recall that the matching field polytope Q^K_Λ is defined as $(\hat{\phi}^K_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(Q^K)$.

We are now ready to prove the Minkowski sum property of toric subvarieties of (partial) flag varieties.

Proposition 4.2.10 (Minkowski sum property)**.** *Let $K \subseteq [n]$ be a non-empty subset and let Λ be a coherent matching field for $\mathcal{F}\ell(K;n)$. Then the matching field polytope Q^K_Λ is equal to the Minkowski sum of the polytopes Q^k_Λ associated to the matching fields Λ^k for each $k \in K$.*

Proof. Observe that the polytope $Q = \Delta_{\binom{n}{1}-1} \times \cdots \times \Delta_{\binom{n}{n-1}-1} \subseteq \mathbb{R}^{\binom{n}{1}+\cdots+\binom{n}{n-1}}$ is a Minkowski sum of simplices

$$Q = \sum_{k=1}^{n-1} \{0\} \times \cdots \times \{0\} \times \Delta_{\binom{n}{k}-1} \times \{0\} \times \cdots \times \{0\}.$$

Since linear maps commute with taking Minkowski sums, we have that the matching field polytope $Q^{[n]}_\Lambda = (\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(Q)$ is the Minkowski sum of polytopes $(\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})\left(\{0\} \times \cdots \times \Delta_{\binom{n}{k}-1} \times \cdots \times \{0\}\right)$ for each $k \in [n-1]$. By definition, each summand is the matching field polytope $Q^k_\Lambda \subseteq \mathbb{R}^{(n-1)\times n}$ associated to the Grassmannian embedded into a product of projective spaces $\mathrm{Gr}(k,n) \subseteq \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$. □

Remark 4.2.11. We note that Proposition 4.2.10 may be generalized as follows. For each $\lambda \in \mathbb{N}^{n-1}$ define the polytope $\lambda Q := \lambda_1 \Delta_{\binom{n}{1}-1} \times \cdots \times \lambda_{n-1}\Delta_{\binom{n}{n-1}-1}$. If $\lambda_i > 0$ for all i then λQ corresponds to a choice of a very ample line bundle on $\mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ which defines a different embedding into projective space. Otherwise, λQ corresponds to a choice of a very ample line bundle on the subspace $\mathbb{P}^{\binom{n}{k_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{k_t}-1} \subseteq \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$, where $K = \{k_1 < \cdots < k_t\} \subseteq [n-1]$ indexes the nonzero entries of λ.

Following Algorithm 1, the H-description of each scaled simplex is given by

$$\lambda_k \Delta_{\binom{n}{k}-1} = \left\{ x \in \mathbb{R}^{\binom{n}{k}-1} \mid \mathbf{J}_{\binom{n}{k}-1} \cdot x \geq -\rho'_{\binom{n}{k}} \right\}$$

where $\rho'_{\binom{n}{k}} = (0, \ldots, 0, \lambda_k)^t \in \mathbb{Z}^{\binom{n}{k}}$. So, the H-description of λQ is given by the same matrix $\mathbf{B}$; hence the grading matrix is also given by the same matrix $\mathbf{D}$. Similarly to Section 4.2.3, define $\rho' = \left(\rho'_{\binom{n}{1}} | \ldots | \rho'_{\binom{n}{n-1}} \right)^t$. Notice that $\mathbf{D}$ sends the vector ρ' to λ.

The polytope $\lambda Q_\Lambda := (\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(\lambda Q)$ is different from the matching field polytope Q_Λ. However, the proof of Proposition 4.2.10 may be applied to show that λQ_Λ is a Minkowski sum of polytopes associated to Grassmannians. In this case, the summands are given by $(\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(\lambda_k \Delta_{\binom{n}{k}-1})$. In particular, if λ is the indicator vector for the subset $K \subseteq [n-1]$ then we have shown that the matching field polytope Q_Λ^K for the partial flag variety $\mathcal{F}\ell(K; n)$ is the Minkowski sum of the polytopes Q_Λ^k where $k \in K$.

4.2.4 Newton–Okounkov Bodies and Matching Fields

Matching field polytopes are examples of Newton–Okounkov bodies. Consider the total coordinate ring $R_Q = \mathbb{C}[p_I | \varnothing \subsetneq I \subsetneq [n]]$ of $\mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$. Fix a matching field Λ and assume that the kernel of ϕ_Λ is a toric degeneration of $\mathcal{F}\ell_n$.

Recall the lattice map $\hat{\phi}_\Lambda$ described in the 3rd step of the application of Algorithm 1 to ϕ_Λ. After an arbitrary choice of linear ordering $\succ$ on $\mathbb{Z}S$, we can derive from Λ a valuation:

$$\widetilde{\mathfrak{v}}_\Lambda : \mathbb{C}[p_I] \setminus \{0\} \to \mathbb{Z}S,$$

$$\sum_{\alpha \in \mathbb{Z}^{\binom{n}{1}+\cdots+\binom{n}{n-1}}} c_\alpha \mathbf{p}^\alpha \mapsto \min_{\succ} \{ \hat{\phi}_\Lambda(\alpha) | c_\alpha \neq 0 \}.$$

In particular, the monomials $\mathbf{p}^v$ in $\mathbb{C}[p_I]$ are sent to $\hat{\phi}_\Lambda(v)$. So the image of $\widetilde{\mathfrak{v}}_\Lambda$ agrees with the image of $\hat{\phi}_\Lambda$. Let $A^{[n]} = \mathbb{C}[p_I]/(\ker\phi)$ be the homogeneous coordinate ring of $\mathcal{F}\ell_n$. We obtain a quasi-valuation $\mathfrak{v}_\Lambda$ on $A^{[n]}$ given by:

$$\mathfrak{v}_\Lambda(\widetilde{f}) = \max\{\widetilde{\mathfrak{v}}_\Lambda(f) \mid f \equiv \widetilde{f} \mod \ker\phi_\Lambda\}.$$

Since Λ induces a toric degeneration of $\mathcal{F}\ell_n$, it follows that $\mathfrak{v}_\Lambda$ is a valuation and (the image under the quotient map of) the Plücker variables form a Khovanskii basis for $A^{[n]}$. See [16, Corollary 2]. Furthermore, $Q_\Lambda^{[n]}$ is the *Newton–Okounkov body* $\Delta(A^{[n]}, \mathfrak{v}_\Lambda)$ of the pair $(A^{[n]}, \mathfrak{v}_\Lambda)$. By [16, Corollary 1], we have that $\Delta(A^{[n]}, \mathfrak{v}_\Lambda)$ is a Minkowski sum of polytopes. Moreover, by Proposition 4.2.10, we have that $\Delta(A^{[n]}, \mathfrak{v}_\Lambda)$ is the Minkowski sum of the polytopes Q_Λ^k, which are themselves Newton–Okounkov bodies for the Grassmannian $Q_\Lambda^k = \Delta(A^k, \mathfrak{v}_\Lambda)$, where A^k denotes the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(k, n)$.

4.2.5 Mutations of Matching Field Polytopes

The main goal of this section is to prove that a subclass of the matching fields labeled by permutations introduced in Section 4.2.2 are combinatorial mutation equivalent. To describe the permutations that we consider, we make use of the language of permutation patterns.

Definition 4.2.12 (Pattern avoidance)**.** A permutation $\sigma \in S_n$ written in single-line notation $(\sigma(1), \ldots, \sigma(n))$ is said to *contain the pattern* $\pi = (\pi_1, \ldots, \pi_k) \in S_k$ for some $k \leq n$ if there is a subsequence $(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k))$ for some $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, such that the relative order of the subsequence coincides with that of π. Explicitly, for all $p, q \in [k]$ we have that $\sigma_{i_p} < \sigma_{i_q}$ if and only if $\pi_p < \pi_q$. If σ does not contain the pattern π, then we say that σ is free of pattern π.

For example, the only permutation in S_n that is free of the pattern 12 is $w_0 = (n, n-1, \ldots, 1)$.

Recall that for $\sigma \in S_n$, B^σ is the matching field defined in Definition 4.2.3. When seen as a matching field for the partial flag variety $\mathcal{F}\ell(K; n)$, we denote the corresponding matching field polytope by P_σ^K. We can now state the main theorem of this section.

Theorem 4.2.13. *Fix $n \in \mathbb{Z}$, $n > 0$ and a subset $K \subseteq [n]$. If $\sigma \in S_n$ is a permutation that avoids the patterns 4123, 3124, 1423 and 1324, then the polytope P_σ^K is combinatorial mutation equivalent to the Gelfand–Tsetlin polytope $P_{w_0}^K$.*

We prove Theorem 4.2.13 by explicitly constructing a sequence of combinatorial mutations between matching field polytopes. Throughout this section, unless otherwise specified, we fix the following setup.

Setup 4.2.14. Fix n, a permutation $\sigma \in S_n$ and a positive integer $\ell \in [n-1]$. We define $\lambda = \sigma^{-1}(\ell)$ and $\mu = \sigma^{-1}(\ell+1)$. We assume that $\lambda < \mu$ and define $\tau = (\ell\ \ell+1)\sigma$. For each $k \in [n]$, the matching field polytope for the Grassmannian $\mathrm{Gr}(k,n)$ associated to the matching fields B^σ and B^τ will be denoted by P^k_σ and P^k_τ respectively. Each such polytope is assumed to live in $\mathbb{R}^{k\times n} \subseteq \mathbb{R}^{(n-1)\times n}$. Explicitly, this inclusion is given by extending the standard basis $\{e_{i,j} \mid i \in [k],\ j \in [n]\}$ of $\mathbb{R}^{k\times n}$ to the standard basis of $\{e_{i,j} \mid i \in [n-1],\ j \in [n]\}$ of $\mathbb{R}^{(n-1)\times n}$. From Proposition 4.2.10, it follows that for each non-empty subset $K \subseteq [n-1]$, the matching field polytope for the partial flag variety $\mathcal{F}\ell(K;n)$ associated to the matching field B^γ is given by $P_\gamma = \sum_{k\in K} P^k_\gamma$ for each $\gamma \in \{\sigma,\tau\}$. We primarily work with the polytopes P^k_σ and so write v^k_I for the vertex of the polytope P^k_σ corresponding to the k-subset $I \subseteq [n]$, i.e.

$$v^k_I = \sum_{a=1}^{k} e_{a,i_{B^\sigma(I)(a)}}, \quad I = \{i_1 < \cdots < i_k\}.$$

Note that if any of ℓ, λ or μ are fixed then the rest are also determined uniquely by σ.

Definition 4.2.15 (Tropical map $\varphi^{\sigma,\ell}$)**.** With the notation of Setup 4.2.14, we define $w \in \mathbb{R}^{(n-1)\times n}$ by

$$w_{i,j} = \begin{cases} 1 & \text{if } (i=1 \text{ and } j=\lambda) \text{ or } (i=2 \text{ and } j=\mu),\\ -1 & \text{if } (i=1 \text{ and } j=\mu) \text{ or } (i=2 \text{ and } j=\lambda),\\ 0 & \text{otherwise.}\end{cases}$$

We define $f \in \mathbb{R}^{(n-1)\times n}$ by

$$(f)_{i,j} = \begin{cases} 1 & \text{if } i=1,\ \sigma(j) > \ell+1,\\ -1 & \text{if } i=2,\ \sigma(j) \geq \ell,\\ 0 & \text{otherwise}\end{cases}$$

We note that $f \in w^\perp$, hence we may define the tropical map $\varphi^{\sigma,\ell} := \varphi_{w,F}$ where $F = \mathrm{conv}\{0,f\}$.

Lemma 4.2.16. *Fix $k < n$ and Setup 4.2.14. For each $v \in V(P^k_\sigma)$ we have $\langle f, v\rangle \in \{-1,0,1\}$. More precisely:*

(1) $\langle f,v\rangle = 1$ if and only if $v = v^k_I$ with $I = \{j_1, j_2 < j_3 < \cdots < j_k\}$ such that ($k=1$ and $\sigma(j_1) > \ell+1$) or ($k \geq 2$ and $\sigma(j_1) > \ell+1 > \ell > \sigma(j_2)$);

(2) $\langle f, v\rangle = -1$ *if and only if* $v = v_I^k$ *with* $k \geq 2$ *and* $I = \{\lambda, \mu < i_3 < \cdots < i_k\}$.

Proof. Note that if $k = 1$, then the entries of $f \in \mathbb{R}^{1\times n} \subseteq \mathbb{R}^{(n-1)\times n}$ are either 0 or 1, hence $\langle v_I, f\rangle \in \{0, 1\}$. Assume now that $k \geq 2$.

Let $v = v_I \in V(P_\sigma^k)$ and suppose that $I = \{\lambda < \mu < i_3 < \cdots < i_k\}$. Note that the only non-zero coordinates of f lie in the first two rows and $\ell = \sigma(\lambda) < \sigma(\mu) = \ell + 1$. Therefore, we have

$$\langle f, v\rangle = f_{1,\mu} + f_{2,\lambda} = 0 - 1 = -1.$$

Suppose that $I = \{i_1 < i_2 < i_3 < \cdots < i_k\}$ with $\{i_1, i_2\} \neq \{\lambda, \mu\}$. Let $a, b \in \{1, 2\}$ be indices such that $\sigma(i_a) < \sigma(i_b)$. It follows that the inner product of f and v is given by

$$\langle f, v\rangle = f_{1,i_b} + f_{2,i_a} = \begin{cases} 0 + 0 = 0 & \text{if } \sigma(i_b) \leq \ell + 1, \\ 1 + 0 = 1 & \text{if } \sigma(i_a) < \ell \text{ and } \sigma(i_b) > \ell + 1, \\ 1 - 1 = 0 & \text{if } \sigma(i_a) \geq \ell. \end{cases}$$

In particular, $\langle f, v_I\rangle \in \{-1, 0, 1\}$ and so (1) and (2) hold. □

Proposition 4.2.17. *Fix* $k < n$ *and Setup 4.2.14. We have* $\varphi^{\sigma,\ell}(V(P_\sigma^k)) = V(P_\tau^k)$ *is a bijection between the vertices of* P_σ^k *and* P_τ^k *that preserves the indexing of the vertices.*

Proof. Recall that the tropical map $\varphi^{\sigma,\ell} := \varphi_{w,F}$ acts as follows

$$\varphi^{\sigma,\ell}(x) = \begin{cases} x & \text{if } \langle f, x\rangle \geq 0, \\ x - \langle f, x\rangle w & \text{if } \langle f, x\rangle < 0 \end{cases}$$

for each $x \in \mathbb{R}^{(n-1)\times n}$. By Lemma 4.2.16, the only vertices of P_σ^k that are not fixed by $\varphi^{\sigma,\ell}$ are indexed by subsets $I = \{\lambda, \mu < i_3 < \cdots < i_k\}$. For each such vertex v_I of P_σ^k, we have that $v_I - w$ is the vertex of P_τ^k indexed by I. Since P_σ^k and P_τ^k have the same number of vertices, it follows that $\varphi^{\sigma,\ell}$ acts as a bijection on the vertices. □

We proceed by showing the following technical lemmas (Lemmas 4.2.18, 4.2.19 and 4.2.20) that we will use to prove Theorem 4.2.13. The overachieving idea is as follows. Assume Setup 4.2.14, let $\varphi_{w,F}$ be the tropical map as in Definition 4.2.15 with $F = \operatorname{conv}\{0, f\}$ and let P_σ be the matching field polytope associated to B^σ. Suppose that for every pair of vertices $v_1, v_2 \in V(P_\sigma)$ with $\langle v_1, f\rangle = 1 = -\langle v_2, f\rangle$ there exist vertices $u_1, u_2 \in V(P_\tau) \cap f^\perp$ such that $v_1 + v_2 = u_1 + u_2$. Then it follows that $\varphi_{w,F}(P_\sigma) \subseteq \operatorname{conv}\{\varphi_{w,F}(v) \mid v \in V(P_\tau)\} = P_\tau$.

