Rational Szegő quadratures associated with Chebyshev weight functions

Adhemar Bultheel *, Ruymán Cruz-Barroso, Karl Deckers [1]

Department of Computer Science, K.U.Leuven, Celestijnenlaan 200 A, B-3001 Leuven, Belgium.

Pablo González-Vera [2]

Department of Mathematical Analysis. La Laguna University. 38271 La Laguna. Tenerife. Canary Islands. Spain

Abstract

In this paper we characterize rational Szegő quadrature formulas associated with Chebyshev weight functions, by giving explicit expressions for the corresponding para-orthogonal rational functions and weights in the quadratures. As an application, we give characterizations for Szegő quadrature formulas associated with rational modifications of Chebyshev weight functions. Some numerical experiments are finally presented.

Key words: Rational Szegő quadrature formulas; Szegő quadrature formulas; orthogonal rational functions; Chebyshev weight functions.

1 Introduction

Szegő quadrature formulas approximate integrals over the complex unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and are the analogs of Gauss quadrature formulas that approximate integrals over an interval $I := [-1, 1]$. That is

$$\int f(z)d\mu(z) \approx \sum_{j=1}^{n} \lambda_j f(z_j),$$

where μ is a positive measure on $\mathbb{T}$, respectively I.

* Corresponding author.
Email address: Adhemar.Bultheel@cs.kuleuven.be (Adhemar Bultheel).

[1] The work of these authors is partially supported by the Fund of Scientific Research (FWO), project "RAM: Rational modelling: optimal conditioning and stable algorithms", grant ♯G.0423.05 and the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with the authors.
[2] The work of this author is partially supported by the research project MTM 2005-08571 of the Spanish Government.

Gauss quadrature formulas are optimal in the sense that nodes $z_j \in I$ and weights $\lambda_j > 0$ are chosen such that the quadrature formula is exact for all functions $f \in \mathbb{P}_{2n-1}$ that are polynomials of degree up to $2n-1$ and it is not possible to construct a quadrature formula of this form that is exact for all polynomials of degree $2n$. They have a maximal domain of validity. Therefore, the nodes have to be chosen as the zeros of the nth polynomial orthogonal with respect to the inner product $\langle f, g\rangle = \int f(z)g(z)d\mu(z)$. These zeros are all simple and in I.

On the unit circle, one needs to define a positive definite Hermitian inner product $\langle f, g\rangle = \int f(z)\overline{g(z)}d\mu(z)$. The nth orthogonal polynomial with respect to this inner product has all its zeros inside the open unit disk, and thus, they are not very useful as nodes of a quadrature formula. Here, the para-orthogonal polynomials are producing the nodes that are needed. Para-orthogonal polynomials are orthogonal to all polynomials of lower degree, except the constants. All their zeros are simple and on $\mathbb{T}$. These para-orthogonal polynomials have one free parameter $\tau \in \mathbb{T}$. When using the zeros of the nth para-orthogonal polynomials, then there exist positive weights such that the quadrature formula is exact for all $f \in \mathcal{R}_{n-1}$, that is the space of all Laurent polynomials of degree at most $n-1$, again a space of dimension $2n-1$. Like for Gaussian formulas, this space is a maximal domain of validity since it is not possible to have an n-point quadrature formula with distinct nodes and positive weights that is exact in a space of Laurent polynomials of a larger degree neither for the positive or the negative powers of z. The Szegő quadrature formulas were first studied by Jones-Njåstad-Thron [25] in connection with the trigonometric moment problem.

The Joukowski transform is a map between the unit circle and the interval $[-1, 1]$. It implies a relationship between the Szegő and the Gauss quadrature formulas. In fact this was studied already by Szegő [28, Section 11.5] and Geronimus [21, Chapter 9]. For $\tau = 1$, a $2n$-point Szegő formula results in an n-point Gauss quadrature formula and for $\tau = -1$, a $(2n+2)$-point Szegő formula can be related to an n-point Gauss-Lobatto formula, having two extra nodes in the end-points of the interval. Also Gauss-Radau formulas can be obtained taking $(2n+1)$-point Szegő formulas and $\tau = \pm 1$, which fixes one of the end-points of the interval; see [1]. For other values of τ, the quadrature on the interval is not optimal; see [2]. Szegő-Lobatto and Szegő-Radau formulas on the unit circle were recently discussed by Jagels and Reichel [24].

This theory has been generalized in a sequence of papers by Bultheel et al to the case where the (orthogonal) polynomials in the previous theory are replaced by (orthogonal) rational functions having prescribed poles outside the closed unit disk. If all these poles are at infinity, the polynomials reappear as a special case. For a comprehensive survey see [3]. See also [5] for a survey.

So the Szegő quadrature formulas are replaced by rational Szegő quadrature formulas. If $\mathcal{L}_n$ is the space of rational functions of degree n at most whose poles are at $\{1/\overline{\alpha}_k : k = 1, \ldots, n\}$ and $\mathcal{L}_{n*}$ is the space of rational functions of degree n at most whose poles are at $\{\alpha_k : k = 1, \ldots, n\}$, then the rational Szegő quadrature formulas are exact in the space $\mathcal{R}_{n-1} = \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)*}$. The nodes are the zeros of the nth rational para-orthogonal function, that depends, as in the polynomial case on a parameter $\tau \in \mathbb{T}$.

A relation between orthogonal rational functions on the interval and on the unit circle has been discussed in [15] and [32]. Rational Szegő-Lobatto and Szegő-Radau formulas have been recently studied in [7].

Gauss-Chebyshev quadrature formulas are associated with a measure that is generated by one of the Chebyshev weights. These weights are among the rare examples where a translation to the unit circle give explicit expressions for the orthogonal polynomials. This is also true for the orthogonal rational functions, which have been under investigation with respect to rational Gauss-Chebyshev quadrature; see e.g. [16], [17], [18], [20], [29] and [30]. To get more general cases where explicit expressions are obtained, a technique of rational modifications of these Chebyshev weights are considered in [14].

In this paper we will continue in this line of developments by considering the construction of explicit expressions for the para-orthogonal rational functions and for the weights in rational Szegő quadrature formulas that result from Chebyshev weights as they are translated into the unit circle. As an application, characterization results for Szegő quadrature formulas associated with rational modifications of these Chebyshev weights are obtained.

The outline of this paper is as follows. After giving the theoretical preliminaries in Section 2, in Section 3 we give explicit expressions for the orthogonal rational functions on the unit circle, associated with Chebyshev weight functions. Next, in Section 4 and 5 we characterize rational Szegő quadrature formulas associated with Chebyshev weight functions, respectively Szegő quadrature formulas associated with rational modifications of Chebyshev weight functions. We conclude with some numerical examples.

2 Preliminaries

Suppose μ is a positive bounded Borel measure on $[-\pi, \pi]$, and consider the general framework of the approximation of integrals on the unit circle $\mathbb{T}$ in the complex plane, i.e. integrals of the form [3]

$$I_\mu(f) := \int_{-\pi}^{\pi} f(e^{\mathbf{i}\theta}) d\mu(\theta) = \int_{\mathbb{T}} f(z) d\mu(z). \tag{1}$$

As usual, estimations of $I_\mu(f)$ are produced when replacing $f(z)$ in (1) by an appropriate approximating (interpolating) function $L(z)$, so that $I_\mu(L)$ can now be easily computed. Let $\Lambda := \mathbb{C}[z, z^{-1}]$ denote the complex vector space of Laurent polynomials in the variable z. We then set $\Lambda_{p,q} := \text{span}\{z^p, \ldots, z^q\}$ for $p, q \in \mathbb{Z}$, with $p \leq q$. Because of the density of Λ in $C(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, f \text{ is continuous}\}$ with respect to the uniform norm (see e.g. [13, pp. 304-305]), it seems reasonable to approximate $f(z)$ in (1) by an appropriate Laurent polynomial. This way, the so-called "quadrature formulas on the unit circle", or "Szegő rules", introduced in [25] (see also [22], [23, Chapter 4] and [27]) and of the form

$$I_n(f) := \sum_{j=1}^{n} \lambda_j f(z_j), \quad z_j \in \mathbb{T}, \quad j = 1, \ldots, n, \quad z_j \neq z_k \text{ if } j \neq k, \tag{2}$$

appear as the analogue on $\mathbb{T}$ of the Gaussian formulas when dealing with the estimation of integrals with respect to a measure supported on an interval $[a, b]$, with $-\infty \leq a < b \leq +\infty$ (see e.g. [19]).

Consider now the Hilbert space $L_2^\mu(\mathbb{T})$ of measurable functions ϕ for which $\int_{-\pi}^{\pi} |\phi(e^{\mathbf{i}\theta})|^2 d\mu(\theta) < +\infty$. Then the inner product induced by μ is given by

$$\langle \phi, \psi \rangle_\mu = \int_{-\pi}^{\pi} \phi\left(e^{\mathbf{i}\theta}\right) \overline{\psi\left(e^{\mathbf{i}\theta}\right)} d\mu(\theta) \quad , \quad \phi, \psi \in L_2^\mu(\mathbb{T}). \tag{3}$$

[3] The measure μ on $[-\pi, \pi]$ induces a measure on $\mathbb{T}$ for which we shall use the same notation μ.

In this paper we will deal with the more general framework of orthogonal rational functions (ORFs). Suppose a sequence of complex numbers $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{D}$ is given, and define the *Blaschke factors*

$$\zeta_k(z) = \frac{z - \alpha_k}{1 - \overline{\alpha_k} z} \ , \quad k = 1, 2, \ldots ,$$

and *Blaschke products*

$$B_0(z) \equiv 1 \ , \quad B_k(z) = B_{k-1}(z)\zeta_k(z) \ , \quad k = 1, 2, \ldots .$$

The space of rational functions with poles in $\{1/\overline{\alpha_1}, \ldots, 1/\overline{\alpha_n}\}$ is then defined as

$$\mathcal{L}_n = \operatorname{span}\{B_0(z), \ldots, B_n(z)\} ,$$

where we set $\mathcal{L}_{-1} = \emptyset$ to be the trivial subspace. This way, the ordinary polynomial situation is recovered by taking $\alpha_k = 0$ for every $k = 1, 2, \ldots$, so that $\zeta_k(z) = z$ and $B_k(z) = z^k$.

We define the substar conjugate of a function f as $f_*(z) = \overline{f(1/\overline{z})}$, and the super-star conjugation of a function $f_n \in \mathcal{L}_n$ as

$$f_n^*(z) = B_n(z) f_{n*}(z).$$

Note that $f_*(z) = \overline{f(z)}$ whenever $z \in \mathbb{T}$. Further, for a given polynomial $P_n(z) = \sum_{k=0}^n c_k z^k$ of exact degree n, the super-star conjugate [4] is given by $P_n^*(z) = \sum_{k=0}^n \overline{c_{n-k}} z^k$. Consequently, define

$$\pi_0(z) \equiv 1 \ , \quad \pi_k(z) = \prod_{j=1}^{k} (1 - \overline{\alpha_j} z) \ , \quad k = 1, 2, \ldots ,$$

and let $\mathbb{P}_n$ represent the space of polynomials of degree less than or equal to n. Then, we equivalently have for every $k = 1, 2, \ldots$ that

$$B_k(z) = \frac{\pi_k^*(z)}{\pi_k(z)}$$

and

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \mathbb{P}_n \right\} .$$

For a fixed natural number n we obtain a set of orthonormal rational functions $\{\chi_k(z)\}_{k=0}^n$ by orthonormalizing the basis $\{B_k(z)\}_{k=0}^n$ (in this order) with respect to the measure μ and inner product given by (3). Note that $\chi_n(z) = \sum_{k=0}^n a_k B_k(z)$ is uniquely determined if we assume the *leading coefficient* a_n to be strictly positive. Repeating the process for every n, an orthonormal system $\{\chi_k(z)\}_{k=0}^\infty$ is obtained, so that $\chi_n \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$, $\chi_n \perp \mathcal{L}_{n-1}$ and $\langle \chi_n, \chi_n \rangle_\mu = 1$ for every $n \geq 0$. We now have the following lemma.

Lemma 2.1 *Let $\chi_n(z) = \frac{P_n(z)}{\pi_n(z)} \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$. Then P_n has exactly n zeros in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.*

Proof.- Since all the zeros of χ_n are in $\mathbb{D}$ because of [3, Corollary 3.2.2(3)], a zero of P_n that is not a zero of χ_n is only possible if it cancels a zero of π_n. Suppose there exists an $\alpha \in \mathbb{D}$ such that $1/\overline{\alpha}$ is a zero of P_n with multiplicity $m \geq 1$. For the special case in which $\alpha = 0$, we say that P_n has a zero at infinity with multiplicity m iff $P_n \in \mathbb{P}_{n-m}$. Hence it follows that there are at least m indices $j \in \{1, \ldots, n\}$ for which $\alpha_j = \alpha$.

[4] In literature, this is also referred to as the reversed or reciprocal polynomial.

Whatever the choice of the sequence of α_k is, it should always hold that $\chi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$. Now take all α_k that are different from α fixed, and consider α variable. Note that χ_n depends continuously on α, as do P_n and π_n. So let us make this explicit by writing $\chi_n(z,\alpha) = P_n(z,\alpha)/\pi_n(z,\alpha)$. Since $P_n(z,\alpha)$ is of the form

$$P_n(z,\alpha) = (1-\overline{\alpha}z)^m \hat{P}_{n-m}(z,\alpha) \quad \text{with} \quad \hat{P}_{n-m} \in \mathbb{P}_{n-m},$$

we obtain for $\alpha \to \alpha_n$

$$\begin{aligned}\chi_n(z,\alpha_n) &= \lim_{\alpha\to\alpha_n} \frac{(1-\overline{\alpha}z)^m \hat{P}_{n-m}(z,\alpha)}{\pi_n(z,\alpha)} \\ &= \frac{(1-\overline{\alpha_n}z)^{m-1} \hat{P}_{n-m}(z,\alpha_n)}{\pi_{n-1}(z,\alpha_n)} \in \mathcal{L}_{n-1},\end{aligned}$$

contradicting our assumption that $\chi_n(z,\alpha) \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ for all possible choices of $\alpha \in \mathbb{D}$. □

Next, consider for every $n,p,q \geq 0$ the sets

$$\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\},$$

$$\mathcal{L}_{p,q} = \mathcal{L}_q \cdot \mathcal{L}_{p*} = \{fg : f \in \mathcal{L}_q, g \in \mathcal{L}_{p*}\} = \left\{ \frac{P(z)}{\pi_q(z)\pi_p^*(z)} : P \in \mathbb{P}_{p+q} \right\}$$

and $\mathcal{R}_n = \mathcal{L}_{n,n}$. Following the ordinary polynomial situation (see [25]), we say that a sequence of functions $\Psi_n(z) \in \mathcal{L}_n$ is *para-orthogonal* whenever $\Psi_n(z) \perp \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, where $\mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}$, and $\langle \Psi_n(z), 1\rangle_\mu \cdot \langle \Psi_n(z), B_n(z)\rangle_\mu \neq 0$. Furthermore, $\Psi_n(z)$ is called "κ_n-invariant" iff there exists a $\kappa_n \in \mathbb{T}$ so that $\Psi_n^*(z) = \kappa_n \Psi_n(z)$ for every $z \in \mathbb{C}$. Note that the concept of κ_n-invariance is usually defined in literature for $\kappa_n \in \mathbb{C}\backslash\{0\}$. However, there can only exist κ_n-invariant rational functions whenever $\kappa_n \in \mathbb{T}$. Indeed, if $\Psi_n(z) = \frac{P_n(z)}{\pi_n(z)} \in \mathcal{L}_n$ is κ_n-invariant, then $\Psi_n^*(z) = \frac{P_n^*(z)}{\pi_n(z)} = \frac{\kappa_n P_n(z)}{\pi_n(z)}$. Consequently, it remains to prove the statement for ordinary polynomials. If $P_n(z) = \sum_{k=0}^n c_k z^k$, with $c_n \neq 0$, is κ_n-invariant, then it holds that $c_0 = P(0) \neq 0$, $\overline{c_n} = \kappa_n c_0$ and $\overline{c_0} = \kappa_n c_n$. Consequently, $\kappa_n = \frac{\overline{c_n}}{c_0} = \frac{\overline{c_0}}{c_n}$, which implies that $|c_n| = |c_0|$, and hence, that $\kappa_n \in \mathbb{T}$.

