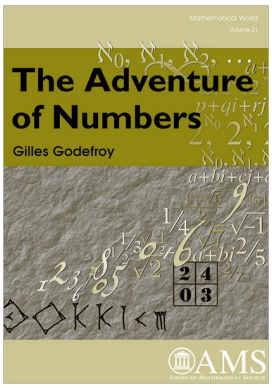
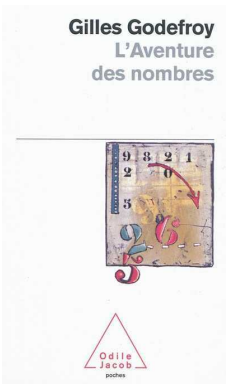
L'aventure des nombres *Gilles Godefroy* Éditions Odile Jacob , Paris, 1997, (237 p.), ISBN 978-2-7381-0422-9, and in pocket edition, 2011 (ix+234 p.), ISBN 978-2-7381-2551-4.
The adventure of numbers *Gilles Godefroy* Vol. 21 of Mathematical World, AMS, 2004 (ix+194 p.), soft cover, ISBN 978-0-8218-3304-9.

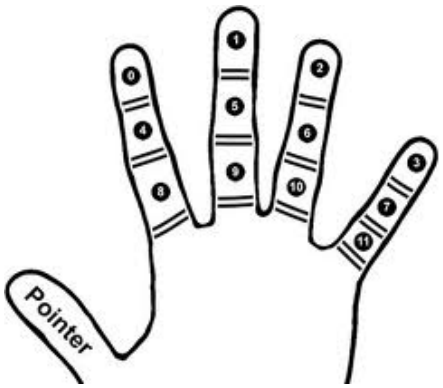




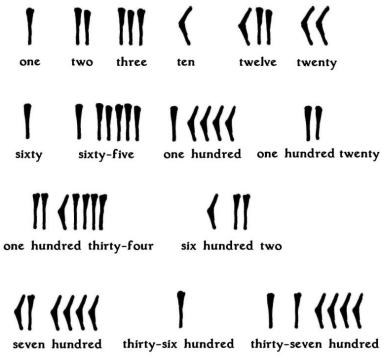
Although the original version dates back to 1997, and the English translation was published in 2004, the release of the French version in the series *Poches Odile Jacob* in 2011 is a good reason to take another look at this book.

The title may suggest that this is one of these books popularizing mathematics for the layman. However, I do not think that a non-mathematician will reach the end of the book. Godefroy sketches the history of numbers and number systems, starting with the most primitive attempts of man to count on their fingers but it ends with all the major questions and challenges that mathematics is faced with at the end of the twentieth century.

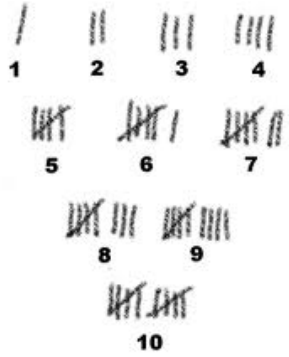
The start is easy and smooth and will be appreciated by the mathematician as well as the non-mathematician. Although humans usually have ten fingers (digits!), the decimal system seems to be the most natural, but also the binary system (one-two-many) is in our digital computer era a widespread habit. Nevertheless, we also live with several traces of the number system in base 12. In a sense, this is also natural if you count on your fingers. Use your thumb as a pointer at the twelve finger bones of the four other fingers. If you count the multiples of 12 with the fingers of your other hand, you end up with 60. In a sense 12 is more convenient for calculating since it has more divisors that 10.



counting to 12



Sumerian numbers



tallies

Counting is one thing, writing down numbers is another. With commerce flourishing in old Sumeria (about 3500 BC) there was a need to have a physical way to denote quantities that were bought or sold. The first numbers were written on clay, with little marks each representing one unit. This is what we still do when we keep scores writing tallies. Cutting notches in a stick for commercial reasons was common until relatively recently. The French word for notches is *tailles*, and this crossed to English as *tallies*. One cuts notches on tally sticks. The longest stick (stock) is kept by the lender and the other is taken by the borrower. And this gave rise to the word *stockholder*. But sticks with notches were not the only way of course. Also stones (small pebbles) were used (calculus is Latin for small pebble). Basically this is what is done on an abacus, an instrument still very much in use in China. Whether it are stones or notches, a systematic way had to be invented to group them in larger quantities. That required a major step towards a positional number system and brings us to the Sumerians and Babylonians. To write numbers Sumerians used the *nail* and the *wedge* as the only symbols. Their system gradually evolved into a positional

system in base 60, much like we denote our decimal numbers. But a positional system also needed an indication that some power of 60 is missing. So there is already a need for the number zero.