Lemma 4.2.18. *Assume that σ is free of the patterns* 1423 *and* 4123. *Furthermore, assume that $\mu = \max\{i \in [n] \mid \sigma(i) > n+1-i\}$. Then for every $v_I^k \in V(P_\sigma^k)$ and $v_J^h \in V(P_\sigma^h)$ with $\langle f, v_I^k\rangle = -1$, $\langle f, v_J^h\rangle = 1$, there exist $u_1 \in V(P_\sigma^k) \cap f^\perp$, $u_2 \in V(P_\sigma^h) \cap f^\perp$ such that $v_I^k + v_J^h = u_1 + u_2$.*

Proof. Let $v_I^k \in V(P_\sigma^k)$ and $v_J^h \in V(P_\sigma^h)$ such that $\langle f, v_I^k\rangle = -1$, $\langle f, v_J^h\rangle = 1$. Note that $k \geq 2$. Moreover, by Lemma 4.2.16 it follows that $I = \{\lambda, \mu < i_3 < \cdots < i_k\}$ and $J = \{j_1, j_2 < j_3 < \cdots < j_h\}$ such that $\sigma(j_1) > \ell + 1$ and if $h \geq 2$ then we also have $\ell > \sigma(j_2)$. By the assumption on μ, it follows that $j_1 < \mu$. Since σ is free of 1423 and 4123, if $h \geq 2$ then it follows that $\max\{j_1, j_2\} > \lambda$. We proceed by taking cases on h.

Case 1. Assume $h = 1$. Then $J = \{j_1\}$ with $\sigma(j_1) > \ell + 1$. Since $j_1 < \mu < i_3$, we have

$$\begin{aligned} v_I^k + v_J^h &= \left(e_{1,\mu} + e_{2,\lambda} + \sum_{a=3}^{k} e_{a,i_a}\right) + e_{1,j_1} \\ &= \left(e_{1,j_1} + e_{2,\lambda} + \sum_{a=3}^{k} e_{a,i_a}\right) + e_{1,\mu} = u_1 + u_2 \end{aligned}$$

where $u_1 = v^k_{\{j_1, \lambda < i_3 < \cdots < i_k\}} \in V(P_\sigma^k)$ and $u_2 = v^h_{\{\mu\}} \in V(P_\sigma^h)$, which satisfy $\langle f, u_1\rangle = \langle f, u_2\rangle = 0$.

Case 2. Assume $h = k = 2$. Then $I = \{\lambda, \mu\}$ and $J = \{j_1, j_2\}$ with $\sigma(j_2) < \ell < \ell + 1 < \sigma(j_1)$. We can write

$$v_I^k + v_J^h = (e_{1,\mu} + e_{2,\lambda}) + (e_{1,j_1} + e_{2,j_2}) = (e_{1,j_1} + e_{2,\lambda}) + (e_{1,\mu} + e_{2,j_2}) = u_1 + u_2$$

with $u_1 = e_{1,j_1} + e_{2,\lambda}$ and $u_2 = e_{1,\mu} + e_{2,j_2}$.

Case 3. Assume $h = 2$ and $k \geq 3$. By assumption, we have that $J = \{j_1, j_2\}$ with $\sigma(j_2) < \ell < \ell + 1 < \sigma(j_1)$ and $j_1 < \mu < i_3$, hence

$$\begin{aligned} v_I^k + v_J^h &= \left(e_{1,\mu} + e_{2,\lambda} + \sum_{a=3}^{k} e_{a,i_a}\right) + (e_{1,j_1} + e_{2,j_2}) \\ &= \left(e_{1,j_1} + e_{2,\lambda} + \sum_{a=3}^{k} e_{a,i_a}\right) + (e_{1,\mu} + e_{2,j_2}) = u_1 + u_2 \end{aligned}$$

where $u_1 = v^k_{\{j_1, \lambda < i_3 < \cdots < i_k\}} \in V(P_\sigma^k)$ and $u_2 = v^h_{\{\mu, j_2\}} \in V(P_\sigma^h)$, which satisfy $\langle f, u_1\rangle = \langle f, u_2\rangle = 0$.

Case 4. Assume $h \geq 3$. Then $J = \{j_1, j_2 < j_3 < \cdots < j_h\}$ with $\sigma(j_2) < \ell < \ell + 1 < \sigma(j_1)$. If $k = 2$ then note that $\max\{j_1, j_2\} > \lambda$, and so we have

$$\begin{aligned} v_I^k + v_J^h &= (e_{1,\mu} + e_{2,\lambda}) + \left(\sum_{b=1}^{h} e_{b,j_b} \right) \\ &= (e_{1,\mu} + e_{2,j_2}) + \left(e_{1,j_1} + e_{2,\lambda} + \sum_{b=3}^{h} e_{b,j_b} \right) = u_1 + u_2 \end{aligned}$$

where $u_1 = v^k_{\mu,j_2} \in V(P^k_\sigma)$ and $u_2 = v^h_{j_1,\lambda<j_3<\cdots<j_h} \in V(P^h_\sigma)$, which satisfy $\langle f, u_1 \rangle = \langle f, u_2 \rangle = 0$. Otherwise, if $k \geq 3$ then we take further cases with either $j_2 < \mu$ or $j_2 > \mu$.

Case 4.1. Assume that $j_2 < \mu$. In this case we have that

$$\begin{aligned} v_I^k + v_J^h &= \left(e_{1,\mu} + e_{2,\lambda} + \sum_{a=1}^{k} e_{a,i_a} \right) + \left(\sum_{b=1}^{h} e_{b,j_b} \right) \\ &= \left(e_{1,\mu} + e_{2,j_2} + \sum_{a=3}^{k} e_{a,i_a} \right) + \left(e_{1,j_1} + e_{2,\lambda} + \sum_{b=3}^{h} e_{b,j_b} \right) = u_1 + u_2 \end{aligned}$$

where $u_1 = v^k_{\mu,j_2<i_3<\cdots<i_k} \in V(P^k_\sigma)$ and $u_2 = v^h_{j_1,\lambda<j_3<\cdots<j_h} \in V(P^h_\sigma)$, which satisfy $\langle f, u_1 \rangle = \langle f, u_2 \rangle = 0$.

Case 4.2. Assume that $j_2 > \mu$. In this case we have that

$$\begin{aligned} v_I^k + v_J^h &= \left(e_{1,\mu} + e_{2,\lambda} + \sum_{a=1}^{k} e_{a,i_a} \right) + \left(\sum_{b=1}^{h} e_{b,j_b} \right) \\ &= \left(e_{1,j_1} + e_{2,\lambda} + \sum_{a=3}^{k} e_{a,i_a} \right) + \left(e_{1,\mu} + e_{2,j_2} + \sum_{b=3}^{h} e_{b,j_b} \right) = u_1 + u_2 \end{aligned}$$

where $u_1 = v^k_{j_1,\lambda<i_3<\cdots<i_k} \in V(P^k_\sigma)$ and $u_2 = v^h_{\mu,j_2<j_3<\cdots<j_h} \in V(P^h_\sigma)$, which satisfy $\langle f, u_1 \rangle = \langle f, u_2 \rangle = 0$.

In each case we have constructed $u_1 \in V(P^k_\sigma)$ and $u_2 \in V(P^h_\sigma)$ such that $v_I^k + v_J^h = u_1 + u_2$ and $\langle f, u_1 \rangle = \langle f, u_2 \rangle = 0$, so we are done. □

An easy consequence of the lemma above is the following result which shows that every point in the Minkowski sum P^K_σ can be written as a sum of vertices that all lie in the same half-space defined by f.

Lemma 4.2.19. *Assume Setup 4.2.14 holds and that* $\mu = \max\{i \in [n] \mid \sigma(i) > n+1-i\}$. *If* σ *is free of patterns* $1423, 4123, 1324$ *and* 3124, *then so is* $\tau = (\ell\ \ell+1)\sigma$.

Proof. Assume by contradiction that τ contains the pattern 1423. Then it follows that ℓ and $\ell+1$ correspond to elements of the pattern, otherwise the same pattern exists in σ. Since $\lambda < \mu$ by assumption, ℓ and $\ell+1$ correspond to 3 and 4 in the pattern respectively. However, σ contains the pattern $(\ell\ \ell+1)1423 = 1324$, a contradiction. The same argument shows that τ is also free of the pattern 4123.

Assume by contradiction that τ contains the pattern 3124. As before, if ℓ and $\ell+1$ are not contained in the pattern then the same pattern appears in σ. Therefore, ℓ and $\ell+1$ correspond to 2 and 3 in the pattern, respectively. By assumption, for all $i > \mu$ we have $\sigma(i) < \ell+1$. However, the 4 in the pattern 3124 contradicts this assumption since $\ell+1$ corresponds to 3. Therefore, τ is free of the pattern 3124. A similar argument shows that τ is free of the pattern 1324. □

Lemma 4.2.20. *Assume Setup 4.2.14 holds. Suppose* σ *is free of patterns* $1423, 4123, 1324$ *and* 3124 *and assume that* $\mu = \max\{i \in [n] \mid \sigma(i) > n+1-i\}$. *Then* $\varphi^{\sigma,\ell}$ *defines a combinatorial mutation from* P_σ^K *to* P_τ^K.

Proof. Write $P_\sigma^{K,+} = P_\sigma^K \cap \{x \in \mathbb{R}^{(n-1)\times n} \mid \langle f, x\rangle \geq 0\}$ and similarly $P_\sigma^{K,-} = P_\sigma^K \cap \{x \in \mathbb{R}^{(n-1)\times n} \mid \langle f, x\rangle \leq 0\}$. Note that $\sum_{k\in K} P_\sigma^{k,+} \subseteq P_\sigma^{K,+}$. Consider $t \in P_\sigma^{K,+}$. We can write t in the form

$$t = \sum_{k\in K} \sum_{I \in \binom{[n]}{k}} \alpha_I^k v_I^k$$

for some $\alpha_I^k \geq 0$ such that $\sum_I \in \binom{[n]}{k} \alpha_I^k = 1$ for all $k \in K$. Suppose that there exist $k, h \in K$ and $I \in \binom{[n]}{k}$, $J \in \binom{[n]}{h}$ such that $\alpha_I^k \neq 0$, $\alpha_J^h \neq 0$ and v_I^k, v_J^h lie in two different half-spaces defined by f, that is, $\langle f, v_I^k\rangle = -1$ and $\langle f, v_J^h\rangle = 1$. By Lemma 4.2.18, there exists $u_1 \in V(P_\sigma^k) \cap f^\perp$ and $u_2 \in V(P_\sigma^h) \cap f^\perp$ such that $v_I^k + v_J^h = u_1 + u_2$. Then, if $\alpha = \min\{\alpha_I^k, \alpha_J^h\}$, we can substitute $\alpha_I^k v_I^k + \alpha_J^h v_J^h$ with $\alpha(u_1+u_2) + (\alpha_I^k - \alpha)v_I^k + (\alpha_J^h - \alpha)v_J^h$. The new expression for t contains a strictly smaller number of pair of vertices lying in opposite half-space defined by f. Since the sum is finite, we can repeat the process until we get an expression such that for all $k \in K$ and $I \in \binom{[n]}{k}$ with $\alpha_I^k \neq 0$, $\langle f, v_I^k\rangle \geq 0$. Then $t \in \sum_{k\in K} P^{k,+}$ and we can conclude that $\sum_{k\in K} P_\sigma^{k,+} = P_\sigma^{K,+}$.

Since the map $\varphi^{\sigma,\ell}$ is linear on $P_\sigma^{K,+}$ and on $P_\sigma^{K,-}$, we have

$$\varphi^{\sigma,\ell}\left(P_\sigma^K\right) = \varphi^{\sigma,\ell}\left(P_\sigma^{K,+}\right) \cup \varphi^{\sigma,\ell}\left(P_\sigma^{K,-}\right) = \sum_{k\in K} \varphi^{\sigma,\ell}\left(P_\sigma^{k,+}\right) \cup \sum_{k\in K} \varphi^{\sigma,\ell}\left(P_\sigma^{k,-}\right).$$

By Proposition 4.2.17, it follows that

$$\sum_{k\in K} \varphi^{\sigma,\ell}\left(P_\sigma^{k,+}\right) \cup \sum_{k\in K} \varphi^{\sigma,\ell}\left(P_\sigma^{k,-}\right) = \sum_{k\in K} P_\tau^{k,+} \cup \sum_{k\in K} P_\tau^{k,-}.$$

To conclude, it suffices to prove that $P_\tau^{K,+} = \sum_{k\in K} P_\tau^{k,+}$ and $P_\tau^{K,-} = \sum_{k\in K} P_\tau^{k,-}$. To do this, it suffices to show that an analogous statement to Lemma 4.2.18 holds for τ. The claim will follow analogously to the case for σ at the beginning of this proof. Fix $v_I^k \in V(P_\tau^k)$ and $v_J^h \in V(P_\tau^h)$ with $\langle f, v_I^k\rangle = -1$, $\langle f, v_J^h\rangle = 1$, and $k \neq h$. We will construct $u_1 \in V(P_\tau^k) \cap f^\perp$ and $u_2 \in V(P_\tau^h) \cap f^\perp$ such that $v_I^k + v_J^h = u_1 + u_2$. By choice of v_I^k and v_J^h and by Lemma 4.2.16, it follows immediately that $I = \{\lambda, \mu < i_3 < \cdots < i_k\}$ and $J = \{j_1, j_2 < j_3 < \cdots < j_h\}$ with $\sigma(j_1) > \ell + 1$ and if $h \geq 2$ then we also have $\ell > \sigma(j_2)$. By the assumption on μ, it follows that $j_1 < \mu$. Since τ is free of 1423 and 4123, if $h \geq 2$ then it follows that $\max\{j_1, j_2\} > \lambda$. We proceed by taking cases on h.

Case 1. Assume $h = 1$. Since $j_1 < \mu$ we have

$$\begin{aligned} v_I^k + v_J^h &= \left(e_{1,\lambda} + e_{2,\mu} + \sum_{a=3}^{k} e_{a,i_a}\right) + e_{1,j} \\ &= \left(e_{1,j_1} + e_{2,\mu} + \sum_{a=3}^{k} e_{a,i_a}\right) + e_{1,\lambda} = v^k_{\{j_1,\mu<i_3<\cdots<i_k\}} + v^h_{\{\lambda\}}. \end{aligned}$$

Case 2. Assume $h = 2$. It follows that $k \geq 3$. Since $j_1 < \mu$ we have that

$$\begin{aligned} v_I^k + v_J^h &= \left(e_{1,\lambda} + e_{2,\mu} + \sum_{a=3}^{k} e_{a,i_a}\right) + (e_{1,j_1} + e_{2,j_2}) \\ &= \left(e_{1,j_1} + e_{2,\mu} + \sum_{a=3}^{k} e_{a,i_a}\right) + (e_{1,\lambda} + e_{2,j_2}) = u_1 + u_2 \end{aligned}$$

where $u_1 = v^k_{\{j_1,\mu<i_3<\cdots<i_k\}}$ and $u_2 = v^h_{\{\lambda,j_2\}}$.

Case 3. Assume $h \geq 3$. If $k = 2$ then we have that $\max\{j_1, j_2\} \geq \lambda$, and so we have

$$v_I^k + v_J^h = (e_{1,\lambda} + e_{2,\mu}) + \left(\sum_{b=1}^{h} e_{b,j_b}\right)$$

$$= (e_{1,\lambda} + e_{2,j_2}) + \left(e_{1,j_1} + e_{2,\mu} + \sum_{b=3}^{h} e_{b,j_b}\right) = u_1 + u_2$$

where $u_1 = v^k_{\lambda,j_2}$ and $u_2 = v^h_{j_1,\mu<j_3<\cdots<j_h}$. Otherwise, if $k \geq 3$ we take cases on whether $j_2 < \mu$ or $j_2 > \mu$.

Case 3.1. Assume that $j_2 < \mu$. Since $\max\{j_1, j_2\} \geq \lambda$ we have

$$v_I^k + v_J^h = \left(e_{1,\lambda} + e_{2,\mu} + \sum_{a=1}^{k} e_{a,i_a}\right) + \left(\sum_{b=1}^{h} e_{b,j_b}\right)$$

$$= \left(e_{1,\lambda} + e_{2,j_2} + \sum_{a=3}^{k} e_{a,i_a}\right) + \left(e_{1,j_1} + e_{2,\mu} + \sum_{b=3}^{h} e_{b,j_b}\right) = u_1 + u_2$$

where $u_1 = v^k_{\lambda,j_2<i_3<\cdots<i_k}$ and $u_2 = v^h_{j_1,\mu<j_3<\cdots<j_h}$.