κ_n-Invariant para-orthogonal rational functions for μ are characterized in [4] as

$$\Psi_n(z) = \Psi_n(z,\tau_n) = C_n\left[\phi_n(z) + \tau_n \phi_n^*(z)\right] \in \mathcal{L}_n \ ; \ n \geq 1, \ C_n \in \mathbb{C}\backslash\{0\}, \tag{4}$$

where $\tau_n = \frac{\overline{C_n \kappa_n}}{C_n} \in \mathbb{T}$, and $\phi_n(z)$ is an nth orthogonal rational function for μ. Furthermore, it is proved in [7, Theorem 2.4] that it suffices to compute $\phi_{n-1}(z)$ for the computation of $\Psi_n(z)$.

The following result, proved by Bultheel et. al. (see [3, Chapter 5]), is an extension of a well known characterization for Szegő quadrature formulas (see [25]) to the rational case.

Theorem 2.2 (Rational Szegő quadrature) *Let $n \geq 1$ and $\tau_n \in \mathbb{T}$. Then,*

(1) $\Psi_n(z,\tau_n)$, given by (4), has exactly n distinct zeros on $\mathbb{T}$,
(2) there exist positive numbers $\lambda_1, \ldots, \lambda_n$ so that

$$I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) = I_\mu(f) = \int_{-\pi}^{\pi} f(e^{\mathrm{i}\theta}) d\mu(\theta), \ \forall f \in \mathcal{R}_{n-1}, \tag{5}$$

where $z_1, \ldots, z_n$ *are the zeros of* $\Psi_n(z, \tau_n)$,

(3) $\mathcal{R}_{n-1}$ *is a maximal domain of validity, i.e. there can not be exactness neither in* $\mathcal{L}_{n-1,n}$ *nor in* $\mathcal{L}_{n,n-1}$.

□

A connection between quadrature formulas on the unit circle and the interval $[-1, 1]$ is given in [1]. If $\sigma(x)$ is a weight function on $[-1, 1]$, we obtain a weight function on $\mathbb{T}$ by setting $\mu'(\theta) = \omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ (see [28]), where μ' denotes the Radon-Nikodym derivative of the measure μ with respect to the Lebesgue measure. In the special case in which $\sigma(x)$ is a Jacobi weight function, i.e. $\sigma(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1$, the corresponding weight function on $\mathbb{T}$ is given by

$$\begin{aligned}\omega(\theta) &= (1-\cos\theta)^\alpha(1+\cos\theta)^\beta\,|1-\cos^2\theta|^{\frac{1}{2}} \\ &= (1-\cos\theta)^a(1+\cos\theta)^b\,;\ \ a = \alpha+\tfrac{1}{2} > -\tfrac{1}{2},\ b = \beta+\tfrac{1}{2} > -\tfrac{1}{2}.\end{aligned}$$

Finally, if $a, b \in \{0, 1\}$, the so-called Chebyshev weight functions appear. Therefore, we set

$$\omega_1(\theta) \equiv 1 \quad , \quad \omega_2(\theta) = \sin^2\theta\ , \quad \omega_3(\theta) = 1-\cos\theta \quad , \quad \omega_4(\theta) = 1+\cos\theta. \tag{6}$$

In the remainder of this paper we then shall be concerned with rational Szegő quadratures associated with the Chebyshev weight functions $\omega(\theta) = \omega_i(\theta)$, $i = 1, \ldots, 4$ given by (6). We start in the next section with giving explicit expressions for the corresponding ORFs.

3 ORFs associated with Chebyshev weight functions

As it is known, few measures give rise to explicit expressions for orthogonal polynomials and even less for ORFs; generally, the computation of such a family proceeds by using a recursive process (see e.g. [3, Theorem 4.1.1]).

It is well known that the so-called *Malmquist basis*, given by

$$\phi_0^{(1)}(z) \equiv 1 \quad \text{and} \quad \phi_n^{(1)}(z) = \frac{zB_{n-1}(z)}{1-\overline{\alpha_n}z}\ , \ \ n > 0, \tag{7}$$

is an orthogonal basis for the Lebesgue measure $d\mu(\theta) = \omega_1(\theta)d\theta \equiv d\theta$ (see e.g. [3, pp. 51]). Recently, explicit expressions are derived in [15] for ORFs associated with the weight functions $\omega_i(\theta)$, $i \in \{3, 4\}$, given by (6). Let $Q_n(z)$ be defined as

$$Q_n(z) = 1 + z\frac{B_{n-1}'(z)}{B_{n-1}(z)} = 1 + \sum_{k=1}^{n-1}\frac{1-|\alpha_k|^2}{|z-\alpha_k|^2} \quad \text{for} \quad z \in \mathbb{T}. \tag{8}$$

We then have the following theorem ([15]).

Theorem 3.1 *Let* $i \in \{3, 4\}$ *be fixed and set* $\nu_i = (-1)^{i-1}$. *Next, define*

$$X_n^{(i)}(z) = a_n^{(i)} + z^2\left(z - b_n^{(i)}\right)\frac{B_{n-1}(z)}{1-\overline{\alpha_n}z}\ , \tag{9}$$

where

$$a_n^{(i)} = \frac{\nu_i B_{n-1}(\nu_i)}{(1-\nu_i\overline{\alpha_n})Q_n^{(i)}+1}\ ,\quad b_n^{(i)} = \nu_i + \frac{1-\nu_i\overline{\alpha_n}}{B_{n-1}(\nu_i)}a_n^{(i)} \tag{10}$$

and $Q_n^{(i)} = Q_n(\nu_i)$ *(note that this way,* $X_n^{(i)}(\nu_i) = X_n^{(i)'}(\nu_i) = 0$*). Then the sequence of rational functions* $\left\{\phi_n^{(i)}(z)\right\}_{n=0}^{\infty}$*, where*

$$\phi_0^{(i)}(z) \equiv 1 \quad \textit{and} \quad \phi_n^{(i)}(z) = \frac{X_n^{(i)}(z)}{(z-\nu_i)^2}\ ,\ \ n>0, \tag{11}$$

forms a set of ORFs with respect to $\omega_i(\theta) = 1-\nu_i\cos\theta$*. Furthermore, the sequence* $\{\chi_n^{(i)}(z)\}_{n=0}^{\infty}$*, with* $\chi_n^{(i)}(z) = c_n^{(i)}\phi_n^{(i)}(z)$*, where*

$$\left|c_0^{(i)}\right|^2 = \frac{1}{2\pi} \quad \textit{and} \quad \left|c_n^{(i)}\right|^2 = \nu_i\frac{(1-|\alpha_n|^2)(1-\nu_i\alpha_n)^2}{\pi\left(b_n^{(i)}-\alpha_n\right)\left(1-\alpha_n\overline{b_n^{(i)}}\right)}\ ,\ \ n>0, \tag{12}$$

forms a set of orthonormal rational functions with respect to $\omega_i(\theta)$*.*

□

So far, explicit expressions for ORFs associated with the weight function $\omega_2(\theta)$ in (6) are still not known (as indicated in [15]). So, in the remainder of this section we will deal with this open problem.

Note that

$$\begin{aligned}\omega_2(\theta) &= \sin^2\theta = (1+\cos\theta)(1-\cos\theta) = \left(1+\frac{z^2+1}{2z}\right)(1-\cos\theta)\\ &= \frac{1}{2}|z+1|^2(1-\cos\theta) = \frac{1}{2}|z+1|^2\omega_3(\theta), \quad z = e^{\mathbf{i}\theta}.\end{aligned}$$

Hence, suppose $\phi_{n+1}^{(3)}$ is a rational function with poles in $\{0,\alpha_1,\ldots,\alpha_n\}$ that is orthogonal on the unit circle with respect to the weight function $\omega_3(\theta)$. Further, let $\phi_n^{(2)}$ be a rational function with poles in $\{\alpha_1,\ldots,\alpha_n\}$ that is orthogonal on the unit circle with respect to the weight function $\omega_2(\theta)$. Then for $n>0$, it follows from [14, Theorem 6] that there exist constants u_n, t_n and v_n so that

$$(z+1)^2\phi_n^{(2)}(z) = (u_nz+t_n)\phi_{n+1}^{(3)}(z) + v_n(1-\overline{\alpha_n}z)\phi_{n+1}^{(3)*}(z). \tag{13}$$

We now have the following two theorems.

Theorem 3.2 *Suppose*

$$\begin{aligned} a^+ &= \tfrac{B_{n-1}(1)}{1-\overline{\alpha_n}}, \qquad a^- = \tfrac{B_{n-1}(-1)}{1+\overline{\alpha_n}},\\ b^+ &= 1+Q_n^{(3)}+\tfrac{1}{1-\overline{\alpha_n}},\ b^- = 1+Q_n^{(4)}+\tfrac{1}{1+\overline{\alpha_n}},\end{aligned} \tag{14}$$

where $Q_n^{(3)}$ *and* $Q_n^{(4)}$ *are as defined before in Theorem 3.1. Next, let* $\underline{x} = (d_n,e_n,f_n,g_n)^T$*,* $\underline{y} =$

$(-a^+, a^-, -a^+(b^+ + 2), a^-(b^- + 2))^T$ *and*

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & a^+ & a^+ \\ 1 & -1 & -a^- & a^- \\ 0 & 1 & a^+b^+ & a^+(b^+ + 1) \\ 0 & -1 & -a^-b^- & a^-(b^- + 1) \end{pmatrix},$$

and assume

$$\underline{x} = \boldsymbol{A}^{-1}\underline{y} = \frac{1}{\varrho_n} \begin{pmatrix} 4a^+a^-(b^+a^- - b^-a^+) \\ -4a^+a^-[a^+(b^- + 1) + a^-(b^+ + 1)] \\ (a^+ + a^-)^2 + 4a^+a^-(b^+ + b^- + b^+b^-) \\ -2[2a^+a^-(b^+ - b^-) + (a^+ + a^-)(a^+ - a^-)] \end{pmatrix}, \tag{15}$$

where ϱ_n *is given by*

$$\varrho_n = (a^+ + a^-)^2 - 4a^+a^-b^+b^- \ . \tag{16}$$

Define $X_n^{(2)}$ *by*

$$X_n^{(2)}(z) = d_n + e_n z + z^3 \left(f_n + g_n z + z^2\right) \frac{B_{n-1}(z)}{1 - \overline{\alpha_n} z} \ , \tag{17}$$

and set

$$\phi_0^{(2)}(z) \equiv 1 \quad \textit{and} \quad \phi_n^{(2)}(z) = \frac{X_n^{(2)}(z)}{(z^2 - 1)^2}.$$

Then the sequence of rational functions $\left\{\phi_n^{(2)}(z)\right\}_{n=0}^{\infty}$ *forms a set of ORFs with respect to* $\omega_2(\theta) = \sin^2\theta$.

Proof.- (The computations are cumbersome; therefore, we will only give the outline of the proof.) From Theorem 3.1 and (13) it follows that for $n > 0$, $\phi_n^{(2)}(z)$ should be of the form

$$\phi_n^{(2)}(z) = C_n \frac{X_n^{(2)}(z)}{(z^2 - 1)^2}, \quad C_n \neq 0,$$

where $X_n^{(2)}$ is given by (17). For the sake of simplicity, we may as well assume that $C_n = 1$. Furthermore, we should have that $\phi_n^{(2)} \in \mathcal{L}_n$. Consequently, it must hold that $X_n^{(2)}(\pm 1) = X_n^{(2)\prime}(\pm 1) = 0$. This leaves us with a system of four equations in the four unknowns d_n, e_n, f_n and g_n. Solving this system for the coefficients d_n, e_n, f_n and g_n then gives (15). The analytic solution of (15), given by the second equality in (15) and (16), has been computed with the aid of Maple 10 [5] .

□

Theorem 3.3 *The sequence* $\{\chi_n^{(2)}(z)\}_{n=0}^{\infty}$, *with* $\chi_n^{(2)}(z) = h_n\phi_n^{(2)}(z)$, *where*

$$|h_0|^2 = \frac{1}{\pi}, \quad |h_n|^2 = \frac{-2(1 - |\alpha_n|^2)(1 - \alpha_n^2)^2}{\pi(f_n + g_n\alpha_n + \alpha_n^2)(\overline{f_n}\alpha_n^2 + \overline{g_n}\alpha_n + 1)} \ , \quad n > 0, \tag{18}$$

and $\phi_n^{(2)}(z)$ *is as defined before in Theorem 3.2, forms a set of orthonormal rational functions with respect to* $\omega_2(\theta) = \sin^2\theta$.

[5] Maple and Maple V are registered trademarks of Waterloo Maple, Inc.

