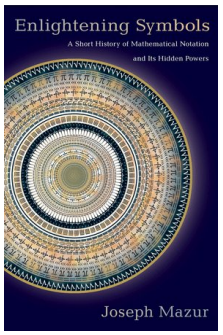
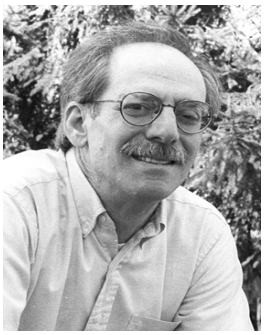
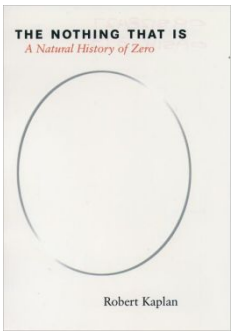
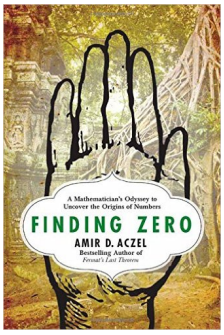
Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers. by *Joseph Mazur*. Princeton University Press, 2014, ISBN 978-0691-1-5463-3 (hbk), 285 pp.
The Nothing that Is: A Natural History of Zero. by *Robert Kaplan*. Oxford University Press, 2000, ISBN 978-0195142372 (hbk), 240 pp.
Finding Zero. A Mathematician's Odyssey to Uncover the Origins of Numbers by *Amir D. Aczel*. Palgrave Macmillan Trade, 2015, ISBN 978-1137279842 (hbk), 256 pp.




Joseph Mazur




Robert Kaplan



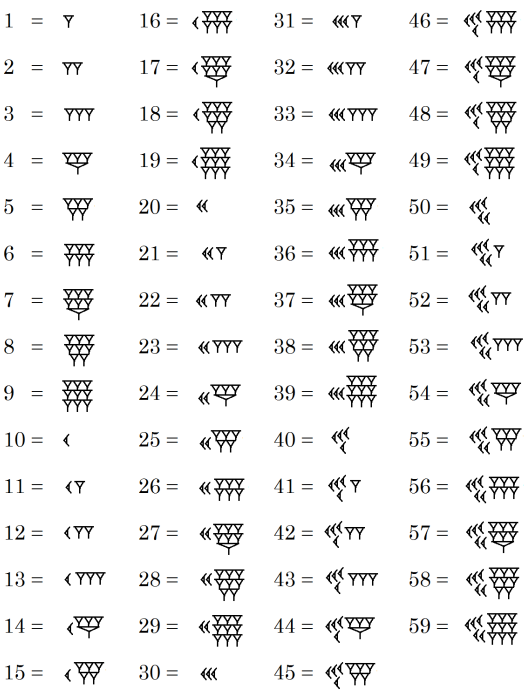

Amir Aczel

When you are a LaTeX user, you will know that there are several counters to number sections, definitions, footnotes etc. You can select the way in which they are represented. One possibility is to select `\arabic` to get the usual numbers. After reading these books, you know that these Arabic numbers should actually be called Hindu numerals or perhaps Hindu-Arabic because they originate from India and were brought to other societies directly such a the Khmer, and to our regions though the Arabs. *Lam Lay Yong* and *Ang Tian Se* even suggest that the system was originally Chinese[1]. The book by Aczel is all about a forgotten paper by *Georges Cœdès*[2] (pronounce *sédès*) of 1931 who discovered decimal notation, including the zero, in a Cambodian stele that was 2 centuries older than was accepted at that time. In another paper by Cœdès[3] of 1968, he explains how the Southeast Asian states were influenced by the Hindu society. Hence proving that our digits, even including the zero that had to be re-invented in the West much later, has an Hindu origin. Kaplan also discusses the origin of the different notations for numbers, but in particular the origin of zero. About this zero, there is a lot of folklore and speculation circulating. Kaplan discusses many of them, but is not very specific concerning the actual origin. Perhaps there are many possible explanations because people did really talk or discuss about there being none or nothing and there were several ways to denote a zero, but whether this was a way of indicating that there was no number or whether there was indeed a number, namely zero, will probably always be speculation.

