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Moving frames for the numerical solution of partial differential equations in complex domains. Computation using orthonormal basis vectors. (English) Zbl 08132501

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When solving an initial value problem describing for example a particle moving on a curved surface, and you obtained the solution at some point, then you want to predict where the particle will be a moment later, which requires to know the local situation (speed, curvature, etc.). That can be best described by a local, preferably orthogonal or even orthonormal reference frame, an idea dating back to Cartan. As the particle moves on, one wants the reference frame to move along. For a 2D surface, an obvious choice is to choose one vector as the normal to the surface and the other two in the tangent plane, one as a tangent vector to the trajectory of the particle, and the other vector in the tangent plane can be obtained as a vector product of the normal and the tangent to the curve.

This idea of moving frames and all the technicalities of representing and transforming the coordinates of vectors in different frames is the subject of Chapter 1, and the construction described above is one of the construction methods described in Chapter 2.

The variation of the local frames give information about the shape of the surface the properties of the medium (like anisotropy) that influences the trajectory. The (Levi-Civita) connection captures the rate of angular changes and defines properties of the wave trajectories (e.g. diverging or not). In finite element methods, piecewise smooth frames can be defined per element with possible discontinuities at the element boundaries. Spherical elements are given as an example.

Since the instantaneous change and the rate of change is important, one need to be able to compute covariant derivatives of scalars and vectors where 'covariant' means 'as one moves along a curve on the surface'. How to do that with moving frames is explained in Chapter 3. The coordinate vector fields of the coordinate system (x^1, x^2) in the tangent plane has as basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ representing tangent vectors. Their dot product is the geometric tensor g_{ij}. Taking covariant derivatives ∇_j, $j = 1, 2, 3$ of these will generate second order derivatives, captured by the Christoffel symbols Γ^k_{ij}. The sphere is worked out as an example, including an implementation in matlab code.

The last basic element missing to solve the PDEs is to connect differential operators with moving frames, which is discussed in Chapter 4.

With all the preparations being done, Chapter 5 then applies these tools for the solution of PDEs with as examples: conservation laws, diffusion-reaction processes, shallow water equations on a sphere, and Maxwell equations. Several pages of matlab code are included that are included in the open source spectral/hp element package NEKTAR++.

In the final Chapter 6, problems of wave propagation are discussed. The two vectors in the tangent plane are aligned with wave trajectory and wave front respectively. The relative acceleration and its connection with the Riemann curvature tensor is explained and conditions are obtained that may result in a conduction block. Several practical examples are included.

The material requires some knowledge of differential geometry and topology, and of course numerical methods for solving PDEs (Galerkin, spectral methods, finite elements,...) A small set of exercises is provided for every chapter, but the main challenge is to experiment with the NEKTAR++ package for some benchmark problems. That package includes the code that was listed and discussed in this book.

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MSC:

65-02 Research exposition (monographs, survey articles) pertaining to numerical analysis
65-04 Software, source code, etc. for problems pertaining to numerical analysis
65M50 Mesh generation, refinement, and adaptive methods for the numerical solution of initial value and initial-boundary value problems involving PDEs
58E15 Variational problems concerning extremal problems in several variables; Yang-Mills functionals

Keywords:

finite elements; initial value problem; differential geometry; differentiable manifold; covariant derivative; vector field; vector bundle; geometric tensor; Christoffel symbols; Riemann curvature tensor; Levi-Civita connection; shallow water equations; Maxwell equations; diffusion-reaction process; conservation law; wave front; matlab code; Nektar++;

Software:

Nektar++; Matlab

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