Proof.- The expression for $n = 0$ is easily verified; so, we continue for $n > 0$. First, note that

$$\sin^2(\theta)d\theta = -(z^2-1)^2\frac{dz}{4\mathbf{i}z^3}, \quad z = e^{\mathbf{i}\theta}.$$

Hence, we have that

$$\begin{aligned}
\left\langle \chi_n^{(2)}, B_n \right\rangle_{\omega_2} &= -\frac{1}{4\mathbf{i}}\int_{\mathbb{T}} \chi_n^{(2)}(z)B_{n*}(z)(z^2-1)^2\frac{dz}{z^3} \\
&= -\frac{h_n}{4\mathbf{i}}\left[\int_{\mathbb{T}}\frac{(d_n+e_nz)B_{n*}(z)}{z^3}dz + \int_{\mathbb{T}}\frac{(f_n+g_nz+z^2)B_{n-1}(z)}{(1-\overline{\alpha_n}z)B_n(z)}dz\right] \\
&= -\frac{h_n}{4\mathbf{i}}\left[-\int_{\mathbb{T}}\frac{(d_n+e_n\overline{z})B_{n*}(\overline{z})}{\overline{z}^3}d\overline{z} + \int_{\mathbb{T}}\frac{f_n+g_nz+z^2}{z-\alpha_n}dz\right] \\
&= -\frac{h_n}{4\mathbf{i}}\left[\int_{\mathbb{T}}(d_nz+e_n)\overline{B_n(\overline{z})}dz + \int_{\mathbb{T}}G_n(z)dz\right] \\
&= -\frac{h_n}{4\mathbf{i}}\left[\int_{\mathbb{T}}F_n(z)dz + \int_{\mathbb{T}}G_n(z)dz\right].
\end{aligned}$$

Clearly, F_n is analytic in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, so that

$$\begin{aligned}
\left\langle \chi_n^{(2)}, B_n \right\rangle_{\omega_2} &= -\frac{h_n}{4\mathbf{i}}\int_{\mathbb{T}}G_n(z)dz = -\frac{h_n}{4\mathbf{i}}\int_{\mathbb{T}}\frac{f_n+g_nz+z^2}{z-\alpha_n}dz \\
&= -\frac{h_n\pi}{2}\mathrm{Res}\left\{\frac{f_n+g_nz+z^2}{z-\alpha_n}, \alpha_n\right\} = -\frac{h_n\pi(f_n+g_n\alpha_n+\alpha_n^2)}{2}.
\end{aligned}$$

Finally, suppose $\chi_n^{(2)}$ is of the form

$$\chi_n^{(2)}(z) = \sum_{k=0}^{n} a_k B_k(z).$$

Then it is easily verified that $\left\langle \chi_n^{(2)}, B_n \right\rangle_{\omega_2} = \frac{1}{\overline{a_n}}$, with

$$\overline{a_n} = \chi_n^{(2)*}(\alpha_n) = \frac{\overline{h_n}(\overline{f_n}\alpha_n^2 + \overline{g_n}\alpha_n + 1)}{(1-|\alpha_n|^2)(1-\alpha_n^2)^2}.$$

Consequently,

$$|h_n|^2 = -\frac{2(1-|\alpha_n|^2)(1-\alpha_n^2)^2}{\pi(f_n+g_n\alpha_n+\alpha_n^2)(\overline{f_n}\alpha_n^2+\overline{g_n}\alpha_n+1)},$$

which ends the proof.

□

Finally, we will also need the following lemma.

Lemma 3.4 *It holds that*

$$d_n + e_n\epsilon \neq 0,$$

where $\epsilon \in \{\pm 1\}$, and d_n and e_n are as defined before in Theorem 3.2.

Proof.- Suppose $d_n + e_n\epsilon = 0$ for a fixed $\epsilon \in \{\pm 1\}$. We then have that

$$d_n + e_n z = e_n(z - \epsilon).$$

Since a^+ and a^-, given by (14), are different from zero, and $X_n^{(2)}(\epsilon) = 0$, we also have that

$$(f_n + g_n z + z^2) = (z - \epsilon)(z - f_n\epsilon).$$

Consequently,

$$X_n^{(2)}(z) = (z - \epsilon)\left[e_n + z^3(z - f_n\epsilon)\frac{B_{n-1}(z)}{1 - \overline{\alpha_n} z}\right] = (z - \epsilon)\hat{X}_n(z).$$

Note that for $z \in \mathbb{T}$ it holds that

$$\hat{X}_n'(z) = z^2\frac{B_{n-1}(z)}{1 - \overline{\alpha_n} z}\left[z\left(2 + Q_n(z) + \frac{1}{1 - \overline{\alpha_n}}\right) - f_n\epsilon\left(1 + Q_n(z) + \frac{1}{1 - \overline{\alpha_n}}\right)\right],$$

where $Q_n(z)$ is as defined before in (8), so that

$$\begin{cases} X_n^{(2)'}(1) &= 0 \\ X_n^{(2)'}(-1) &= 0 \end{cases} \Leftrightarrow \begin{cases} \hat{X}_n'(1) &= 0 \\ \hat{X}_n'(-1) &= 0 \end{cases} \Leftrightarrow \begin{cases} a^+\left[(1 + b^+) - f_n\epsilon b^+\right] = 0 \\ a^-\left[(1 + b^-) + f_n\epsilon b^-\right] = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} f_n\epsilon = 1 + \frac{1}{b^+} \\ f_n\epsilon = -\left(1 + \frac{1}{b^-}\right) \end{cases} \Leftrightarrow \begin{cases} f_n\epsilon &= 1 + \frac{1}{b^+} \\ \frac{1}{b^+} + \frac{1}{b^-} &= -2 \end{cases}.$$

Furthermore, since $\alpha_n \in \mathbb{D}$, we have that $0 < \Re\left\{\frac{1}{1 \pm \alpha_n}\right\} < \infty$. From (8) and (14) we then deduce that $2 < \Re\{b^+\}, \Re\{b^-\} < \infty$, so that

$$-2 = \Re\left\{\frac{1}{b^+} + \frac{1}{b^-}\right\} = \Re\left\{\frac{1}{b^+}\right\} + \Re\left\{\frac{1}{b^-}\right\} > 0.$$

This is a contradiction, and hence, $d_n + e_n\epsilon \neq 0$.

□

4 Rational Szegő quadratures associated with Chebyshev weight functions

The aim of this section is to characterize rational Szegő formulas $I_n(f)$, given by (5), associated with the Chebyshev weight functions $\omega_i(\theta)$, $i = 1, \ldots, 4$, given by (6). Therefore, we will derive explicit expressions for the associated para-orthogonal rational functions $\Psi_n(z, \tau_n)$ by means of (4) along with the results provided in the previous section. The zeros $\{z_j\}_{j=1}^n$ of $\Psi_n(z, \tau_n)$ are the nodes we need for $I_n(f)$. First, let us consider the reproducing kernel function for $\mathcal{L}_n$ associated with a general measure μ, namely

$$\mathcal{K}_n(z, \xi) = \sum_{k=0}^{n} \chi_k(z)\overline{\chi_k(\xi)}.$$

The following Christoffel-Darboux formula has been proved in [3, Theorem 3.1.3]:

$$\mathcal{K}_n(z,\xi) = \frac{\chi^*_{n+1}(z)\overline{\chi^*_{n+1}(\xi)} - \chi_{n+1}(z)\overline{\chi_{n+1}(\xi)}}{1-\zeta_{n+1}(z)\overline{\zeta_{n+1}(\xi)}},\ n \geq 1. \tag{19}$$

Moreover, the following well known expression for the weights in a Szegő quadrature formula has been proved for the rational case in [3, Theorem 5.4.2]:

$$\lambda_j = \frac{1}{\sum_{k=0}^{n-1} |\chi_k(z_j)|^2}\ ;\ \ j = 1,\ldots,n. \tag{20}$$

We are now able to prove the following proposition.

Proposition 4.1 *Let $\{z_j\}_{j=1}^n$ and $\{\lambda_j\}_{j=1}^n$ represent the set of nodes and weights of a rational Szegő quadrature formula (5) for μ, and suppose $\chi_n(z)$ is the corresponding n^{th} orthonormal rational function. It then holds that*

$$\lambda_j^{-1} = \frac{z_j\,|z_j-\alpha_n|^2}{1-|\alpha_n|^2}\begin{vmatrix} \overline{\chi_n(z_j)} & \overline{\chi^*_n(z_j)} \\ (\chi^*_n)'(z_j) & \chi'_n(z_j) \end{vmatrix},\ \ j = 1,\ldots,n. \tag{21}$$

Proof.- From (19) and (20) it follows that

$$\lambda_j^{-1} = \lim_{\substack{z\to z_j \\ z\in\mathbb{T}}} \mathcal{K}_{n-1}(z,z_j) = \frac{(\chi^*_n)'(z_j)\overline{\chi^*_n(z_j)} - \chi'_n(z_j)\overline{\chi_n(z_j)}}{-\zeta'_n(z_j)\overline{\zeta_n(z_j)}}. \tag{22}$$

The statement is now easily verified by taking into account that $\overline{\zeta_n(z)} = 1/\zeta_n(z)$ and $\zeta'_n(z) = \frac{1-|\alpha_n|^2}{z|z-\alpha_n|^2}\zeta_n(z)$ for $z \in \mathbb{T}$.

□

We start with the Lebesgue measure $d\mu(\theta) = \omega_1(\theta)d\theta \equiv d\theta$. From (4) and (7) it follows that, up to a multiplicative factor,

$$\Psi_n^{(1)}(z,\tau_n) = \frac{V_n^{(1)}(z,\tau_n)}{\pi_n(z)};\quad z,\tau_n \in \mathbb{T},$$

where

$$V_n^{(1)}(z,\tau_n) = p_0^{(1)}(z)\pi_{n-1}(z) + \tau_n z p_0^{(1)*}(z)\pi^*_{n-1}(z) \tag{23}$$

and

$$p_0^{(1)}(z) \equiv \tau_n. \tag{24}$$

Moreover, for this special case it is well known that (see [6])

$$\lambda_j^{(1)} = \frac{2\pi}{Q_n\left(z_j^{(1)}\right)}\ ,\ \ j = 1,\ldots,n, \tag{25}$$

with $Q_n(z)$ given by (8). Indeed, let $\chi_n^{(1)}(z) = k_n\phi_n^{(1)}(z)$ with $\phi_n^{(1)}(z)$ given by (7) and $|k_n|^2 = \frac{1-|\alpha_n|^2}{2\pi}$ for $n \geq 1$. Then it follows from Proposition 4.1 that

$$\lambda_j^{(1)} = \frac{1-|\alpha_n|^2}{z_j^{(1)}\left|z_j^{(1)}-\alpha_n\right|^2} \times$$
$$\left| \begin{array}{cc} \frac{\overline{k_n}}{\left(z_j^{(1)}-\alpha_n\right)B_{n-1}\left(z_j^{(1)}\right)} & \frac{\overline{k_n}z_j^{(1)}}{\left(z_j^{(1)}-\alpha_n\right)} \\ -\frac{k_n\overline{\alpha_n}}{\left(1-\overline{\alpha_n}z_j^{(1)}\right)^2} & \frac{k_nB_{n-1}\left(z_j^{(1)}\right)Q_n\left(z_j^{(1)}\right)}{\left(1-\overline{\alpha_n}z_j^{(1)}\right)} - \frac{k_n\overline{\alpha_n}z_j^{(1)}B_{n-1}\left(z_j^{(1)}\right)}{\left(1-\overline{\alpha_n}z_j^{(1)}\right)^2} \end{array} \right|^{-1}$$
$$= \frac{1-|\alpha_n|^2}{z_j^{(1)}\left|z_j^{(1)}-\alpha_n\right|^2} \times \left[\frac{|k_n|^2Q_n\left(z_j^{(1)}\right)}{\left(z_j^{(1)}-\alpha_n\right)\left(1-\overline{\alpha_n}z_j^{(1)}\right)}\right]^{-1} = \frac{1-|\alpha_n|^2}{|k_n|^2Q_n\left(z_j^{(1)}\right)}.$$

Thus, we have the following theorem.

Theorem 4.2 *Let $n \geq 1$ and $\tau_n \in \mathbb{T}$. The nodes of an n-point rational Szegő quadrature $I_n(f)$ for $I_{\omega_1}(f)$, given by (5) and (6), are then the zeros of $V_n^{(1)}(z,\tau_n)$ given by (23)–(24), while the weights are given by (25).*

□

Next, consider the weight functions $\omega_i(\theta) = 1-\nu_i\cos\theta$, with $i \in \{3,4\}$ and $\nu_i = (-1)^{i-1}$. From (4) and Theorem 3.1 it follows that, up to a multiplicative constant,

$$\Psi_n^{(i)}(z,\tau_n) = \frac{V_n^{(i)}(z,\tau_n)}{(z-\nu_i)^2\pi_n(z)}\ ;\ \ i=3,4\ ,\ \ z,\tau_n \in \mathbb{T},$$

where

$$V_n^{(i)}(z,\tau_n) = p_1^{(i)}(z)\pi_{n-1}(z) + \tau_n z^2 p_1^{(i)*}(z)\pi_{n-1}^*(z) \tag{26}$$

and

$$p_1^{(i)}(z) = a_n^{(i)}(1-\overline{\alpha_n}z) + \tau_n(1-\overline{b_n^{(i)}}z). \tag{27}$$

Explicit expressions for the weights can be deduced as well from Theorem 3.1 and (21). Suppose $z_j = z_j^{(i)} \neq \nu_i$ for every $j \in \{1,\ldots,n\}$ and fixed $i \in \{3,4\}$. Next, let

$$T_n^{(i)} = \overline{X_n^{(i)}(z_j)}\left[\left(Q_{n+1}(z_j)+1\right)\left(1-\nu_iz_j\right)+2\nu_iz_j\right] + z_j\left(\overline{X_n^{(i)}(z_j)}\right)'\left(1-\nu_iz_j\right).$$

Then, with $\lambda_j = \lambda_j^{(i)}$ and $\left|c_n^{(i)}\right|^2$ given by (12), it follows that

$$
\begin{aligned}
\lambda_j^{-1} &= \left|c_n^{(i)}\right|^2 \frac{z_j\left|z_j-\alpha_n\right|^2}{1-|\alpha_n|^2}\begin{vmatrix} \overline{\phi_n^{(i)}(z_j)} & \overline{\phi_n^{(i)*}(z_j)} \\ \left(\phi_n^{(i)*}\right)'(z_j) & \phi_n^{(i)\prime}(z_j)\end{vmatrix} \\
&= \left|c_n^{(i)}\right|^2 \frac{z_j\left|z_j-\alpha_n\right|^2}{1-|\alpha_n|^2}\begin{vmatrix} \frac{z_j^2}{(1-\nu_i z_j)^2}\overline{X_n^{(i)}(z_j)} & \frac{X_n^{(i)}(z_j)}{B_n(z_j)(z_j-\nu_i)^2} \\ \frac{z_j B_n(z_j)}{(1-\nu_i z_j)^3}T_n^{(i)} & \frac{X_n^{(i)'}(z_j)}{(z_j-\nu_i)^2}-\frac{2X_n^{(i)}(z_j)}{(z_j-\nu_i)^3}\end{vmatrix} \\
&= \left|c_n^{(i)}\right|^2 \frac{z_j\left|z_j-\alpha_n\right|^2}{(1-|\alpha_n|^2)(z_j-\nu_i)^4}\begin{vmatrix} z_j^2\overline{X_n^{(i)}(z_j)} & \frac{X_n^{(i)}(z_j)}{B_n(z_j)} \\ \frac{z_j B_n(z_j)}{1-\nu_i z_j}T_n^{(i)} & X_n^{(i)\prime}(z_j)-\frac{2X_n^{(i)}(z_j)}{z_j-\nu_i}\end{vmatrix} \\
&= \left|c_n^{(i)}\right|^2 \frac{\left|z_j-\alpha_n\right|^2}{(1-|\alpha_n|^2)\left|z_j-\nu_i\right|^4}\times \\
&\qquad \left\{z_j\left[\overline{X_n^{(i)}(z_j)}X_n^{(i)\prime}(z_j)-X_n^{(i)}(z_j)\left(\overline{X_n^{(i)}(z_j)}\right)'\right]\right. \\
&\qquad\qquad \left.-\left|X_n^{(i)}(z_j)\right|^2(Q_{n+1}(z_j)+1)\right\} \\
&= \left|c_n^{(i)}\right|^2 \frac{\left|z_j-\alpha_n\right|^2}{(1-|\alpha_n|^2)\left|z_j-\nu_i\right|^4}\times \\
&\qquad \left\{2\Re\left[z_j\overline{X_n^{(i)}(z_j)}X_n^{(i)\prime}(z_j)\right]-\left|X_n^{(i)}(z_j)\right|^2(Q_{n+1}(z_j)+1)\right\}.
\end{aligned}
$$