LaTeX has also the possibility to select `\Roman` to get the familiar Roman numerals, but also you can select `\alph` for an alphabetic numbering or as sometimes used for footnotes some symbols can be selected like †, ‡, §, etc. with `\fnsymbol`. The latter system using a different symbol for every number obviously only works for small numbers. In principle, they could be used to generate a number system This is what the Roman numeral also do: only a few symbols were selected and used as numeral so compose a larger number. The Roman system uses I, V, X, L, C, D, M to denote 1, 5, 10, 50, 100, 500, 1000, which suffices to denote current years, but it would be a problem to use them in cosmology or nano-science. The Egyptians had hieroglyphics

[1] Lam Lay Yong and Ang Tian Se. *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China*, World Scientific 2004.
[2] G. Cœdès. *A propos de l'origine des chiffres arabes*, Bull. School of Oriental Studies **6**(2), 1931
[3] G. Cœdès. *The Indianized States of Southeast Asia*, Univ. of Hawaii Press, 1968.

for 10^k, $k = 0, 1, \ldots, 6$. The Aztecs had symbols for 1, 10, 20, 100, 200, 300, 400, and 8000. Some Greek and Hebrew systems used letters of the alphabet like in `\alph` to denote numbers but combine it with a decimal system to enlarge the representable numbers. For example the Hebrew latter have 22 letters to denote 1-9, 10-90, and 100-400. They added 5 more letters for 500-900.

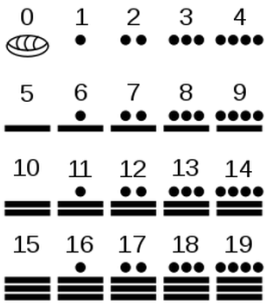


Babylonian numbers

Of course the oldest and really simple systems had only one symbol: a bar, and one had to count tally marks. If we group them in packages of five or ten, these packages can be represented by a second symbol. The Babylonian cuneiform system represented a unit and a ten, some Chinese and Mayan systems also had two marks: a unit and a group of five. These were enough to form numbers from 1 to 9 or 10, but the ingenuity came from the way they were composed to form larger numbers without inventing new symbols. Sometimes decorations were added to the symbols like dots or bars to multiply them with a power of 10. Archimedes had a way to describe really large numbers 'exceeding the the number of grains of sand in the whole universe'. First order were myriad (10^4) and a myriad-myriad (10^8). Putting order upon order and periods upon periods and unit upon units, he had for example a myriad-myriad units of the myriad-myriadth order of the myriad-myriadth period was actually $10^{8\times10^{15}}$. He wanted to be able to count all the grains of sand on the beach or even in the world. Knuth's up-arrow notation for iterated exponentation, now common in Ramsey theory, uses a similar idea, but is more structured.



Haab calendar



Aztec numerals

Since we are used to our system, it is obvious that the positional decimal system has won, but this did not come about overnight. The problem was that the zero was not part of the counting system. If you count horses, you will count at least one horse, but you cannot count horses when there is no horse around. So one left a space at a certain position if needed, but it was difficult to distinguish between one, two or three spaces. Thus some placeholder was invented to mark the spaces like the Babylonians or the Maya did. Note that as a placeholder it did not appear at the end of a number. It was definitely not considered to be a number until Bramagupta ($\pm$ 628 AD) had it with the arithmetic rules and negative numbers. Stone tables of Gwalior (South of Dehli, 876 AD) had a round indication of zero and copper plates go back to the 6th century but these could be forgeries. But Kaplan discusses many other possible notations of what can be indicated as zero or perhaps as a place holder. There was a dot, that could also be a separator. Some zeros look like a theta. Or other possibilities that look less familiar to us. According to Mazur, the positional system came to Europe via the Arab translation in Al-Khwārizmī's *al-jabr* (825 AD) together with the Hindu-Arab numerals. Fibonacci's *Liber abbaci* helped distributing the computational rules into Europe did miss out the number zero and spoke of only 9 digits. The zero as a number was re-introduced some 200 years later in work by N. Chuquet while solving quadratic equations. Although the Mayan Haab