Case 3.2. Assume that $j_2 > \mu$. Since $j_1 < \mu$ we have

$$v_I^k + v_J^h = \left(e_{1,\lambda} + e_{2,\mu} + \sum_{a=1}^{k} e_{a,i_a}\right) + \left(\sum_{b=1}^{h} e_{b,j_b}\right)$$

$$= \left(e_{1,j_1} + e_{2,\mu} + \sum_{a=3}^{k} e_{a,i_a}\right) + \left(e_{1,\lambda} + e_{2,j_2} + \sum_{b=3}^{h} e_{b,j_b}\right) = u_1 + u_2$$

where $u_1 = v^k_{j_1,\mu<i_3<\cdots<i_k}$ and $u_2 = v^h_{\lambda,j_2<j_3<\cdots<j_h}$.

So in each case we have constructed vertices $u_1 \in V(P^k_\tau)$ and $u_2 \in V(P^h_\tau)$ such that $v_I^k + v_J^h = u_1 + u_2$ and $\langle f, u_1\rangle = \langle f, u_2\rangle = 0$. Hence, we can conclude that

$$\varphi^{\sigma,\ell}\left(P^K_\sigma\right) = \sum_{k\in K} P^{k,+}_\tau \cup \sum_{k\in K} P^{k,-}_\tau = P^{K,+}_\tau \cup P^{K,-}_\tau = P^K_\tau. \qquad \square$$

We can now use the lemma to prove the main result of this section.

Proof of Theorem 4.2.13. Let $\sigma \in S_n$ be a permutation free of the patterns $1423, 4123, 3124$ and 1324. We prove that there is a sequence of combinatorial mutations from P^K_σ to $P^K_{w_0}$ by reverse induction on the inversion number of σ.

Since w_0 has the maximum inversion number, the base case is trivial. Assume now $\sigma \neq w_0$. Let $\mu = \max\{i \in [n] \mid \sigma(i) > n + 1 - i\}$. Define $\ell = \sigma^{-1}(\mu) - 1$ and $\lambda = \sigma(\ell)$. By the definition of μ, it immediately follows that $\lambda < \mu$ and so σ and ℓ satisfy the conditions of Setup 4.2.14. By Lemma 4.2.19, we have that $\tau = (\ell\ \ell + 1)\sigma$ is free of patterns $1423, 4123, 1324$ and 3124. Since the inversion number of τ is one more than the inversion number of σ, by induction there is a sequence of combinatorial mutations from P^K_τ to $P^K_{w_0}$. By Lemma 4.2.20 there is a combinatorial mutation from P^K_σ to P^K_τ, which completes the proof. □

4.3 Chain-Order Polytopes

We now move our attention to a different family of polytopes, arising from partially ordered sets. The following results are contained in [28], which is joint work with Oliver Clarke and Akihiro Higashitani.

4.3.1 Posets and Polytopes

Let $(\mathcal{X}, <)$ be a partially ordered set, which we usually denote as $\mathcal{X}$. A subset $U \subseteq \mathcal{X}$ is called an *up-set* if for all $x \in U$ and $y \in \mathcal{X}$, we have that $x < y$ implies that $y \in U$. A subset $D \subseteq \mathcal{X}$ is a *down-set* if its complement $\mathcal{X}\backslash D$ is an up-set. A subset $S \subseteq \mathcal{X}$ is called a *chain* if each pair of elements in S is comparable. A subset $S \subseteq \mathcal{X}$ is an *antichain* if no pair of elements in S is comparable. Given a subset $S \subseteq \mathcal{X}$, the collections of minimal and maximal elements are respectively $\min(S) = \{s \in S \mid x \not< s \text{ for all } x \in S\}$ and $\max(S) = \{s \in S \mid x \not> s \text{ for all } x \in S\}$.

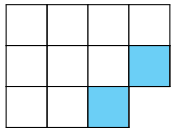
We equip $\mathbb{Z}^2_{>0}$ with the component-wise partial order: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. A *Young diagram* $\lambda \subseteq \mathbb{Z}^2_{>0}$ is a finite down-set, which we take to be a sub-poset of $\mathbb{Z}^2_{>0}$. Young diagrams are often defined as partitions of natural numbers. Explicitly, a partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of $n \in \mathbb{Z}_{>0}$ is naturally associated to the down-set

$$\lambda = \{(1,1), \ldots, (1,\lambda_1),\ (2,1), \ldots, (2,\lambda_2), \ldots\ , (k,1), \ldots, (k,\lambda_k)\} \subseteq \mathbb{Z}^2_{>0}.$$

Typically, a Young diagram λ is depicted as a collection of boxes, with $(1, 1)$ located in the top-left, $(1, 2)$ to the immediate right of $(1, 1)$, $(2, 1)$ immediately

below $(1,1)$, and so on. We say that a box $r \in \lambda$ is a *corner* if $r \in \max(\lambda)$ is a maximal element.

Example 4.3.1. The Young diagram given by the partition $(4,4,3)$, is represented via the following collection of boxes.



The Young diagram has two corners: $(2,4)$ and $(3,3)$, which are the shaded boxes above.

We write $\mathbb{R}^\lambda$ for the real vector space with distinguished basis $\{e_p \mid p \in \lambda\}$. Given $x, y \in \mathbb{R}^\lambda$ we write $x \cdot y = \sum_{p\in\lambda} x_p y_p$ for the standard dot-product. For all $x \in \mathbb{R}^\lambda$ and $p \notin \lambda$, we take the convention that $x_p = 0 \in \mathbb{R}$ and $e_p = 0 \in \mathbb{R}^\lambda$.

We recall the definition of two polytopes classically associated to a poset.

Definition 4.3.2 (Order polytope and chain polytope)**.** The *order polytope* of the Young diagram λ is the polytope

$$\mathcal{O}(\lambda) = \left\{x \in \mathbb{R}^\lambda \mid 0 \le x_p \le x_q \le 1 \text{ for all } p \le q \text{ in } \lambda\right\}.$$

The *chain polytope* of λ is the polytope

$$\mathcal{C}(\lambda) = \left\{x \in \mathbb{R}^\lambda \mid 0 \le x_{p_1} + \cdots + x_{p_n} \le 1 \text{ for all } p_1 < \cdots < p_n \text{ in } \lambda\right\}.$$

Each point $x \in \mathbb{R}^\lambda_{\ge 0}$ can be thought of as a non-negative *filling* of the Young diagram, given by writing the value x_p in the box $p \in \lambda$. A point $x \in \mathbb{R}^\lambda_{\ge 0}$ lies in the kth dilate $k\mathcal{O}(\lambda)$ if and only if the values in each box do not exceed k and increase when moving down and to the right. Similarly, a point $x \in \mathbb{R}^\lambda_{\ge 0}$ lies in $k\mathcal{C}(\lambda)$ if and only if the sum of values along any path in x that starts at $(1,1)$ and moves down and to the right is at most k. The vertices of these polytopes have the following descriptions.

Proposition 4.3.3 ([110, Corollary 1.3 and Theorem 2.2])**.** *Fix a Young diagram λ and for each subset $S \subseteq \lambda$ define the characteristic vector $\chi(S) \in \mathbb{R}^\lambda$ by $\chi(S)_p = 1$ if $p \in S$ and $\chi(S)_p = 0$ if $p \notin S$. The vertices of $\mathcal{O}(\lambda)$ and $\mathcal{C}(\lambda)$ are respectively*

$$V(\mathcal{O}(\lambda)) = \{\chi(S) \mid S \text{ an up-set of } \lambda\},$$

$$V(\mathcal{C}(\lambda)) = \{\chi(S) \mid S \text{ an antichain of } \lambda\}.$$

The order polytope and chain polytope are mutation equivalent [63]. The sequence of mutations can be realized by a decomposition of a piece-wise linear map called the *transfer map*, introduced in [110], which interpolates between the polytopes. Intermediate polytopes of this sequence of mutations are given by the so-called chain-order polytope.

Definition 4.3.4 (Chain-order polytope)**.** Let λ be a Young diagram and let $C \subsetneq \lambda$ be a proper up-set. The *chain-order polytope* of λ with respect to C is the polytope $\mathcal{O}_C(\lambda)$ defined by the following inequalities:

$$\left\{ x \in \mathbb{R}^\lambda \;\middle|\; \begin{array}{ll} 0 \leq x_p \leq 1 & \text{for all } p \in \lambda, \\ x_p \leq x_q & \text{for all } p \leq q \text{ in } \lambda \backslash C, \\ x_p + x_{q_1} + \cdots + x_{q_n} \leq 1 & \text{for all } p \in \lambda \backslash C \text{ and } q_1, \ldots, q_n \in C \\ & \qquad \text{such that } p < q_1 < \cdots < q_n. \end{array} \right\}.$$

If $C = \emptyset$, then $\mathcal{O}_C(\lambda) = \mathcal{O}(\lambda)$ coincides with the order polytope. If $C = \lambda \backslash \{(1,1)\}$, then $\mathcal{O}_C(\lambda) = \mathcal{C}(\lambda)$ coincides with the chain polytope. For $C = \lambda$, we define $\mathcal{O}_C(\lambda) = \mathcal{C}(\lambda)$ to be the chain polytope.

Example 4.3.5. Consider the Young diagram λ given by the partition $(3,2)$ and the up-set $C = \{(1,2),(1,3),(2,2)\}$ corresponding to the highlighted boxes below. The chain-order polytope of λ with respect to C has vertices:

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 1&0\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 1&0\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 0&1&0\\\hline 0&0\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 0&1&0\\\hline 1&0\\\cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1&0\\\cline{1-2}\end{array} \text{ and } \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&1\\\cline{1-2}\end{array}.$$

Restricted Chain and Order Polytopes

We now consider restricted versions of the order and chain polytopes; that is, the intersection of these polytopes with certain hyperplanes. To define them, we fix the following setup that will be used throughout the rest of this section.

Setup 4.3.6. Fix a Young diagram λ. Let $m_1, m_2 \in \mathbb{Z}_{>0}$ be the smallest natural numbers such that $\lambda \subseteq [m_1] \times [m_2]$. That is, if λ is given by the partition $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s\}$, then $(m_1, m_2) = (s, \lambda_1)$. Fix a vector $\underline{d} = (d_{m_1-1}, d_{m_1-2}, \ldots, d_0, d_{-1}, \ldots, d_{1-m_2}) \in \mathbb{Z}_{>0}^{m_1+m_2-1}$ and a natural number $k \in \mathbb{Z}_{>0}$. Define the set of integers $\mathrm{Diag}(\lambda) := \{1-m_2, 2-m_2, \ldots, m_1-1\}$, which label the *diagonals* of λ. For each $\ell \in \mathrm{Diag}(\lambda)$, denote by r_ℓ the maximal element in the *ℓth diagonal of λ*: $\{(i,j) \in \lambda \mid i - j = \ell\}$. Visually, the box r_ℓ is the bottom-right-most box of the Young diagram on the ℓth diagonal. See Figure 4.2.

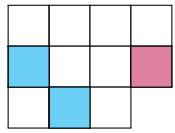


Figure 4.2: Highlighted in blue, the 1st diagonal given by the boxes $(2,1)$ and $r_1 = (3,2)$. Highlighted in purple, the box $r_{-2} = (2,4)$.

Definition 4.3.7 (Restricted order polytope and restricted chain polytope)**.** Fix Setup 4.3.6. The *restricted order polytope of* λ with respect to $\underline{d}$ and k is the restriction of the kth dilate of the order polytope of λ defined by:

$$\mathcal{O}(\lambda)^k_{\underline{d}} = \left\{ x \in k\mathcal{O}(\lambda) \;\middle|\; \sum_{i-j=\ell} x_{(i,j)} = d_\ell \text{ for all } \ell \in \mathrm{Diag}(\lambda) \right\}.$$

The *restricted chain polytope of* λ with respect to $\underline{d}$ and k is the restriction of the kth dilate of the chain polytope of λ defined by:

$$\mathcal{C}(\lambda)^k_{\underline{d}} = \left\{ x \in k\mathcal{C}(\lambda) \;\middle|\; \sum_{p \le r_\ell} x_p = d_\ell \text{ for all } \ell \in \mathrm{Diag}(\lambda) \right\}.$$

Remark 4.3.8. If $\lambda = \{(1,1), \ldots, (n,n)\}$ is square and $\underline{d} = (1, 2, \ldots, n-1, n, n-1, \ldots, 1)$, then the restricted order polytope is equal to the restricted Gelfand–Tsetlin polytope and the restricted chain polytope is equal to the restricted Birkhoff polytope. The period-collapse phenomenon of these polytopes is studied in [3]. In Section 4.3.4, we show that period-collapse also occurs for some other families of polytopes.

4.3.2 Chain-Order Newton–Okounkov Bodies

It is possible to generalize the concept of chain-order polytopes to that of marked chain order polytopes. The construction is as follows.

Let $\mathcal{X}$ be a poset and let $\mathcal{X}^*$ be a subposet containing all the maximal and minimal elements of $\mathcal{X}$. Let $\mathbf{m} \in \mathbb{R}^{\mathcal{X}^*}$ be such that $m_a \le m_b$ whenever $a \le b$ in $\mathcal{X}$.

Definition 4.3.9 (Marked chain-order polytope)**.** Fix a proper up-set C of $\mathcal{X} \setminus \mathcal{X}^*$. The *marked chain-order polytope* of $(\mathcal{X}, \mathcal{X}^*, \mathbf{m}, C)$ is the polytope

$\mathcal{O}_C(\mathcal{X}, \mathcal{X}^*, \mathbf{m})$ defined by

$$\left\{ x \in \mathbb{R}^{\mathcal{X}\setminus\mathcal{X}^*} \;\middle|\; \begin{array}{ll} x_p \geq 0 & \text{for all } p \in C \\ \sum_{i=1}^{n} x_{q_i} \leq y_b - y_a & \text{for all } a < q_1 < \cdots < q_n < b, \\ & q_i \in C,\ a, b \in \mathcal{X} \setminus C \end{array} \right\}$$

where $y_p = m_p$ if $p \in \mathcal{X}^*$ and $y_p = x_p$ if $p \in \mathcal{X} \setminus (\mathcal{X}^* \cup C)$.

If $C = \emptyset$, then $\mathcal{O}_C(\mathcal{X}, \mathcal{X}^*, \mathbf{m}) = \mathcal{O}(\mathcal{X}, \mathcal{X}^*, \mathbf{m})$ is called the *marked order polytope*. If C consists of $\mathcal{X} \setminus \mathcal{X}^*$ minus one element, then $\mathcal{O}_C(\mathcal{X}, \mathcal{X}^*, \mathbf{m}) = \mathcal{C}(\mathcal{X}, \mathcal{X}^*, \mathbf{m})$ is called the *marked chain polytope*. For $C = \mathcal{X} \setminus \mathcal{X}^*$, we set $\mathcal{O}_{\mathcal{X}\setminus\mathcal{X}^*}(\mathcal{X}, \mathcal{X}^*, \mathbf{m}) = \mathcal{C}(\mathcal{X}, \mathcal{X}^*, \mathbf{m})$.

Ardila, Bliem and Salazar showed that it is possible to generalize Stanley's transfer map to the marked order and marked chain polytopes [5]. Fujita [50] showed that if we consider posets given by specific Young diagrams λ, marked chain-order polytopes are Newton–Okounkov bodies for flag varieties.

Theorem 4.3.10 ([50, Theorem 6.1])**.** *Let λ be the Young diagram given by $(n, n-1, \ldots, 2, 1)$ and let $\lambda^* = \{(0,0), (1,n), (2,n-1), \ldots, (n,1)\}$. For specific choices of the marking $\mathbf{m}$ and for any proper up-set $C \subseteq \lambda \setminus \lambda^*$, the marked chain-order polytope $\mathcal{O}_C(\lambda, \lambda^*, \mathbf{m})$, is a Newton–Okounkov body for the full flag variety $\mathcal{F}\ell_n$ up to translation by an integer vector.*

4.3.3 Mutations of Chain-Order Polytopes

We now want to prove the main result of this section, which shows that restricted order polytopes and restricted chain polytopes are connected by a sequence of combinatorial mutations.

Theorem 4.3.11. *Fix Setup 4.3.6. The restricted order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ and the restricted chain polytope $\mathcal{C}(\lambda)^k_{\underline{d}}$ are mutation equivalent.*

In particular, by Proposition 4.1.1, if a polytope exhibits period collapse, then all mutation equivalent polytopes simultaneously exhibit period collapse.

We construct a sequence of mutation equivalent polytopes that interpolate between the restricted order and restricted chain polytopes. The intermediate polytopes will be the restricted version of the chain-order polytopes. We show that all piece-wise linear maps that connect these polytopes are combinatorial mutations. The foundation for our construction is the piece-wise linear map introduced by Pak [99].