Here, the last equality is due to the fact that for $z\in\mathbb{T}$, and for every $f(z)$ that is analytic in a small annulus containing the complex unit circle, it holds that (see e.g. [31, Lem. 3.3])

$$
\left(\overline{f(z)}\right)' = \frac{d\overline{f}}{dz} = \frac{d\overline{f}}{d\overline{z}}\cdot\frac{d\overline{z}}{dz} = -\overline{\frac{f'(z)}{z^2}}. \tag{28}
$$

From (9) we deduce that

$$
X_n^{(i)\prime}(z) = \frac{z^2(z-b_n^{(i)})B_{n-1}(z)}{1-\overline{\alpha_n}z}\left[\frac{2}{z}+\frac{B_{n-1}'(z)}{B_{n-1}(z)}+\frac{1}{z-b_n^{(i)}}+\frac{\overline{\alpha_n}}{1-\overline{\alpha_n}z}\right],
$$

so that for $z\in\mathbb{T}$ it holds that

$$
\begin{aligned}
zX_n^{(i)\prime}(z) &= \left(X_n^{(i)}(z)-a_n^{(i)}\right)\left(1+Q_n(z)+\frac{z}{z-b_n^{(i)}}+\frac{\overline{\alpha_n}z}{1-\overline{\alpha_n}z}\right) \\
&= \left(X_n^{(i)}(z)-a_n^{(i)}\right)\left(1+Q_{n+1}(z)+\frac{z(b_n^{(i)}-\alpha_n)}{(z-b_n^{(i)})(z-\alpha_n)}\right). \qquad (29)
\end{aligned}
$$

Consequently,

$$\lambda_j^{-1} = \left|c_n^{(i)}\right|^2 \frac{|z_j-\alpha_n|^2}{(1-|\alpha_n|^2)\,|z_j-\nu_i|^4}\times \left[\left(|X_n^{(i)}(z_j)|^2 - 2\Re\{a_n^{(i)}\overline{X_n^{(i)}(z_j)}\}\right)[1+Q_{n+1}(z_j)] + 2\Re\left\{\frac{z_j(b_n^{(i)}-\alpha_n)\left(\left|X_n^{(i)}(z_j)\right|^2 - a_n^{(i)}\overline{X_n^{(i)}(z_j)}\right)}{(z_j-b_n^{(i)})(z_j-\alpha_n)}\right\}\right].$$

Finally, we have that

$$|X_n^{(i)}(z)|^2 - 2\Re\{a_n^{(i)}\overline{X_n^{(i)}(z)}\} = \left|\frac{z-b_n^{(i)}}{z-\alpha_n}\right|^2 - |a_n^{(i)}|^2\,,$$
$$|X_n^{(i)}(z)|^2 - a_n^{(i)}\overline{X_n^{(i)}(z)} = \left|\frac{z-b_n^{(i)}}{z-\alpha_n}\right|^2 + \overline{a_n^{(i)}}\frac{z^2(z-b_n^{(i)})B_{n-1}(z)}{1-\overline{\alpha_n}z}$$

and [6]

$$z_j^2 B_{n-1}(z_j) = -\overline{\tau_n}\frac{p_1^{(i)}(z_j)}{p_1^{(i)*}(z_j)}, \tag{30}$$

where $p_1^{(i)}(z)$ is given by (27), so that

$$\lambda_j^{-1} = \frac{\left|c_n^{(i)}\right|^2}{(1-|\alpha_n|^2)\,|z_j-\nu_i|^4}\left[\left(|z_j-b_n^{(i)}|^2 - |a_n^{(i)}|^2|z_j-\alpha_n|^2\right)[1+Q_{n+1}(z_j)] + 2\Re\left\{(b_n^{(i)}-\alpha_n)\left(\frac{1-\overline{b_n^{(i)}}z_j}{z_j-\alpha_n} - \overline{a_n^{(i)}}\,\overline{\tau_n}\frac{p_1^{(i)}(z_j)}{p_1^{(i)*}(z_j)}\right)\right\}\right]. \tag{31}$$

Whenever $z_j = \nu_i$ for a certain $j \in \{1,\dots,n\}$, computing λ_j by the aid of (21) requires two times the application of l'Hôpital's rule to compute $\overline{\chi_n^{(i)}(\nu_i)}$, $\overline{\chi_n^{(i)*}(\nu_i)}$, $\left(\chi_n^{(i)*}\right)'(\nu_i)$ and $\chi_n^{(i)\prime}(\nu_i)$. This gives rise to tedious calculations along with an inappropriate expression for computational purposes. Since $z_j \neq z_k$ for $j \neq k$, there can be at the most one index j for which $z_j = \nu_i$. Therefore, λ_j can then be computed as follows:

$$\lambda_j = \int_{-\pi}^{\pi}\omega_i(\theta)d\theta - \sum_{\substack{k=1\\k\neq j}}^{n}\lambda_k = 2\pi - \sum_{\substack{k=1\\k\neq j}}^{n}\lambda_k. \tag{32}$$

Note that there exists an index j so that $z_j = \nu_i$ iff $\tau_n = -\frac{\phi_n^{(i)}(\nu_i)}{\phi_n^{(i)*}(\nu_i)}$. From (11) and (28) it follows

[6] There are no $\tau_n \in \mathbb{T}$ and $\gamma \in \mathbb{T}\setminus\{\nu_i\}$ so that $p_1^{(i)*}(\gamma) = 0$; hence, the right hand side of (30) is well defined. A proof for this statement is given in the Appendix.

that

$$-\frac{\phi_n^{(i)}(\nu_i)}{\phi_n^{(i)*}(\nu_i)} = \frac{-1}{B_n(\nu_i)} \lim_{\substack{z\to\nu_i \\ z\in\mathbb{T}}} \frac{X_n^{(i)}(z)}{\overline{X_n^{(i)}(z)}}$$

$$= \frac{1}{B_n(\nu_i)} \lim_{\substack{z\to\nu_i \\ z\in\mathbb{T}}} \frac{X_n^{(i)'}(z)}{\overline{X_n^{(i)'}(z)}} = \frac{-1}{B_n(\nu_i)} \frac{X_n^{(i)''}(\nu_i)}{\overline{X_n^{(i)''}(\nu_i)}}. \quad (33)$$

Let

$$Y_n^{(i)}(z) = 1 + Q_{n+1}(z) + \frac{z(b_n^{(i)} - \alpha_n)}{(z - b_n^{(i)})(z - \alpha_n)}.$$

Then it follows from (29) that

$$zX_n^{(i)''}(z) = X_n^{(i)'}(z)\left(Y_n^{(i)}(z) - 1\right) + \left(X_n^{(i)}(z) - a_n^{(i)}\right) Y_n^{(i)'}(z),$$

and hence,

$$\begin{aligned} X_n^{(i)''}(\nu_i) &= -a_n^{(i)} \nu_i Y_n^{(i)'}(\nu_i) = a_n^{(i)} \left[\frac{\nu_i (b_n^{(i)} - \alpha_n)(1 - \alpha_n b_n^{(i)})}{(\nu_i - b_n^{(i)})^2 (\nu_i - \alpha_n)^2} - \nu_i Q'_{n+1}(\nu_i) \right] \\ &= R_n^{(i)} e^{\mathbf{i}\gamma_n^{(i)}}, \quad R_n^{(i)} > 0 \text{ and } \gamma_n^{(i)} \in [0, 2\pi), \end{aligned}$$

where

$$zQ'_{n+1}(z) = 2\mathbf{i} \sum_{k=1}^{n} \frac{\Im\{\overline{\alpha_k} z\}(1 - |\alpha_k|^2)}{|z - \alpha_k|^4} \text{ for } z \in \mathbb{T}. \quad (34)$$

Consequently, there exists an index j so that $z_j = \nu_i$ iff $\tau_n = \frac{e^{\mathbf{i}(2\gamma_n^{(i)} - \pi)}}{B_n(\nu_i)}$.

We now have proved the following theorem.

Theorem 4.3 *Let $i \in \{3, 4\}$ be fixed and set $\nu_i = (-1)^{i-1}$. Further, assume $\tau_n \in \mathbb{T}$, for $n \geq 1$. An n-point rational Szegő quadrature $I_n(f)$ for $I_{\omega_i}(f)$, given by* (5) *and* (6)*, is then characterized as follows:*

(1) Denote by $\{\nu_i, \nu_i, z_1, \ldots, z_n\}$ the set of zeros of $V_n^{(i)}(z, \tau_n)$, given by (26)–(27)*. Then $\{z_1, \ldots, z_n\}$ is the set of nodes for $I_n(f)$.*

(2) For every $j \in \{1, \ldots, n\}$ for which $z_j \neq \nu_i$, the jth weight is given by (31)*. Whenever $z_j = \nu_i$ for a certain $j \in \{1, \ldots, n\}$, the associated weight is given by* (32)*.*

□

To conclude this section, we consider the weight function $\omega_2(\theta)$. From (4) and Theorem 3.2 it now follows that, up to a multiplicative factor,

$$\Psi_n^{(2)}(z, \tau_n) = \frac{V_n^{(2)}(z, \tau_n)}{(z^2 - 1)^2 \pi_n(z)} \; ; \quad z, \tau_n \in \mathbb{T},$$

where

$$V_n^{(2)}(z, \tau_n) = p_2^{(2)}(z) \pi_{n-1}(z) + \tau_n z^3 p_2^{(2)*}(z) \pi_{n-1}^*(z) \quad (35)$$

and

$$p_2^{(2)}(z) = (d_n + e_n z)(1 - \overline{\alpha_n} z) + \tau_n(\overline{f_n} z^2 + \overline{g_n} z + 1). \tag{36}$$

Explicit expressions for the weights can be deduced as well from (21) and Theorems 3.2–3.3. Assume $\epsilon \in \{\pm 1\}$ and suppose $z_j = z_j^{(2)} \neq \epsilon$ for every $j \in \{1, \ldots, n\}$. Next, let

$$T_n^{(2)} = \left[Q_{n+1}(z_j) + 3 - \frac{4z_j^2}{z_j^2 - 1} \right] z_j \overline{X_n^{(2)}(z_j)} - \overline{X_n^{(2)'}(z_j)}.$$

Then, with $\lambda_j = \lambda_j^{(2)}$ and $|h_n|^2$ given by (18), it now follows that

$$\begin{aligned}
\lambda_j^{-1} &= |h_n|^2 \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \begin{vmatrix} \overline{\phi_n^{(2)}(z_j)} & \overline{\phi_n^{(2)*}(z_j)} \\ \left(\phi_n^{(2)*}\right)'(z_j) & \phi_n^{(2)'}(z_j) \end{vmatrix} \\
&= |h_n|^2 \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \begin{vmatrix} \frac{z_j^4}{(z_j^2-1)^2} \overline{X_n^{(2)}(z_j)} & \frac{X_n^{(2)}(z_j)}{B_n(z_j)(z_j^2-1)^2} \\ \frac{z_j^2 B_n(z_j)}{(z_j^2-1)^2} T_n^{(2)} & \frac{X_n^{(2)'}(z_j)}{(z_j^2-1)^2} - \frac{4z_j X_n^{(2)}(z_j)}{(z_j^2-1)^3} \end{vmatrix} \\
&= |h_n|^2 \frac{z_j |z_j - \alpha_n|^2}{(1 - |\alpha_n|^2)\left(z_j^2 - 1\right)^4} \begin{vmatrix} z_j^4 \overline{X_n^{(2)}(z_j)} & \frac{X_n^{(2)}(z_j)}{B_n(z_j)} \\ z_j^2 B_n(z_j) T_n^{(2)} & X_n^{(2)'}(z_j) - \frac{4z_j X_n^{(2)}(z_j)}{z_j^2 - 1} \end{vmatrix} \\
&= |h_n|^2 \frac{|z_j - \alpha_n|^2}{(1 - |\alpha_n|^2)\left|z_j^2 - 1\right|^4} \times \\
&\qquad \left\{ 2\Re\left[z_j \overline{X_n^{(2)}(z_j)} X_n^{(2)'}(z_j) \right] - \left| X_n^{(2)}(z_j) \right|^2 (Q_{n+1}(z_j) + 3) \right\}.
\end{aligned}$$

From (17) we deduce that

$$\begin{aligned}
X_n^{(2)'}(z) = \frac{z^3(f_n + g_n z + z^2) B_{n-1}(z)}{1 - \overline{\alpha_n} z} \times \\
\left[\frac{3}{z} + \frac{B_{n-1}'(z)}{B_{n-1}(z)} + \frac{g_n + 2z + e_n \frac{1 - \overline{\alpha_n} z}{z^3 B_{n-1}(z)}}{f_n + g_n z + z^2} + \frac{\overline{\alpha_n}}{1 - \overline{\alpha_n} z} \right],
\end{aligned}$$

so that for $z \in \mathbb{T}$ it holds that

$$\begin{aligned}
z X_n^{(2)'}(z) &= \left[X_n^{(2)}(z) - (d_n + e_n z) \right] \times \\
&\qquad \left\{ 2 + Q_n(z) + \frac{z\left(g_n + 2z + e_n \frac{1 - \overline{\alpha_n} z}{z^3 B_{n-1}(z)} \right)}{f_n + g_n z + z^2} + \frac{\overline{\alpha_n} z}{1 - \overline{\alpha_n} z} \right\} \\
&= \left[X_n^{(2)}(z) - (d_n + e_n z) \right] \times \\
&\qquad \left\{ 3 + Q_{n+1}(z) + \frac{z^2 G_n(z)}{(f_n + g_n z + z^2)(z - \alpha_n)} \right\},
\end{aligned} \tag{37}$$

where

$$G_n(z) = \alpha_n f_n \overline{z}^2 - 2 f_n \overline{z} - (\alpha_n + g_n) + e_n |z - \alpha_n|^2 \overline{z^3 B_{n-1}(z)}.$$

