Prescribed-time Unknown Input Observers Design for Linear Systems with Commensurate Delays

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***Abstract*— This paper investigates the problem of prescribed-time observer design for a class of linear time-delay systems with commensurate delays and unknown inputs. Based on ring theory and the periodic delayed feedback approach, Luenberger-like observers are designed such that the estimation error converges to zero within a finite time. As a by-product, it is demonstrated how a previously proposed generalized coordinate transformation allows to extend the scope of periodic delayed output feedback, originally developed for systems with input-output delay, to a class of systems with state-delays.**

***Index Terms*— Time-delay system, commensurate delays, prescribed-time observer, unknown inputs**

I. Introduction

THE observer design for delayed systems is an important and complex problem in the field of control theory Observer design plays a crucial role in addressing the state estimation problem in delayed systems, with the objective of providing effective state estimation for control systems with unmeasurable state, thereby enabling the tracking and control of the state of the system. This has widespread applications such as observer-based prescribed-time control [26], H_2/H_∞ control [16] and fault detection [1], [2].

Moreover, the issue of designing unknown input observers has garnered widespread attention in recent years. In particular, the case of linear systems without delays has been thoroughly addressed, with various methods, such as sliding-mode observers [3], [4], [8]–[10], [15], system inversion [20], the consensus-based approach [22] and several other approaches [13], [14], being extensively developed and refined. Articles [6], [19], [23] also presented the design of a prescribed-time observer for systems with unknown inputs. Moreover, a unifying framework for unknown input observer design of linear time-invariant systems was proposed in [27]. However, the problem of observer design for linear time-delay systems remains an important research topic. Researchers have introduced various tools to address this problem. The commonly used approaches developed in recent decades are based on H_∞ control theory [28], [29], Lyapunov-Krasovskii functionals [25] / Lyapunov-Razumikhin functions [24], on geometric ideas [5] and on the interpretation as dynamical system on a polynomial ring [7], [12], [30].

In [12], based on the ring theory, a generalized coordinate change such that in the transformed system the time-delay terms are only in the output injection was proposed. Inspired by which, [31] designed an exponentially stable observer for general time-delay systems with commensurate delays in the state, the input and the output. Article [17] employed a similar method and achieved prescribed-time state observation by designing a pair of coupled Luenberger observers. Different from [17], this paper adopts the technique of polynomial ring and designs a prescribed-time Luenberger-like observer for general time-delay systems by using periodic delayed feedback, where the observer gain is continuously differentiable and the convergence time is independent of the initial condition.

This paper is organized as follows: In Section II, some properties of polynomial matrices are reviewed, and the periodic delayed feedback method, which is one of the core aspects of this paper, is briefly introduced. In Section III, a generalized coordinate transformation is applied and a prescribed-time unknown input observer design is presented. In Section IV, a numerical example is carried out to verify the effectiveness of the prescribed-time observer design.

Notation: We use $\mathbf{R}[d], \mathbf{R}^{m\times n}[d]$ to denote the polynomial ring over the real field $\mathbf{R}$ and the set of matrices of dimension $m\times n$ whose entries are in $\mathbf{R}[d]$, respectively. For a polynomial matrix $A(d)$, $\deg(A(d))$ means the maximal degree of all the entries of $A(d)$. I_n means the identity matrix of dimension $n\times n$. Define the observability matrix $Q_l(A(d),C(d))$ of the polynomial matrix pair $(A(d),C(d))$ as $Q_l(A(d),C(d)) = [C(d)^{\mathrm{T}},(C(d)A(d))^{\mathrm{T}},\cdots,(C(d)A^{l-1}(d))^{\mathrm{T}}]^{\mathrm{T}}$.

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II. Preliminaries and Problem Statement

Consider the following linear system with commensurate delays:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=0}^{k_1} A_i x(t-i\tau) + \sum\limits_{i=0}^{k_2} B_i u(t-i\tau) \\ \qquad + \sum\limits_{i=0}^{k_3} E_i w(t-i\tau), \\ y(t) = \sum\limits_{i=0}^{k_4} C_i x(t-i\tau) + \sum\limits_{i=0}^{k_5} D_i u(t-i\tau), \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $y(t) \in \mathbf{R}^p$ is the output vector, $u(t) \in \mathbf{R}^m$ is the known input, $w(t) \in \mathbf{R}^l$ is the unknown input and $\tau > 0$ is a given delay. The initial condition $\varphi(s)$ of (1) is a piece-wise continuous function $\varphi : [-k_0\tau, 0] \to \mathbf{R}^n$, where $k_0 = \max\{k_1, k_2, k_3, k_4, k_5\}$. A_i, B_i, C_i, D_i, E_i are the constant matrices of appropriate dimension. Let d be the operator defined as $dx(t) = x(t-\tau)$. Then the system (1) can be rewritten as

