Prescribed-Time Unknown Input Observers Design for Singular Systems: A Periodic Delayed Output Approach

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***Abstract*—This paper studies the design of prescribed-time full- and reduced-order unknown input observers (UIOs) for linear singular systems with input delay, satisfying the observer matching condition and the strong observability condition. To this end, a condition in terms of a matrix equation is firstly used to transform the considered linear singular system into a regular one with unknown input. Then, prescribed-time full- and reduced-order UIOs are established by employing periodic delayed output approach such that the estimation error converges to zero in prescribed time. Finally, comparative simulations are shown to demonstrate the effectiveness and superiority of the proposed approaches.**

***Index Terms*—Linear singular systems, Unknown input, Prescribed-time observer, Input time-delay, Periodic delayed output.**

I. Introduction

In practical situations, some inputs might be inaccessible. The unknown output observer (UIO) serves as an effective estimation tool, which is able to observe the system's state by exploiting the measurable output. The design of UIOs has received extensive attention [4], [6]. In dynamic systems, actuator failures, external disturbances [27], particular types of model uncertainty [16], and network attacks can all be seen as unknown input signals. Therefore, the UIO technique has broad prospects and great value in theoretical analysis and synthesis, such as fault tolerant control [5], fault estimation [8], and attack detection [23]. On the other hand, fruitful results have been reported in areas where the UIO technique is applicable, see, for example, [1], [11], [20], and [21].

Singular systems, also known as descriptor systems and differential-algebraic systems, constitute a more general class of dynamical systems that exhibits both dynamic and static properties of the system. However, traditional state-space models can only characterize the dynamics of the system. Therefore, in certain scenarios, singular systems can describe a real-time response of a physical system more accurately and vividly than standard state-space models. In addition, they naturally arise in modeling systems with a network structure, where the differential equations typically describe the dynamics of subsystems and the delay-difference equations their interconnections. Singular systems have been used to characterize various physical systems such as circuit systems [22], mechanical systems [12], and biological systems [26]. Due to their special structure, a slight change in the input may cause adverse effects on the system [9]. As a consequence, the state estimation of singular systems with unknown input is one of the hot issues to be considered and studied. Until now, a large number of achievements on the design of the UIO for singular systems have been reported [2], [7], [10], [14], [24]. Existence conditions and design procedures of UIO were given in [24]. By solving a generalized constrained Sylvester equation, the authors of [7] designed a reduced-order UIO to estimate some unknown states of the system under unmeasurable inputs and gave criteria for the existence of the observer. Different from the above literature, article [14] considered a more complex model, namely, a singular system with multiple unknown inputs and unknown faults, and achieved an efficient estimation of faults and states by developing a proportional multiple-integral observer. All of the above design methods estimate system states by constructing differential equations. Unlike their method, based on the strong detectability condition, [2] used higher-order derivatives of measurable output signals to establish relationships with system states and to achieve asymptotical estimation.

It is noticed that the UIOs designed for singular systems in the above-mentioned results are asymptotically converging observers. The convergence time for such an observer to achieve acceptable convergence accuracy is difficult to determine, since it is related with the initial states of the original system and the convergence rate of the observer. An improvement consists of configuring the spectrum of the observer to fall as far as possible on the leftmost of the complex plane, so as to achieve a faster convergence rate. However, this would require a large observer gain, rendering the observer difficult to implement. Therefore, an interesting and meaningful task is to find new UIO design methods that can achieve fast and accurate state estimation. Recently, few works [25], [28] have investigated the finite-time UIOs design for continuous-time singular systems. These methods need to solve matrix equations to determine the observer gains. With increasing matrix dimension, it may difficult to balance accuracy and

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calculation time to obtain a solution. Recently, a simple and straightforward UIO design method proposed in [33] has been used to solve the above problem in [17]. This motivates us to apply that method to design a prescribed-time UIO for singular systems. In contrast to [17], the system considered in this paper is singular and contains input delay. In most cases, time delays tend to have a negative impact in control systems [18].

In this paper, we show that, for a broad class of singular systems with input delay, periodic delayed output approach can be used to achieve prescribed-time full- and reduced-order UIOs. Its main contributions can be stated as:

- Based on a transformed linear system, the full- and reduced-order UIOs are developed to estimate the states of the underlying singular systems in prescribed time.
- The observer gains designed in this paper are only related to the known system parameter matrices and are continuous, continuously differentiable, and even smooth.

The rest parts of this work are list as follows. The problem formulation and some preliminaries are stated in Section II. Both prescribed-time full- and reduced-order UIOs are designed in Section III. Two simulation examples are carried out to substantiate the feasibility of the proposed techniques in Section IV and some conclusions are summarised in Section V.

Notation: The n dimensional Euclidean space is denoted as $\mathbf{R}^n$. The sets of positive integer numbers, positive real numbers, and $n \times l$ real matrices are denoted by, $\mathbf{Z}_+, \mathbf{R}_+$, and $\mathbf{R}^{n\times l}$, respectively. For a (square) matrix $\mathcal{A}$, we employ $\mathcal{A}^{\mathrm{T}}$, $\mathcal{A}^{-1}$ and $\mathcal{A}^{-}$ to stand its transpose, inverse and generalized inverse, respectively. We use I_n to indicate the $n \times n$-dimensional identity matrix.

II. Problem introduction and preliminaries

Consider a linear singular system with unknown input

$$\begin{cases} F\dot{x} &= Ax + Bu(t-h) + Ed, \\ y &= Cx + Du(t-h), \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the known input, $d \in \mathbf{R}^l$ is the unknown input, $y \in \mathbf{R}^p$ is the measurable output, $h \in \mathbf{R}_+$ is a known scalar representing the delay, and (A, B, C, D, E, F) is a matrix group with $\mathrm{rank}(E) = l$, $\mathrm{rank}(C) = p$, and $\mathrm{rank}(F) = \bar{n} < n$. Let the initial conditions be $x(0) = x_0$ and $u(\theta) = \nu(\theta), \theta \in [-h, 0)$, where $\nu(\theta)$ is a piecewise continuous function. We omit the time variables in state/input/output vectors, for example, $x = x(t)$.

The goal of the work is to design UIOs such that the estimation errors converge to zero in the prescribed time T, as shown below.

Problem 1: Design full- and reduced-order prescribed-time observers with the form

$$\begin{cases} \dot{\chi} = g(\chi_t, u_t, y_t), \\ \hat{x} = f(\chi_t, u_t, y_t), \end{cases}$$

such that $\hat{x} - x = 0, \forall t \geq T$.

For solving Problem 1, two assumptions are firstly introduced below.

Assumption 1: [15] Matrices E, F, and C satisfy

$$\mathrm{rank}\begin{bmatrix} F & E \\ C & 0 \end{bmatrix} = n + \mathrm{rank}(E). \tag{2}$$

Assumption 2: [15] Matrices F, A, E and C satisfy

$$\mathrm{rank}\begin{bmatrix} sF - A & E \\ C & 0 \end{bmatrix} = n + \mathrm{rank}(E), \ \forall s \in \mathbf{C}. \tag{3}$$

Remark 1: Assumptions 1 and 2 are the so-called observer matching condition (OMC) and strongly observability condition (SOC), respectively. In this paper, inspired by [32], Assumption 1 is taken to remove the unknown input term by utilizing the output signal, and Assumption 2 is to ensure that the observation errors converge to zero in prescribed time. Moreover, under Assumption 2, the column regularity of system (1) can be guaranteed, which means that the system has a unique solution [13].