Consequently,

$$\lambda_j^{-1} = |h_n|^2 \frac{|z_j - \alpha_n|^2}{(1-|\alpha_n|^2)\left|z_j^2-1\right|^4} \times \left[\left(|X_n^{(2)}(z_j)|^2 - 2\Re\{(d_n + e_n z_j)\overline{X_n^{(2)}(z_j)}\} \right) [3 + Q_{n+1}(z_j)] + 2\Re\left\{ \frac{z_j^2 G_n(z_j) \left(|X_n^{(2)}(z_j)|^2 - (d_n + e_n z_j)\overline{X_n^{(2)}(z_j)} \right)}{(f_n + g_n z_j + z_j^2)(z_j - \alpha_n)} \right\} \right].$$

Finally, we have that

$$|X_n^{(2)}(z)|^2 - 2\Re\{(d_n + e_n z)\overline{X_n^{(2)}(z)}\} = \left| \frac{f_n + g_n z + z^2}{z - \alpha_n} \right|^2 - |d_n + e_n z|^2 ,$$

$$|X_n^{(2)}(z)|^2 - (d_n + e_n z)\overline{X_n^{(2)}(z)} = \left| \frac{f_n + g_n z + z^2}{z - \alpha_n} \right|^2 + \overline{d_n + e_n z} \frac{z^3 (f_n + g_n z + z^2) B_{n-1}(z)}{1 - \overline{\alpha_n} z}$$

and [7]

$$z_j^3 B_{n-1}(z_j) = -\overline{\tau_n} \frac{p_2^{(2)}(z_j)}{p_2^{(2)*}(z_j)}, \tag{38}$$

where $p_2^{(2)}(z)$ is given by (36). Consequently,

$$G_n(z_j) = \alpha_n f_n \overline{z_j}^2 - 2 f_n \overline{z_j} - (\alpha_n + g_n) - \tau_n e_n |z_j - \alpha_n|^2 \frac{p_2^{(2)*}(z_j)}{p_2^{(2)}(z_j)},$$

so that

$$\lambda_j^{-1} = \frac{|h_n|^2}{(1-|\alpha_n|^2)\left|z_j^2-1\right|^4} \times \left[\left(|f_n + g_n z_j + z_j^2|^2 - |d_n + e_n z_j|^2 |z_j - \alpha_n|^2 \right) [3 + Q_{n+1}(z_j)] + 2\Re\left\{ G_n(z_j) \left[\frac{\overline{f_n} z_j^2 + \overline{g_n} z_j + 1}{z_j - \alpha_n} - \overline{\tau_n}(\overline{d_n} z_j + \overline{e_n}) \frac{p_2^{(2)}(z_j)}{p_2^{(2)*}(z_j)} \right] \right\} \right]. \tag{39}$$

Whenever $z_j = \epsilon$ for a certain $j \in \{1, \ldots, n\}$, computing λ_j by the aid of (21) again requires two times the application of l'Hôpital's rule to compute $\overline{\chi_n^{(2)}(\epsilon)}$, $\overline{\chi_n^{(2)*}(\epsilon)}$, $\left(\chi_n^{(2)*}\right)'(\epsilon)$ and $\chi_n^{(2)\prime}(\epsilon)$.

[7] If there are no $\tau_n \in \mathbb{T}$ and $\gamma \in \mathbb{T} \setminus \{\pm 1\}$ so that $p_2^{(2)*}(\gamma) = 0$, then the right hand side of (38) is well defined. At this moment of writing no proof has been found for this statement in general, but a proof is given in the Appendix for the special case in which $n = 1$ or $\alpha_n = 0$.

This gives rise to tedious calculations along with an inappropriate expression for computational purposes. Since $z_j \neq z_k$ for $j \neq k$, there can be at the most one index j for which $z_j = \epsilon$. Therefore, if $z_k \neq -\epsilon$ for every $k \in \{1, \ldots, n\} \setminus \{j\}$, λ_j can be computed as follows:

$$\lambda_j = \int_{-\pi}^{\pi} \omega_2(\theta) d\theta - \sum_{\substack{k=1 \\ k \neq j}}^{n} \lambda_k = \pi - \sum_{\substack{k=1 \\ k \neq j}}^{n} \lambda_k. \tag{40}$$

If, on the other hand, there exist indices j and k in $\{1, \ldots, n\}$ so that $z_j = -z_k = 1$, then λ_j and λ_k can be computed by solving the following system of equations:

$$\begin{cases} \lambda_j + \lambda_k & = \pi - \sum\limits_{\substack{s=1 \\ s \notin \{j,k\}}}^{n} \lambda_s \\ \lambda_j \left|\chi_1^{(2)}(1)\right|^2 + \lambda_k \left|\chi_1^{(2)}(-1)\right|^2 & = 1 - \sum\limits_{\substack{s=1 \\ s \notin \{j,k\}}}^{n} \lambda_s \left|\chi_1^{(2)}(z_s)\right|^2 \end{cases}, \tag{41}$$

where we computed $\chi_1^{(2)}(z)$ with the aid of Maple 10 to find that

$$\left|\chi_1^{(2)}(z)\right|^2 = \frac{1}{\pi} \left(\frac{1 - |\alpha_1|^2}{1 - \left|\frac{\alpha_1}{2 - \alpha_1^2}\right|^2} \right) \left| \frac{z + \frac{\overline{\alpha_1}}{2 - \overline{\alpha_1}^2}}{1 - \overline{\alpha_1} z} \right|^2. \tag{42}$$

Note that there exists an index j so that $z_j = \epsilon$ iff $\tau_n = -\frac{\phi_n^{(2)}(\epsilon)}{\phi_n^{(2)*}(\epsilon)}$. From Theorem 3.2 and proceeding as in (33) it follows that

$$-\frac{\phi_n^{(2)}(\epsilon)}{\phi_n^{(2)*}(\epsilon)} = \frac{-1}{B_n(\epsilon)} \frac{X_n^{(2)''}(\epsilon)}{\overline{X_n^{(2)''}(\epsilon)}}.$$

Let

$$Y_n^{(2)}(z) = 3 + Q_{n+1}(z) + \frac{z^2 G_n(z)}{(f_n + g_n z + z^2)(z - \alpha_n)}.$$

Then it follows from (37) that

$$z X_n^{(2)''}(z) = X_n^{(2)'}(z) \left(Y_n^{(2)}(z) - 1 \right) - e_n Y_n^{(2)}(z) + \left(X_n^{(2)}(z) - (d_n + e_n z) \right) Y_n^{(2)'}(z).$$

It also follows from (37) that

$$0 = \epsilon X_n^{(2)'}(\epsilon) = \left(X_n^{(2)}(\epsilon) - (d_n + e_n \epsilon) \right) Y_n^{(2)}(\epsilon) = -(d_n + e_n \epsilon) Y_n^{(2)}(\epsilon),$$

so that $Y_n^{(2)}(\epsilon) = 0$ due to Lemma 3.4. Thus,

$$\begin{aligned} X_n^{(2)''}(\epsilon) &= -(d_n + e_n \epsilon) \epsilon Y_n^{(2)'}(\epsilon) \\ &= (d_n + e_n \epsilon) \left[\frac{\epsilon \left(k_n(\epsilon) + e_n \overline{B_{n-1}(\epsilon)} l_n(\epsilon) \right)}{(1 + f_n + g_n \epsilon)^2 (\epsilon - \alpha_n)^2} - \epsilon Q'_{n+1}(\epsilon) \right] \\ &= R_{n,\epsilon}^{(2)} e^{\mathbf{i} \gamma_{n,\epsilon}^{(2)}}; \quad \epsilon \in \{\pm 1\},\ R_{n,\epsilon}^{(2)} > 0 \text{ and } \gamma_{n,\epsilon}^{(2)} \in [0, 2\pi), \end{aligned}$$

where

$$k_n(\epsilon) = f_n\alpha_n(6-f_n) - (1+g_n\alpha_n)(\alpha_n+g_n) - f_n(g_n+4\epsilon)(1+\alpha_n^2)$$
$$l_n(\epsilon) = |\epsilon-\alpha_n|^2\left[2(g_n-\alpha_n)+\epsilon(3+f_n-g_n\alpha_n)\right] + \left[|\epsilon-\alpha_n|^2 Q_n(\epsilon) - \epsilon 2\mathbf{i}\Im\{\alpha_n\}\right](1+f_n+g_n\epsilon)(\epsilon-\alpha_n),$$

and with $Q_n(z)$ and $zQ'_{n+1}(z)$ given by respectively (8) and (34). Consequently, there exists an index j so that $z_j = \epsilon$ iff $\tau_n = \frac{e^{\mathbf{i}(2\gamma^{(2)}_{n,\epsilon}-\pi)}}{B_n(\epsilon)}$.

We finally have proved the following theorem.

Theorem 4.4 *Let $\tau_n \in \mathbb{T}$ for $n \geq 1$ and suppose $\epsilon \in \{\pm 1\}$. An n-point rational Szegő quadrature $I_n(f)$ for $I_{\omega_2}(f)$, given by* (5) *and* (6)*, is then characterized as follows:*

(1) Denote by $\{1,1,-1,-1,z_1,\ldots,z_n\}$ the set of zeros of $V_n^{(2)}(z,\tau_n)$, given by (35)–(36)*. Then $\{z_1,\ldots,z_n\}$ is the set of nodes for $I_n(f)$.*

(2) For every $j \in \{1,\ldots,n\}$ for which $z_j \neq \epsilon$, the jth weight is given by (39)*. Whenever $z_j = \epsilon$ for a certain $j \in \{1,\ldots,n\}$ and $z_k \neq -\epsilon$ for every $k \in \{1,\ldots,n\}\setminus\{j\}$, the jth weight is given by* (40)*. Finally, if there exist two distinct indices j and k in $\{1,\ldots,n\}$ so that $z_j = -z_k = 1$, the associated weights are given by* (41)*.*

□

5 Szegő quadratures associated with rational modifications of Chebyshev weight functions

In this section we will consider a rational modification of a measure μ that is a measure of the form

$$d\tilde{\mu}(\theta) = \frac{d\mu(\theta)}{|h(z)|^2}\ ;\ \ z = e^{\mathbf{i}\theta}, \tag{43}$$

with $h(z)$ a given polynomial of degree m whose zeros can not be on $\mathbb{T}$. Without loss of generality we can assume

$$h(z) = \prod_{l=1}^{m}(z-\alpha_l)\ ;\ \ |\alpha_l| < 1,\ \alpha_l \neq 0,\ \ l = 1,\ldots,m. \tag{44}$$

We will then consider Szegő quadrature formulas with respect to the measure $\tilde{\mu}$. In this respect, we recall that an n-point Szegő rule (2) for a measure μ has maximal domain of validity $\Lambda_{-(n-1),n-1} = \text{span}\{z^{-(n-1)},\ldots,z^{n-1}\}$, with dimension $2n-1$, and that the nodes are the zeros of a para-orthogonal polynomial associated with μ; see e.g. [1,8–10,12,22,24,25]. We also recall that for $\kappa_n \in \mathbb{T}$, κ_n-invariant para-orthogonal polynomials associated with μ are characterized in [25] as

$$\Phi_n(z) = C_n\left[\rho_n(z) + \tau_n\rho_n^*(z)\right] \in \mathbb{P}_n\ ;\ n \geq 1,\ C_n \in \mathbb{C}\setminus\{0\}, \tag{45}$$

where $\tau_n = \frac{\overline{C_n\kappa_n}}{C_n} \in \mathbb{T}$ and $\{\rho_k(z)\}_{k=0}^{\infty}$ is a sequence of Szegő polynomials associated with μ; see e.g. [26], [28].

When $d\mu(\theta) = d\theta$, rational modifications of the Lebesgue measure appear so that Szegő polynomials for $\tilde{\mu}$ were earlier considered by Szegő in [28]. Observe that when $m = 1$, $\tilde{\mu}$ gives rise to the

so-called Poisson kernel. In relation to this, Waadeland considered in [33] for the first time Szegő quadrature formulas for the measure $d\tilde{\mu}(\theta) = \frac{d\theta}{|z-\alpha|^2}$, providing one of the first examples of this type of quadratures in the literature.

Rational modifications of the Lebesgue measure are very suitable in order to approximate other measures. For instance, the following important result is given in [26, Theorem 1.7.8]: Let η be a probability measure supported on $\mathbb{T}$, and suppose $\{\varphi_n(z)\}_{n=0}^{\infty}$ is a sequence of orthonormal polynomials associated with η. For every n it then holds that

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi \left|\varphi_n(e^{\mathbf{i}\theta})\right|^2} = 1$$

and

$$d\eta_n(\theta) = \frac{d\theta}{2\pi \left|\varphi_n(e^{\mathbf{i}\theta})\right|^2} \longrightarrow d\eta \ \text{ weakly when } n \to \infty.$$

Consider now the polynomial $h(z)$ given by (44) and set $\alpha_k = 0$ for every $k > m$. For every $n \geq m$ we then have that

$$\pi_n^*(z) = \prod_{j=1}^{n}(z-\alpha_j) = z^{n-m}h(z), \ \ \pi_n(z) = \prod_{j=1}^{n}(1-\overline{\alpha_j}z) = \prod_{j=1}^{m}(1-\overline{\alpha_j}z) = h^*(z).$$

So, let $\chi_n(z)$ denote the nth orthonormal rational function associated with μ. Then for every $n \geq m$, $\chi_n(z) = \frac{P_n(z)}{h^*(z)}$ with $P_n(z) \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$. Moreover, if $n > m$, we have that $\langle \chi_n(z), u_k(z) \rangle_\mu = 0$, where $u_k(z) = \frac{z^k}{h^*(z)}$ with $k = 0, 1, \ldots, n-1$, implying that

$$0 = \int_{-\pi}^{\pi} \frac{P_n(z)}{h^*(z)} \overline{\left(\frac{z^k}{h^*(z)}\right)} d\mu(\theta) = \int_{-\pi}^{\pi} P_n(z)\overline{z^k} \frac{d\mu(\theta)}{|h^*(z)|^2} = \int_{-\pi}^{\pi} P_n(z)\overline{z^k} \frac{d\mu(\theta)}{|h(z)|^2}$$

with $z = e^{i\theta}$ and $k = 0, 1, \ldots, n-1$. Consequently, the numerator of the nth orthonormal rational function associated with the measure μ coincides for $n > m$ with an nth orthonormal polynomial associated with the measure $\tilde{\mu}$ given by (43).