Pak's piece-wise linear map. Fix Setup 4.3.6. We recall, from [99, Section 4], the piece-wise linear map $\xi_\lambda : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ which bijectively maps $\mathcal{O}(\lambda)^k_{\underline{d}}$ to $\mathcal{C}(\lambda)^k_{\underline{d}}$. The map is defined inductively on $|\lambda|$. If $|\lambda| = 1$, then $\xi_\lambda = \mathrm{id}$.

If $|\lambda| > 1$, then let $r = r_\ell$ be a corner of λ. We recall the convention that for any $x \in \mathbb{R}^\lambda$, we have $x_p = 0$ if $p \notin \lambda$. Define the map $\chi_r : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ as follows:

$$\chi_r(x)_{(i,j)} = \begin{cases} x_{(i,j)} & \text{if } i-j \neq \ell, \\ x_{(i,j)} - \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } (i,j) = r, \\ \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \\ \quad + \min\{x_{(i+1,j)}, x_{(i,j+1)}\} - x_{(i,j)} & \text{if } i-j = \ell \text{ and } (i,j) \neq r. \end{cases}$$

By induction, we define $\xi_\lambda = (\xi_{\lambda\setminus\{r\}} \times \mathrm{id}) \circ \chi_r$. By [99, Theorem 4], the map ξ_λ is well-defined, in particular, it does not depend on the choice of corner r above.

We naturally extend the definition of the map χ_r to non-corners $\alpha \in \lambda$. This is done by defining $\chi_\alpha(x)_{(i,j)} = x_{(i,j)}$ for all $(i,j) \not\leq \alpha$ and taking the above definition for all other coordinates $(i,j) \leq \alpha$. So we may express Pak's map as a composition:

$$\xi_\lambda = \chi_{\alpha_1} \circ \chi_{\alpha_2} \circ \cdots \circ \chi_{\alpha_n}, \tag{4.3}$$

for any permutation $(\alpha_1, \ldots, \alpha_n)$ of the elements of λ that satisfies $\alpha_i < \alpha_j \implies i < j$.

A decomposition of Pak's map. Fix $\alpha = (a,b) \in \lambda$ in the ℓth diagonal of λ. We express the map χ_α as the composition of tropical maps and a unimodular map. We will show that this gives a sequence of combinatorial mutations. The tropical maps and unimodular map are defined as follows:

- For each $0 \leq i \leq i_{\max}(\alpha) := \min\{a,b\} - 1$, let

$$w_i = e_{(a-i-1,\, b-i-1)} - e_{(a-i,\, b-i)}$$

$$F_i = \mathrm{conv}\left\{e_{(a-i-1,\, b-i)},\, e_{(a-i,\, b-i-1)}\right\}.$$

  Define the tropical map as the combinatorial mutation given by w_i, F_i, i.e. $\varphi_i := \varphi_{w_i, F_i}$.

- Let ψ be the unimodular map:

$$\psi(x)_{(i,j)} = \begin{cases} x_{(i,j)} & \text{if } i-j \neq \ell, \\ x_{(i,j)} - (x_{i-1,j} + x_{i,j-1}) & \text{if } (i,j) = \alpha, \\ -x_{(i,j)} + (x_{i-1,j} + x_{i,j-1}) & \text{if } i-j = \ell \text{ and } (i,j) \neq r\alpha. \end{cases}$$

Proposition 4.3.12. *With the notation above,* $\chi_\alpha = \psi \circ \varphi_{i_{\max}(\alpha)} \circ \cdots \circ \varphi_0$.

Proof. It is straightforward to show that the composition of the tropical maps $(\varphi_{i_{\max}(\alpha)} \circ \cdots \circ \varphi_0(x))_{(i,j)}$is

$$\begin{cases} x_{(i,j)} & \text{if } i-j \neq \ell, \\ x_{(i,j)} + \min\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } (i,j) = \alpha, \\ x_{(i,j)} + \min\{x_{(i-1,j)}, x_{(i,j-1)}\} & \\ \quad - \min\{x_{(i+1,j)}, x_{(i,j+1)}\} & \text{if } i-j = \ell \text{ and } (i,j) < \alpha. \end{cases}$$

For all $u, v \in \mathbb{R}$ we have $\min\{u,v\} - u - v = -\max\{u,v\}$, so it follows that

$$\psi \circ \varphi_{i_{\max}(\alpha)} \circ \cdots \circ \varphi_0(x) = \chi_\alpha(x)$$

for every $x \in \mathbb{R}^\lambda$. □

To prove that these tropical maps are combinatorial mutations, we proceed inductively. For each $0 \leq i \leq i_{\max}(\alpha)$, we show that the image of the polytope under φ_i is convex. As a result, we obtain a collection of mutation equivalent polytopes. Recall that Pak's map ξ_λ can be expressed as a composition of maps χ_α, as in Equation (4.3). Then, for each $1 \leq s \leq n$, the polytope

$$\mathcal{O}_C(\lambda)^k_{\underline{d}} = \chi_{\alpha_s} \circ \chi_{\alpha_{s+1}} \circ \cdots \circ \chi_{\alpha_n}(\mathcal{O}(\lambda)^k_{\underline{d}})$$

can be regarded as the restricted analogue of the chain-order polytope $\mathcal{O}_C(\lambda)$ with respect to the up-set $C = \{\alpha_s, \alpha_{s+1}, \ldots, \alpha_n\}$. We give an abstract definition of the polytope $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ and prove that the above equality holds in Section 4.3.3.

Definition 4.3.13 (Restricted chain-order polytope)**.** Fix Setup 4.3.6 and let $C \subseteq \lambda$ be an up-set. For each $\ell \in \mathrm{Diag}(\lambda)$, we define:

- $R_\ell = \{(a,b) \in \lambda \mid (a,b) \leq r_\ell\}$ the rectangular sub-Young diagram of λ,
- $S_\ell = \max((\lambda \backslash C) \cap R_\ell)$ the corners of $(\lambda \backslash C) \cap R_\ell$,
- $\overline{S_\ell} = \{(a,b) \in R_\ell \mid (a+i, b+i) \in S_\ell \text{ for some } i \geq 0\}$ the diagonals of R_ℓ ending in S_ℓ,
- $T_\ell = \min(C \cap R_\ell)$ the minimal elements of $C \cap R_\ell$,
- $\overline{T_\ell} = \{(a,b) \in R_\ell \mid (a+i, b+i) \in T_\ell \text{ for some } i \geq 1\}$ the diagonals in R_ℓ that end one step before T_ℓ.

The *restricted chain-order polytope* $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ with respect to $\underline{d} \in \mathbb{Z}_{>0}^{m_1+m_2-1}$ and $k \in \mathbb{Z}_{>0}$ is

$$\left\{ x \in k\mathcal{O}_C(\lambda) \;\middle|\; \sum_{s \in \overline{S_\ell}} x_s - \sum_{t \in \overline{T_\ell}} x_t + \sum_{u \in C \cap R_\ell} x_u = d_\ell \text{ for each } \ell \in \mathrm{Diag}(\lambda) \right\}.$$

Example 4.3.14. Let $\lambda = \{(1,1), \ldots, (5,6)\}$ be the rectangular Young diagram with 4 rows and 5 columns. Let $C \subset \lambda$ be the up-set with minimal elements $(4,3)$ and $(3,5)$. Fix $\ell = -1$, then $r_\ell = (5,6)$ and the set $R_\ell = \lambda$ is the entire poset. The sets S_ℓ, $\overline{S}_\ell$, T_ℓ and $\overline{T}_\ell$ are shown in Figure 4.3.

	$\overline{s}$	$\overline{t}$		$\overline{s}$	
$\overline{t}$		$\overline{s}$	$\overline{t}$		s
	$\overline{t}$		s	t	
$\overline{s}$		t			
	s				

Figure 4.3: Depiction of the Young diagram in Example 4.3.14. Shaded boxes represent the up-set C. Fix $\ell = -1$. The set S_ℓ is given by the boxes labelled with s, the set $\overline{S}_\ell$ are the boxes s and $\overline{s}$, the set T_ℓ are the boxes t, and the set $\overline{T}_\ell$ are the boxes $\overline{t}$.

Proof of Mutation Equivalence

We now prove Theorem 4.3.11, which follow from Lemmas 4.3.16 and 4.3.17. We begin by fixing the notation for the affine hyperplanes that define the restricted chain-order polytopes and then proceed to prove the two lemmas followed by the main theorem.

Notation 4.3.15. Following the notation from Definition 4.3.13, given a restricted chain-order polytope $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ and $\ell \in \mathrm{Diag}(\lambda)$, we define the hyperplane

$$H_\ell = \left\{ x \in \mathbb{R}^\lambda \;\middle|\; \sum_{s \in \overline{S_\ell}} x_s - \sum_{t \in \overline{T_\ell}} x_t + \sum_{u \in C \cap R_\ell} x_u = d_\ell \right\}.$$

In particular, we have that $\mathcal{O}_C(\lambda)^k_{\underline{d}} = k\mathcal{O}_C(\lambda) \cap \bigcap_\ell H_\ell$, where the intersection runs over $\mathrm{Diag}(\lambda)$. Whenever we work with hyperplanes H_ℓ with respect to

different up-sets C, we write H_ℓ^C for H_ℓ to avoid ambiguity. Similarly, we write S_ℓ^C, $\overline{S_\ell^C}$, T_ℓ^C and $\overline{T_\ell^C}$ for the sets in Definition 4.3.13.

Lemma 4.3.16. *Fix Setup 4.3.6 and let $C \subsetneq \lambda$ be an up-set. Let $r = (a, b)$ be a corner of $\lambda\backslash C$ and $\ell = a - b$. We denote by $\chi_r : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ the extension of Pak's map defined on $\mathbb{R}^{\lambda\backslash C}$. Then $\chi_r : \mathcal{O}_C(\lambda) \to \mathcal{O}_{C\cup\{r\}}(\lambda)$ is a bijection. Recall the decomposition $\chi_r = \psi \circ \varphi_{i_{\max}(r)} \circ \cdots \circ \varphi_0$ from Proposition 4.3.12. For all $i \in \{0, \ldots, i_{\max}(r)\}$ and $\ell \in \mathrm{Diag}(\lambda)$, we have that $\varphi_i(H_\ell^C) = H_\ell^C$ and $\psi(H_\ell^C) = H_\ell^{C\cup\{r\}}$. Hence, $\chi_r : \mathcal{O}_C(\lambda)_{\underline{d}}^k \to \mathcal{O}_{C\cup\{r\}}(\lambda)_{\underline{d}}^k$ is a bijection.*

Proof. By [99], we have that χ_r admits an inverse given by:

$$\chi_r^{-1}(x)_{(i,j)} = \begin{cases} x_{(i,j)} + \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } (i,j) = r, \\ \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } i - j = \ell \text{ and} \\ \qquad + \min\{x_{(i+1,j)}, x_{(i,j+1)}\} - x_{(i,j)} & \qquad (i,j) < r, \\ x_{(i,j)} & \text{otherwise.} \end{cases}$$

Hence, to show that $\chi_r : \mathcal{O}_C(\lambda) \to \mathcal{O}_{C\cup\{r\}}(\lambda)$ is a bijection, it suffices to prove that $\chi_r(\mathcal{O}_C(\lambda)) = \mathcal{O}_{C\cup\{r\}}(\lambda)$. Fix $x \in \mathcal{O}_C(\lambda)$. We show that $\chi_r(x) \in \mathcal{O}_{C\cup\{r\}}(\lambda)$ by showing that $\chi_r(x)$ satisfies all the defining inequalities of $\mathcal{O}_{C\cup\{r\}}(\lambda)$:

- If $i - j \neq \ell$ or both $i - j = \ell$ and $(i, j) > r$, then $0 \leq \chi_r(x)_{(i,j)} = x_{(i,j)} \leq 1$. Since r is a corner of $\lambda\backslash C$, for every $(i, j) \leq r$ with $i - j = \ell$, it follows that $\max\{x_{(i-1,j)}, x_{(i,j-1)}\} \leq x_{(i,j)} \leq \min\{x_{(i+1,j)}, x_{(i,j+1)}\}$, hence $0 \leq \chi_r(x)_{(i,j)} \leq 1$.
- Let $(i_1, j_1) \leq (i_2, j_2)$ in $\lambda\backslash(C \cup \{r\})$. Since, for every $(i, j) \in \lambda\backslash(C \cup \{r\})$ with $i - j = \ell$ we have that $\max\{x_{(i-1,j)}, x_{(i,j-1)}\} \leq \chi_r(x)_{(i,j)} \leq \min\{x_{(i+1,j)}, x_{(i,j+1)}\}$, it follows that $\chi_r(x)_{(i_1,j_1)} \leq \chi_r(x)_{(i_2,j_2)}$.
- It remains to show that if $p \in \lambda\backslash(C \cup \{r\})$ and $q_1, \ldots, q_n \in C \cup \{r\}$ with $p < q_1 < \cdots < q_n$, then $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} \leq 1$. First assume that $q_i \neq r$ for every $i \in \{1, \ldots, n\}$. If p does not lie on the ℓth diagonal, then $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} = x_p + x_{q_1} + \cdots + x_{q_n} \leq 1$. Otherwise, if p lies on the ℓth diagonal, then $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} \leq x_r + x_{q_1} + \cdots + x_{q_n} \leq 1$.

  Suppose that $r \in \{q_1, \ldots, q_n\}$. It follows that $r = q_1$. Since $p < q_1$ and $x_p \leq \max\{x_{(a-1,b)}, x_{(a,b-1)}\}$, we have $\chi_r(x)_p + \chi_r(x)_r = \chi_r(x)_p + x_{(a,b)} - \max\{x_{(a-1,b)}, x_{(a,b-1)}\} \leq x_r$. Hence, $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} \leq x_r + x_{q_2} + \cdots + x_{q_n} \leq 1$.

So, we have shown that $\chi_r(\mathcal{O}_C(\lambda)) \subseteq \mathcal{O}_{C\cup\{r\}}(\lambda)$. Using the explicit formula for χ_r^{-1}, the reverse inclusion $\chi_r^{-1}(\mathcal{O}_{C\cup\{r\}}(\lambda)) \subseteq \mathcal{O}_C(\lambda)$ follows similarly.

Recall the decomposition $\chi_r = \psi \circ \varphi_{i_{\max}(r)} \circ \cdots \circ \varphi_0$ from Proposition 4.3.12 and the defining equation of the hyperplane H_ℓ^C given by

$$H_\ell^C : \quad \sum_{s\in\overline{S_\ell^C}} x_s - \sum_{t\in\overline{T_\ell^C}} x_t + \sum_{u\in C\cap R_\ell} x_u = d_\ell.$$

Consider the tropical map $\varphi_i = \varphi_{w_i,F_i}$. We have that $w_i = e_\alpha - e_\beta$ for some α and β in λ that lie on the same diagonal of $\lambda\backslash C$. Since the sets $\overline{S_\ell^C}$ and $\overline{T_\ell^C}$ are unions of diagonals of $\lambda\backslash C$, it follows that $\varphi_i(H_\ell^C) = H_\ell^C$ for each $i \in \{0,\dots,i_{\max}(r)\}$ and $\ell \in \mathrm{Diag}(\lambda)$.