$$\begin{cases} \dot{x}\,(t) = A(d)x(t) + B(d)u(t) + E(d)w(t) \\ y\,(t) = C(d)x(t) + D(d)u(t), \end{cases} \tag{2}$$

where $A(d) = \sum_{i=0}^{k_1} A_i d^i, B(d) = \sum_{i=0}^{k_2} B_i d^i, C(d) = \sum_{i=0}^{k_4} C_i d^i$, $D(d) = \sum_{i=0}^{k_5} D_i d^i$ and $E(d) = \sum_{i=0}^{k_3} E_i d^i$ are the matrices over the ring $\mathbf{R}[d]$ of real polynomials.

For polynomial matrices, we recall the following definitions: a polynomial matrix $P(s) \in \mathbf{R}^{p\times m}[s]$ is said to be left unimodular, if $\operatorname{rank}(P(s)) = m, \forall s \in \mathbf{C}$; a polynomial matrix $P(s) \in \mathbf{R}^{p\times m}[s]$ is said to be right unimodular, if $\operatorname{rank}(P(s)) = p, \forall s \in \mathbf{C}$; a polynomial matrix $P(s) \in \mathbf{R}^{p\times m}[s]$ is said to be unimodular, if it is both right unimodular and left unimodular.

In this paper, our objective is to design a prescribed-time observer for system (1) with unknown inputs. The methodology relies on transformations, which are possible under the following assumptions.

Assumption 1: The polynomial matrix $C(d)E(d)$ is left unimodular.

Assumption 2: The polynomial matrices $A(d), B(d)$ and $E(d)$ satisfy

$$\operatorname{rank}\begin{bmatrix} -sI_n + A(d) & E(d) \\ C(d) & 0 \end{bmatrix} = n + l, \forall s, d \in \mathbf{C}.$$

Remark 1: It is worth noting that when there is no time delay in the linear system (2), i.e., when $A(d) = A, B(d) = B, C(d) = C, D(d) = D, E(d) = E$ are constant matrices, Assumption 1 and Assumption 2 become $\operatorname{rank}(CE) = l$ and

$$\operatorname{rank}\begin{bmatrix} sI_n - A & E \\ C & 0 \end{bmatrix} = n + l, \forall s \in \mathbf{C},$$

which are precisely the assumptions made in [11] (see also [13]). Therefore, the two assumptions in this paper can be seen as a natural extension of the two assumptions in [11].

Furthermore, in Assumption 1, it suffices to require that the invariant factors of the matrices $E(d)$ and $C(d)E(d)$ are identical. If $E(d)$ is not left unimodular, we can replace $E(d)w(t)$ by $\tilde{E}(d)\tilde{w}(t)$ with $\tilde{E}(d)$ left unimodular in a preliminary step (based on the Smith form).

Moreover, we introduce the following two important lemmas, which provide the theoretical foundation for the subsequent observer design.

Lemma 1: (See [12]) Consider the time-delay system represented by

$$\begin{cases} \dot{x}(t) = A(d)x(t), \\ y(t) = C(d)x(t). \end{cases} \tag{3}$$

There exist a left unimodular matrix $T(d) \in \mathbf{R}^{pl^*\times n}[d]$ and matrix $F(d) = [F_1^{\mathrm{T}}(d), \cdots, F_{l^*}^{\mathrm{T}}(d)]^{\mathrm{T}} \in \mathbf{R}^{pl^*\times p}[d]$ such that $T(d)A(d) = (\bar{A} + F(d)\bar{C})T(d)$, $C(d)T_L(d) = \bar{C}$, where $T_L(d)$ is the left inverse of $T(d)$ satisfying $T_L(d)T(d) = I_n$ and

$$\begin{aligned} \bar{A} &= \begin{bmatrix} 0 & I_p & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & I_p \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{pl^*\times pl^*} \\ \bar{C} &= \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{p\times pl^*}, \end{aligned} \tag{4}$$

if and only if there exists an integer l^* such that the observability matrix $Q_{l^*}(A(d), C(d))$ is left unimodular. In this case, the system (3) can be transformed into the following form

$$\begin{cases} \dot{z}(t) = \bar{A}z(t) + F(d)y(t) \\ y(t) = \bar{C}z(t) \\ z(t) = T(d)x(t). \end{cases} \tag{5}$$