Example 5.1 *Take the Lebesgue measure $d\mu(\theta) = \omega_1(\theta)d\theta \equiv d\theta$; see* (6)*. Then for every $n > m$ it follows from* (7) *that*

$$\begin{aligned} \chi_n(z) &= c_n z \prod_{j=1}^{n-1} \left(\tfrac{z-\alpha_j}{1-\overline{\alpha_j}z}\right) = c_n z \prod_{j=1}^{m} \left(\tfrac{z-\alpha_j}{1-\overline{\alpha_j}z}\right) \prod_{j=m+1}^{n-1} \left(\tfrac{z-\alpha_j}{1-\overline{\alpha_j}z}\right) \\ &= c_n \tfrac{z^{n-m}h(z)}{h^*(z)} \ , \ \ c_n \neq 0. \end{aligned}$$

Hence, $P_n(z) = z^{n-m}h(z)$ represents the nth monic Szegő polynomial for the measure $d\tilde{\mu}(\theta) = \frac{d\theta}{|h(z)|^2}$ with $z = e^{\mathbf{i}\theta}$ (compare with the first approach given in [28, pp. 289-290]; actually, this expression for $P_n(z)$ also holds for $n = m$).

Let us see next what happens with the quadrature formulas. Assume as above $\alpha_k = 0$ for $k > m$ and $\alpha_l \in \mathbb{D}\backslash\{0\}$ for $l = 1, \ldots, m$. Let $h(z) = \prod_{l=1}^{m}(z-\alpha_l) \in \mathbb{P}_m \backslash \mathbb{P}_{m-1}$, and suppose $n > m$. From Theorem 2.2 we then have that

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_\mu(f) = \int_{-\pi}^{\pi} f(e^{\mathbf{i}\theta}) d\mu(\theta) \ , \ \ \forall f \in \mathcal{R}_{n-1}.$$

Consider now the case in which $f(z)=\frac{z^k}{\pi^*_{n-1}(z)\pi_{n-1}(z)}\in\mathcal{R}_{n-1}$ for $k=0,1,\ldots,2n-2$, and observe that $\pi_{n-1}(z)=z^{n-1}\overline{\pi^*_{n-1}(z)}$ for $z\in\mathbb{T}$. Taking into account that $k=0,1,\ldots,2n-2$, it then follows that

$$f(z)=\frac{z^{k-(n-1)}}{|\pi^*_{n-1}(z)|^2}=\frac{z^{k+1-n}}{|h(z)|^2}=\frac{z^s}{|h(z)|^2}\ ;\ \ -(n-1)\le s\le n-1\ ,\ \ z\in\mathbb{T}.$$

Hence,

$$\begin{aligned}\int_{-\pi}^{\pi}f(z)d\mu(\theta)&=\int_{-\pi}^{\pi}\frac{z^s}{|h(z)|^2}d\mu(\theta)=\sum_{j=1}^n\lambda_j\frac{z_j^s}{|h(z_j)|^2}=\sum_{j=1}^n\tilde{\lambda}_jz_j^s\\&=\int_{-\pi}^{\pi}z^sd\tilde{\mu}(\theta)\ ;\ \ z=e^{i\theta}\ ,\ -(n-1)\le s\le n-1,\end{aligned}$$

where $\tilde{\lambda}_j=\frac{\lambda_j}{|h(z_j)|^2}$ for $j=1,\ldots,n$ and $d\tilde{\mu}(\theta)$ is given by (43). It follows from Theorem 2.2 that $\{z_j\}_{j=1}^n$ are the zeros of $\Psi_n(z,\tau_n)=\chi_n(z)+\tau_n\chi_n^*(z)$. Furthermore, setting $\chi_n(z)=\frac{P_n(z)}{\pi_n(z)}$ and

$$\chi_n^*(z)=B_n(z)\chi_{n*}(z)=B_n(z)\frac{\overline{P_n(1/\bar{z})}}{\overline{\pi_n(1/\bar{z})}}=\frac{B_n(z)P_n^*(z)}{z^n\overline{\pi_n(1/\bar{z})}}=\frac{P_n^*(z)}{\pi_n(z)},$$

we obtain that

$$\Psi_n(z,\tau_n)=\frac{P_n(z)+\tau_nP_n^*(z)}{\pi_n(z)}=\frac{P_n(z)+\tau_nP_n^*(z)}{h^*(z)}.$$

Thus, we have proved the following theorem.

Theorem 5.2 *Let $h(z)=\prod_{l=1}^m(z-\alpha_l)$ with $\alpha_l\in\mathbb{D}\backslash\{0\}$ for every $l=1,\ldots,m$, and set $\alpha_k=0$ for every $k>m$. Further, let $n>m$ and suppose $\chi_n(z)=\frac{P_n(z)}{h^*(z)}$ with $P_n(z)\in\mathbb{P}_n\backslash\mathbb{P}_{n-1}$ is the nth orthonormal rational function associated with the measure μ. Then for a given $\tau_n\in\mathbb{T}$, $I_n(f)=\sum_{j=1}^n\tilde{\lambda}_jf(z_j)$ is an n-point Szegő quadrature formula for $d\tilde{\mu}(\theta)=\frac{d\mu(\theta)}{\left|h(e^{\mathbf{i}\theta})\right|^2}$, if and only if*

(1) the nodes $\{z_j\}_{j=1}^n$ are the zeros of $P_n(z)+\tau_nP_n^*(z)$, and
(2) $\tilde{\lambda}_j=\frac{\lambda_j}{|h(z_j)|^2}$, where $\{\lambda_j\}_{j=1}^n$ is the set of weights corresponding to the n-point rational Szegő quadrature for μ.

□

As an application of the previous theorem, we will now deduce characterization theorems for Szegő quadrature formulas associated the measures $d\tilde{\mu}(\theta)=\frac{\omega_i(\theta)d\theta}{|h(z)|^2}$, where $z=e^{\mathbf{i}\theta}$ and $\omega_i(\theta)$ is given by (6) for $i\in\{1,\ldots,4\}$.

We start with the Lebesgue measure (compare with [11, Theorem 4.2]).

Theorem 5.3 *Assume $h(z)=\prod_{l=1}^m(z-\alpha_l)$ with $\alpha_l\in\mathbb{D}\backslash\{0\}$ for every $l=1,\ldots,m$. Let $\tau_n\in\mathbb{T}$ and set $d\tilde{\mu}(\theta)=\frac{d\theta}{\left|h(e^{\mathbf{i}\theta})\right|^2}$. An n-point Szegő formula for $\tilde{\mu}$ and $n>m$ is then given by*

$$I_n(f)=\sum_{j=1}^n\tilde{\lambda}_jf(z_j),$$

where the nodes $\{z_j\}_{j=1}^n$ *are the zeros of* $z^{n-m}h(z)+\tau_n h^*(z)$ *and the weights* $\{\tilde{\lambda}_j\}_{j=1}^n$ *are given by*

$$\tilde{\lambda}_j = \frac{2\pi}{|h(z_j)|^2\left[n-m+\sum_{l=1}^m \frac{1-|\alpha_l|^2}{|z_j-\alpha_l|^2}\right]}\ ;\ \ j=1,\ldots,n.$$

□

In the following theorem we characterize a Szegő quadrature formula associated with a rational modification of the Chebyshev weight functions $1\pm\cos\theta$.

Theorem 5.4 *Suppose* $h(z)=\prod_{l=1}^m(z-\alpha_l)$ *with* $\{\alpha_l\}_{l=1}^m\subset\mathbb{D}\backslash\{0\}$, $n>m$, $\tau_n\in\mathbb{T}$, $\nu_i=(-1)^{i-1}$ *for fixed* $i\in\{3,4\}$, *and let*

$$d\tilde{\mu}(\theta)=\frac{1-\nu_i\cos\theta}{|h(e^{\mathbf{i}\theta})|^2}d\theta.$$

Next, let $Q_n(z)=n-m+\sum_{l=1}^m\frac{1-|\alpha_l|^2}{|z-\alpha_l|^2}$ *and define* $\tilde{Q}_n^{(i)}=[1+Q_n(\nu_i)]^{-1}$. *Further, assume* $a_n^{(i)}=\nu_i^n\frac{h(\nu_i)}{\overline{h(\nu_i)}}\tilde{Q}_n^{(i)}$ *and* $b_n^{(i)}=\nu_i\left(1+\tilde{Q}_n^{(i)}\right)$. *An* n*-point Szegő formula for* $\tilde{\mu}$ *is then given by* $I_n(f)=\sum_{j=1}^n\tilde{\lambda}_j f(z_j)$, *where the set of nodes and weights are determined as follows:*

(1) Denote by $\{\nu_i,\nu_i,z_1,\ldots,z_n\}$ *the set of zeros of*

$$\begin{aligned}V_n^{(i)}(z,\tau_n)=z^{n+1-m}&\left[-b_n^{(i)}+\left(1+\tau_n\overline{a_n^{(i)}}\right)z\right]h(z)\\&+z^m\left[a_n^{(i)}+\tau_n\left(1-b_n^{(i)}z\right)\right]\overline{h(z)}.\end{aligned}$$

Then $\{z_1,\ldots,z_n\}$ *is the set of nodes for* $I_n(f)$.

(2) For every $j\in\{1,\ldots,n\}$ *for which* $z_j\neq\nu_i$, *the* j^{th} *weight is given by*

$$\tilde{\lambda}_j^{-1}=\nu_i\frac{|h(z_j)|^2\left(|z_j-b_n^{(i)}|^2-|a_n^{(i)}|^2\right)}{\pi b_n^{(i)}|z_j-\nu_i|^4}\times\left[1+Q_{n+1}(z_j)+2b_n^{(i)}\Re\left\{\left[-b_n^{(i)}+\left(1+\tau_n\overline{a_n^{(i)}}\right)z_j\right]^{-1}\right\}\right].$$

Whenever $z_j=\nu_i$ *for a certain* $j\in\{1,\ldots,n\}$, *the associated weight is given by* $\tilde{\lambda}_j=\lambda_j/|h(z_j)|^2$, *where* λ_j *is given by* (32).

□

Finally, for the remainder Chebyshev weight function we have the following theorem.

Theorem 5.5 *Suppose* $h(z)=\prod_{l=1}^m(z-\alpha_l)$ *with* $\{\alpha_l\}_{l=1}^m\subset\mathbb{D}\backslash\{0\}$, $n>m$, $\tau_n\in\mathbb{T}$, $\epsilon\in\{\pm1\}$, *and let*

$$d\tilde{\mu}(\theta)=\frac{\sin^2\theta}{|h(e^{\mathbf{i}\theta})|^2}d\theta.$$

Next, let $Q_n(z)=n-m+\sum_{l=1}^m\frac{1-|\alpha_l|^2}{|z-\alpha_l|^2}$ *and define*

$$a^+=\frac{h(1)}{\overline{h(1)}}\ ,\ \ a^-=(-1)^{n-1}\frac{h(-1)}{\overline{h(-1)}}\ ,\ \ b^+=2+Q_n(1)\ ,\ \ b^-=2+Q_n(-1).$$

Further, assume $\underline{x} = (d_n, e_n, f_n, g_n)^T$ *is the solution of the system given by* (15)–(16). *An* n*-point Szegő formula for* $\tilde{\mu}$ *is then given by* $I_n(f) = \sum_{j=1}^{n} \tilde{\lambda}_j f(z_j)$, *where the set of nodes and weights are determined as follows:*

(1) Let $\{+1+1, -1, -1, z_1, \ldots, z_n\}$ *be the set of zeros of*

$$V_n^{(2)}(z, \tau_n) = z^{n+2-m}\left[\left(1+\tau_n\overline{d_n}\right)z^2 + \left(g_n + \tau_n\overline{e_n}\right)z + f_n\right]h(z) + z^m\left[\tau_n\overline{f_n}z^2 + \left(e_n+\tau_n\overline{g_n}\right)z + \tau_n + d_n\right]\overline{h(z)}.$$

Then $\{z_1, \ldots, z_n\}$ *is the set of nodes of* $I_n(f)$.

(2) For every $j \in \{1, \ldots, n\}$ *for which* $z_j \neq \epsilon$, *the* j^{th} *weight is given by*

$$\tilde{\lambda}_j^{-1} = \frac{2|h(z_j)|^2\left(|d_n+e_nz_j|^2 - |f_n+g_nz_j+z_j^2|^2\right)}{\pi f_n\left|z_j^2-1\right|^4} \times \left[3 + Q_{n+1}(z_j) - \frac{2\Re\left\{\overline{\tau_n}(2f_n\overline{z_j}+g_n)\overline{z_j}p_2^{(2)}(z_j) + e_nz_j\overline{p_2^{(2)}(z_j)}\right\}}{\left|p_2^{(2)}(z_j)\right|^2}\right],$$

where $p_2^{(2)}(z_j) = (d_n + e_nz_j) + \tau_n\left(\overline{f_n}z_j^2 + \overline{g_n}z_j + 1\right)$. *Whenever* $z_j = \epsilon$ *for a certain* $j \in \{1, \ldots, n\}$ *and* $z_k \neq -\epsilon$ *for every* $k \in \{1, \ldots, n\} \setminus \{j\}$, *the* j^{th} *weight is given by* $\tilde{\lambda}_j = \lambda_j / |h(z_j)|^2$, *where* λ_j *is given by* (40). *Finally, if there exist two distinct indices* j *and* k *in* $\{1, \ldots, n\}$ *so that* $z_j = -z_k = 1$, *the associated weights are given by* $\tilde{\lambda}_j = \lambda_j / |h(z_j)|^2$ *and* $\tilde{\lambda}_k = \lambda_k / |h(z_k)|^2$, *where* λ_j *and* λ_k *are given by* (41)-(42).

□

6 Numerical examples

The aim of this final section is to present some numerical illustrations of the results given in Section 5 in the construction of Szegő-type quadrature formulas with respect to rational modifications of Chebyshev weight functions.

We start by recalling that the parameter $\tau_n \in \mathbb{T}$ in (45) can be choosen freely to fix a complex number $\lambda \in \mathbb{T}$ as a node of the Szegő quadrature formula. By setting $\tau_n = -\frac{\rho_n(\lambda)}{\lambda^n\overline{\rho_n(\lambda)}} \in \mathbb{T}$, the so-called "Szegő-Radau rules" arise (see e.g. [10, Proposition 2.8]). Recently, Jagels and Reichel have characterized Szegő-Lobatto quadrature formulas in [24], i.e. Szegő rules with two prescribed nodes on $\mathbb{T}$. Suppose z_α and z_β are two distinct points on $\mathbb{T}$. Let $N \geq 2$ be fixed, and assume there exist $N-2$ distinct nodes $z_1, \ldots, z_{N-2}$ on $\mathbb{T}$ such that $z_j \neq z_\alpha$ and $z_j \neq z_\beta$ for $1 \leq j \leq N-2$. Furthermore, suppose there exist positive weights $A_1, A_2, \lambda_1, \ldots, \lambda_{N-2}$ so that

$$\forall f \in \Lambda_{-(N-2),N-2}: \quad I_N(f) := A_1f(z_\alpha) + A_2f(z_\beta) + \sum_{j=1}^{N-2}\lambda_jf(z_j) = \int_{\mathbb{T}} f(z)d\mu(z) =: I_\mu(f)\ .$$

Then, $I_N(f)$ is called an N-point Szegő-Lobatto quadrature formula for μ with prescribed nodes z_α and z_β. We now have the following theorem (for the proof, see [24]).