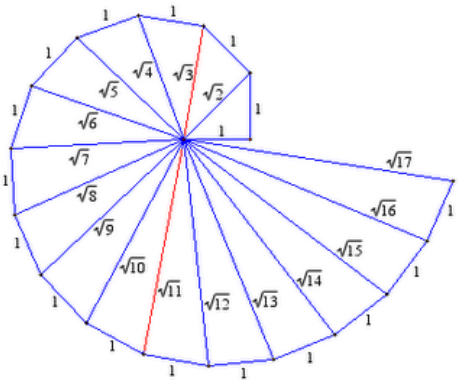
tally sticks

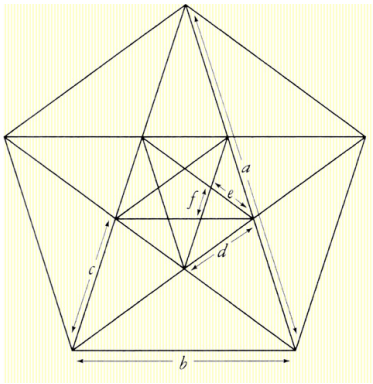


battle of Marathon

The Babylonian kingdom ended when it fell to the Persian emperor Cyrus the Great in 539 BC. The Greek were shielded away from the Persian expansionism when they won the battle of Marathon in 490 BC. It was in that Greece that the Pythagorean school that was founded around 600 BC came to full expansion. For them integers were the foundation of knowledge, leading to an atomistic view of space. All lengths were (finite) integer multiples of units (*nomads*). Taking the ratio of two lengths then gave rise to a rational number that was the ratio of two (finite) integers. So all quantities became commensurable, i.e., were integer multiples of a common unit. Although the Pythagorean theorem was probably known by the Chinese centuries earlier, it is considered to be their main mathematical fortress, and yet, it bears in it the irrationality of $\sqrt{2}$, contradicting their credo of commensurability.

And irrationals were everywhere. The spiral of Theodorus constructs all $\sqrt{n}$ for integer n and the golden ratio was embedded in the pentagram, to which the Pythagoreans attributed a mystic value. Plato founded his school shortly after the discovery of the irrationals. One of its members, Eudoxus, actually succeeded in constructing geometric objects whose magnitude (length, area, volume) had a ratio m/n that was just larger or just smaller than the irrational number. This in fact precedes Dedekind cuts by 22 centuries.







scholars at Abbasid library

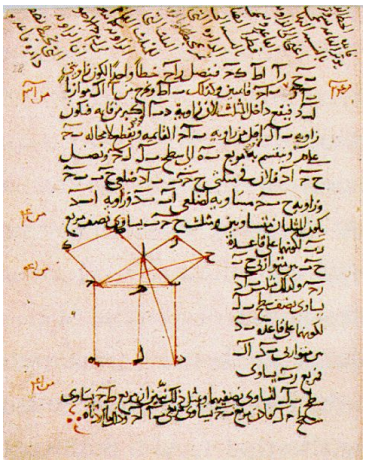
from Maqamat of al-Hariri, XIIth

Eudoxus is in fact also the author of what we now refer to as the Archimedean axiom (the existence of infinitely small and infinitely large numbers). Archimedes used this idea to approximate a circle by regular n-gons for increasing n. Using inscribed and circumscribed approximations, converging to each other, we may recognize integral calculus here. Note also that this implies an implicit notion of infinity, it was never used explicitly by the Greek.