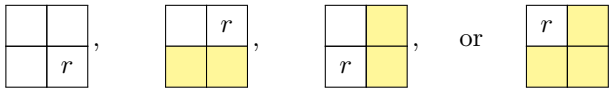
It remains to show that $\psi(H_\ell^C) = H_\ell^{C\cup\{r\}}$. Let $x \in H_\ell^C$, then we have

$$\begin{aligned}\sum_{\substack{i-j=\ell\\(i,j)\le r}} x_{(i,j)} =& \psi(x)_{(a,b)} + \psi(x)_{(a-1,b)} + \psi(x)_{(a,b-1)} + \\ &+ \sum_{\substack{i-j=\ell\\(i,j)<r}} (-\psi(x)_{(i,j)} + \psi(x)_{(i-1,j)} + \psi(x)_{(i,j-1)}).\end{aligned}$$

Since $\psi(x)_{(i,j)} = x_{(i,j)}$ for every (i,j) with $i-j \neq \ell$, using the previous equality, the following holds:

$$\begin{aligned} d_\ell &= \sum_{s\in\overline{S_\ell^C}} x_s - \sum_{t\in\overline{T_\ell^C}} x_t + \sum_{u\in C\cap R_\ell} x_u \\ &= \sum_{\substack{s\in\overline{S_\ell^C}\\ i-j\neq\ell}} x_s - \sum_{t\in\overline{T_\ell^C}} x_t + \sum_{u\in C\cap R_\ell} x_u + \sum_{\substack{i-j=\ell\\(i,j)\le r}} x_{(i,j)} \\ &= \sum_{\substack{s\in\overline{S_\ell^C}\\ i-j\neq\ell}} \psi(x)_s - \sum_{t\in\overline{T_\ell^C}} \psi(x)_t + \sum_{u\in C\cap R_\ell} \psi(x)_u \\ &\quad + \psi(x)_{(a,b)} - \sum_{\substack{i-j=\ell\\(i,j)<r}} \psi(x)_{(i,j)} + \sum_{\substack{i-j=\ell\\(i,j)\le r}} \psi(x)_{(i-1,j)} + \psi(x)_{(i,j-1)} \end{aligned}$$

Note that $r \in T_\ell^{C\cup\{r\}}$. There are four possible behaviors of C near the corner r of $\lambda\backslash C$ as shown below, where the highlighted boxes lie in C:



In particular, since r is a corner of $\lambda\backslash C$, we have that $(a-1,b)$ and $(a,b-1)$ are not in S_ℓ^C. Moreover, $(a+1,b) \in T_\ell^C$ if and only if $(a,b-1) \notin S_\ell^{C\cup\{r\}}$. Similarly, $(a,b+1) \in T_\ell^C$ if and only if $(a-1,b) \notin S_\ell^{C\cup\{r\}}$. So, the previous expression can be written as:

$$\sum_{s\in S_\ell^{C\cup\{r\}}} \psi(x)_s - \sum_{t\in T_\ell^{C\cup\{r\}}} \psi(x)_t + \sum_{u\in(C\cup\{r_\ell\})\cap R_\ell} \psi(x)_u = d_\ell$$

that is, $\psi(H_\ell^C) = H_\ell^{C\cup\{r\}}$. □

Lemma 4.3.17. *Fix Setup 4.3.6 and let $C \subsetneq \lambda$ be an up-set. Let $r = (a,b)$ be a corner of $\lambda\backslash C$ and recall the decomposition $\chi_r = \psi \circ \varphi_{i_{\max}(r)} \circ \cdots \circ \varphi_0$ from Proposition 4.3.12. For every $i \in \{0,\dots,i_{\max}(r)\}$, the tropical map $\varphi_i = \varphi_{w_i,F_i}$ gives a combinatorial mutation of the polytope $(\varphi_{i-1}\circ\cdots\circ\varphi_0)(\mathcal{O}_C(\lambda)^k_{\underline{d}})$.*

Proof. By Lemma 4.3.16, for each $i \in \{0,\dots,i_{\max}(r)\}$ and $\ell \in \{1-m_2,\dots,m_1-1\}$, the hyperplane H_ℓ^C is invariant under φ_i. Hence, it is enough to prove that φ_i gives a combinatorial mutation of the polytope $P_i := (\varphi_{i-1}\circ\cdots\circ\varphi_0)(\mathcal{O}_C(\lambda))$.

We introduce the following notation for readability:

$$A = (a-i-1,b-i-1),\ B = (a-i-1,b-i),\ C = (a-i,b-i-1),\ D = (a-i,b-i).$$

Recall that $F_i = \operatorname{conv}\{e_B, e_C\}$. We may assume that both B and C lie in λ, otherwise φ_i is a unimodular map and the result follows immediately. So, the regions of linearity of the tropical map φ_i are given by

$$U_\geq := \{x \in \mathbb{R}^\lambda \mid x_B \geq x_C\} \text{ and } U_\leq := \{x \in \mathbb{R}^\lambda \mid x_B \leq x_C\}.$$

Similarly, we define $U_>$ and $U_<$ to be the interiors of $U_\geq$ and $U_\leq$ respectively. Let $U_= := U_\geq \cap U_\leq$ be the intersection of the regions of linearity.

Let $p \in U_> \cap P_i$ and $q \in U_< \cap P_i$ be two points that lie in the interiors of the regions of linearity. To show that $P_{i+1} := \varphi_i(P_i)$ is convex, it suffices to show that the line segment between $\varphi_i(p)$ and $\varphi_i(q)$ is contained in P_{i+1}. Without loss of generality, we may assume that p and q are vertices of P_i. For each $i \in \{0,\dots,i_{\max}(r)\}$, we will show that the line segment $[\varphi_i(p),\varphi_i(q)]$ is

contained in P_{i+1}. Moreover, we will show that φ_i gives a bijection between the vertices of P_i and P_{i+1}. We proceed by induction on i.

Fix $i = 0$. Since $p_B > p_C$, $q_B > q_C$, and p and q are vertices of $\mathcal{O}_C(\lambda)$, we have that

$$\begin{cases} p_A = p_C = 0 \\ p_B = p_D = 1 \end{cases} \quad \text{and} \quad \begin{cases} q_A = q_B = 0 \\ q_C = q_D = 1. \end{cases}$$

Note that, by the definition of the chain-order polytope, we have that $p_s = q_s = 0$ for all $s \leq A$. We define the points $\alpha, \beta, \gamma, \delta \in \mathbb{R}^\lambda$ as follows:

- $\alpha_s = 0$ for all $s \leq D$ and $\alpha_s = \min\{p_s, q_s\}$ for all $s \not\leq D$, in particular $\alpha_A = \alpha_B = \alpha_C = \alpha_D = 0$,
- $\beta_s = 0$ for all $s < D$ and $\beta_s = \min\{p_s, q_s\}$ for all $s \not< D$, in particular $\beta_A = \beta_B = \beta_C = 0$ and $\beta_D = 1$,
- $\gamma_s = 0$ for all $s \leq A$ and $\gamma_s = \max\{p_s, q_s\}$ for all $s \not\leq A$, in particular $\gamma_A = 0$ and $\gamma_B = \gamma_C = \gamma_D = 1$,
- $\delta_s = 0$ for all $s < A$, $\delta_A = 1$, and $\delta_s = \max\{p_s, q_s\}$ for all $s \not\leq A$, in particular $\delta_A = \delta_B = \delta_C = \delta_D = 1$.

It is straightforward to check that the points α, β, γ and δ lie in $\mathcal{O}_C(\lambda)$. Observe that $\alpha, \beta, \gamma, \delta \in U_=$ and $p + q = \beta + \gamma$. In particular, the line segment $[p, q]$ is not an edge of $\mathcal{O}_C(\lambda)$. By the definition of the tropical map φ_0, we have that $\varphi_0(p) + \varphi_0(q) = \varphi_0(\alpha) + \varphi_0(\delta)$. To see this, note that φ_0 fixes all coordinates except A and D, which vary based only on the values of B and C. Writing out the coordinates A, B, C, D of these points we have:

$$\varphi_0(p) + \varphi(q) = \varphi_0 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \varphi_0 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \varphi_0 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \varphi_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \varphi_0(\alpha) + \varphi(\delta)$$

where order of the coordinates is given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For the remaining coordinates, the equality follows immediately from the definition of α and δ above. Therefore, the line segment $[\varphi_0(p), \varphi(q)]$ is contained in the convex hull $\operatorname{conv}\{\varphi_0(p), \varphi_0(q), \varphi_0(\alpha), \varphi_0(\delta)\}$. Hence, P_1 is convex. Observe that the line segment $[\varphi_0(p), \varphi(q)]$ is not an edge of P_1.

We now show that φ_0 gives a bijection between the vertices of $\mathcal{O}_C(\lambda)$ and P_1. Clearly, φ_0 is a bijection between the vertices of P_1 and $\mathcal{O}_C(\lambda)$ that lie in the

interiors $U_>$ and $U_<$. Let v be any vertex of $\mathcal{O}_C(\lambda)$ that lies in $U_=$. Assume by contradiction that $\varphi_0(v)$ is not a vertex of P_1, then $\varphi_0(v)$ lies in strict interior of an edge of P_1. Therefore, there exist vertices $p \in P_1 \cap U_>$ and $q \in P_0 \cap U_<$ such that $\varphi_0(v)$ lies on the line segment between them, which is an edge of P_1. However, we have already observed that all such line segments are not edges of P_1, a contradiction. Therefore, $\varphi_0(v)$ is a vertex of P_0.

Now, let v be a non-vertex of $\mathcal{O}_C(\lambda)$ and assume by contradiction that $\varphi_0(v)$ is a vertex of P_1. Since $\varphi_0(v)$ is a vertex, it follows that v lies on an edge of $\mathcal{O}_C(\lambda)$ that intersects the interiors of both regions of linearity $U_>$ and $U_<$. Hence, there exist vertices $p \in U_> \cap \mathcal{O}_C(\lambda)$ and $q \in U_< \cap \mathcal{O}_C(\lambda)$ such that $v \in [p, q]$. However, we have already observed that there are no such edges, a contradiction. Therefore, $\varphi_0(v)$ is not a vertex of P_1. And so we have shown that v is a vertex of $\mathcal{O}_C(\lambda)$ if and only if $\varphi_0(v)$ is a vertex of P_0.

The inductive step, for $i > 0$, follows almost identically to the case $i = 0$. The only difference is that, for each $x \in \mathbb{R}^\lambda$, we do not work with the value $x_{(a-i,b-i)}$. Instead, we use the value $x_{(a-i,b-i)} + \min\{x_{(a-i+1,b-i)}, x_{(a-i,b-i+1)}\}$. □

Proof of Theorem 4.3.11. Fix a maximal flag of up-sets of λ: $\emptyset = C_0 \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_u = \lambda$. Hence, for each $i \in \{1, \ldots, u\}$, we have that $C_i \backslash C_{i-1}$ is a singleton $\{r\}$ such that r is a corner of $\lambda \backslash C_{i-1}$. By Lemma 4.3.16, the map χ_r is a bijection between the restricted chain-order polytopes $\chi_r : \mathcal{O}_{C_{i-1}}(\lambda)^k_{\underline{d}} \to \mathcal{O}_{C_i}(\lambda)^k_{\underline{d}}$. From Proposition 4.3.12 we have that $\chi_r = \psi \circ \varphi_{i_{\max}(r)} \circ \cdots \circ \varphi_0$ is the composition tropical maps φ_i and a unimodular map ψ. By Lemma 4.3.17, we have that for each $j \in \{0, \ldots, i_{\max}(r)\}$, the tropical map φ_j gives a combinatorial mutation of the polytope $\varphi_{j-1} \circ \cdots \circ \varphi_0(\mathcal{O}_{C_{i-1}}(\lambda)^k_{\underline{d}})$. So $\mathcal{O}_{C_{i-1}}(\lambda)^k_{\underline{d}}$ and $\mathcal{O}_{C_i}(\lambda)^k_{\underline{d}}$ are mutation equivalent for each i. Hence, $\mathcal{O}(\lambda)^k_{\underline{d}} = \mathcal{O}_{C_0}(\lambda)^k_{\underline{d}}$ and $\mathcal{C}(\lambda)^k_{\underline{d}} = \mathcal{O}_{C_u}(\lambda)^k_{\underline{d}}$ are mutation equivalent. □

Remark 4.3.18. The proof of Lemma 4.3.17 shows that all chain-order polytopes $\mathcal{O}_C(\lambda)$ have the same number of vertices. However, this does not hold for the restricted polytopes. See [3, Examples 2.1 and 2.3].

4.3.4 Observations about Period Collapse

In this section we focus on the phenomenon of period collapse, as studied in [3]. We provide some small computations, straightforward corollaries of our main results, and further questions. We begin with the following observation.

Proposition 4.3.19 ([3, Theorem 2.1 and Lemma 2.2])**.** *If λ is rectangular then the Ehrhart quasi-polynomial of $\mathcal{O}(\lambda)^k_{\underline{d}}$ is a polynomial.*

Proof. The proof follows identically to [3, Theorem 2.1 and Lemma 2.2] with a slight modification for the rectangular case. Explicitly, if $\lambda = [m_1] \times [m_2]$, then $\mathcal{O}(\lambda)^k_{\underline{d}}$ is integrally equivalent to the Gelfand–Tsetlin polytope $\mathrm{GT}_{\alpha,\beta}$ of shape α and content β where $\alpha = (k^{(m_1)}, 0^{(m_2)})$ and

$$\beta = (d_{m_1-1},\, d_{m_1-2} - d_{m_1-1},\, d_{m_1-3} - d_{m_1-2},\, \ldots,\, d_{m_1-m_2} - d_{m_1-m_2+1},$$
$$k + d_{m_1-m_2-1} - d_{m_1-m_2},\, \ldots,\, k + d_{-m_2+1} - d_{-m_2+2}).$$

The notation $k^{(m_1)}$ above means k repeated m_1 times. □

By Proposition 4.1.1, combinatorial mutations preserve the Ehrhart (quasi)-polynomial, so we immediately obtain the following corollary of Theorem 4.3.11.

Corollary 4.3.20. *If λ is a rectangular Young diagram then the Ehrhart quasi-polynomial of the restricted chain-order polytope $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ is a polynomial, for any $\underline{d}$, k and up-set $C \subseteq \lambda$.*

We observe that some non-rectangular Young diagrams give rise to restricted chain-order polytopes with period collapse. For the next proposition, we require the following definition. Let λ be a Young diagram and fix a pair of adjacent diagonals ℓ and $\ell+1$ where $\ell \geq 0$ (resp. ℓ and $\ell-1$ where $\ell \leq 0$). Assume the diagonals have the same length. We define *the Young diagram $\lambda\backslash\ell$ with diagonal ℓ removed* as follows:

$$\begin{cases} \{(i,j) \in \lambda \mid i-j < \ell\} \cup \{(i-1,j) \mid (i,j) \in \lambda \text{ and } i-j > \ell\} & \text{if } \ell \geq 0, \\ \{(i,j) \in \lambda \mid i-j > \ell\} \cup \{(i,j-1) \mid (i,j) \in \lambda \text{ and } i-j < \ell\} & \text{resp. if } \ell \leq 0. \end{cases}$$

Given a vector $\underline{d}$, as in Setup 4.3.6 for the Young diagram $\lambda \subseteq [m_1] \times [m_2]$, we also define $\underline{d}\backslash\ell = (d_{m_1-1}, \ldots, d_{\ell+1}, d_{\ell-1}, \ldots, d_{1-m_2})$. We think of $\underline{d}\backslash\ell$ as the corresponding vector for the Young diagram $\lambda\backslash\ell$. See Example 4.3.24.

Proposition 4.3.21. *If λ contains two adjacent diagonals of the same length, say ℓ and $\ell+1$ with $\ell \geq 0$ (resp. ℓ and $\ell-1$ with $\ell \leq 0$), and $d_\ell = d_{\ell+1}$ (resp. $d_\ell = d_{\ell-1}$) then the polytopes $\mathcal{O}(\lambda)^k_{\underline{d}}$ and $\mathcal{O}(\lambda\backslash\ell)^k_{\underline{d}\backslash\ell}$ are integrally equivalent. Also, if $d_{\ell+1} < d_\ell$ (resp. $d_{\ell-1} < d_\ell$) then $\mathcal{O}(\lambda)^k_{\underline{d}}$ is empty.*

Proof. Assume $\ell \geq 0$ and take any point $x \in \mathcal{O}(\lambda)^k_{\underline{d}}$. Write $\alpha_1, \ldots, \alpha_t$ for the boxes along the ℓth diagonal in λ and $\beta_1, \ldots, \beta_t$ for the coordinates along the

$(\ell+1)$th diagonal. By definition of the restricted order polytope, we have that $x_{\alpha_1}+\cdots+x_{\alpha_t}=d_\ell=d_{\ell+1}=x_{\beta_1}+\cdots+x_{\beta_t}$ and for each $i\in[t]$ we have $x_{\alpha_i}\leq x_{\beta_i}$. It follows immediately that $x_{\alpha_i}=x_{\beta_i}$ for all $i\in[t]$. Therefore, the projection from $\mathcal{O}(\lambda)^k_{\underline{d}}$ to $\mathcal{O}(\lambda\backslash\ell)^k_{\underline{d}\backslash\ell}$ which removes the ℓth diagonal of λ gives an integral equivalence of polytopes.