Moreover, the left unimodular matrix $T(d) = [T_1^{\mathrm{T}}(d), T_2^{\mathrm{T}}(d), \cdots, T_{l^*}^{\mathrm{T}}(d)]^{\mathrm{T}}$ is given by

$$\begin{cases} T_1(d) = C(d), \\ T_{i+1}(d) = T_i(d)A(d) - F_i(d)C(d), \quad 1 \le i \le l^* - 1, \end{cases} \tag{6}$$

and matrices $F_i(d)$ are determined through

$$[F_{l^*}(d), \cdots, F_1(d)] = C(d)A^{l^*}(d)\left[Q_{l^*}(A(d), C(d))\right]_L, \tag{7}$$

where $[Q_{l^*}(A(d), C(d))]_L$ is the left inverse of $Q_{l^*}(A(d), C(d))$.

Lemma 2: (See [18]) There exists an integer $l^* > 0$ such that the observability matrix $Q_{l^*}(A(d), C(d))$ is left unimodular if and only if

$$\operatorname{rank}\begin{bmatrix} sI_n - A(d) \\ C(d) \end{bmatrix} = n, \forall s, d \in \mathbf{C}.$$

Finally, we recall the periodic delayed feedback approach proposed in [32], which is crucial for the design of the prescribed-time observer. Firstly, some definitions need to be provided, which will be referenced later.

Definition 1: ([32]) For a given constant $r > 0$ and an integer $k \ge 0$, a symmetric matrix function $H_r : \mathbf{R} \to \mathbf{R}^{p\times p}$ is said to be an $\mathbf{S}_p^{(k)}(r)$ function if

(1) $H_r(t)$ is $2r$-periodic and $H_r(t) = 0, \forall t \in [0, r]$.

(2) $H_r(t) \ge 0, \forall t \in [r, 2r]$ and there exists a constant $r^* \in (r, 2r)$ such that $H_r(r^*) > 0$.

(3) $H_r \in \mathbf{C}_{p\times p}^k$, which implies that $H_r^{(i)}(r) = H_r^{(i)}(2r) = 0, i = 0, 1, \cdots, k$.

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It is easy to construct an $\mathbf{S}_p^{(k)}(r)$ function. For instance, for any integer $k > 0$, an $\mathbf{S}_p^{(k)}(r)$ function $H_r(t)$ can be constructed as

$$H_r(t) = \begin{cases} 0, & t \in [0, r] \\ \left(\left(\mathrm{e}^{a_1(t-r)} - 1\right)\left(1 - \mathrm{e}^{a_2(t-2r)}\right)\right)^{k+1} I_p, & t \in (r, 2r) \end{cases} \tag{8}$$

where $a_1 > 0$ and $a_2 > 0$ are given constants.

Finally, we give the following important proposition.

Proposition 1: (See [32]) Consider the time-delay system

$$\dot{x}(t) = P(t)x(t) + Q(t)x(t-r), t \geq 0, \tag{9}$$

where $r > 0$ is a given constant and $\varphi(s), s \in [-r, 0]$ is the initial condition of $x(t)$. The matrix pair $(P(t), Q(t)) : (\mathbf{R}, \mathbf{R}) \to (\mathbf{R}^{n\times n}, \mathbf{R}^{n\times n})$ is bounded piecewise continuous and $2r$-periodic with $Q(t)$ satisfying

$$Q(t) = \begin{cases} 0, & t \in [t_0 + 2kr, t_0 + (2k+1)r), \\ Q_0(t), & t \in [t_0 + (2k+1)r, t_0 + 2(k+1)r), \end{cases}$$

where $k = 0, 1, \cdots$. Let $\Phi(t, s)$ be the state transition matrix of the linear system $\dot{x}(t) = P(t)x(t)$. Define

$$\begin{aligned} \Delta(r, t_0) =& \Phi(t_0 + 2r, t_0) \\ &+ \int_{t_0+r}^{t_0+2r} \Phi(t_0 + 2r, s) Q_0(s) \Phi(s - r, t_0) \mathrm{d}s, \end{aligned} \tag{10}$$

where t_0 is a known constant. Then system (9) is prescribed-time stable if and only if $\Delta(r, 0)$ is nilpotent. Moreover, if $\Delta(r, t_0) = 0$, then there holds $x(t) = 0, \forall t \geq t_0 + 2r$.