Theorem 6.1 (Szegő-Lobatto quadrature) *Let z_α and z_β be two distinct fixed points on $\mathbb{T}$ and set $a = z_\alpha^{n-1}\frac{\overline{\rho_n(z_\alpha)}}{\rho_n(z_\alpha)} \in \mathbb{T}$ and $b = z_\beta^{n-1}\frac{\overline{\rho_n(z_\beta)}}{\rho_n(z_\beta)} \in \mathbb{T}$. Then one of the following statements holds:*

(1) If $az_\alpha = bz_\beta$, we have that an n-point Szegő formula $I_n(f)$ has z_α and z_β as nodes through setting $\tau_n = -\frac{\rho_n(z_\alpha)}{\rho_n^*(z_\alpha)} = -\frac{\rho_n(z_\beta)}{\rho_n^*(z_\beta)} \in \mathbb{T}$.

(2) If $a = b$, an $(n+1)$-point Szegő formula $I_{n+1}(f)$ has z_α and z_β as nodes through setting $\tau_{n+1} = -z_\alpha\frac{\rho_n(z_\alpha)}{\rho_n^*(z_\alpha)} = -z_\beta\frac{\rho_n(z_\beta)}{\rho_n^*(z_\beta)} \in \mathbb{T}$.

(3) Suppose $az_\alpha \neq bz_\beta$ and $a \neq b$, and denote by Γ the circle with center $c = -\frac{z_\alpha - z_\beta}{az_\alpha - bz_\beta}$ and radius $r = \left|\frac{a-b}{az_\alpha - bz_\beta}\right|$. Let $\tilde{\delta}_{n+1} \in \Gamma \cap \mathbb{D}$, a circular arc that is proved to be a non-empty set, and set $\tau_{n+2} = -\frac{az_\alpha - bz_\beta}{a-b}\tilde{\delta}_{n+1} - \frac{z_\alpha - z_\beta}{a-b} \in \mathbb{T}$. Then, there exists an $(n+2)$-point Szegő-Lobatto quadrature formula whose nodes are the zeros of $\tilde{\Phi}_{n+2}(z, \tau_{n+2}) = z\tilde{\rho}_{n+1}(z) + \tau_{n+2}\tilde{\rho}_{n+1}^*(z)$ with $\tilde{\rho}_{n+1}(z) = z\rho_n(z) + \tilde{\delta}_{n+1}\rho_n^*(z)$.

□

Remark 6.2 *In the special case in which $\tilde{\delta}_{n+1} = \rho_{n+1}(0)$ in Theorem 6.1(3), the $(n+2)$-point Szegő-Lobatto formula is actually an $(n+2)$-point Szegő formula and consequently, exact in $\Lambda_{-(n+1),n+1}$.*

In analogy with the real line situation (see e.g. [19]), the set of Szegő, Szegő-Radau and Szegő-Lobatto rules we call Szegő-type quadratures.

Example 6.3 *Consider the rational modification of the Lebesgue measure given by*

$$d\tilde{\mu}(\theta) = \frac{d\theta}{5/4 - \sin(2\theta)} = \frac{d\theta}{|z^2 - \mathbf{i}/2|^2}, \quad z = e^{\mathbf{i}\theta}.$$

From Example 5.1 it then follows that the corresponding monic Szegő polynomials are explicitly given by $\rho_n(z) = z^{n-2}(z^2 - \mathbf{i}/2)$ for every $n \geq 2$. Next, let $n = 10$, and suppose $z_\alpha = 1$ and $z_\beta = \mathbf{i}$. From Theorem 6.1 we then deduce that

$$a = \frac{3+4\mathbf{i}}{5} \quad \textit{and} \quad b = \frac{4+3\mathbf{i}}{5},$$

where clearly $a - b = \frac{-1+\mathbf{i}}{5} \neq 0$ and $az_\alpha - bz_\beta = \frac{6}{5} \neq 0$. Consequently, we are dealing with the third situation of Theorem 6.1. Elementary calculation yields that Γ is the circle with center $c = \frac{5}{6}(-1+\mathbf{i})$ and radius $r = \frac{\sqrt{2}}{6}$, and $\tilde{\delta}_{11} \in \Gamma \cap \mathbb{D}$ iff (see also Figure 1)

$$\tilde{\delta}_{11} = c + re^{\mathbf{i}\theta}, \quad \textit{with} \quad -\theta_0 < \theta < -\theta_1, \quad \theta_0 = \arctan(7), \quad \theta_1 = \frac{\pi}{2} - \theta_0. \tag{46}$$

Since $\rho_{11}(0) = 0 \notin \Gamma \cap \mathbb{D}$, it follows that the 12-point Szegő-Lobatto formula associated with $\tilde{\delta}_{11}$ is not a 12-point Szegő formula (see Remark 6.2); thus, the domain of exactness is $\Lambda_{-10,10}$. Therefore, the nodes in the 12-point Szegő-Lobatto formula associated with $\tilde{\delta}_{11}$ are the zeros of $\tilde{\Phi}_{12}(z, \tau_{12})$, given by

$$\tilde{\Phi}_{12}(z, \tau_{12}) = z\tilde{\rho}_{11}(z) + \tau_{12}\tilde{\rho}_{11}^*(z),$$

where

$$\tilde{\rho}_{11}(z) = z\rho_{10}(z) + \tilde{\delta}_{11}\rho_{10}^*(z) = z^9(z^2 - \mathbf{i}/2) + \tilde{\delta}_{11}(1 + z^2\mathbf{i}/2),$$

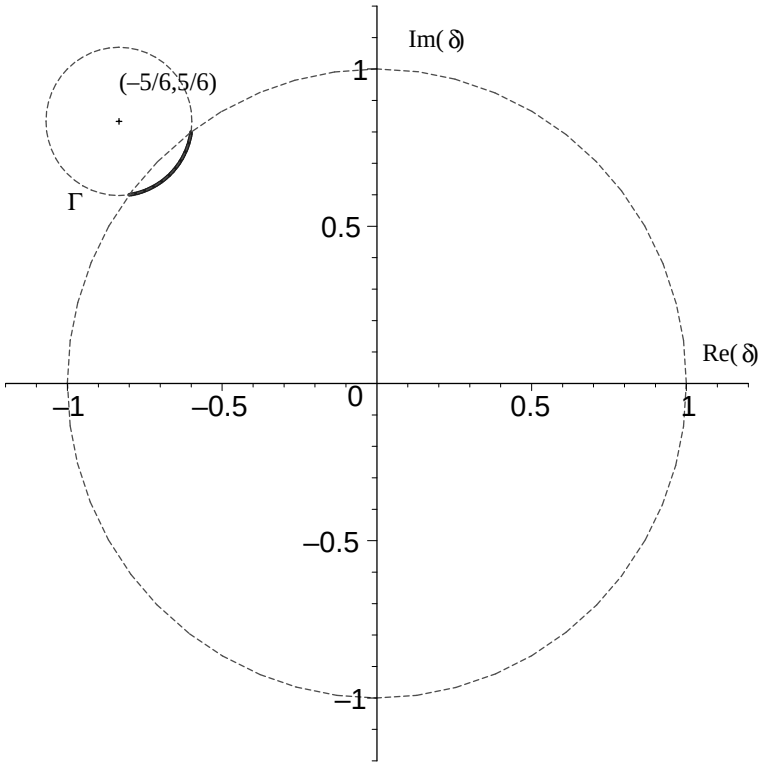

Fig. 1. The circular arc $\Gamma \cap \mathbb{D}$.

and $\tau_{12} = 3(1+\mathbf{i})\tilde{\delta}_{11}+5$. *Let* z_j, $j = 1, \ldots, 12$, *denote the zeros of* $\tilde{\Phi}_{12}(z, \tau_{12})$. *We have used Maple 10 with 40 digits to compute these zeros for the case in which* $\tilde{\delta}_{11} = -\frac{3}{4} + \frac{10-\sqrt{7}}{12}\mathbf{i}$ *(that is (46) with* $\theta = -\arctan(\sqrt{7})$*) and* $\tau_{12} = \frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}\mathbf{i}$, *respectively for the case in which* $\tilde{\delta}_{11} = \frac{2}{3}(-1+\mathbf{i})$ *(that is (46) with* $\theta = -\pi/4$*) and* $\tau_{12} = 1$. *The results are given in Table 1, respectively Table 2.*

Finally, since $\tilde{\delta}_{11} \neq \rho_{11}(0)$, *we are dealing with a modified measure so that we can not use Theorem 5.3 to compute the corresponding weights. Let* $\tilde{\varphi}_{11}(z)$ *denote the polynomial of degree* 11 *that is orthonormal with respect to this modified measure. We then have the following Christoffel-Darboux formula in the ordinary polynomial situation (see also [3, Theorem 3.1.3]):*

$$\mathcal{K}_{11}(z,\xi) = \frac{\tilde{\varphi}^*_{11}(z)\overline{\tilde{\varphi}^*_{11}(\xi)} - z\overline{\xi}\tilde{\varphi}_{11}(z)\overline{\tilde{\varphi}_{11}(\xi)}}{1 - z\overline{\xi}}.$$

Setting $\lambda_j^{-1} = \mathcal{K}_{11}(z_j, z_j)$ *and proceeding as in Section 4, it is easily verified that*

$$\lambda_j^{-1} = 2\Re\left\{z_j\overline{\tilde{\varphi}_{11}(z_j)}\tilde{\varphi}'_{11}(z_j)\right\} - 10\left|\tilde{\varphi}_{11}(z_j)\right|^2.$$

Further, with $\tilde{\varphi}_{11}(z) = c\tilde{\rho}_{11}(z)$ *we obtain that*

$$\lambda_j^{-1} = |c|^2\mu_j^{-1}, \quad \mu_j^{-1} := 2\Re\left\{z_j\overline{\tilde{\rho}_{11}(z_j)}\tilde{\rho}'_{11}(z_j)\right\} - 10\left|\tilde{\rho}_{11}(z_j)\right|^2,$$

so that it remains to determine $|c|^2$. *It should now hold that*

$$\sum_{j=1}^{12}\lambda_j = \int_{-\pi}^{\pi}\frac{d\theta}{\left|e^{2\mathbf{i}\theta} - \mathbf{i}/2\right|^2} = \frac{8\pi}{3} = \frac{\sum_{j=1}^{12}\mu_j}{|c|^2},$$

Table 1
The nodes z_j and weights λ_j in the 12-point Szegő-Lobatto formula for the case in which $\tilde{\delta}_{11} = -\frac{3}{4}+\frac{10-\sqrt{7}}{12}\mathbf{i}$ and $\tau_{12} = \frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}\mathbf{i}$.

j	z_j	λ_j
1	1	0.4516560678
2	$0.9638145989 + 0.2665734776\mathbf{i}$	0.1232452240
3	$0.8316896358 + 0.5552408033\mathbf{i}$	1.3564329045
4	$0.5255107494 + 0.8507869605\mathbf{i}$	1.3244867113
5	$\mathbf{i}$	0.4907276839
6	$-0.5936593393 + 0.8047164649\mathbf{i}$	0.2935974363
7	$-0.9584388804 + 0.2852979365\mathbf{i}$	0.3549235398
8	$-0.9499460287 - 0.3124140564\mathbf{i}$	0.8469595298
9	$-0.7027746634 - 0.7114125192\mathbf{i}$	1.6733022550
10	$-0.2969879126 - 0.9548812386\mathbf{i}$	0.8202977524
11	$0.3024297035 - 0.9531716920\mathbf{i}$	0.3490662051
12	$0.8140918496 - 0.5807361367\mathbf{i}$	0.2928850997

and hence,

$$|c|^2 = \frac{3}{8\pi}\sum_{j=1}^{12}\mu_j.$$

The resulting weights are given in Table 1 and 2 for the case in which $\tilde{\delta}_{11} = -\frac{3}{4} + \frac{10-\sqrt{7}}{12}\mathbf{i}$ *and* $\tau_{12} = \frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}\mathbf{i}$*, respectively for the case in which* $\tilde{\delta}_{11} = \frac{2}{3}(-1+\mathbf{i})$ *and* $\tau_{12} = 1$*.*

With these nodes and weights, the integral is computed to within the numerical precision for

$$\int_{-\pi}^{\pi}\left(\frac{5}{4} - \sin(2\theta)\right)d\tilde{\mu}(\theta), \quad \textit{and for} \quad \int_{-\pi}^{\pi} e^{\mathbf{i}k\theta}d\tilde{\mu}(\theta), \quad k = -10,\ldots,10.$$

Example 6.4 *Consider the rational modifications*

$$d\tilde{\mu}(\theta) = \frac{\omega_i(\theta)d\theta}{|z-1/4|^2}, \quad i = 2,3,4, \quad z = e^{\mathbf{i}\theta},$$

where $\omega_i(\theta)$ *is given by* (6)*, and let*

$$f_1(z) = \frac{(z-1/4)\sin(z)}{8\pi z}, \quad f_2(z) = \frac{(4-z)\sin(1/z)}{8\pi z} \quad \textit{and} \quad f_3(z) = \frac{\sin(z)}{8\pi z}.$$

For these functions, we have used Maple 10 with 40 digits to compute the absolute error of the n-point Szegő quadrature formula, i.e.

$$e_{\tilde{\mu}}(f_j) = |I_{\tilde{\mu}}(f_j) - I_n(f_j)|, \quad j = 1,2,3,$$

for $n = 8, 16, 24$ *with* $\tau_n = 1$*. The results are given in Tables 3–5.*

Table 2
The nodes z_j and weights λ_j in the 12-point Szegő-Lobatto formula for the case in which $\tilde{\delta}_{11} = \frac{2}{3}(-1+\mathbf{i})$ and $\tau_{12} = 1$.

j	z_j	λ_j
1	1	0.4833219467
2	$0.8615961242 + 0.5075944433\mathbf{i}$	1.1348377814
3	$(1+\mathbf{i})/\sqrt{2}$	0.5132214093
4	$0.5075944433 + 0.8615961242\mathbf{i}$	1.1348377814
5	$\mathbf{i}$	0.4833219467
6	$-0.5892211604 + 0.8079717966\mathbf{i}$	0.2926150925
7	$-0.9562662716 + 0.2924975517\mathbf{i}$	0.3518577037
8	$-0.9522800017 - 0.3052258155\mathbf{i}$	0.8334298010
9	$-(1+\mathbf{i})/\sqrt{2}$	1.6722343497
10	$-0.3052258155 - 0.9522800017\mathbf{i}$	0.8334298010
11	$0.2924975517 - 0.9562662716\mathbf{i}$	0.3518577037
12	$0.8079717966 - 0.5892211604\mathbf{i}$	0.2926150925

Table 3
The absolute error $e_{\tilde{\mu}}(f_1)$ for the n-point Szegő quadrature formula with respect to the rational modification of the weight functions $\omega_i(\theta)$.