Since $x_{\alpha_i}\leq x_{\beta_i}$ for all $i\in[t]$, it follows that $d_\ell\leq d_{\ell+1}$. So, if $d_{\ell+1}<d_\ell$ then we have that $\mathcal{O}(\lambda)^k_{\underline{d}}$ is empty. A similar argument proves the result when $\ell\leq 0$. □

So for any Young diagram λ, we may apply Propositions 4.3.19 and 4.3.21 to construct restricted order polytopes that exhibit period collapse. We obtain the following corollary of Theorem 4.3.11.

Corollary 4.3.22. *Let $\underline{d}$ be the vector given by*

$$(\underline{d})_\ell=|\{\textit{boxes of the }\ell\textit{th diagonal of }\lambda\}|.$$

Then the Ehrhart quasi-polynomial of $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ is a polynomial.

Remark 4.3.23. We note that Proposition 4.3.19 does not immediately generalize to the non-rectangular cases. Unless the vector $\underline{d}$ contains repeated entries, as in Proposition 4.3.21, it is not clear whether restricted order polytopes $\mathcal{O}(\lambda)^k_{\underline{d}}$ for non-rectangular posets λ are related to the Gelfand–Tsetlin polytope $\mathrm{GT}_{\alpha,\beta}$. Restricted order polytopes can be realized as the intersection of a GT-polytope with certain hyperplanes. However, it is not clear why such an intersection would give a polytope with period collapse.

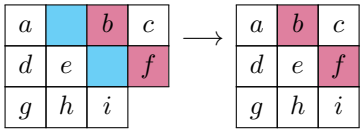
Example 4.3.24. Consider the partition of 11 given by $(4,4,3)$ and let λ be the corresponding Young diagram. Let $k=2$ and $\underline{d}=(1,2,3,2,2,1)$. Then the vertices of restricted order $\mathcal{O}(\lambda)^k_{\underline{d}}$ are

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1&1&1&1\\\hline 1&1&1\\\cline{1-3}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 1&1&1&1\\\hline 1&1&2\\\cline{1-3}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 0&1&1&1\\\hline 1&2&2\\\cline{1-3}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 1&1&2&2\\\hline 1&1&2\\\cline{1-3}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&1&2&2\\\hline 1&2&2\\\cline{1-3}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 0&\frac{1}{2}&\frac{1}{2}&1\\\hline \frac{1}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}\\\hline 1&\frac{3}{2}&\frac{3}{2}\\\cline{1-3}\end{array},\quad \text{and}\quad \begin{array}{|c|c|c|c|}\hline \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&1\\\hline \frac{1}{2}&\frac{1}{2}&\frac{3}{2}&\frac{3}{2}\\\hline 1&\frac{3}{2}&2\\\cline{1-3}\end{array}.$$

The shaded adjacent diagonals $\ell=-1$ (in blue) and $\ell-1=-2$ (in purple) have the same values since $d_{-1}=d_{-2}=2$. Let $\lambda'=\lambda\backslash\ell$ be the square Young diagram given by the partition $(3,3,3)$ and $\underline{d}'=\underline{d}\backslash\ell=(1,2,3,2,1)$. By

Proposition 4.3.21, the vertices of the restricted order polytope $\mathcal{O}(\lambda')^k_{\underline{d}'}$ can be obtained from the above vertices by applying the projection:



This projection gives an integral equivalence between the two restricted order polytopes. The Ehrhart polynomial of this polytope is given by:

$$\frac{1}{12}t^4 + \frac{1}{2}t^3 + \frac{17}{12}t^2 + 2t + 1$$

and its h^*-vector is $(1, 0, 1, 0, 0)$.

Example 4.3.25. We give an example of a restricted order polytope for some non-rectangular Young diagram λ that does not satisfy the assumptions Propositions 4.1.1 or 4.3.21. Consider the partition $\lambda = (4, 4, 3)$, $k = 3$ and $\underline{d} = (1, 2, 3, 2, 3, 1)$. We index the coordinates of $\mathbb{R}^\lambda$ as follows:

1	2	3	4
5	6	7	8
9	10	11	

and let $V = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&1&0&0&0&0&\frac{1}{2}&\frac{1}{2}\\
1&1&1&0&1&1&0&1&0&1&\frac{1}{2}\\
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0&0&\frac{1}{2}&\frac{1}{2}\\
1&1&1&1&1&1&1&0&0&\frac{3}{2}&\frac{3}{2}\\
1&1&2&2&1&2&2&2&2&\frac{3}{2}&\frac{3}{2}\\
2&2&2&3&2&2&3&2&3&2&\frac{5}{2}\\
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&2&2&2&2&2&\frac{3}{2}&\frac{3}{2}\\
1&2&2&2&2&2&2&3&3&\frac{3}{2}&\frac{3}{2}
\end{pmatrix} \in (\mathbb{R}^\lambda)^{11}$

The vertices of the restricted chain-order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ are the columns of V. The Ehrhart quasi-polynomial of this polytope is the polynomial given by

$$\frac{7}{120}t^5 + \frac{1}{2}t^4 + \frac{41}{24}t^3 + 3t^2 + \frac{41}{15}t + 1.$$

The h^*-vector for this polytope is $(1, 3, 3, 0, 0, 0)$.

Example 4.3.26. Let λ be the Young diagram associated to the partition $(4, 4, 3, 2)$. Let $k = 3$ and $\underline{d} = (1, 3, 2, 3, 2, 3, 1)$. Label the coordinates of $\mathbb{R}^\lambda$ as

follows:

1	2	3	4
5	6	7	8
9	10	11	
12	13		

and let $V = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&0&0&0&0&1&0&0&0&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2} \\ 1&1&1&0&1&1&0&1&0&1&1&0&1&0&1&\frac{1}{2}&1&\frac{1}{2} \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2} \\ 1&1&1&1&1&1&1&0&0&1&1&1&0&0&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2} \\ 1&1&2&2&1&2&2&2&2&1&2&2&2&2&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2} \\ 2&2&2&3&2&2&3&2&3&2&2&3&2&3&2&\frac{5}{2}&2&\frac{5}{2} \\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&1&1&\frac{1}{2}&\frac{1}{2} \\ 1&1&1&1&2&2&2&2&2&2&2&2&2&2&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2} \\ 1&2&2&2&2&2&2&3&3&2&2&2&3&3&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2} \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 2&2&2&2&2&2&2&2&2&3&3&3&3&3&2&2&\frac{5}{2}&\frac{5}{2} \end{pmatrix} \in (\mathbb{R}^{\lambda})^{18}$

The restricted order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ has 18 vertices which are given by the columns of V. Similarly, to the previous examples, the polytope exhibits period collapse. Its Ehrhart polynomial is

$$\frac{7}{240}t^6 + \frac{77}{240}t^5 + \frac{23}{16}t^4 + \frac{163}{48}t^3 + \frac{68}{15}t^2 + \frac{197}{60}t + 1$$

and its h^*-vector is $(1, 7, 11, 2, 0, 0, 0)$.

Question 9. Is the Ehrhart quasi-polynomial of the restricted chain-order polytope $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ a polynomial for any up-set $C \subseteq \lambda$, vector $\underline{d}$ and positive integer k?

By Theorem 4.3.11, it suffices to consider only the restricted order polytopes. Also, using the combinatorial mutations defined in Section 4.3.3, the question is simultaneously answered for all other intermediate polytopes that appear in the sequence of combinatorial mutations between restricted chain-order polytopes.

Question 10. What is the degree of the h^*-polynomial of a restricted chain-order polytope?

The examples above seem to suggest that the degree of the h^*-polynomial is bounded above by half the dimension of the polytope. Given a Young diagram with n boxes and ℓ diagonals, for sufficiently generic k and $\underline{d}$, the dimension of the restricted order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ is $n - \ell$.

Question 11. Which non-lattice restricted order polytopes are mutation equivalent to lattice polytopes?

Consider the restricted order polytope in Example 4.3.24. It turns out that all intermediate polytopes, which appear in the sequence of mutations constructed in Section 4.3.3, are non-lattice polytopes. For other non-lattice restricted

order polytopes, we ask whether any of the intermediate polytopes are lattice polytopes. We note that if a restricted order polytope is mutation equivalent to a lattice polytope, then this gives an alternative proof that the Ehrhart quasi-polynomial is a polynomial.

Chapter 5

Toric Degenerations and Tropical Grassmannians

This chapter will focus on the study of tropical Grassmannians and tropical flag varieties as described in Section 2.4.3. In this chapter we will focus on the study of these varieties over the complex field $\mathbb{C}$, which we will omit to avoid heavy notation. In particular, in Section 5.1 we will show how matching fields give a computational construction of toric degenerations of Grassmannians and partial flag varieties corresponding to prime cones in the tropicalization. In Section 5.3, the computational results from the previous sections are shown for $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$.

This chapter is based on joint work with Oliver Clarke and Fatemeh Mohammadi [35]. Section 5.2 is based on work performed by Cas Proost in his Master's thesis [104].

5.1 Toric Degenerations from Matching Fields

In Section 4.2.3, we studied how matching fields give rise to matching field polytopes and toric varieties. We now want to focus on describing when these toric varieties are toric degenerations of the corresponding partial flag variety. We mostly refer to the approach described in [85, 81].

As in the previous sections, we restrict ourselves to the complete flag variety $\mathcal{F}\ell_n$ and note that all arguments apply to partial flag varieties and, in particular,

Grassmannians. Consider the polynomial ring $\mathbb{C}[\mathbf{p}] = \mathbb{C}[p_I \mid I \subseteq [n]]$ and the Plücker ideal $\mathcal{I} = \ker(\phi)$ that defines $\mathcal{F}\ell_n$ as a subvariety of multiprojective space, where ϕ is as defined in eq. (4.2). Fix a coherent matching field Λ for the flag variety $\mathcal{F}\ell_n$ which is induced by the weight matrix M_Λ. The *weight vector* $w_\Lambda \in \mathbb{R}^{\binom{n}{1}+\cdots+\binom{n}{n-1}}$ associated to Λ and M_Λ is given by

$$w_\Lambda = \left(\min_{\sigma \in S_n} \left\{ \sum_{a=1}^{|I|} (M_\Lambda)_{a,i_{\sigma(a)}} \right\} \right)_{\varnothing \subsetneq I \subsetneq [n]},$$

where $I = \{i_1 < \cdots < i_{|I|}\}$. In other words, the term corresponding to $I \subseteq [n]$ of w_Λ is the tropical determinant of the matrix $(M_\Lambda)_I$.

We use Gröbner degeneration, see Section 2.4.1, to construct a flat family $\pi : \mathcal{F} \to \mathbb{A}^1$ over $\mathbb{A}^1$ such that the initial ideal $\mathrm{in}_{w_\Lambda}(\mathcal{I})$ is associated to the fiber over $\{0\}$. Explicitly, for each $t \in \mathbb{A}^1 \backslash \{0\}$ the fiber $\pi^{-1}(t)$ is the subvariety of multiprojective space defined by the ideal

$$\overline{\mathcal{I}_t} = \left\langle t^{-\min_{\boldsymbol{u}}\{w_\Lambda \cdot \boldsymbol{u}\}} f(t^{w_{\Lambda \cdot I_1}} p_{I_1}, \ldots, t^{w_{\Lambda \cdot I_d}} p_{I_d}) \mid f = \sum a_{\boldsymbol{u}} \mathbf{p}^{\boldsymbol{u}} \in \mathcal{I} \right\rangle,$$

where $d = \binom{n}{1} + \cdots + \binom{n}{n-1}$ is the number of Plücker coordinates in $\mathbb{C}[\mathbf{p}]$. If $\mathrm{in}_{w_\Lambda}(\mathcal{I})$ is toric, then we say that Λ produces a toric degeneration of $\mathcal{F}\ell_n$. Whether $\mathrm{in}_{w_\Lambda}(\mathcal{I})$ is toric is treated in [111, Chapter 11]. In particular, by [111, Theorem 11.3], we always have the inclusion

$$\mathrm{in}_{w_\Lambda}(\mathcal{I}) \subseteq \mathcal{I}_\Lambda \tag{5.1}$$

where $\mathcal{I}_\Lambda$ is the toric ideal $\ker(\phi_\Lambda)$. On the other hand, by [111, Theorem 11.4], the ideal in_{w_Λ} is toric if and only if the reverse inclusion holds.

Remark 5.1.1. Suppose that M and M' are two weight matrices that induce the same matching field Λ. Equivalently, the initial term of each Plücker form with respect to either M or M' is the same. If Λ gives rise to a toric degeneration, then $\mathrm{in}_w(\mathcal{I}) = \mathcal{I}_\Lambda = \mathrm{in}_{w'}(\mathcal{I})$, where w and w' are the weight vectors associated to M and M'. In particular, w and w' lie in the same cone of the Gröbner fan of the Plücker ideal $\mathcal{I}$. So, given a coherent matching field Λ, by a slight abuse of notation we write M_Λ for a weight matrix that induces Λ, and we write w_Λ for the weight vector induced by M_Λ. If a matching field does not give rise to a toric degeneration then we specify M_Λ and w_Λ explicitly.

Example 5.1.2. We give an example of two different weight matrices that induce the same matching field, hence they give same toric degeneration of $\mathrm{Gr}(2,4)$. Note that for any weight matrix $M \in \mathbb{R}^{2\times 4}$, the matching field induced by M is invariant under adding a constant to any column of M. So, we may

always assume that the first row of M is all zeros. Let M_1 and M_2 be as follows:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 9 & 5 & 4 & 0 \end{bmatrix}.$$

In this case, we note that the induced matching field is well-defined because the entries in the second row of the weight matrices are distinct. The induced weights on the Plücker variables p_{12}, p_{13}, p_{23}, p_{14}, p_{24}, p_{34} are given by

$$w_1 = (2, 1, 1, 0, 0, 0) \quad \text{and} \quad w_2 = (5, 4, 4, 0, 0, 0)$$

respectively. The corresponding initial ideals are: $\mathrm{in}_{w_1}(\mathcal{I}) = \langle p_{13}p_{24} - p_{14}p_{23} \rangle = \mathrm{in}_{w_2}(\mathcal{I})$.

To prove the reverse inclusion to eq. (5.1), we follow a similar strategy to [26] and consider the Hilbert function of $\mathrm{in}_{w_\Lambda}(\mathcal{I})$ and the Ehrhart function of the matching field polytope, see Definition 2.3.3.

We now show that all coherent matching field polytopes for all partial flag varieties are normal. Recall from Remark 4.2.11 that the polytope λQ is a product of dilated simplices for each $\lambda \in \mathbb{N}^{n-1}$.

Proposition 5.1.3. *Fix $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{N}^{n-1}$ and consider the polytope $\lambda Q_\Lambda := (\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(\lambda Q)$. Then any lattice point $\alpha \in \lambda Q_\Lambda$ can be written as*

$$\alpha = \sum_{k=1}^{n-1} \sum_{j=1}^{\lambda_k} v_{I_{(k,j)},\Lambda}$$

for some vertices $v_{I_{k,j},\Lambda}$ of Q^k_Λ. In particular, for any $\ell \in \mathbb{N}$ and $K = \{k_1 < \cdots < k_\ell\}$ the matching field polytopes Q^K_Λ is normal, when considered as a lattice polytope with respect to the lattice $\mathbb{Z}S$.

Proof. Fix a lattice point $\alpha \in \lambda Q_\Lambda$. Recall that $\mathbb{Z}S$ is generated by $\{s_I := \hat{\phi}_\Lambda(e_I) \mid \varnothing \subsetneq I \subsetneq [n]\}$ of where $e_I \in \mathbb{Z}^{\binom{n}{1}+\cdots+\binom{n}{n-1}}$ is a standard lattice generator. Each lattice point s_I is a vertex of the polytope $\hat{\phi}_\Lambda(\Delta_k) = Q^k_\Lambda$ where $k = |I|$. Therefore, we can write α as a sum of lattice generators $\alpha = \sum_I a_I s_I$ for some $a_I \in \mathbb{N}$. By Lemma 4.2.6, we have that ϕ_Λ is concurrent with respect to the $\mathrm{Cl}(Y_\mathcal{P})$-grading of R_Q. That is, if $\hat{\phi}$ is the map defined in eq. (2.2), $\alpha = \hat{\phi}_\Lambda(\beta)$ for some lattice point β, and we write $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ for the $\mathrm{Cl}(Y_\mathcal{P})$-degree of corresponding monomial $x^\alpha \in \mathbb{C}[x_{i,j}]$. For each lattice generator $s_I = \hat{\phi}_\Lambda(e_I)$, we have that the corresponding element p_I has degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 appears in position $k = |I|$. It follows that $\lambda_k = \sum_{|I|=k} a_I$, and so we may rewrite the above sum so that $\alpha = \sum_{k=1}^{n-1} \sum_{j=1}^{\lambda_k} v_{I_{(k,j)},\Lambda}$ where $I_{(k,j)} \subseteq [n]$ is a k-subset.