III. PRESCRIBED-TIME OBSERVER DESIGN

Now we consider system (1) and then apply the transformations to it. Since $C(d)E(d)$ and $E(d)$ are both left unimodular under Assumptions 1 and 2, there exist unimodular matrices $R(d) \in \mathbf{R}^{n\times n}[d]$ and $S(d) \in \mathbf{R}^{p\times p}[d]$ such that

$$R(d)E(d) = \begin{bmatrix} I_l \\ 0 \end{bmatrix}, S(d)C(d)E(d) = \begin{bmatrix} I_l \\ 0 \end{bmatrix}.$$

Partition the matrices as follows

$$R(d) = \begin{bmatrix} R_1(d) \\ R_2(d) \end{bmatrix}, R^{-1}(d) = [R_3(d), R_4(d)], S(d) = \begin{bmatrix} S_1(d) \\ S_2(d) \end{bmatrix},$$

where $R_1(d) \in \mathbf{R}^{l\times n}[d]$, $R_3(d) \in \mathbf{R}^{n\times l}[d]$ and $S_1(d) \in \mathbf{R}^{l\times p}[d]$. Then we have

$$\begin{aligned} R_1(d)E(d) = I_l, R_2(d)E(d) = 0, \\ S_1(d)C(d)E(d) = I_l, S_2(d)C(d)E(d) = 0, \\ E(d) = R^{-1}(d)\begin{bmatrix} I_l \\ 0 \end{bmatrix} = R_3(d). \end{aligned} \tag{11}$$

Consider the state transformation

$$\begin{bmatrix} \eta_1(t)^{\mathrm{T}} & \eta_2(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = R(d)x(t), \eta_1(t) \in \mathbf{R}^l$$

and output transformation

$$\begin{bmatrix} y_1(t)^{\mathrm{T}} & y_2(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = S(d)y(t), y_1(t) \in \mathbf{R}^l.$$

Partition the transformed system matrices as follows

$$\begin{aligned} \tilde{A}(d) &= \begin{bmatrix} A_{11}(d) & A_{12}(d) \\ A_{21}(d) & A_{22}(d) \end{bmatrix}, R(d)B(d) = \begin{bmatrix} B_1(d) \\ B_2(d) \end{bmatrix}, \\ \tilde{C}(d) &= \begin{bmatrix} C_1(d) & C_2(d) \\ C_3(d) & C_4(d) \end{bmatrix}, S(d)D(d) = \begin{bmatrix} D_1(d) \\ D_2(d) \end{bmatrix}, \end{aligned}$$

where $\tilde{A}(d) = R(d)A(d)R^{-1}(d)$, $\tilde{C}(d) = S(d)C(d)R^{-1}(d)$, $A_{11}(d) \in \mathbf{R}^{l\times l}[d]$, $B_1(d) \in \mathbf{R}^{l\times m}[d]$, $C_1(d) \in \mathbf{R}^{l\times l}[d]$ and $D_1(d) \in \mathbf{R}^{l\times m}[d]$. Moreover, it can be derived from (11) that

$$\begin{aligned} C_1(d) = S_1(d)C(d)R_3(d) = I_l, C_2(d) = S_1(d)C(d)R_4(d), \\ C_3(d) = S_2(d)C(d)R_3(d) = 0, C_4(d) = S_2(d)C(d)R_4(d). \end{aligned}$$

Thus, system (2) can be rewritten as

$$\begin{cases} \dot{\eta}_1(t) = A_{11}(d)\eta_1(t) + A_{12}(d)\eta_2(t) + B_1(d)u(t) + w(t) \\ \dot{\eta}_2(t) = A_{21}(d)\eta_1(t) + A_{22}(d)\eta_2(t) + B_2(d)u(t) \\ y_1(t) = \eta_1(t) + C_2(d)\eta_2(t) + D_1(d)u(t) \\ y_2(t) = C_4(d)\eta_2(t) + D_2(d)u(t) \end{cases} \tag{12}$$