Secondly, we recall an important proposition that originates from [33]

Proposition 1: Consider the system

$$\dot{\zeta} = \bar{A}(t)\zeta + \bar{B}(t)\zeta(t-\tau), \ t \geq 0, \tag{4}$$

where $\tau \in \mathbf{R}_+, \zeta(\theta) = \varphi(\theta), \theta \in [-\tau, 0]$, $\left(\bar{A}(t), \bar{B}(t)\right) \in (\mathbf{R}^{n\times n}, \mathbf{S}^{(k)}_{n\times n}(\tau))$ are 2τ-periodic functions and assume that $\bar{B}(t) \neq 0$ for $t \in [0, \tau]$. Let the state transition matrix of system $\dot{\zeta} = \bar{A}(t)\zeta$ be given by $\Phi(t, s)$, and the monodromy matrix [1] of system (4) be expressed by

$$\Delta(\tau) = \Phi(2\tau, 0) + \int_{\tau}^{2\tau} \Phi(2\tau, s)\bar{B}(s)\Phi(s-\tau, 0)\,\mathrm{d}s.$$

If $\iota = \min\{\bar{\iota} \geq 1 | (\Delta(\tau))^{\bar{\iota}} = 0, \bar{\iota} \in \mathbf{N}\}$ exists, then it holds that

$$\zeta = 0, \ \forall t \geq 2\iota\tau. \tag{5}$$

Thirdly, we give two lemmas. The first lemma is used to transform the system (1) into a regular linear system with unknown input, whereas the second lemma acts to remove d such that the transformed linear system has a form amenable for observer design.

Lemma 1: There exists a nonsingular matrix P and a matrix Q satisfying

$$PF + QC = I_n, \tag{6}$$

if Assumption 1 is satisfied.

Appendix A contains the proof.

Lemma 2: If condition (2) hold, then

$$\tilde{W}PE = 0, \tag{7}$$

where P is defined in (55) in Appendix A, and

$$\tilde{W} = I_n - WC, \ W = PE(CPE)^{-}. \tag{8}$$

Proof: From [12], we get that

$$\mathrm{rank}\begin{bmatrix} F & E \\ C & 0 \end{bmatrix} = n + \mathrm{rank}(CPE).$$

[1] This matrix maps $\zeta(0)$ to $\zeta(2\tau)$.

Note that

$$\operatorname{rank}(E)=\operatorname{rank}\left[\begin{array}{cc}-I & C\\ 0 & I\end{array}\right]\left[\begin{array}{c}CPE\\ PE\end{array}\right]=\operatorname{rank}\left[\begin{array}{c}CPE\\ PE\end{array}\right].$$

Thus if condition (2) hold, then

$$\operatorname{rank}\left[\begin{array}{c}CPE\\ PE\end{array}\right]=\operatorname{rank}(CPE).$$

The remaining proof is similar to the proof of Lemma 1 in [17] and thus is omitted. ■

At the end of this section, we introduce a definition.

Definition 1: [33] Given $\tau\in\mathbf{R}_+$ and $k\in\mathbf{Z}_+$. If a matrix function $H_\tau(t):\mathbf{R}\to\mathbf{R}^{p_1\times p_2}$ satisfies

1) $H_\tau(t)$ is continuously differentiable with order k, which means $H_r^{(j)}(\tau)=H_r^{(j)}(2\tau)=0,\ j=0,1,\ldots,k$,

2) $H_\tau(t)$ is 2τ-periodic and $H_\tau(t)=0,\ \forall t\in[2q\tau,(2q+1)\tau],q\in\mathbf{N}$.

then we call that $H_\tau(t)$ is an $\mathbf{S}^{(k)}_{p_1\times p_2}(\tau)$ function. In addition, if $p_1=p_2=p$, $H_\tau(t)\geq 0,\ \forall t\in[(2q+1)\tau,2(q+1)\tau]$ and there exists a $\tau^*\in(\tau,2\tau)$ such that $H_\tau(\tau^*)>0$, we call that $H_\tau(t)$ is an $\mathbf{S}^{(k)}_p(\tau)$ function.

Note that an $\mathbf{S}^{(k)}_p(\tau)$ function is easy to construct. For example, an $\mathbf{S}^{(k)}_p(\tau),k<\infty$ function $H_\tau(t)$ can be respectively constructed as [33]

$$H_\tau(t)=\begin{cases}0,\ t\in[0,\tau],\\ \left(\left(\mathrm{e}^{a_1(t-\tau)}-1\right)\left(1-\mathrm{e}^{a_2(t-2\tau)}\right)\right)^{k+1}I_p,t\in(\tau,2\tau),\end{cases}\tag{9}$$

where $a_1,\ a_2\in\mathbf{R}_+$.

III. Main results

A. Full-order observer design

In this subsection, Problem 1 will be solved. Let $L_0\in\mathbf{R}^{n\times p}$ be a constant matrix, and $L_{(A_0,\tau)}(t)\in\mathbf{S}^{(k)}_{n\times p}(\tau)$ be defined as

$$L_{(A_0,\tau)}(t)=\mathrm{e}^{A_0t}M_0\mathrm{e}^{A_0^{\mathrm{T}}(t-\tau)}C^{\mathrm{T}}H_\tau(t),\tag{10}$$

where the constant matrix $M_0\in\mathbf{R}^{n\times n}$ will be designed later, $H_\tau(t)\in\mathbf{S}^{(k)}_p(\tau)$ and $A_0=\tilde{A}-L_0C$ with

$$\tilde{A}=\tilde{W}PA,\tag{11}$$

in which P is defined in (55) and $\tilde{W}$ in (8). To simplify the notations, we use $L_{A_0}(t)$ to replace $L_{(A_0,\tau)}(t)$ in the remainder of this paper. We also define $W_{\mathrm{o}}(A_0,\tau)$ as

$$W_{\mathrm{o}}(A_0,\tau)=\int_0^\tau\mathrm{e}^{A_0^{\mathrm{T}}s}C^{\mathrm{T}}H_\tau(s+\tau)C\mathrm{e}^{A_0s}\mathrm{d}s.\tag{12}$$

Now we present a lemma with conditions ensuring that matrix $W_{\mathrm{o}}(A_0,\tau)$ is invertible whose proof is presented in Appendix B.

Lemma 3: Let Assumption 1 hold. Then (A_0,C) is observable if and only if Assumption 2 is satisfied. In such case, the matrix $W_{\mathrm{o}}(A_0,\tau)$ is invertible.

Let's denote some matrices as follows

$$\begin{aligned}G_1&=A_0(\tilde{W}Q+W)+L_0,\\ N_1&=\tilde{W}Q+W,\\ H_1&=\tilde{W}PB-G_1D,\\ G_1(t)&=L_{A_0}(t)(W_0-C\tilde{W}Q),\end{aligned}\tag{13}$$

where $W_0=I_p-CW$, while P, Q, W and $\tilde{W}$ are respectively defined in (55), (56) and (8).

Theorem 1: Let Assumptions 1-2 be satisfied, T be a prescribed constant satisfying $T>h$, and the matrix group $\{G_1,N_1,H_1,G_1(t)\}$ be represented in (13), where the gain $L_{A_0}(t)$ is described in (10). If the matrix M_0 is taken as

$$M_0=W_{\mathrm{o}}^{-1}(A_0,\tau),\tag{14}$$

where $\tau=\frac{1}{2}(T-h)$ and matrix $W_{\mathrm{o}}(A_0,\tau)$ is defined in (12), then a solution to Problem 1 is obtained as

$$\begin{cases}\dot{\chi}=A_0\chi-L_{A_0}(t-h)C\chi(t-\tau)+G_1y\\ \qquad\quad+H_1u(t-h)+G_1(t-h)y(t-\tau)\\ \qquad\quad-G_1(t-h)Du(t-\tau-h),\\ \hat{x}=\chi+N_1y-N_1Du(t-h),\end{cases}\tag{15}$$

where the initial conditions can be arbitrarily chosen.

Proof: This proof can be divided into four parts.

Part 1: Transformation. Based on (6), we can express

$$\dot{x}=PEd+PA+Q\dot{y}+PBu(t-h)-QD\dot{u}(t-h).\tag{16}$$

Part 2: Elimination of the unknown input. (16) together with system (1) gives

$$\begin{aligned}\dot{y}=&CPBu(t-h)+CPAx+CPEd\\ &+CQ\dot{y}+(I_p-CQ)D\dot{u}(t-h),\end{aligned}$$

which further implies

$$\begin{aligned}CPEd=&((I_p-CQ)\dot{y}-CPAx\\ &-CPBu(t-h)-(I_p-CQ)D\dot{u}(t-h))\triangleq\lambda.\end{aligned}$$

Then, in view of Lemma 1 in [17], there must exist some $\delta\in\mathbf{R}^l$ such that

$$d=(CPE)^-\lambda+(I_l-(CPE)^-CPE)\delta.\tag{17}$$

Substituting (17) into (16) yields

$$\begin{aligned}\dot{x}=&PBu(t-h)+Q\dot{y}+PAx-QD\dot{u}(t-h)\\ &+PE(CPE)^-\lambda+PE(I_l-(CPE)^-CPE)\delta\\ =&(PA-WCPA)-(QD+WCPB-PB)u(t-h)\\ &+(W(I_p-CQ)+Q)\dot{y}-W(I_p-CQ)D\dot{u}(t-h)\\ &+PE(I_l-(CPE)^-CPE)\delta\\ =&\tilde{W}PAx+(W+\tilde{W}Q)\dot{y}\\ &+\tilde{W}PBu(t-h)-(W+\tilde{W}Q)D\dot{u}(t-h),\end{aligned}\tag{18}$$

where we have used (7).