Fix $K = \{k_1 < \cdots < k_\ell\}$. The polytope $Q := Q_\Lambda^K$ is $(\hat{\phi}_\Lambda \otimes_{\mathbb{Z}} \mathbb{R})(\lambda Q)$, where λ is the indicator vector of K. That is, for all $i \in [n-1]$ we have $\lambda_i = 1$ if $i \in K$, otherwise $\lambda_i = 0$. To prove that Q is normal let $\alpha \in mQ \cap \mathbb{Z}S$ be any lattice point. By linearity, α is a lattice point of the polytope $(\hat{\phi}_\Lambda \otimes \mathbb{Z})(\lambda' Q)$ where $\lambda' = m\lambda$. By the previous part, we have that $\alpha = \sum_{k\in K} \sum_{i=1}^{\lambda'_k} v_{I_{(k,j)},\Lambda}$, where $v_{I_{(k,j)},\Lambda}$ is a vertex of Q_Λ^k. For each $j \in [m]$ define $\alpha_j = \sum_{k\in K} v_{I_{(k,j)},\Lambda} \in Q \cap \mathbb{Z}S$. Therefore, $\alpha = \alpha_1 + \cdots + \alpha_m$ and so Q is normal. □

We now want to show that for all matching fields whose polytopes are combinatorial mutation equivalent to the Gelfand–Tsetlin polytope, the reverse inclusion to eq. (5.1) holds. As a consequence, we get a large family of toric degenerations of Grassmannians and flag varieties.

Theorem 5.1.4. *Fix $n \in \mathbb{Z}$, $n > 0$ and a subset $K \subseteq [n]$. Let Λ be a matching field for the partial flag variety $\mathcal{F}\ell(K;n)$. If the matching field polytope $\mathcal{P}_\Lambda$ is combinatorial mutation equivalent to the Gelfand–Tsetlin polytope, then Λ gives rise to a toric degeneration of $\mathcal{F}\ell(K;n)$.*

Proof. We prove the result for the complete flag variety $\mathcal{F}\ell_n$, and we note that the general case is completely analogous. Recall that the diagonal matching field B^{w_0} induces a toric degeneration of the flag variety which is associated to the Gelfand–Tsetlin polytope. Therefore, the Hilbert function of $\mathcal{F}\ell_n$ agrees with the Hilbert function of $\mathbb{C}[p_I]/\mathcal{I}_{B^{w_0}}$. Since $\mathcal{P}_{w_0}$ is normal by Proposition 5.1.3, we have that the Hilbert function of $\mathbb{C}[p_I]/\mathcal{I}_{B^{w_0}} = \mathbb{C}[\mathrm{Cone}(\mathcal{P}_{w_0}) \cap \mathbb{Z}S \times \mathbb{Z}]$ is equal to the Ehrhart polynomial of $\mathcal{P}_{w_0}$. By [2], tropical maps preserve the Ehrhart function and so the Ehrhart polynomial of $\mathcal{P}_\Lambda$ coincides with the Hilbert function of $\mathcal{F}\ell_n$. Moreover, $\mathcal{P}_\Lambda$ is normal by Proposition 5.1.3, so we have that the Hilbert function of $\mathbb{C}[p_I]/\mathcal{I}_\Lambda = \mathbb{C}[\mathrm{Cone}(\mathcal{P}_\Lambda) \cap \mathbb{Z}S \times \mathbb{Z}]$ is equal to the Ehrhart polynomial of $\mathcal{P}_\Lambda$. Hence, the Hilbert function of $\mathcal{F}\ell_n$ and the Hilbert function of $\mathbb{C}[p_I]/\mathcal{I}_\Lambda$ agree. Since Gröbner degenerations preserve the Hilbert function, $\mathbb{C}[p_I]/\mathrm{in}_{w_\Lambda}(\mathcal{I})$ and $\mathbb{C}[p_I]/\mathcal{I}_\Lambda$ have the same Hilbert function. Since $\mathrm{in}_{w_\Lambda}(\mathcal{I}) \subseteq \mathcal{I}_\Lambda$, it follows that the ideals are equal. In particular, $\mathrm{in}_{w_\Lambda}(\mathcal{I})$ is a toric ideal, as desired. □

From Theorem 4.2.13, the following result follows.

Corollary 5.1.5. *Fix $n \in \mathbb{Z}$, $n > 0$ and a subset $K \subseteq [n]$. If $\sigma \in S_n$ is a permutation that avoids the patterns 4123, 3124, 1423 and 1324, then B^σ gives rise to a toric degeneration of $\mathcal{F}\ell(K;n)$. In particular, by taking $K = \{k\}$ or $K = [n]$, the matching field B^σ gives rise to a toric degeneration of $\mathrm{Gr}(k,n)$ or $\mathcal{F}\ell_n$, respectively.*

Remark 5.1.6. It is proven in [30] that the ideal $\mathcal{I}_\Lambda$ is quadratically generated for any so-called block diagonal matching field. Our computations show that this is true for every matching field of the form B^σ up to $n = 8$. We expect this to be true for all matching fields B^σ. Notice that, providing a bound for the degree of the generators of a general ideal is a difficult problem. Such problem is usually studied for special families of combinatorial ideals; see, e.g., [62, 95, 41, 39, 88].

5.2 Combinatorial mutations through non-prime cones

In this section we will study a particular example of combinatorial mutations between matching field polytopes that were not included in Theorem 4.2.13. This section is based on research done by Cas Proost in his Master's thesis [104].

The combinatorial mutation equivalence between matching field polytopes P_σ was proved in the restricted family of permutations that avoid the patterns 4123, 3124, 1423, 1324. The following example shows how we can construct a sequence of combinatorial mutations between two matching field polytopes from permutations containing the forbidden patterns.

Example 5.2.1 ([35, Example 4])**.** Consider the matching field polytope P_σ given by $\sigma = (6\,2\,4\,3\,5\,1)$ for $\mathrm{Gr}(3,6)$. Note that σ contains the forbidden pattern 1324 on the indices $2,3,4,5$. Following the same process as for the matching field polytopes considered in Section 4.2.5, we need to construct a sequence of combinatorial mutations from the polytope P_σ to P_τ where $\tau = (6\,2\,5\,3\,4\,1)$. Let

$$f_1 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix},$$

$$f_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and let $F_i = \mathrm{conv}(0, f_i)$ for $i = 1, 2$. It is possible to show that $P_\tau = \varphi_{w_1,F_1} \circ \varphi_{w_2,F_2} \circ \varphi_{-w_1,F_1}(P_\sigma)$ and this is a sequence of combinatorial mutations. Note that the polytope Q obtained after applying the two mutations, $\varphi_{w_2,F_2} \circ \varphi_{w_1,F_1}(P_\sigma)$, is not a lattice polytope, since it has one rational vertex:

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix} = \frac{1}{2}(v_{134} + v_{526}),$$

where v_{134} and v_{526} are vertices of P_σ. Our computation shows that the polytope Q can be obtained from the polytope of the hexagonal matching field in Figure 5.3, by adding the extra rational point above.

Note that Theorem 4.2.13 does not apply to the matching field B^τ as it contains the pattern 1423 on the indices $2, 3, 4, 5$. Using the *MatchingFields* package [24] for *Macaulay2* [59], it is straightforward to check that B^σ gives rise to a toric degeneration of $\mathrm{Gr}(3, 6)$. By the sequence of mutations above, we have that B^τ also gives rise to a toric degeneration of $\mathrm{Gr}(3, 6)$.

The meaning of the rational polytope Q was studied by Cas Proost in his Master's thesis [104]. The description given in there is constructed, following an algorithm introduced in [20], as follows. Let the cones C_σ and C_τ of $\mathrm{trop}(\mathrm{Gr}(3, 6))$ be, respectively, the cones containing the weights defined by the matching fields B^σ and B^τ. Let C_Λ be the cone corresponding to the matching field Λ whose polytope is $\varphi_{-w_1,F_1}(P_\sigma)$. The first mutation from Example 5.2.1 allows us to move from the cone C_σ to the cone C_Λ. We want to understand the meaning of the second mutation.

There is a unique non-prime cone adjacent to C_Λ, which we will denote by C_1. The primary decomposition of the initial ideal of the Plücker ideal defining $\mathrm{Gr}(3, 6)$ with respect to a weight in C_1 contains a unique toric component. After fixing a binomial basis of the latter ideal, it is possible to find a unique such binomial that does not lie in $\mathrm{in}_{C_1}(I_{3,6})$, given by $f_1 = p_{234}p_{156} + p_{124}p_{356}$. The ideal

$$I'_{3,6} = I_{3,6} + \langle f_1 - y_1 \rangle,$$

defines a larger embedding of $\mathrm{Gr}(3, 6)$. There is a surjective map from $\mathrm{trop}(I'_{3,6})$ to $\mathrm{trop}(I_{3,6})$ and $\mathrm{trop}(I'_{3,6})$ contains two prime cones, C_a and C_b whose projection to $\mathrm{trop}(I_{3,6})$ intersects the relative interior of C_1.

With this construction, the rational polytope Q has the following interpretation.

Proposition 5.2.2 ([104, Section 4.2.3])**.** *The Newton–Okounkov polytope associated to C_a is isomorphic to the rational polytope Q appearing in Example 5.2.1.*

Remark 5.2.3. From a combinatorial point of view, $\mathrm{trop}(\mathrm{Gr}(3, 6))$ is the fan over a simplicial complex that contains 15 bipyramids. The cones over the triangulations into 3 simplices of these bipyramids are exactly the non-prime cones of $\mathrm{trop}(\mathrm{Gr}(3, 6))$ [108]. When computing the new embedding with respect to a non-prime cone lying in a given bipyramid, we obtain the cones over the triangulation into two simplices of the fixed bipyramid, see Figure 5.2. However,

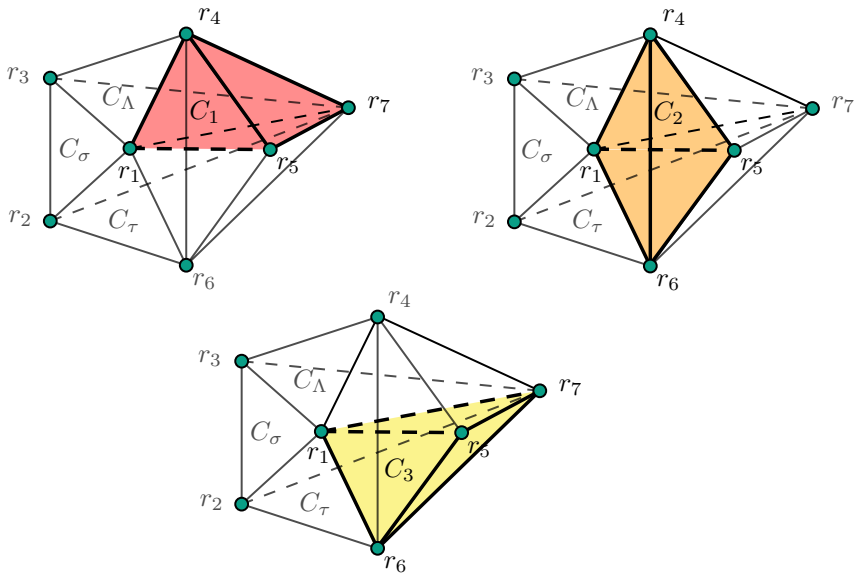


Figure 5.1: The local structure of $\mathrm{trop}(\mathrm{Gr}(3,6))$. In the ambient space modulo the lineality space, these are 4 dimensional cones and should be thought of as the cones over the simplices in the figure. The three pictures highlight the three non-prime cones C_1, C_2 and C_3.

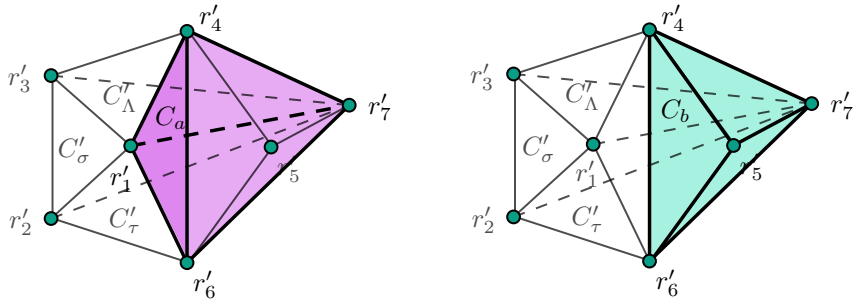


Figure 5.2: The local structure of $\mathrm{trop}(I'_{3,6})$. Each cone should be thought of as the cone over the simplex representing it in the ambient space modulo the lineality.

we also create new non-prime cones. The tropicalization of $I'_{3,6}$ contains 1337 maximal cones, 92 of which are non-prime [104, Theorem 7]. Of these 92, 44 have image under π which is not maximal, while the remaining map to

non-prime cones in $\mathrm{trop}(\mathrm{Gr}(3,6))$.

See Figure 5.1 and Figure 5.2 for a representation of the respective positions of the non-prime cones in $\mathrm{trop}(I_{3,6})$ and of the two new prime cones in $\mathrm{trop}(I'_{3,6})$, respectively.

5.3 Computational Results

We have seen in Example 5.2.1 that there exist matching fields B^σ that give rise to toric degenerations of partial flag varieties that are not covered by Theorem 4.2.13. However, in such cases, the matching field polytopes may be mutable. That is, the polytopes can be transformed into one another by a sequence of combinatorial mutations.

In this section, we consider a class of matching fields that generalize those in Definition 4.2.3. Moreover, we provide a complete computational classification of all toric degenerations obtained from these matching fields for small Grassmannians and flag varieties, namely $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, $\mathcal{F}\ell_4$, and $\mathcal{F}\ell_5$. In Examples 5.3.3, 5.3.6, 5.3.7 and 5.3.8, the matching field polytopes are constructed using the *MatchingFields* package [24] for *Macaulay2* [59], and their properties and isomorphism classes are computed using *Polymake* [54].

Definition 5.3.1 (Matrix M_c^σ)**.** Fix $n \geq 2$ and let $p \geq n+1$ be a prime number. For each natural number $c \geq 1$ coprime to p and permutation $\sigma \in S_n$ we define the weight matrix

$$M_c^\sigma = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ c\sigma(1) & c\sigma(2) & \cdots & c\sigma(n-1) & c\sigma(n) \\ np & (n-1)p & \cdots & 2p & p \\ np^2 & (n-1)p^2 & \cdots & 2p^2 & p^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ np^{n-2} & (n-1)p^{n-2} & \cdots & 2p^{n-2} & p^{n-2} \end{bmatrix}.$$

We now show that each M_c^σ induces a coherent matching field which we denote by B_c^σ.

Proposition 5.3.2. *For all $n \geq 2$, $c \in [p-1]$ and permutations $\sigma \in S_n$, the weight matrix M_c^σ induces a coherent matching field for the flag variety.*

Proof. Fix a subset $I = \{j_1, \ldots, j_k\} \subseteq [n]$. For each $\tau \in S_k$, we define $w_\tau := \sum_{i=1}^k M_{i,j_{\tau(i)}}$. We proceed by showing that the values w_τ are distinct for each

$\tau \in S_k$. Let $\tau, \tau' \in S_k$ be permutations and assume that $w_\tau = w_{\tau'}$. By definition of the weight matrix M_c^σ, we may write

$$w_\tau = \sum_{i=1}^{k-2} a_i p^i + ca_0 \quad \text{and} \quad w_{\tau'} = \sum_{i=1}^{k-2} b_i p^i + cb_0$$

with $a_i, b_i \in [n]$ for all $i \in \{0, 1, \ldots, k-2\}$. Note that p is a prime number and c, a_0, b_0 are all coprime to p. Since we have that $0 = w_\tau - w_{\tau'} = c(a_0 - b_0) \mod p$, it follows that $a_0 = b_0$.