The Romans did not care too much about mathematics. It is a surprising for how long their impossible archaic number system has survived. After Plato's school was closed, the interesting scene moved to the East. It is at this point in history mainly thanks to the Islamic culture that the Greek mathematical heritage has been saved. Al-Mamoun, an Abbasidian caliph, founded a house of wisdom (the *Baid Al-Hikma*) in Baghdad. It was a library and an intellectual center where the Greek scriptures were translated. Indian and Iranian scholars joined their knowledge and it are the Indians that gave us the decimal positional notation and, another major breakthrough: the zero. Where geometry was dominating

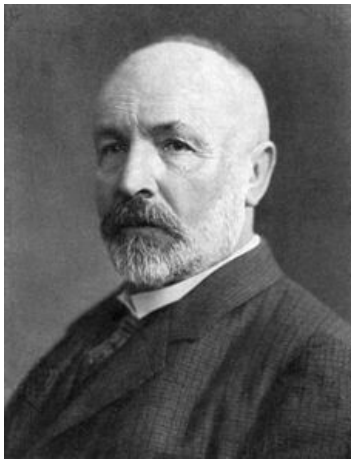
mathematics for the Greek, here in the Abbasidian caliphate lies the birthplace of algebra. It was Mohammed Ibn Musa Al-Khwarizmi who described the Indian number system. Of this description only a Latin translation survived. Since he claimed to be original, the Indian system became known to us as the *arabic numerals*. Also our words 'algebra' is drawn from the title of his book *Hisab al-jabr wa'l muqabalah* and the latinization of his name led to our word 'algorithm'. The Arabic word *sifr* for zero became our 'cypher'. His fundamental conceptual contribution though is that he starts from the equations. The Arabs study polynomials and their roots, i.e., the algebraic irrationals. The Arabic texts, and hence also the Greek mathematics that had been translated into Arabic, were not translated into Latin but until the twelfth century. This contact between Christian and Arab cultures was not via the Middle East because the crusades did not allow a friendly exchange of scientific ideas, but it happened mainly in Spain where a relative coexistence was possible, until in 1492, when Baobdil, the last Arab king of Granada, was chased away and left for Africa.

That brings us back to Italy of the 15th century. Solving the cubic and quartic equation was a study already started by the Arabs, but although it is accepted that some Italians kept the formulas as a secret, it was Gerolamo Cardano, a professor in Bologna (yes, the very one whose name is associated with the Cardan joint in your car) who published the formulas in 1545. The formulas for solving the quadratic equation gave rise to square roots of negative numbers. Cardano did not know how to deal with this. His student Bombelli came close to an interpretation in 1579. However, society did not seem ripe for the complex numbers. The fact that they are not ordered gave rise to much confusion.



Nasrudeen Tusi's record of Euclid's proof of the Pythagorean Theorem

The translations of the Greek and Arabic texts that became generally available were quickly spread with the printing technique that had just been invented. The evolution of science and in particular mathematics now took off at an exponential rate. Newton and Leibniz invented calculus, which introduced the infinitesimals, not exactly numbers, but still.... Two numbers were considered equal not only when their difference is zero, but also when their difference is 'incomparably small'. This tacitly introduced the concept of infinity. Besides the good old rationals and the algebraic numbers studied by the Italians before, now also the irrationals were entering the scene. If the complex roots of the cubic had perplexed Bombelli and Cardan, the periodicity of the complex exponential has even more confused the early workers of analysis. It was Euler in 1748 who introduced the complex exponential through its series expansion, which led to the famous relation $e^{i\pi} = -1$.



Georg Cantor (1845-1918)

It is difficult to bring the adventure of numbers at the same speed and detail as it was summarized till now. And we are only at about one third through the book. Also Godefroy needs to resort to higher mathematics since in the 19th century scholars were shaking the foundations of mathematics itself, and it all started with the real numbers that needed a precise axiomatic definition. Richard Dedekind introduced the reals as cuts (*Schnitts*): a real number is a partition of the rationals into those sets A and B such that every element of A is less than an element of B. Now, finally, the Archimedian axiom can be proved for the reals. Meanwhile, the algebraists introduced the quaternions (Hamilton) and later octonions (Graves and Cayley) as a further generalization of the complex numbers.