Part 3: Construction of a UIO in the form of (15). From (18) we get

$$\begin{cases}\dot{x}=\tilde{A}x+\tilde{B}\tilde{u}(t-h),\\ y=Cx+\tilde{D}\tilde{u}(t-h),\end{cases}\tag{19}$$

where $\tilde{A}$ is shown in (11),

$$\tilde{B}=\left[\begin{array}{ccc}\tilde{W}PB & -(\tilde{W}Q+W)D & \tilde{W}Q+W\end{array}\right],$$
$$\tilde{D}=\left[\begin{array}{ccc}D & 0 & 0\end{array}\right],\quad\tilde{u}=[u^{\mathrm{T}},\dot{u}^{\mathrm{T}},\dot{y}^{\mathrm{T}}(t+h)]^{\mathrm{T}}$$

By Lemma 3 we know that $(\tilde{A}, C)$ is observable. Then, following the observer designed in [33], we design the observer of system (19) as

$$\begin{cases} \dot{\hat{x}} &= \tilde{A}\hat{x} + \tilde{B}\tilde{u}(t-h) + L_0(y-\hat{y}) \\ &\quad + L_{A_0}(t-h)(y(t-\tau) - \hat{y}(t-\tau)), \\ \hat{y} &= C\hat{x} + \tilde{D}\tilde{u}(t-h), \end{cases} \tag{20}$$

where

$$\hat{y} = \begin{cases} y, & t \geq 0, \\ \psi_1, & \forall t \in [-\tau, 0), \end{cases}$$

in which ψ_1 is a piecewise continuous function.

Let $e = x - \hat{x}$. Then it is obtained from (15) and (20) that

$$\dot{e} = A_0 e - L_{A_0}(t-h)\left(y(t-\tau) - \hat{y}(t-\tau)\right). \tag{21}$$

Consideration is given to two cases:

Case I: $0 < \tau \leq h$. From (21), we have, for $t \in [0, \tau)$,

$$\dot{e} = A_0 e - L_{A_0}(t-h)\left(\psi_0(t-\tau) - C\psi_1(t-\tau)\right), \tag{22}$$

where ψ_0 is the initial condition for y, and, for $t \geq \tau$,

$$\dot{e} = A_0 e - L_{A_0}(t-h)Ce(t-\tau).$$

Case II: $\tau > h$. For $t \in [0, h)$, we obtain (22) and, for $t \in [h, \tau)$, we get from (21) that

$$\dot{e} = A_0 e, \tag{23}$$

where we have applied

$$L_{A_0}(t-h) = 0, \forall t \in [h, \tau). \tag{24}$$

Moreover, for $t \geq \tau$, system (21) turns into

$$\dot{e} = A_0 e - L_{A_0}(t-h)Ce(t-\tau). \tag{25}$$

In light of (24), both systems (23) and (25) can be expressed as

$$\dot{e} = A_0 e - L_{A_0}(t-h)Ce(t-\tau), t \geq h. \tag{26}$$

Denote $\mu = t - h$ and $\epsilon(\mu) = e(\tau + h)$. Then system (26) can be rewritten as

$$\dot{\epsilon}(\mu) = A_0\epsilon(\mu) - L_{A_0}(\tau)C\epsilon(\mu - \tau), \tau \geq 0, \tag{27}$$

which is in the form of (4) with $\bar{A}(t) = A_0$, $\bar{B}(t) = -L_{A_0}(\tau)C$ and $q = \tau$. Then, in view of Proposition 1, the monodromy matrix for system (27) is given by

$$\Delta(\tau) = \mathrm{e}^{2A_0\tau}\left(I_n - M_0 W_{\mathrm{o}}(A_0, \tau)\right).$$

According to Lemma 3 and (5), we can get that $e = 0, \forall t \geq h + 2\tau$, if and only if (14) is satisfied. Moreover, from (20), it is clear that

$$\begin{aligned} \dot{\hat{x}} =& \tilde{A}\hat{x} + \tilde{B}\tilde{u}(t-h) + L_0 y - L_0\hat{y} \\ &+ L_{A_0}(t-h)y(t-\tau) - L_{A_0}(t-h)\hat{y}(t-\tau) \\ =& A_0\hat{x} - L_{A_0}(t-h)C\hat{x}(t-\tau) + L_0 y \\ &+ L_{A_0}(t-h)y(t-\tau) - L_{A_0}(t-h)\tilde{D}\tilde{u}(t-\tau-h) \\ &+ (\tilde{B} - L_0\tilde{D})\tilde{u}(t-h) \\ =& A_0\hat{x} - L_{A_0}(t-h)C\hat{x}(t-\tau) + L_0 y \\ &+ L_{A_0}(t-h)y(t-\tau) - L_{A_0}(t-h)Du(t-\tau-h) \\ &+ (\tilde{W}PB - L_0 D)u(t-h) \\ &- (\tilde{W}Q + W)D\dot{u}(t-h) + (\tilde{W}Q + W)\dot{y}. \end{aligned} \tag{28}$$

Part 4: Elimination of the derivatives of input and output. Let

$$\chi = \hat{x} - (W + \tilde{W}Q)y + (\tilde{W}Q + W)Du(t-h). \tag{29}$$

Then, system (28) can be represented as

$$\begin{aligned} \dot{\chi} =& \dot{\hat{x}} - (W + \tilde{W}Q)\dot{y} + (\tilde{W}Q + W)D\dot{u}(t-h) \\ =& A_0\hat{x} - L_{A_0}(t-h)C\hat{x}(t-\tau) + L_0 y \\ &+ L_{A_0}(t-h)y(t-\tau) - L_{A_0}(t-h)Du(t-\tau-h) \\ &+ (\tilde{W}PB - L_0 D)u(t-h) \\ =& A_0\chi - L_{A_0}(t-h)C\chi(t-\tau) \\ &+ (L_0 + A_0(\tilde{W}Q + W))y \\ &+ L_{A_0}(t-h)(W_0 - C\tilde{W}Q)y(t-\tau) \\ &+ (\tilde{W}PB - L_0 D - A_0(\tilde{W}Q + W)D)u(t-h) \\ &+ L_{A_0}(t-h)(C\tilde{W}Q - W_0)Du(t-\tau-h). \end{aligned}$$

The expression for $\hat{x}$ in (15) is finally obtained from (29). ■

Remark 2: Note that the lower bound on T is due to the input delay, and if there is no input delay, any positive number T can be allowed.

Remark 3: If there is no artificial delay τ in the observer (20), the finite time observation cannot be achieved. The reason has been explained in Remark 1 in [17]. And kindly notice that the artificial delay τ depends on T, but not the converse.

Remark 4: According to Assumption 2 we have $d = (CPE)^{-}\lambda$, which implies that a estimation of $d(t)$ can be reconstructed as

$$\begin{aligned} \hat{d} =& (CPE)^{-}((I_p - CQ)\dot{y} - CPA\hat{x} \\ &- CPBu(t-h) - (I_p - CQ)D\dot{u}(t-h)). \end{aligned}$$

Note that the right-hand side of the above equation contains the control input and its derivative, which may render an implementation difficult.