The next step is to show that $a_i = b_i$ for all $i \in \{1, \ldots, k-2\}$. Assume by contradiction that $d \in \{1, \ldots, k-2\}$ is the largest index such that $a_d \neq b_d$. Without loss of generality, we may assume that $a_d > b_d$. For each $i \in \{1, \ldots, d\}$ we have that $a_i - b_i \geq -n \geq 1 - p$. It follows that

$$w_\tau - w_{\tau'} = (a_d - b_d)p^d + \sum_{i=1}^{d-1}(a_i - b_i)p^i \geq p^d - \sum_{i=1}^{d-1}(p-1)p^i = p > 0.$$

By assumption, we have $w_\tau = w_{\tau'}$ which is a contradiction. So we have shown that for any pair of permutations $\tau, \tau' \in S_k$, if $w_\tau = w_{\tau'}$ then $\tau = \tau'$. Hence, the values w_τ are distinct for all $\tau \in S_k$. In particular, the minimum value w_τ is attained by a unique permutation $\tau \in S_k$. So, following Definition 4.2.1, the expression $\operatorname{argmin}_\tau(w_\tau)$ gives a well-defined permutation for any subset $I \subseteq [n]$, and so the weight matrix M_c^σ induces a matching field. □

We use *Macaulay2* [24, 59] to confirm that certain matching fields B_c^σ give rise to toric degenerations of $\mathrm{Gr}(k, n)$ and $\mathcal{F}\ell_n$. To do this, we compute the initial ideal of the corresponding Plücker ideal with respect to the weight vector w induced by the matrix M_c^σ. Whenever the matching field gives rise to a toric degeneration, it follows that w lies in the corresponding tropicalization. In the following examples, we describe the maximal cones of the tropical varieties that contain these weight vectors w. We do this by comparing the polytopes. Note that the maximal cones of the tropicalization are listed up to symmetry under the symmetric group S_n. Since the f-vector of a matching field polytope is invariant under S_n, we compute the f-vectors of the matching field polytopes, using *Polymake* [54], and compare them to the already-known f-vectors associated to the maximal cones of the tropicalizations of $\mathrm{Gr}(k, n)$ and $\mathcal{F}\ell_n$. We note that some polytopes associated to non-isomorphic cones may have the same f-vector. In this case, we differentiate the polytopes by comparing their face lattices.

Example 5.3.3 (The Grassmannian $\mathrm{Gr}(3, 6)$)**.** There are seven top-dimensional cones of $\mathrm{Trop}(\mathrm{Gr}(3, 6))$ up to isomorphism [108]. Of these cones, six are prime. Of these six prime cones, five of them may be realized by matching fields B_c^σ.

Cone type	c	σ
EEEG	3	$(6,4,3,5,2,1)$
EEFF (a)	1	$(6,5,4,3,2,1)$
EEFF (b)	1	$(1,6,5,4,3,2)$
EEFG	1	$(4,3,2,1,6,5)$
EFFG	1	$(2,1,6,5,4,3)$
FFFGG	a non-prime cone from Example 5.3.5	
EEEE	a prime cone that cannot be obtained from a matching field	

Table 5.1: Maximal cones of $\mathrm{Trop}(\mathrm{Gr}(3,6))$ and their associated matching field polytopes B_c^σ.

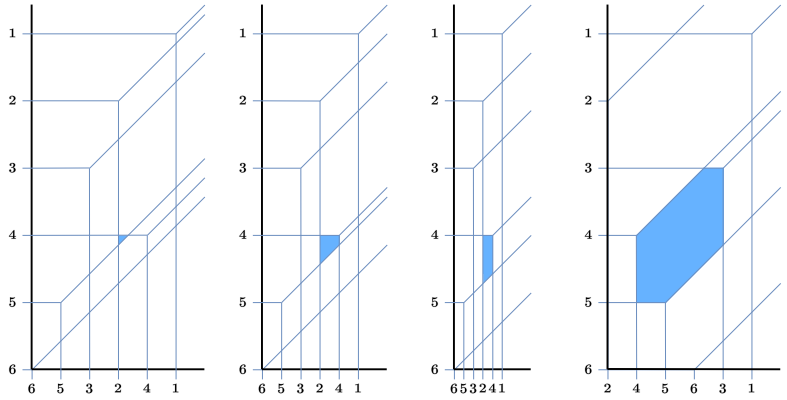


Figure 5.3: Tropical line arrangements associated to B_c^σ with the $(2,2,2)$-cell shaded. The right-most line arrangement corresponds to B_3^{615234}. From left to right the other line arrangements correspond to $\sigma = 643521$ and $c = 1$, 2 and 3 respectively.

In Table 5.1 below we list the type of the maximal cones of $\mathrm{Trop}(\mathrm{Gr}(3,6))$ together with an example of a matching field that gives rise to the same toric degeneration.

The isomorphism type of the toric variety of a matching field Λ is characterized in [90] by its corresponding *tropical line arrangement* $T(\Lambda)$, namely by the shape of the $(2,2,2)$-cell of $T(\Lambda)$. A straightforward computation shows that for $c = 1$, the $(2,2,2)$-cell of $T(B_c^\sigma)$ cannot be a triangle for any permutation $\sigma \in S_6$. However, this is possible for $c = 3$ and $\sigma = 643521$. See Figure 5.3 (left-most). We note that the cones of type EEEE cannot be realized by any matching field. See [90, Table 1].

In this paper, we have shown that, for all Grassmannians, the matching field polytopes of B_c^σ are mutation equivalent when $c = 1$ and σ is free a certain collection of permutations. We believe that our proofs can be extended to all matching fields B_c^σ, for any c. Note that for all $c \in [p-1]$, the matching fields $B_c^{w_0}$ where $w_0 = (n, n-1, \ldots, 1)$ give rise to the Gelfand–Tsetlin polytope.

Conjecture 5.3.4. *The matching field polytopes associated to B_c^σ that give rise to toric degenerations of* $\mathrm{Gr}(k,n)$ *are all mutation equivalent for $\sigma \in S_n$ and $c \in [p-1]$.*

Example 5.3.5. In the case $c = 3$, it is shown in [90] that a matching field for $\mathrm{Gr}(3,6)$ does not give rise to a toric degeneration if and only if the bounded $(2,2,2)$-cell of its tropical line arrangement is hexagonal. We note that if $c = 3$, then hexagonal matching fields appear among matching fields of the form B_c^σ. For example, if $\sigma = 615234$ then the weight matrix M_c^σ gives rise to the tropical line arrangement in Figure 5.3 (right-most). We note that the polytope P associated to this matching field is not mutation equivalent to the Gelfand–Tsetlin polytope. One way to see this is by noting that the normalized volume of P is 38 whereas the normalized volume of the Gelfand–Tsetlin polytope is 42.

Example 5.3.6 (The Grassmannian $\mathrm{Gr}(3,7)$)**.** The tropicalization of $\mathrm{Gr}(3,7)$ has a total of 125 maximal cones, which were computed in [61]. Using *Macaulay2*, we have observed that 69 of these cones are prime. However, 5 of them do not arise from matching fields, as they contain a *sub-weight* of type EEEE. That is, for each such weight vector w there exists an index $i \in [7]$ such that the restriction of w to the coordinates $J \subseteq [6]$ with $i \notin J$, lies in a cone of type EEEE in the tropicalization of $\mathrm{Gr}(3,6)$. Using weight matrices M_c^σ with $c \leq 6$, we can obtain 40 of the prime cones. Following the numbering conventions in [61] listed on the corresponding website, we list the prime cones of $\mathrm{Trop}(\mathrm{Gr}(3,7))$ together with their associated matching fields in Table 5.2. For each prime cone, we compute the toric polytope associated to the corresponding initial ideal. We identify whether that polytope is isomorphic to any matching field polytope $\mathcal{P}_{B_c^\sigma}$ where $c \in [6]$ and $\sigma \in S_7$. For example, the polytope associated to Gröbner cone number 14 is isomorphic to the matching field polytopes for $B_2^{1463725}$, $B_3^{1453627}$, $B_5^{1365724}$ and $B_6^{1365724}$ and is not isomorphic to any matching field polytope B_1^σ.

Gröbner cone index	List of (c,σ) such that B_c^σ gives the same toric degeneration
4	None – has EEEE as a sub-weight
6	None – has EEEE as a sub-weight
8	None
9	None
10	None – has EEEE as a sub-weight

Gröbner cone index	List of (c, σ) such that B_c^σ gives the same toric degeneration
13	None – has EEEE as a sub-weight
14	$(2, 1463725)$, $(3, 1453627)$, $(4, 1465723)$, $(c, 1365724) : c \in \{5, 6\}$
15	$(2, 3452617)$, $(3, 2456713)$, $(4, 1456723)$, $(c, 1356724) : c \in \{5, 6\}$
16	None
17	None
18	$(2, 1364725)$, $(3, 1354627)$, $(c, 1367425) : c \in \{4, 5, 6\}$
19	$(2, 1346725)$, $(3, 1345627)$, $(4, 2345671)$, $(5, 2456713)$, $(6, 1467532)$
21	None
22	None
23	None
24	None
25	$(c, 1354267) : c \in \{1, 2, 3\}$, $(c, 3541267) : c \in \{4, 5, 6\}$
26	$(c, 1345267) : c \in \{1, 2, 3\}$, $(c, 3451267) : c \in \{4, 5, 6\}$
27	None
28	$(c, 1342567) : c \in \{1, 2, 3\}$, $(c, 1236745) : c \in \{4, 5, 6\}$
29	None
30	$(c, 1352467) : c \in \{1, 2, 3\}$, $(c, 1263745) : c \in \{4, 5\}$, $(6, 1263547)$
31	$(c, 1354627) : c \in \{1, 2\}$, $(c, 1453267) : c \in \{3, 4, 5, 6\}$
32	$(c, 1345627) : c \in \{1, 2\}$, $(3, 3452167)$, $(4, 1245637)$, $(c, 4532167) : c \in \{5, 6\}$
33	$(3, 5672413)$, $(c, 1362745) : c \in \{4, 5, 6\}$
34	$(c, 1234567) : c \in [6]$
35	$(c, 1235467) : c \in [6]$
36	None
37	None
38	$(c, 1253467) : c \in [6]$
39	None
41	None
42	None
44	None
45	$(5, 2564713)$, $(6, 5624713)$
46	None – has EEEE as a sub-weight
47	$(2, 2365714)$, $(3, 1365724)$, $(c, 1354627) : c \in \{4, 5, 6\}$
48	$(2, 2356714)$, $(3, 1356724)$, $(c, 1345627) : c \in \{4, 5, 6\}$
49	None
52	$(2, 2367514)$, $(3, 1367524)$, $(4, 1356427)$, $(c, 1456732) : c \in \{5, 6\}$
53	$(2, 3561724)$, $(3, 2367415)$, $(4, 1367524)$, $(c, 1356427) : c \in \{5, 6\}$
54	$(3, 1462735)$, $(4, 1463752)$, $(c, 1453672) : c \in \{5, 6\}$
57	$(2, 2365174)$, $(3, 1365274)$, $(c, 1265734) : c \in \{4, 5, 6\}$
58	$(2, 2356174)$, $(3, 1356274)$, $(c, 1256734) : c \in \{4, 5, 6\}$
59	None
60	None
61	$(3, 1264735)$, $(c, 1254637) : c \in \{4, 5, 6\}$
62	$(2, 3456127)$, $(3, 1246735)$, $(c, 1245637) : c \in \{4, 5, 6\}$
63	$(2, 3561274)$, $(3, 3451267)$, $(4, 1563274)$, $(c, 1267534) : c \in \{5, 6\}$

Gröbner cone index	List of (c, σ) such that B_c^σ gives the same toric degeneration
64	$(c, 1356247) : c \in \{1, 2\}$, $(3, 1452637)$, $(4, 1453672)$, $(c, 4536172) : c \in \{5, 6\}$
65	$(4, 1267534)$, $(5, 1367542)$, $(6, 1563274)$
68	$(c, 1235647) : c \in [6]$
69	None
70	$(c, 1253647) : c \in [6]$
71	$(c, 1254637) : c \in \{1, 2, 3\}$, $(4, 1354672)$, $(c, 5613247) : c \in \{5, 6\}$
72	$(c, 1245637) : c \in \{1, 2, 3\}$, $(4, 1345672)$, $(c, 4563271) : c \in \{5, 6\}$
73	$(c, 1352647) : c \in [6]$
74	$(c, 1254367) : c \in [6]$
75	$(c, 1245367) : c \in [6]$
76	$(c, 1356427) : c \in \{1, 2, 3\}$, $(4, 2356471)$, $(5, 1675423)$, $(6, 3674251)$
77	None
78	None
82	$(3, 2564173)$, $(4, 2367451)$, $(c, 1267435) : c \in \{5, 6\}$
83	None
84	None
89	$(2, 4561237), (3, 2461753), (c, 1452673) : c \in \{4, 5, 6\}$
90	$(c, 1256347) : c \in \{1, 2, 3\}$, $(c, 1256473) : c \in \{4, 5, 6\}$
94	$(c, 1256437) : c \in [6]$
98	$(3, 3564172)$, $(4, 1267435)$, $(c, 2453167) : c \in \{5, 6\}$

Table 5.2: Maximal prime cones of $\mathrm{Trop}(\mathrm{Gr}(3, 7))$ and their associated matching fields B_c^σ.

Example 5.3.7 (The flag variety $\mathcal{F}\ell_4$)**.** In [20], it is shown that $\mathrm{Trop}(\mathcal{F}\ell_4)$ has 78 maximal cones, of which 72 are prime. They are partitioned into 4 orbits. In Table 5.3 we list the prime cone orbits along with matching fields B_c^σ which give rise to it.

Orbit	Size	(c, σ) such that B_c^σ induces the degeneration	Combinatorial equiv.
1	24	$(c, 2341) : c \in \{3, 4\}$	S2
2	12	$(c, 2341) : c \in \{1, 2\}$, $(c, 3421) : c \in \{3, 4\}$	S1 (Gelfand–Tsetlin)
3	12	$(c, 1342) : c \in [4]$	S3 (FFLV)
4	24	$(c, 1234) : c \in [4]$	

Table 5.3: Maximal prime cone orbits in $\mathrm{Trop}(\mathcal{F}\ell_4)$ and their associated matching fields B_c^σ.

Example 5.3.8 (The flag variety $\mathcal{F}\ell_5$)**.** In [20], it is shown that $\mathrm{Trop}(\mathcal{F}\ell_5)$ has 69780 maximal cones which are partitioned into 536 orbits under the action of $S_5 \times \mathbb{Z}_2$. There are 180 prime cone orbits which give rise to non-isomorphic toric degenerations. In Table 5.4 we list the prime cones that correspond to toric degenerations via matching fields B_c^σ. The table follows the notation in [20],

giving the *orbit index* and *combinatorial equivalences* that include the string polytopes Sn and the Gelfand–Tsetlin polytope.

Orbit	(c, σ) s.t. B_c^σ induces the degeneration	Combinatorial equiv.
32	$(c, 14532) : c \in [6]$	S22
37	$(c, 24531) : c \in \{1, 2, 3\}$	S21
38	$(c, 45312) : c \in \{1, 2, 3\}$	S27, S28
40	$(c, 45321) : c \in [6]$	S1, S18, S26, S29 (Gelfand–Tsetlin)
65	$(c, 12435) : c \in [6]$	
67	$(c, 12345) : c \in [6]$	
68	$(c, 13452) : c \in \{1, 2, 3\}$, $(c, 34512) : c \in \{4, 5, 6\}$	
69	$(c, 23451) : c \in \{1, 2\}$	
86	$(3, 34512)$	
87	$(4, 34521)$	S5, S31
94	$(c, 13425) : c \in [6]$	
95	$(c, 13452) : c \in \{4, 5, 6\}$	
97	$(c, 34215) : c \in [6]$	
98	$(3, 23451)$	
115	$(c, 45123) : c \in [6]$	
118	$(c, 45213) : c \in \{1, 2, 3, 4\}$	
119	$(c, 34521) : c \in \{5, 6\}$	S6
126	$(c, 23415) : c \in \{4, 5, 6\}$	
137	$(c, 23451) : c \in \{4, 5, 6\}$	
145	$(c, 45231) : c \in \{4, 5, 6\}$	
147	$(c, 24531) : c \in \{5, 6\}$	
151	$(c, 12453) : c \in [6]$	

Table 5.4: Maximal prime cones of $\mathrm{Trop}(\mathcal{F}\ell_5)$ and their associated matching fields B_c^σ.

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