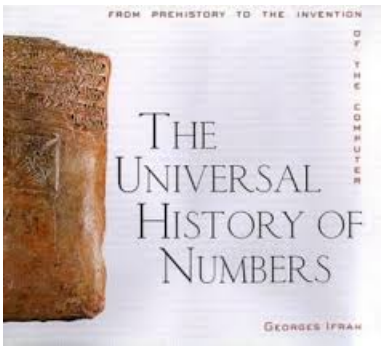
calendar had months of 20 days, numbered from 0 to 19. So they certainly had the zero. It had been uttered that the Hindu got the numbers and/ the zero from the Babylonians through the Greek astronomy. They had certainly ways to talk about zero: *kha* (space), *ambara* (sky) *sunya* (empty). There are also many speculations about the origin of the symbol (something circle-like) to denote a zero. Some say that it is the mark left in the sand when stones were used for counting and a stone was taken away.

European	0	1	2	3	4	5	6	7	8	9
Arabic-Indic	٠	١	٢	٣	٤	٥	٦	٧	٨	٩
Eastern Arabic-Indic (Persian and Urdu)	٠	١	٢	٣	۴	۵	۶	٧	٨	٩
Devanagari (Hindi)	०	१	२	३	४	५	६	७	८	९
Tamil		க	உ	ங	ச	ரு	சா	எ	அ	கூ

Kaplan goes on discussing the early development of mathematics, once the Hindu system for numbers were introduced with the notation for the unknown, the way equations were solved, and how Leibniz and Newton developed calculus. For those who are not familiar with mathematical history, it will probably be astonishing to learn how recent our current notation of mathematical formulas is. It is almost unbelievable that mathematics had evolved till the 15th-16th century on a rhetorical basis, not even a plus sign or an equal sign existed so that equations were described with plain words. Since around 1500 symbols were introduced increasingly and we see an exponential increase in mathematical knowledge ever since. So how did one arrive at these symbols? Where did they come from? What were the mechanisms that shaped them into their now familiar form? Mazur gives an entertaining history of this evolution. In a first part he deals with the notational systems of the numerals as was explained above. Then he considers the usual mathematical symbols like plus, minus, exponentials, square root, etc. in an algebraic context. In a third part he leaves history behind and ponders on the influence of our symbolic notation on the psychology of mathematicians, how symbolic patterns are stored in our brain and how formulas trigger unconscious associations. In retrospect, this may explain how symbolic notation has influenced the evolution of mathematics.