	$i=2$	$i=3$	$i=4$
$n=8$	$4.381892940 \times 10^{-8}$	$6.032565503 \times 10^{-7}$	$7.738268455 \times 10^{-7}$
$n=16$	$4.417815525 \times 10^{-17}$	$6.151507562 \times 10^{-16}$	$7.903210143 \times 10^{-16}$
$n=24$	$1.010117754 \times 10^{-27}$	$1.410431794 \times 10^{-26}$	$1.812756380 \times 10^{-26}$

Table 4
The absolute error $e_{\tilde{\mu}}(f_2)$ for the n-point Szegő quadrature formula with respect to the rational modification of the weight functions $\omega_i(\theta)$.

	$i=2$	$i=3$	$i=4$
$n=8$	$5.094348466 \times 10^{-5}$	$9.843173970 \times 10^{-5}$	$9.705465650 \times 10^{-5}$
$n=16$	$1.925426989 \times 10^{-13}$	$3.815676087 \times 10^{-13}$	$3.801621369 \times 10^{-13}$
$n=24$	$9.701673019 \times 10^{-24}$	$1.932272042 \times 10^{-23}$	$1.92904885 \times 10^{-23}$

7 Appendix

Theorem 7.1 *There are no $\tau_n \in \mathbb{T}$ and $\gamma \in \mathbb{T} \setminus \{\nu_i\}$ so that $p_1^{(i)*}(\gamma) = 0$.*

Proof.- Suppose there are $\tau_n \in \mathbb{T}$ and $\gamma \in \mathbb{T} \setminus \{\nu_i\}$ so that $\tau_n p_1^{(i)*}(\gamma) = 0$. Clearly, it then holds

Table 5
The absolute error $e_{\tilde{\mu}}(f_3)$ for the n-point Szegő quadrature formula with respect to the rational modification of the weight functions $\omega_i(\theta)$.

	$i=2$	$i=3$	$i=4$
$n=8$	1.750776412×10^{-7}	3.438809772×10^{-7}	3.376229665×10^{-7}
$n=16$	$1.766481103\times10^{-16}$	$3.512396863\times10^{-16}$	$3.491851390\times10^{-16}$
$n=24$	$4.039751881\times10^{-27}$	$8.056536588\times10^{-27}$	$8.033580897\times10^{-27}$

that $p_1^{(i)}(\gamma)=0$ as well. Hence, from (27) we then deduce that

$$\tau_n=-\frac{a_n^{(i)}(1-\overline{\alpha_n}\gamma)}{1-\overline{b_n^{(i)}}\gamma}=-\frac{\gamma-b_n(i)}{\overline{a_n^{(i)}}(\gamma-\alpha_n)},\quad \gamma\in\mathbb{T}\setminus\{\nu_i\}.$$

Therefore,

$$\begin{aligned}0&=\gamma\left(|\gamma-b_n^{(i)}|^2-|a_n^{(i)}|^2|\gamma-\alpha_n|^2\right)\\&=(|a_n^{(i)}|^2\overline{\alpha_n}-\overline{b_n^{(i)}})\gamma^2+[1+|b_n^{(i)}|^2-|a_n^{(i)}|^2(1+|\alpha_n|^2)]\gamma+(|a_n^{(i)}|^2\alpha_n-b_n^{(i)})\\&=(\gamma-\nu_i)\left[(|a_n^{(i)}|^2\overline{\alpha_n}-\overline{b_n^{(i)}})\gamma-(|a_n^{(i)}|^2\alpha_n-b_n^{(i)})\right],\quad \gamma\in\mathbb{T}\setminus\{\nu_i\},\end{aligned}\tag{47}$$

where the last equality is due to the fact that the equality in (47) clearly holds for $\gamma=\nu_i$. Since we assumed $\gamma\neq\nu_i$, it follows that

$$\gamma=\nu_i\frac{|a_n^{(i)}|^2\alpha_n-b_n^{(i)}}{|a_n^{(i)}|^2\overline{\alpha_n}-\overline{b_n^{(i)}}}.$$

But from (10) we deduce that

$$|a_n^{(i)}|^2\alpha_n-b_n^{(i)}=\frac{2\Re\{\alpha_n\}-\nu_i[|1-\nu_i\overline{\alpha_n}|^2Q_n^{(i)}+1]}{|(1-\nu_i\overline{\alpha_n})Q_n^{(i)}+1|^2}-\nu_i\in\mathbb{R},$$

so that we again find that γ should equal ν_i.

□

Theorem 7.2 *If $n=1$ or $\alpha_n=0$, there are no $\tau_n\in\mathbb{T}$ and $\gamma\in\mathbb{T}\setminus\{\pm1\}$ so that $\tau_n p_2^{(2)*}(\gamma)=0$.*

Proof.- Similarly as in the proof of Theorem 7.1, we find that there are $\tau_n\in\mathbb{T}$ and $\gamma\in\mathbb{T}\setminus\{\pm1\}$ so that $\tau_n p_2^{(2)*}(\gamma)=0$ iff

$$\begin{aligned}0&=\gamma^2\left(|f_n+g_n\gamma+\gamma^2|^2-|d_n+e_n\gamma|^2|\gamma-\alpha_n|^2\right)\\&=(\gamma^2-1)\left[(e_n\overline{d_n}\overline{\alpha_n}+\overline{f_n})\gamma^2+\varpi_n\gamma-(\overline{e_n}d_n\alpha_n+f_n)\right],\quad \gamma\in\mathbb{T}\setminus\{\pm1\},\end{aligned}\tag{48}$$

where the last equality is due to the fact that the equality in (48) clearly holds for $\gamma=\pm1$, and $\varpi_n\in\mathbb{R}$ is given by

$$\begin{aligned}\varpi_n &= g_n + f_n\overline{g_n} + \alpha_n(|d_n|^2 + |e_n|^2) - d_n\overline{e_n}(1 + |\alpha_n|^2)\\ &= \overline{g_n} + g_n\overline{f_n} + \overline{\alpha_n}(|d_n|^2 + |e_n|^2) - e_n\overline{d_n}(1 + |\alpha_n|^2).\end{aligned}$$

Since we assumed $\gamma \notin \{\pm 1\}$, it should hold that

$$(e_n\overline{d_n}\overline{\alpha_n} + \overline{f_n})\gamma^2 + \varpi_n\gamma - (\overline{e_n}d_n\alpha_n + f_n) = 0.$$

Note that for the parameters a^+, a^-, b^+ and b^-, given by (14), we have that

$$\begin{aligned}&\overline{a^+} = \tfrac{1}{a^+|1-\alpha_n|^2}, \qquad \overline{a^-} = \tfrac{1}{a^-|1+\alpha_n|^2},\\ &\overline{b^+} = b^+ + \tfrac{2\mathbf{i}\Im\{\alpha_n\}}{|1-\alpha_n|^2}, \; \overline{b^-} = b^- - \tfrac{2\mathbf{i}\Im\{\alpha_n\}}{|1+\alpha_n|^2}.\end{aligned}$$

With this observation in mind, we computed ϖ_n with the aid of Maple 10 to find that $\varpi_n = 0$, so that it should hold that

$$\gamma^2 = \frac{\overline{e_n}d_n\alpha_n + f_n}{e_n\overline{d_n}\overline{\alpha_n} + \overline{f_n}}. \tag{49}$$

For the special case in which $\alpha_n = 0$, it follows from (18) that $f_n = \frac{-2}{\pi|h_n|^2} \in \mathbb{R}$, so that we again find that γ should equal ± 1. For $n = 1$, on the other hand, we have that

$$a^+ = \frac{1}{1-\overline{\alpha_1}}, \quad a^- = \frac{1}{1+\overline{\alpha_1}}, \quad b^+ = 2 + a^+ \quad \text{and} \quad b^- = 2 + a^-.$$

We then computed γ^2, given by (49) with $n = 1$, to find that $\gamma^2 = 1$. At this moment of writing, however, we could not verify whether γ^2, given by (49), equals one for the more general case of $\alpha_n \neq 0$ and $n > 1$.

□

References

[1] **A. Bultheel, L. Daruis and P. González-Vera**.- *A connection between quadrature formulas on the unit circle and the interval* $[-1, 1]$, Appl. Numer. Math. 132 (2001), 1-14.

[2] **A. Bultheel, L. Daruis and P. González-Vera**.- *Positive interpolatory quadrature formulas and para-orthogonal polynomials*, J. Comp. Appl. Math. 179 (2005), 97-119.

[3] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad**.- *Orthogonal rational functions*, Cambridge Monographs on Applied and Computational Mathematics, Vol. 5, Cambridge University Press, 1999.

[4] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad**.- *Orthogonal rational functions and quadrature on the unit circle*, Numer. Algor. 3 (1992), 105-116.

[5] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad**.- *Quadrature and orthogonal rational functions*, J. Comp. Appl. Math. 127 (2001), 67-91.

[6] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad**.- *Quadrature formulas on the unit circle based on rational functions*, J. Comp. Appl. Math. 50 (1994), 159-170.

[7] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad**.- *Rational quadrature formulas on the unit circle with prescribed nodes and maximal domain of validity*, Submitted 2007.

[8] **M.J. Cantero, R. Cruz-Barroso and P. González-Vera**.- *A matrix approach to the computation of quadrature formulas on the unit circle*, Appl. Numer. Math. 58 (2008), 296-318.

[9] **R. Cruz-Barroso, L. Daruis, P. González-Vera and O. Njåstad**.- *Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle*, J. Comp. Appl. Math. 200 (2007), 424-440.

[10] **L. Daruis and P. González-Vera**.- *Szegő polynomials and quadrature formulas on the unit circle*, Appl. Numer. Math. 36 (2000), 79-112.

[11] **L. Daruis, P. González-Vera and M. Jiménez-Paiz**.- *Quadrature formulas associated with rational modifications of the Chebyshev weight functions*, Comp. Math. Appl. 51 (2006), 419-430.

[12] **L. Daruis, P. González-Vera and O. Njåstad**.- *Szegő quadrature formulas for certain Jacobi-type weight functions*, Math. Comp. 71 (2002), 683-701.

[13] **P.J. Davis**.- Interpolation and Approximation. Dover Publications, New York 1975.

[14] **K. Deckers and A. Bultheel**.- *Orthogonal rational functions and rational modifications of a measure on the unit circle*, J. Approx. Theory, accepted 2008.

[15] **K. Deckers, J. Van Deun and A. Bultheel**.- *An extended relation between orthogonal rational functions on the unit circle and the interval* $[-1, 1]$, J. Math. Anal. Appl. 334 (2007), 1260-1275.

[16] **K. Deckers, J. Van Deun and A. Bultheel**.- *Computing rational Gauss-Chebyshev quadrature formulas with complex poles*, In B. H. V. Topping, G. Montero, and R. Montenegro, editors, Proceedings of the Fifth International Conference on Engineering Computational Technology, Kippen, Stirlingshire, United Kingdom, 2006. Civil-Comp Press. Paper 30.

[17] **K. Deckers, J. Van Deun and A. Bultheel**.- *Computing rational Gauss-Chebyshev quadrature formulas with complex poles: the algorithm*. Advances in Engineering Software, 2006. Submitted.

[18] **K. Deckers, J. Van Deun and A. Bultheel**.- *Rational Gauss-Chebyshev quadrature formulas for complex poles outside* $[-1, 1]$, Math. Comp., 77(262) (2008), 967-983, 2008.

[19] **W. Gautschi**.- *A survey of Gauss-Christoffel quadrature formulae*. In E.B. Christoffel; The Influence of his Work in Mathematics and the Physical Sciences, P. L. Eatzer and F. Fehér, Birkhäuser Eds., Basel (1981), 72-147.

[20] **W. Gautschi**.- *Quadrature rules for rational functions*, Numer. Math. 86(4) (2000), 617-633.

[21] **Ya. L. Geronimus**.- *Polynomials orthogonal on a circle and interval*, International Series of Monographs in Pure and Applied Mathematics, Pergamon Press, Oxford, 1960.

[22] **W.B. Gragg**.- *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators and Gaussian quadrature on the unit circle*, J. Comput. Appl. Math. 46 (1993), 183-198. This is a slightly revised version of a paper by the same author and published in Russian in: E. S. Nicholaev editor, Numer. Meth. Lin. Alg., Moscow University Press, Moscow (1982), 16-32.

[23] **U. Grenander and G. Szegő**.- *Toeplitz forms and their applications*, Chelsea, New York NY, 1984.

[24] **C. Jagels and L. Reichel**.- *Szegő-Lobatto quadrature rules*, J. Comp. Appl. Math. 200 (2007), 116-126.

[25] **W.B. Jones, O. Njåstad and W.J. Thron**.- *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. 21 (1989), 113-152.

[26] **B. Simon**.- *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, Amer. Math. Soc. Coll. Publ. Vol. 54.1, Amer. Math. Soc. Providence, R.I. 2005.

[27] **G. Szegő**.- *On bi-orthogonal systems of trigonometric polynomials*, Magyar Tud. Akad. Kutató Int. Közl 8 (1963), 255-273.

[28] **G. Szegő**.- *Orthogonal polynomials*, Amer. Math. Soc. Coll. Publ. Vol 23, Amer. Math. Soc. Providence, R.I. 1975.

[29] **W. Van Assche and I. Vanherwegen**.- *Quadrature formulas based on rational interpolation*, Math. Comp. 61(204) (1993), 765-783.

[30] **J. Van Deun, A. Bultheel and P. González-Vera**.- *On computing rational Gauss-Chebyshev quadrature formulas*, Math. Comp. 75(253) (2007), 307-326.

[31] **J. Van Deun, K. Deckers, A. Bultheel and J. A. C. Weideman**.- *Algorithm 882: Near best fixed pole rational interpolation with applications in spectral methods*, ACM Transactions on Mathematical Software, accepted 2008.

[32] **P. Van gucht and A. Bultheel**.- *A relation between orthogonal rational functions on the unit circle and the interval* $[-1, 1]$, Comm. Anal. Th. Continued Fractions, 8 (2000), 170-182.

[33] **H. Waadeland**.- *A Szegő quadrature formula for the Poisson integral*, J. Comp. Appl. Math. In: C. Brezinski and U. Kulish editors, Elsevier Science Publishers B. V. (North-Holland) IMACS (1992), 479-486.