Remark 5: L_0, the instantaneous observation gain matrix, stabilizes the observer error system when it is in the "open-loop phase". In order to render A_0 Hurwitz, we can choose

$$L_0 = \tilde{Q}(\gamma)C^{\mathrm{T}}, \tag{30}$$

where $\tilde{Q}(\gamma) > 0$ satisfies the following equation [31]:

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} - \tilde{Q}C^{\mathrm{T}}C\tilde{Q} = -\gamma\tilde{Q}, \tag{31}$$

which has a unique positive definite solution if and only if

$$\gamma > -2\min_{i=1,2,\ldots,n}\{\mathrm{Re}\{\lambda_i(\tilde{A})\}\}. \tag{32}$$

Remark 6: Both our paper and [25] consider the design of prescribed-time UIOs for singular systems. The similarity is that the observers proposed in our paper and [25] can accurately estimate the states of the system at a prescribed time. The differences can be described as follows:

- The models studied in the current paper and [25] are different, namely, this paper considers the case of input delay while [25] does not.

- Under the OMC and SOC condition, the prescribed-time unknown input observer can be designed as in our paper. However, except for these two conditions, an additional condition is required on the eigenvalues of the observer gains $N_i, i = 1, 2$, designed in [25].
- Both this paper and [25] achieve accurate estimation at a prescribed time by introducing artificial delay. However, the core idea is different. Specifically, our paper uses periodic time-varying observer gains, such that the monodromy matrix of the periodic delay error system can be configured to 0, hence achieving prescribed-time observation. In contrast to this, the authors of [25] adopted the Engel-Kreisselmeier method, which uses the states of two Luenberger observers combined with delay to achieve prescribed-time observation. It should be pointed out that the dimensionality of the observer proposed in [25] is twice that of the observers proposed in our paper.
- Both full- and reduced-order prescribed-time observers are considered in our paper while only a full-order observers is designed in [25].

Finally, we give the following algorithm to construct observer (15) for system (1).

Algorithm 1

1: Compute matrices P and Q according to (55) and (56) and construct the matrix $\tilde{A}$ based on (8) and (11).
2: Set the parameter γ satisfying (32), and compute the matrix L_0 according to (31) and (30).
3: Compute matrices A_0, G_1, H_1 and N_1 according to (13).
4: Compute the matrix $W_{\mathrm{o}}(A_0, \tau)$ according to (9) and (12), compute the time-varying gain $L_{(A_0,\tau)}(t)$ according to (10), and compute the time-varying gain $G_1(t)$ according to (13).

B. Reduced-order observer design

In this subsection, recalling that $\mathrm{rank}(C) = p$, we construct a prescribed-time UIO with dimension $n - p$ for system (1). Let

$$z = Kx, \tag{33}$$

where K is defined in (48). Then we have from (1) and (54) that

$$\begin{cases} \dot{z} = V^{-1}AK^{-1}z - \bar{F}D\dot{u}(t-h) \\ \qquad\quad + V^{-1}E + \bar{F}\dot{y} + V^{-1}Bu\,(t-h)\,, \\ y = CK^{-1}z + Du(t-h), \end{cases} \tag{34}$$

where V^{-1}, K^{-1} and $\bar{F}$ are respectively defined in (52), (49) and (53). With the idea in the proof in Theorem 1, we apply the same method to remove the unknown input d. Then system (34) can be continued as

$$\begin{cases} \dot{z} = \mathcal{A}z + \mathcal{B}u(t-h) + \mathcal{C}\dot{y} + \mathcal{D}\dot{u}(t-h), \\ y = CK^{-1}z + Du(t-h), \end{cases} \tag{35}$$

where

$$\begin{aligned} &\mathcal{A} = \tilde{W}_1V^{-1}AK^{-1}, \mathcal{B} = \tilde{W}_1V^{-1}B, \\ &\mathcal{C} = W_1 + \tilde{W}_1\bar{F}, \mathcal{D} = -(W_1 + \tilde{W}_1\bar{F})D, \end{aligned}$$

in which $\tilde{W}_1 = I_n - W_1CK^{-1}, W_1 = V^{-1}E(CPE)^{-}$. We partition the coefficient matrices of system (34) as

$$V^{-1}AK^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, V^{-1}B = \begin{bmatrix} \bar{B}_1^{\mathrm{T}} & \bar{B}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

and $V^{-1}E = \left[\bar{E}_1^{\mathrm{T}}, \bar{E}_2^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\bar{A}_{11} \in \mathbf{R}^{p\times p}$, $\bar{B}_1 \in \mathbf{R}^{p\times m}$, $\bar{E}_1 \in \mathbf{R}^{p\times l}$, and $\bar{E}_2 \in \mathbf{R}^{(n-p)\times l}$. Note that

$$\tilde{W}_1 = I_n - W_1CK^{-1} = \begin{bmatrix} I_p - \Upsilon_1 & 0 \\ -\Upsilon_2 & I_{n-p} \end{bmatrix}, \tag{36}$$

where $\Upsilon_i = \bar{E}_i(CPE)^{-}$, $i = 1, 2$. As a result, the coefficient matrices of system (35) can be calculated as

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} (I_p - \Upsilon_1)\bar{A}_{11} & (I_p - \Upsilon_1)\bar{A}_{12} \\ \bar{A}_{21} - \Upsilon_2\bar{A}_{11} & \bar{A}_{22} - \Upsilon_2\bar{A}_{12} \end{bmatrix} \triangleq \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \\ \mathcal{B} &= \begin{bmatrix} (I_p - \Upsilon_1)\bar{B}_1 \\ \bar{B}_2 - \Upsilon_2\bar{B}_1 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{B}_1^{\mathrm{T}} & \tilde{B}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{C} &= \begin{bmatrix} I_p + (\Upsilon_1 - I_p)V_1F_1 \\ (\Upsilon_2V_1 - V_2)F_1 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{C}_1^{\mathrm{T}} & \tilde{C}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{D} &= -\begin{bmatrix} (I_p + (\Upsilon_1 - I_p)V_1F_1)D \\ (\Upsilon_2V_1 - V_2)F_1D \end{bmatrix} \triangleq \begin{bmatrix} \tilde{D}_1^{\mathrm{T}} & \tilde{D}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

and system (34) can be written as

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}u(t-h) \\ \qquad\qquad + \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}\dot{y} + \begin{bmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{bmatrix}\dot{u}(t-h), \\ y = z_1 + Du(t-h). \end{cases} \tag{37}$$

Let $L_{(A_{\mathrm{o}},\tau)} \in \mathbf{S}^{(k)}_{(n-p)\times p}(\tau)$ be defined by

$$L_{(A_{\mathrm{o}},\tau)}(t) = \mathrm{e}^{A_{\mathrm{o}}t}M_{\mathrm{o}}\mathrm{e}^{A_{\mathrm{o}}^{\mathrm{T}}(t-\tau)}\tilde{A}_{12}^{\mathrm{T}}H_\tau(t), \tag{38}$$

where $A_{\mathrm{o}} = \tilde{A}_{22} - L_{\mathrm{o}}\tilde{A}_{12}$ with $L_{\mathrm{o}} \in \mathbf{R}^{(n-p)\times p}$, $M_{\mathrm{o}} \in \mathbf{R}^{(n-p)\times(n-p)}$ is a constant matrix, and $H_\tau(t) \in \mathbf{S}^{(k)}_{n-p}(\tau)$. We use $L_{A_{\mathrm{o}}}(t)$ to denote $L_{(A_{\mathrm{o}},\tau)}(t)$ for simplicity, and introduce some matrices as follows