As Godefroy continues his story of the numbers, more mathematics enter the picture, and this book review is not the place to summarize them. Being half way in the book, we arrive in the 20th century, and this is probably

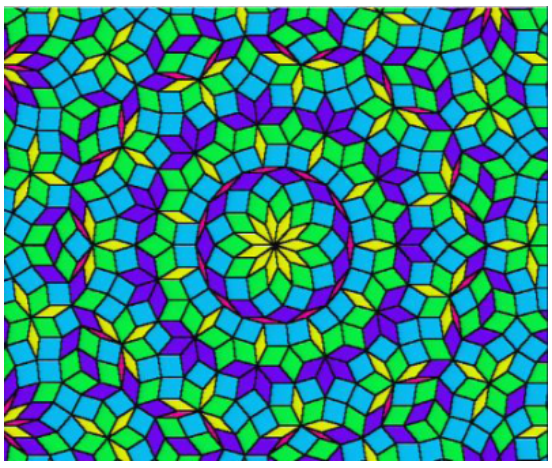
where the non-mathematician will loose track and where only the mathematicians will read on. And it is worth reading on because Godefroy succeeds in introducing the reader to all the major achievements in mathematics of the least century. Fermat's last theorem, Kummer's ideals, p-adic number systems,... A major step was set by Cantor who was far ahead of his time, and a separate chapter is devoted to him. He was so advanced that he was counteracted by many of his contemporaries which had to be silenced by the authority of Hilbert. Nevertheless this hostility led Cantor to depressions and he ended his days in a mental hospital. Cantor is the founder of set theory and Godefroy gives a step by step account of his diagonalization process, the introduction of transfinite numbers, the ordinals $\omega, \omega+1, \omega+2, \ldots, \omega\cdot 2, \omega\cdot 2+1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \varepsilon_0, \ldots$, and the cardinals: $0, 1, \ldots, \aleph_0, \aleph_1, \ldots$. He proved that for any set A, the power set 2^A of all subsets of A had more elements than the set A. This led to a contradiction since the set U of all sets would have a subset 2^U that is larger than U. Numbers had grown out of control.

So Hilbert in his famous speech at the ICM in Paris in 1900 set as one of his challenges to design a formal representation of calculus and set theory to avoid these inconsistencies. In an attempt to construct answers, Zermelo proved the well ordering of sets but thereby relied on the axiom of choice. This gives rise to the Banach-Tarski paradox: one may decompose a ball into finitely many pieces and recompose them to two balls with the same radius. The explanation is that the 'pieces' are abstract entities so that no volume can be assigned to them. Godefroy then moves on to the decision problem (conclude that a system is consistent or not), Gödel's theorem, non-Euclidean geometry, recursively enumerable sets and recursive sets, and the Robinson-Matijasevic theorem proved in 1970. It gives a negative answer to Hilbert's 10th problem asking for an automatic procedure in order to answer whether a diophantine equation has an integer solution. The author introduces the reader to model theory and nonstandard analysis.



Banach-Tarski paradox

In a concluding chapter entitled "And now?", Godefroy gives an update of current research near the end of the 20th century. Aperiodic tilings of the plane, fractals, dynamical systems, computer proofs, cryptography, the fast Fourier transform, Conway games and Conway numbers, quantum mechanics and Alain Connes's noncummutative geometry all get their couple of pages of fame here.



Aperiodic tiling in quasicrystal

A number of appendices and a glossary of terms and definitions contain some basic mathematical stuff to which is often referred in the text. There is however no index, which makes it difficult to look up names and events in the text. Another element that should be mentioned is that the author is regularly challenging the reader to check or elaborate on some formula. The solution of these is given at the end of the corresponding chapter.

Gilles Godefroy has a pleasant style of writing, showing a broad cultural background, although somewhat harmlessly biased towards French heritage. (Understandable given the fact that this is a translation of the French original.) As mentioned before, a mathematical layman will follow easily up to about halfway the book, but will leave the playground to mathematicians when the chapters dive into the foundations of set theory. For the mathematicians it should be clear from our summary that this is not a general history of mathematics. Godefroy clearly focusses on the numbers and the geometry, algebra, analysis, set theory and logic that is needed to pin down their properties.

Adhemar Bultheel