Let $\xi(t) = \eta_1(t) + C_2(d)\eta_2(t)$. Then we can obtain that

$$\begin{aligned} \dot{\xi}(t) =& A_{11}(d)\xi(t) - A_{11}(d)C_2(d)\eta_2(t) + C_2(d)\dot{\eta}_2(t) \\ &+ A_{12}(d)\eta_2(t) + B_1(d)u(t) + w(t) \\ =& (A_{11}(d) + C_2(d)A_{21}(d))\xi(t) \\ &+ (B_1(d) + C_2(d)B_2(d))u(t) + w(t) \\ &+ (C_2(d)A_{22}(d) + A_{12}(d) - A_{11}(d)C_2(d) \\ &- C_2(d)A_{21}(d)C_2(d))\eta_2(t), \end{aligned}$$

and $\dot{\eta}_2(t) = A_{21}(d)\xi(t) + (A_{22}(d) - A_{21}(d)C_2(d))\eta_2(t) + B_2(d)u(t)$. Then (12) can be expressed as

$$\begin{cases} \dot{\xi}(t) = A_1(d)\xi(t) + A_2(d)\eta_2(t) \\ \qquad + (B_1(d) + C_2(d)B_2(d))u(t) + w(t) \\ \dot{\eta}_2(t) = A_{21}(d)\xi(t) + A_3(d)\eta_2(t) + B_2(d)u(t) \\ y_1(t) = \xi(t) + D_1(d)u(t) \\ y_2(t) = C_4(d)\eta_2(t) + D_2(d)u(t) \end{cases} \tag{13}$$

where

$$\begin{aligned} A_1(d) &= A_{11}(d) + C_2(d)A_{21}(d), \\ A_2(d) &= C_2(d)A_{22}(d) + A_{12}(d) - A_{11}(d)C_2(d) \\ &\quad - C_2(d)A_{21}(d)C_2(d), \\ A_3(d) &= A_{22}(d) - A_{21}(d)C_2(d). \end{aligned}$$

Since $\xi(t) = y_1(t) - D_1(d)u(t)$ can be directly obtained from the output, we only need to design an observer for $\eta_2(t)$. The following lemma informs us that Assumptions 1 and 2 ensure the left unimodularness of the observability matrix $Q_k(A_3(d), C_4(d))$.

Lemma 3: Under Assumption 1, there exists an integer $l^* > 0$ such that the observability matrix $Q_{l^*}(A_3(d), C_4(d))$ is left unimodular if and only if Assumption 2 is satisfied.

Proof: It holds that

$$\begin{aligned} &\mathrm{rank}\begin{bmatrix} A(d) - sI_n & E(d) \\ C(d) & 0 \end{bmatrix} \\ =&\mathrm{rank}\begin{bmatrix} R(d)A(d)R^{-1}(d) - sI_n & R(d)E(d) \\ S(d)C(d)R^{-1}(d) & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
=&\operatorname{rank}\begin{bmatrix} A_{11}(d)-sI_l & A_{12}(d) & I_l \\ A_{21}(d) & A_{22}(d)-sI_{n-l} & 0 \\ I_l & C_2(d) & 0 \\ 0 & C_4(d) & 0 \end{bmatrix} \\
=&\operatorname{rank}\begin{bmatrix} 0 & A_{22}(d)-A_{21}(d)C_2(d)-sI_{n-l} \\ I_l & C_2(d) \\ 0 & C_4(d) \end{bmatrix}+l \\
=&\operatorname{rank}\begin{bmatrix} A_3(d)-sI_{n-l} \\ C_4(d) \end{bmatrix}+2l, \forall s,d\in\mathbf{C}.
\end{aligned}
$$

Applying Lemma 2 to $(A_3(d), C_4(d))$ then we can obtain that under Assumption 1, there exists an integer $l^*>0$ such that $Q_{l^*}(A_3(d), C_4(d))$ is left unimodular if and only if Assumption 2 is satisfied. ■

Combining Lemma 1 and Lemma 3, the following theorem is easy to obtain.

Theorem 1: Under Assumptions 1 and 2, there exists a generalized coordinate change $z(t)=T(d)\eta_2(t)$, where $T(d)\in\mathbf{R}^{(p-l)l^*\times(n-l)}$ is left unimodular, such that system (13) can be transformed into the following form

$$
\begin{cases}
\dot{z}(t)=\bar{A}z(t)+F(d)y_2(t)+\tilde{A}_{21}(d)\xi(t) \\
\qquad +(\tilde{B}_2(d)-F(d)D_2(d))u(t) \\
y_2(t)=\bar{C}z(t)+D_2(d)u(t),
\end{cases} \tag{14}
$$

where $\tilde{A}_{21}(d)=T(d)A_{21}(d)$, $\tilde{B}_2(d)=T(d)B_2(d)$ and $\bar{A},\bar{C},T(d),F(d)$ are respectively defined by (4), (6) and (7), in which the matrix pair $(A(d),C(d))$ is replaced by $(A_3(d),C_4(d))$.