$$\begin{aligned} G_1 &= A_{\mathrm{o}}\Omega_0 + \Omega_4, \\ H_1 &= \Omega_1 - L_{\mathrm{o}}\Omega_2 + A_{\mathrm{o}}(\tilde{D}_2 - L_{\mathrm{o}}\Omega_3) \\ G_1(t) &= A_{\mathrm{o}}\Omega\,(t) - \dot{\Omega}\,(t) - L_{A_{\mathrm{o}}}(t)\Omega_5, \\ H_1(t) &= (\dot{L}_{A_{\mathrm{o}}}(t) - A_{\mathrm{o}}L_{A_{\mathrm{o}}}(t))\Omega_3 \\ &\quad - L_{A_{\mathrm{o}}}(t)(\Omega_2 + \tilde{A}_{12}(\tilde{D}_2 - L_{\mathrm{o}}\Omega_3)), \\ M_1 &= K^{-1}\begin{bmatrix} 0^{\mathrm{T}} & I_{n-p} \end{bmatrix}^{\mathrm{T}}, \\ N_1 &= K^{-1}\begin{bmatrix} I_p & \Omega_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \\ N_1(t) &= K^{-1}\begin{bmatrix} 0^{\mathrm{T}} & \Omega^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}, \\ P_1 &= K^{-1}\begin{bmatrix} -D^{\mathrm{T}} & (\tilde{D}_2 - L_{\mathrm{o}}\Omega_3)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \\ P_1(t) &= K^{-1}\begin{bmatrix} 0^{\mathrm{T}} & (L_{A_{\mathrm{o}}}(t)\Omega_3)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} &\Omega_0 = L_{\mathrm{o}} + \tilde{C}_2 - L_{\mathrm{o}}\tilde{C}_1, \quad \Omega_1 = \tilde{B}_2 - \tilde{A}_{21}D, \\ &\Omega_2 = \tilde{B}_1 - \tilde{A}_{11}D, \quad \Omega_3 = D + \tilde{D}_1, \\ &\Omega_4 = \tilde{A}_{21} - L_{\mathrm{o}}\tilde{A}_{11}, \quad \Omega_5 = \tilde{A}_{11} + \tilde{A}_{12}\Omega_0, \\ &\Omega(t) = L_{A_{\mathrm{o}}}(t) - L_{A_{\mathrm{o}}}(t)\tilde{C}_1. \end{aligned}$$

Then we are able to present a solution to Problem 1 as follows.

Theorem 2: Assume that Assumptions 1-2 are hold, $T \in \mathbf{R}_+$ is a constant satisfying $T > h$, and the matrices $\{G_1, H_1, F_1(t), G_1(t), G_2(t), H_1(t)\}$ are shown in (39) in which the observer gain $L_{A_\mathrm{o}}(t)$ is defined in (38). Then, for any $\tau > 0$,

$$W_\mathrm{o}(A_\mathrm{o}, \tau) = \int_0^\tau \mathrm{e}^{A_\mathrm{o}^\mathrm{T} s} \tilde{A}_{12}^\mathrm{T} H_\tau(s+\tau) \tilde{A}_{12} \mathrm{e}^{A_\mathrm{o} s} \mathrm{d}s, \quad (40)$$

is invertible. Furthermore, if we choose M_o as

$$M_\mathrm{o} = W_\mathrm{o}^{-1}(A_\mathrm{o}, \tau), \quad (41)$$

where $\tau = \frac{1}{2}(T-h)$, then a solution to Problem 1 is established as

$$\begin{cases} \dot{\chi} = A_\mathrm{o}\chi - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\chi(t-\tau) + G_1 y \\ \qquad + G_1(t-h)y(t-\tau) + H_1 u(t-h) \\ \qquad + H_1(t-h)u(t-\tau-h), \\ \hat{x} = M_1\chi + N_1 y + N_1(t)\, y(t-\tau) \\ \qquad + P_1 u(t-h) + P_1(t-h)u(t-\tau-h), \end{cases} \quad (42)$$

where the initial conditions can be arbitrarily chosen.

Proof: We first prove that the matrix $W_\mathrm{o}(A_\mathrm{o}, \tau)$ is invertible if Assumption 2 holds under Assumption 1. By Lemma 3, we only need to verify that the observability of $(\mathcal{A}, CK^{-1})$ is equivalent to the observability of $(\tilde{A}, C)$ since $(\mathcal{A}, CK^{-1})$ is observable if and only if $(\tilde{A}_{22}, \tilde{A}_{12})$ is. This is obviously true since

$$\begin{aligned} \mathrm{rank}\begin{bmatrix} sI_n - \mathcal{A} \\ CK^{-1} \end{bmatrix} =& \mathrm{rank}\begin{bmatrix} K^{-1} & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} (sI_n - \mathcal{A})K \\ C \end{bmatrix} \\ =& \mathrm{rank}\begin{bmatrix} sI_n - \tilde{A} \\ C \end{bmatrix}. \end{aligned}$$

According to (37), the state variable z_2 satisfies

$$\begin{aligned} \dot{z}_2 =& \tilde{A}_{22}z_2 + \tilde{A}_{21}z_1 + \tilde{B}_2 u(t-h) + \tilde{C}_2\dot{y} + \tilde{D}_2\dot{u}(t-h) \\ =& \tilde{A}_{22}z_2 + \tilde{A}_{21}y + \tilde{C}_2\dot{y} \\ & + (\tilde{B}_2 - \tilde{A}_{21}D)u(t-h) + \tilde{D}_2\dot{u}(t-h), \end{aligned} \quad (43)$$

and y satisfies

$$\begin{aligned} \dot{y} =& \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 + \tilde{B}_1 u(t-h) \\ & + \tilde{C}_1\dot{y} + (D + \tilde{D}_1)\dot{u}(t-h) \\ =& \tilde{A}_{11}(y - Du(t-h)) + \tilde{A}_{12}z_2 + \tilde{B}_1 u(t-h) \\ & + \tilde{C}_1\dot{y} + (D + \tilde{D}_1)\dot{u}(t-h) \\ =& \tilde{A}_{12}z_2 + \tilde{A}_{11}y + \tilde{C}_1\dot{y} \\ & + (\tilde{B}_1 - \tilde{A}_{11}D)u(t-h) + (D + \tilde{D}_1)\dot{u}(t-h). \end{aligned} \quad (44)$$

Denote

$$y_2 = \dot{y}, u_2 = \begin{bmatrix} y^\mathrm{T}(t+h) & \dot{y}^\mathrm{T}(t+h) & u^\mathrm{T} & \dot{u}^\mathrm{T} \end{bmatrix}^\mathrm{T}.$$

Then, combining (43) and (44) yields

$$\begin{cases} \dot{z}_2 &= \tilde{A}_{22}z_2 + \tilde{\Phi}u_2(t-h), \\ y_2 &= \tilde{A}_{12}z_2 + \tilde{\Psi}u_2(t-h), \end{cases} \quad (45)$$

where

$$\tilde{\Phi} = \begin{bmatrix} \tilde{A}_{21} & \tilde{C}_2 & \Omega_1 & \tilde{D}_2 \end{bmatrix}, \tilde{\Psi} = \begin{bmatrix} \tilde{A}_{11} & \tilde{C}_1 & \Omega_2 & \Omega_3 \end{bmatrix}.$$

Since $(\tilde{A}_{22}, \tilde{A}_{12})$ is observable, we design the following observer for system (45)

$$\begin{cases} \dot{\hat{z}}_2 = \tilde{A}_{22}\hat{z}_2 + \tilde{\Phi}u_2(t-h) + L_\mathrm{o}(y_2 - \hat{y}_2(t)) \\ \qquad + L_{A_\mathrm{o}}(t-h)(y_2(t-\tau) - \hat{y}_2(t-\tau)), \\ \hat{y}_2 = \tilde{A}_{12}\hat{z}_2 + \tilde{\Psi}u_2(t-h), \end{cases} \quad (46)$$

where

$$\hat{y}_2 = \begin{cases} y_2, & t \geq 0, \\ \tilde{\nu}_1, & t \in [-\tau, 0), \end{cases}$$

in which $\tilde{\nu}_1$ is an arbitrarily piecewise continuous function. Denote $e_2 = z_2 - \hat{z}_2$. Then from (45) and (46) we can derive a similar expression as (21). By considering the same two cases as the proof in Theorem 1, we have a similar expression as (27). Hence $e_2 = 0$ is equivalent to condition (40). From (46), we obtain

$$\begin{aligned} \dot{\hat{z}}_2 =& \tilde{A}_{22}\hat{z}_2 + \tilde{\Phi}u_2(t-h) + L_{A_\mathrm{o}}(t-h)y_2(t-\tau) \\ & + L_\mathrm{o}y_2 - L_\mathrm{o}\hat{y}_2 - L_{A_\mathrm{o}}(t-h)\hat{y}_2(t-\tau) \\ =& A_\mathrm{o}\hat{z}_2 + L_\mathrm{o}y_2 - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\hat{z}_2(t-\tau) \\ & + (\tilde{\Phi} - L_\mathrm{o}\tilde{\Psi})u_2(t-h) + L_{A_\mathrm{o}}(t-h)y_2(t-\tau) \\ & - L_{A_\mathrm{o}}(t-h)\tilde{\Psi}u_2(t-\tau-h) \\ =& A_\mathrm{o}\hat{z}_2 - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\hat{z}_2(t-\tau) + \Omega_4 y \\ & + (\Omega_1 - L_\mathrm{o}\Omega_2)u(t-h) - L_{A_\mathrm{o}}(t-h)\tilde{A}_{11}y(t-\tau) \\ & - L_{A_\mathrm{o}}(t-h)\Omega_2 u(t-\tau-h) \\ & + \Omega(t-h)\dot{y}(t-\tau) - L_{A_\mathrm{o}}(t-h)\Omega_3\dot{u}(t-\tau-h) \\ & + (\tilde{D}_2 - L_\mathrm{o}\Omega_3)\dot{u}(t-h) + \Omega_0\dot{y}. \end{aligned}$$