Georges Cœdès



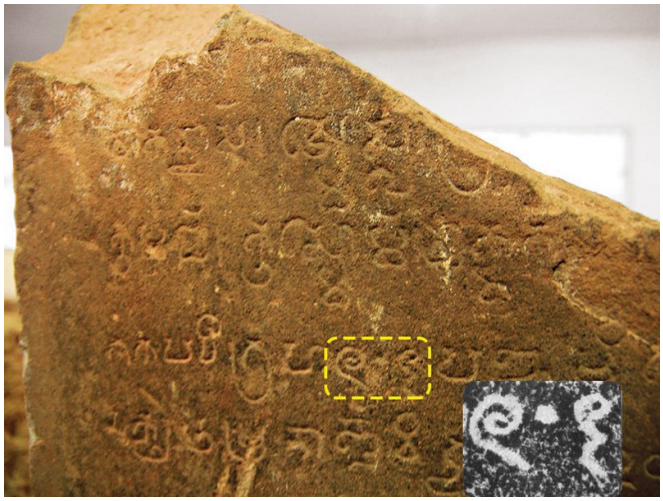
George Ifrah's book



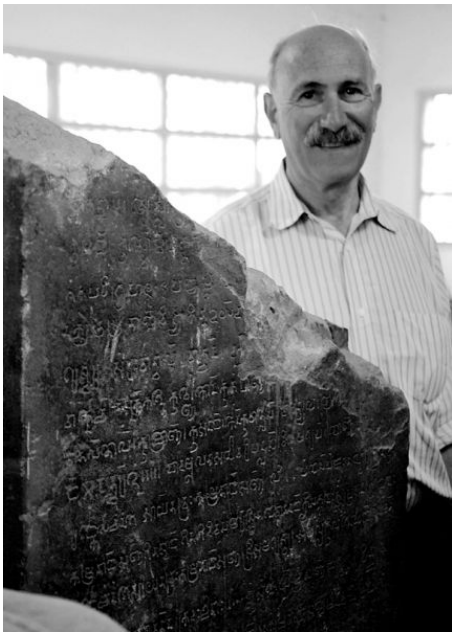
George Ifrah

But let me spend some more words on the book by Aczel. It is all about mathematics and re-establishing the work of Cœd'es mentioned above and in particular about finding out the whereabouts of the stone stele in Cambodia that was numbered K-127 (a prime number) by Cœdès on which he found the proof that the zero was known to the Indians before the Greek used to denote it. Aczel's father was a captain on a cruise ship and the children were often looked after by Laci (pronounced lotzi), the personal stewart of the captain. This Laci raised Aczel's

interest in numbers and mathematics. So he tells the usual stories about the number systems that can also be found in the other books. Later, when he had become a mathematician, he wanted to find out more about the origin of the Hindu-Arabic number system and travels to India, trying to find out why zero (and infinity) more naturally could arise in a cultural climate of the East. Hinduism, Jainism, Buddhism (Buddha was a mathematician), all differ from the way one used to think in the West with out yes or no, true or not true. They have 4 states: true, not true, both true and untrue, and neither true nor untrue (tetralemma). So the 3rd and the 4th may be the seeds that gave rise to the infinite and the nothing (the zero). Aczel even tries to connect the explicit sexual imaging in some of the Hindu temples with the mathematics. In the West the adagio was that nothing could come from nothing. We had to wait till Cantor defined set theory and hence the number system starting with the empty set for zero, then the set containing only the empty set, then the set with the two previous sets as elements, etc. He finally arrives at infinity and the different orders of infinity.



K-127, the inset = 605
The inscription says: "The Chaka era reached year 605 on the fifth day of the waning moon..." We know that in Cambodia the Chaka era began in the year 78 AD. Thus the date of this zero is 605 + 78 = 683.



Aczel at the moment he found K-127

He then learned about the book by Georges Ifrah *The Universal History of Numbers* who cites the work of Cœdès (1886-1969), something also Laci had already pointed out to him, but that he had forgotten. The K-127 that Cœdès had described that contained the oldest known notation for zero as a dot had been in the Cambodian national museum. However when the Red Khmer took over many artifacts were destroyed and nobody knew what had become of the K-127. So Aczel took it as his mission to rediscover the stele. So much of his book is about his adventurous traveling to the Southeast Asian countries in search of K-127. I think I will not take away the thrill of the story when I disclose that he finally was able to find it eventually in 2013. However right at that moment of euphoria, two Italian ladies enter the emporium to pick up at random some piece for their restoration students to practice on. Aczel over-enthusiast just now that he reaches the end of his quest, tells them about the stele. Unfortunately, as a result they insist on having K-127 for their students to test their abilities, much to the shocking dismay of Aczel who considers this as the most fundamental finding for mathematics that had to be conserved and handled with utmost care. The rest of the book consists of his efforts to save K-127 from the supposedly malicious intentions of the Italians. How that ends, I will not disclose.

A. Bultheel