Let

$$\begin{aligned} \chi =& \hat{z}_2 - \Omega_0 y - (\tilde{D}_2 - L_\mathrm{o}\Omega_3)u(t-h) \\ & - \Omega(t-h)y(t-\tau) + L_{A_\mathrm{o}}(t-h)\Omega_3 u(t-\tau-h). \end{aligned} \quad (47)$$

Then we have

$$\begin{aligned} \dot{\chi} =& \dot{\hat{z}}_2 - \Omega_0\dot{y} + L_{A_\mathrm{o}}(t-h)\Omega_3\dot{u}(t-\tau-h) \\ & - \dot{\Omega}(t-h)y(t-\tau) + \dot{L}_{A_\mathrm{o}}(t-h)\Omega_3\dot{u}(t-\tau-h) \\ & + (L_\mathrm{o}\Omega_3 - \tilde{D}_2)\dot{u}(t-h) - \Omega(t-h)\dot{y}(t-\tau) \\ =& A_\mathrm{o}\hat{z}_2 - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\hat{z}_2(t-\tau) + \Omega_4 y \\ & - (L_{A_\mathrm{o}}(t-h)\tilde{A}_{11} + \dot{\Omega}(t-h))y(t-\tau) \\ & + (\dot{L}_{A_\mathrm{o}}(t-h)\Omega_3 - L_{A_\mathrm{o}}(t-h)\Omega_2)u(t-\tau-h) \\ & + (\Omega_1 - L_\mathrm{o}\Omega_2)u(t-h) \\ =& A_\mathrm{o}\chi - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\chi(t-\tau) + (\Omega_4 + A_\mathrm{o}\Omega_0)y \\ & + (A_\mathrm{o}\Omega(t-h) - L_{A_\mathrm{o}}(t-h)\Omega_5 - \dot{\Omega}(t-h))y(t-\tau) \\ & + (\Omega_1 - L_\mathrm{o}\Omega_2 + A_\mathrm{o}(\tilde{D}_2 - L_\mathrm{o}\Omega_3))u(t-h) \\ & + (\dot{L}_{A_\mathrm{o}}(t-h) - A_\mathrm{o}L_{A_\mathrm{o}}(t-h))\Omega_3 u(t-\tau-h) \\ & - L_{A_\mathrm{o}}(t-h)(\Omega_2 + \tilde{A}_{12}(\tilde{D}_2 - L_\mathrm{o}\Omega_3))u(t-\tau-h) \\ & - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\Omega(t-h-\tau)y(t-2\tau) \\ & + L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}L_{A_\mathrm{o}}(t-\tau-h)\Omega_3 u(t-2\tau-h) \\ =& A_\mathrm{o}\chi - L_{A_\mathrm{o}}(t-h)\tilde{A}_{12}\chi(t-\tau) + (\Omega_4 + A_\mathrm{o}\Omega_0)y \\ & + (A_\mathrm{o}\Omega(t-h) - \dot{\Omega}(t-h) - L_{A_\mathrm{o}}(t-h)\Omega_5)y(t-\tau) \\ & + (\Omega_1 - L_\mathrm{o}\Omega_2 + A_\mathrm{o}(\tilde{D}_2 - L_\mathrm{o}\Omega_3))u(t-h) \end{aligned}$$

$$+(\dot{L}_{A_o}(t-h)-A_oL_{A_o}(t-h))\Omega_3 u(t-\tau-h)$$
$$-L_{A_o}(t-h)(\Omega_2+\tilde{A}_{12}(\tilde{D}_2-L_o\Omega_3))u(t-\tau-h),$$

where we have used $L_{A_o}(t-h)=0, \forall t\in[h,h+\tau)$ and $\Omega(t-h-\tau)=0, \forall t\in[h+\tau,h+2\tau)$. Then we obtain the first equation of (42). Moreover, from (37) and (47) we can get

$$\hat{x}=\begin{bmatrix}0\\K^{-1}\end{bmatrix}\chi+K^{-1}\begin{bmatrix}I_p\\\Omega_0\end{bmatrix}y+\begin{bmatrix}0\\K^{-1}\Omega(t-h)\end{bmatrix}y(t-\tau)+K^{-1}\begin{bmatrix}-D\\\tilde{D}_2-L_o\Omega_3\end{bmatrix}u(t-h)-\begin{bmatrix}0\\K^{-1}L_{A_o}(t-h)\Omega_3\end{bmatrix}u(t-\tau-h),$$

which coincides with the second equation of (42). The proof is finished. ■

Remark 7: The observers in (15) and (42) only use the known input and measurable output of system (1), which is different from the one in [2], [3] where unmeasurable higher-order derivatives of the output signals are required. In addition, it should be mentioned that, to implement observer (15) or (42), we will eventually not need $\hat{y}$ or y_2.

Remark 8: Compared with existing results in [19] and [29], the proposed observers can estimate the state at the prescribed time.

Finally, we give the following algorithm to construct observer (42) for system (1).

Algorithm 2

1: Compute matrices P, K, $\bar{F}$ and V^{-1} according to (55), (48), (53) and (52).

2: Set the parameter γ satisfying (32) in which $\tilde{A}$ is replaced by $\tilde{A}_{22}$, and compute the matrix L_o according to (30) and (31) in which C is replaced by $\tilde{A}_{12}$.

3: Compute matrices A_o, G_1, H_1, M_1, N_1 and P_1 according to (39).

4: Compute the matrix $W_o(A_o,\tau)$ according to (12) and (40), compute the time-varying gain $L_{(A_o,\tau)}(t)$ according to (38), and compute the time-varying gains $G_1(t)$, $H_1(t)$, $N_1(t)$ and $P_1(t)$ according to (39).

IV. Simulation examples

In this section, a numerical example and an electronic circuit system model are adopted to demonstrate the effectiveness and applicability of the obtained results.

A. A numerical example

Consider system (1) with matrices given by

$$F=\begin{bmatrix}0.01&0&0\\0&1&0\\0&0&0\end{bmatrix},A=\begin{bmatrix}-0.05&0&0.25\\1&0&0\\0&0.25&2\end{bmatrix},$$
$$E=B=\begin{bmatrix}-20&0&0\end{bmatrix}^{\mathrm{T}},$$
$$C=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix},D=\begin{bmatrix}1\\0\end{bmatrix}.$$

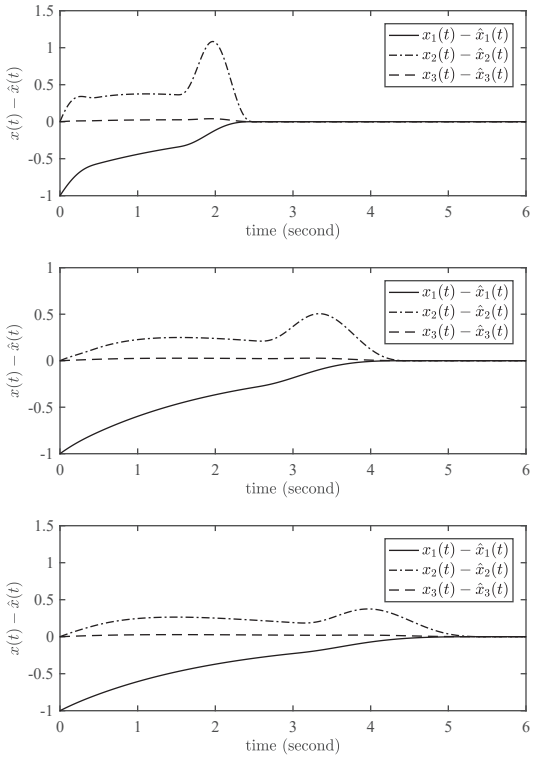


Fig. 1. The error signals for different T.

Besides, we choose the input delay as $h=0.5$ and consider the unknown input $d=\sin(t)$. By direct calculation, we can verify that Assumption 1 holds. Then, according to (55) and (56), we obtain

$$P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{5}\end{bmatrix},Q=\begin{bmatrix}\frac{99}{100}&0\\0&0\\0&1\end{bmatrix}.$$

We select matrix L_0 as (22) in [17] (where matrices $\bar{A}$ is replaced by $\tilde{A}$) such that matrix $\tilde{A}-L_0C$ is Hurwitz. Based on solving the parametric Lyapunov equation (31) with $\gamma=0.5$, we can get the gain matrix

$$L_0=\begin{bmatrix}\frac{81}{164}&\frac{9}{164}\\\frac{63}{41}&\frac{376}{41}\\\frac{9}{164}&\frac{1481}{820}\end{bmatrix}.$$

In the following, an $\mathbf{S}_2^{(k)}(\tau)$ function $H_\tau(t)$ as (9) with $a_1=a_2=k=1$ is considered. By calculation, we have $\mathrm{rank}(\tilde{A},C)=3$, which implies that Assumption 2 is satisfied, where we have used Lemma 3. Assume that the initial states are chosen as $x(0)=[1\ 0\ 1]^{\mathrm{T}}$, $\chi(0)=[1\ 0\ 0]^{\mathrm{T}}$, $u(s)=0, s\in[-\frac{7}{2}\ 0)$ and $y(\theta)=[0\ 0]^{\mathrm{T}}, \theta\in[-3\ 0)$. The known input is chosen as $u=1, t\geq 0$. We consider three cases, namely, $T=2.5\mathrm{s}, T=4.5\mathrm{s}$, and $T=5.5\mathrm{s}$, which, by $\tau=\frac{1}{2}(T-h)$, correspond to $\tau=2, \tau=2.5$, and $\tau=3$. Then, the trajectories of the observer error by using the proposed prescribed-time full-order UIO (15) are plotted in Fig. 1, from which we can observe that, for the different prescribed values

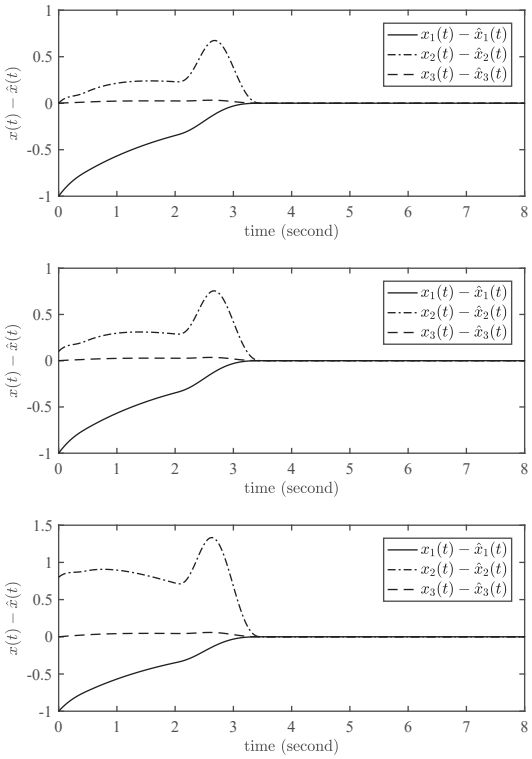


Fig. 2. The error signals for different initial conditions.

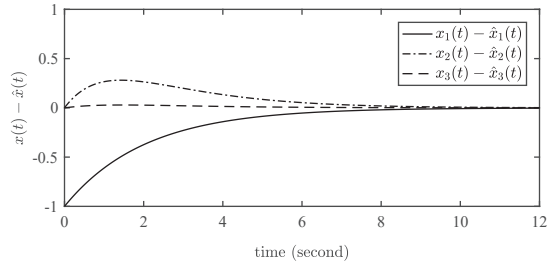


Fig. 3. The error signals associated with asymptotic observer.

of the convergence time T, the observer error converges to zero at $t = T$s. We point out that T is not affected by the initial condition $x(0)$. To illustrate this fact, we set $T = 3.5$s, and different initial conditions $x(0) = [5\ 0\ 2]^{\mathrm{T}}$, $x(0) = [0\ 0.1\ 0.2]^{\mathrm{T}}$, $x(0) = [0.5\ 0.8\ 1]^{\mathrm{T}}$. Then, the trajectories of the observer error by using observer (15) are presented in Fig. 2, which illustrates that for different initial conditions $x(0)$, the observer errors converge to zero at $t = 3.5$s.

In addition, to demonstrate the merit of the estimation performance of the developed approach, we present the error signal using an asymptotic observer obtained by removing the terms containing $L_{A_0}(t-h)$ in (15). The observer errors are depicted in Fig. 3, from which we can see that the observer errors converge asymptotically to zero, with a significant error for $t \geq 12$s. Therefore, the proposed prescribed-time full-order UIO has better estimation performance and converges faster than the asymptotic observer.

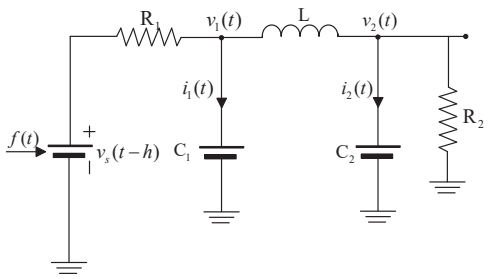


Fig. 4. An electronic circuit.

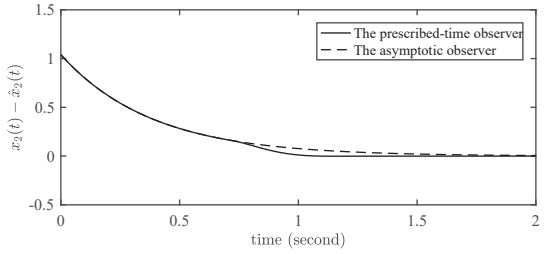


Fig. 5. The error signals associated with different observers.

B. A circuits system

Consider the electronic circuit considered in [28] and depeicted in Fig. 4, where v_s denotes the voltage resource, C_1 and C_2 are capacities, L stands for an inductor, and R_1 and R_2 represent resistors. Let the state be $x = [v_1, v_2, i_2, i_1]^{\mathrm{T}}$, where v_1 and v_2 denote the voltages of C_1 and C_2, i_1 and i_2 represent the currents flowing over inductor L and capacitor C_1, the known input be $u = v_s$, the unknown input be $d = f$, and the output be $y = [v_1, i_2, i_1]^{\mathrm{T}}$. With the help of the Kirchhoff's law, the circuit system with input delay can be characterized by system (1) with matrices:

$$F = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{R_2} & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & R_1 & R_1 \end{bmatrix},$$

$$B = E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let $C_1 = C_2 = 0.1$F, $R_1 = R_2 = 4\Omega$, $L = 0.1$H and $h = 0.2$. It is checked that $\mathrm{rank} \begin{bmatrix} F & E \\ C & 0 \end{bmatrix} = n + \mathrm{rank}\,(E) = 5$, which satisfies Assumption 1. Then, based on (55) and (56), we can get the matrices $P = \mathrm{diag}\{1, 10, 1, 1\}$ and $Q = \mathrm{diag}\{[\frac{9}{10}\ 0]^{\mathrm{T}}, 1.1, 1\}$. We select matrix L_{o} as (22) in [17] (where matrices $\bar{A}$ and C are respectively replaced by $\tilde{A}_{22}$ and $\tilde{A}_{12}$) such that matrix $\tilde{A}_{22} - L_{\mathrm{o}}\tilde{A}_{12}$ is Hurwitz. For this we have used the approach outlines in (21) in [30], based on solving a parametric Lyapunov equations with parameter gamma set to 5.1, the gain matrix $L_{\mathrm{o}} = [0\ 0.1\ 0]$ is obtained. We choose an $\mathbf{S}_2^{(k)}(\tau)$ function $H_\tau(t)$ as (9) with $a_1 = a_2 = k = 1$. Let $T = 1.2$. $\tau = \frac{1}{2}(T - h) = 0.5$. By direct calculation, we can verify that $(\tilde{A}_{22}\ \tilde{A}_{12}) = (-\frac{5}{2}\ [0\ 1\ 0]^{\mathrm{T}})$ is observable, which

means that Assumption 2 holds.

In simulation, we take the initial states as $x(0) = [3\ 1\ 4\ -\frac{9}{2}]^{\mathrm{T}}$, $\chi(0) = 0$, $u(s) = 0, s \in [-\frac{7}{10}\ 0)$ and $y(\theta) = [0\ 0\ 0]^{\mathrm{T}}, \theta \in [-\frac{1}{2}\ 0)$. The known input $u, t \geq 0$ is chosen as $u = Ky$, $K = [0\ 1\ 0]$. Then, the trajectories of the observer error by using the proposed prescribed-time reduced-order UIO (42) and an asymptotic observer (namely (42) with $L_{A_o}(t) = 0$) are recorded in Fig. 5, from which we can see that the observer (42) can achieve accurate estimation in the prescribed time $t = 1.2$s, whereas the asymptotic observer ensures acceptable estimation performance after 2 seconds. Therefore, the proposed prescribed-time reduced-order UIO has better estimation performance and a smaller convergence time than the asymptotic observer.

V. Conclusion

In this paper, both prescribed-time full- and reduced-order UIOs have been designed by applying periodic delayed output approach. Unlike existing gain design methods for solving matrix equations, the main feature of this method is that the resulting time-varying gains depend on the original system matrices and are linear, periodic, continuous, continuously differentiable, and even smooth. The effectiveness of the developed approaches has been substantiated by two simulation examples.

VI. Appendix

A. Proof of Lemma 1

This lemma is well-known for the UIO design of singular systems. Differently from the existing proofs, this paper gives concrete forms of the matrices P and Q. This proof is inspired by [25]. Due to $\mathrm{rank}\,(C) = p$, there exists a matrix $C_0 \in \mathbf{R}^{n\times(n-p)}$ satisfying $\mathrm{rank}\,(C_0) = n - p$ such that $CC_0 = 0$. We construct a matrix

$$K = \left[\begin{array}{cc} C^{\mathrm{T}} & C_0 \end{array}\right]^{\mathrm{T}}. \tag{48}$$

Note that

$$K\left[\begin{array}{cc} C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} & C_0(C_0^{\mathrm{T}}C_0)^{-1} \end{array}\right] = \left[\begin{array}{cc} I_p & 0 \\ 0 & I_{n-p} \end{array}\right],$$

which implies that K is a nonsingular matrix with

$$K^{-1} = \left[\begin{array}{cc} C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} & C_0(C_0^{\mathrm{T}}C_0)^{-1} \end{array}\right]. \tag{49}$$

With this we have

$$\begin{aligned} FK^{-1} &= \left[\begin{array}{cc} FC^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} & FC_0(C_0^{\mathrm{T}}C_0)^{-1} \end{array}\right] \\ &\triangleq \left[\begin{array}{cc} F_1 & F_2 \end{array}\right]. \end{aligned} \tag{50}$$

Thus, we get from (2) that

$$n = \mathrm{rank}\left(\left[\begin{array}{c} F \\ C \end{array}\right] K^{-1}\right) = \mathrm{rank}\left[\begin{array}{cc} F_1 & F_2 \\ I_p & 0 \end{array}\right],$$

which implies that $\mathrm{rank}\,(F_2) = n - p$. Then there exists an $F_0 \in \mathbf{R}^{p\times n}$ satisfying $\mathrm{rank}\,(F_0) = p$ such that $F_0F_2 = 0$. Define $V = [F_0^{\mathrm{T}}, F_2]$. Then, it is clear that

$$\left[\begin{array}{c} (F_0F_0^{\mathrm{T}})^{-1}F_0 \\ (F_2^{\mathrm{T}}F_2)^{-1}F_2^{\mathrm{T}} \end{array}\right]\left[\begin{array}{cc} F_0^{\mathrm{T}} & F_2 \end{array}\right] = \left[\begin{array}{cc} I_p & 0 \\ 0 & I_{n-p} \end{array}\right], \tag{51}$$

which means

$$V^{-1} = \left[\begin{array}{c} (F_0F_0^{\mathrm{T}})^{-1}F_0 \\ (F_2^{\mathrm{T}}F_2)^{-1}F_2^{\mathrm{T}} \end{array}\right] \triangleq \left[\begin{array}{c} V_1 \\ V_2 \end{array}\right]. \tag{52}$$

By defining

$$\bar{F} = \left[\begin{array}{cc} (I_p - V_1F_1)^{\mathrm{T}} & -(V_2F_1)^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}, \tag{53}$$

it is obtained from (50) and (51) that

$$\begin{aligned} &V^{-1}FK^{-1} + \bar{F}CK^{-1} \\ &= \left[\begin{array}{cc} V_1F_1 & 0 \\ V_2F_1 & I_{n-p} \end{array}\right] + \left[\begin{array}{cc} I_p - V_1F_1 & 0 \\ -V_2F_1 & 0 \end{array}\right] = I_n. \end{aligned} \tag{54}$$

As a result, we can choose

$$P = \left[\begin{array}{cc} C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} & C_0(C_0^{\mathrm{T}}C_0)^{-1} \end{array}\right]\left[\begin{array}{c} V_1 \\ V_2 \end{array}\right] \tag{55}$$

and

$$\begin{aligned} Q =& C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}(I_p - (F_0F_0^{\mathrm{T}})^{-1}F_0F_1) \\ &- C_0(C_0^{\mathrm{T}}C_0)^{-1}(F_2^{\mathrm{T}}F_2)^{-1}F_2^{\mathrm{T}}F_1, \end{aligned} \tag{56}$$

to obtain (6).

B. Proof of Lemma 3

The observabilities of two matrix pairs (A_0, C) and $(\tilde{A}, C)$ are equivalent to each other. Thus, under Assumption 1, we only need to prove that $(\tilde{A}, C)$ is observable if and only if Assumption 2 is satisfied. Note that

$$\begin{aligned} &\mathrm{rank}\left[\begin{array}{cc} sF - A & E \\ C & 0 \end{array}\right] = \mathrm{rank}\left[\begin{array}{cc} sPF - PA & PE \\ C & 0 \end{array}\right] \\ =&\mathrm{rank}\left[\begin{array}{cc} I_n & sQ \\ 0 & I_p \end{array}\right]\left[\begin{array}{cc} sPF - PA & PE \\ C & 0 \end{array}\right] \\ =&\mathrm{rank}\left[\begin{array}{cc} sI_n - PA & PE \\ C & 0 \end{array}\right] \\ =&\mathrm{rank}\left[\begin{array}{cc} \tilde{W} & sW \\ 0 & I_p \\ WC & -sW \end{array}\right]\left[\begin{array}{cc} sI_n - PA & PE \\ C & 0 \end{array}\right] \\ =&\mathrm{rank}\left[\begin{array}{cc} sI_n - \tilde{A} & 0 \\ C & 0 \\ -WCPA & PE \end{array}\right]\left[\begin{array}{cc} I_n & 0 \\ (CPE)^{-}CPA & I_l \end{array}\right] \\ =&\mathrm{rank}\left[\begin{array}{cc} sI_n - \tilde{A} & 0 \\ C & 0 \\ 0 & PE \end{array}\right] = \mathrm{rank}\left[\begin{array}{c} sI_n - \tilde{A} \\ C \end{array}\right] + \mathrm{rank}\,(E) \end{aligned}$$

which implies that Assumption 2 is equivalent to the observability of $(\tilde{A}, C)$. This shows that (A_0, C) is observable and thus the invertibility of $W_o(A_0, \tau)$ can be guaranteed.

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