Proof: Applying Lemma 1 yields

$$
\begin{aligned}
\dot{z}(t)=&T(d)A_3(d)\eta_2(t)+T(d)A_{21}(d)\xi(t) \\
&+T(d)B_2(d)u(t) \\
=&(\bar{A}+F(d)\bar{C})z(t)+T(d)A_{21}(d)\xi(t) \\
&+T(d)B_2(d)u(t) \\
=&\bar{A}z(t)+F(d)y_2(t)+T(d)A_{21}(d)\xi(t) \\
&+(T(d)B_2(d)-F(d)D_2(d))u(t),
\end{aligned}
$$

which is precisely Eq. (14). ■

Next, we can proceed with the design of the prescribed-time observer. Denote $n_1=\max\{\deg(F(d)),\deg(\tilde{A}_{21}(d)),\deg(\tilde{B}_2(d)-F(d)D_2(d)),\deg(D_2(d))\}$ and $n_2=\deg((R_4(d)-R_3(d)C_2(d))T_L(d))$. Let $L\in\mathbf{R}^{(p-l)l^*\times(p-l)}$ be a given constant gain matrix such that $A_c=\bar{A}-L\bar{C}$ is Hurwitz and $L_{A_c}(t):\mathbf{R}\to\mathbf{R}^{(p-l)l^*\times(p-l)}$ be defined by

$$
L_{A_c}(t)=\begin{cases}
0, \quad t\in[0,n_1\tau) \text{ or } t\in[n_1\tau+2r,\infty), \\
\mathrm{e}^{A_c(t-n_1\tau)}W\mathrm{e}^{A_c^{\mathrm{T}}(t-n_1\tau-r)}\bar{C}^{\mathrm{T}}H_r(t-n_1\tau), \\
t\in[n_1\tau,n_1\tau+2r],
\end{cases}
$$

where $W\in\mathbf{R}^{(p-l)l^*\times(p-l)l^*}$ is a constant matrix to be determined and $H_r(t)$ is given in Definition 1.

Theorem 2: Let Assumptions 1 and 2 be satisfied and $r>0$ be a prescribed constant. Define the matrix

$$
M(A_c,r)=\int_0^r \mathrm{e}^{A_c^{\mathrm{T}}s}\bar{C}^{\mathrm{T}}H_r(s+r)\bar{C}e^{A_cs}\mathrm{d}s,
$$

which is invertible for $\forall r>0$ since $(A_c,\bar{C})$ is observable. Choose the matrix W as $W=M^{-1}(A_c,r)$. Then the following observer

$$
\begin{cases}
\dot{\hat{z}}(t)=\bar{A}\hat{z}(t)+F(d)y_2(t)+\tilde{A}_{21}(d)\xi(t) \\
\qquad +(\tilde{B}_2(d)-F(d)D_2(d))u(t)+L(y_2(t)-\hat{y}_2(t)) \\
\qquad +L_{A_c}(t)(y_2(t-r)-\hat{y}_2(t-r)) \\
\hat{y}_2(t)=\bar{C}\hat{z}(t)+D_2(d)u(t) \\
\hat{\eta}_2(t)=T_L(d)\hat{z}(t) \\
\hat{x}(t)=R^{-1}(d)\begin{bmatrix}\xi(t)-C_2(d)\hat{\eta}_2(t) \\ \hat{\eta}_2(t)\end{bmatrix}
\end{cases} \tag{15}
$$

is a prescribed-time observer for system (2), which satisfies $x(t)=\hat{x}(t),\forall t\geq(n_1+n_2)\tau+2r$.

Proof: Design the following full-order observer for (14)

$$
\begin{cases}
\dot{\hat{z}}(t)=\bar{A}\hat{z}(t)+F(d)y_2(t)+\tilde{A}_{21}(d)\xi(t) \\
\qquad +(\tilde{B}_2(d)-F(d)D_2(d))u(t)+L(y_2(t)-\hat{y}_2(t)) \\
\qquad +L_{A_c}(t)(y_2(t-r)-\hat{y}_2(t-r)) \\
\hat{y}_2(t)=\bar{C}\hat{z}(t)+D_2(d)u(t)
\end{cases} \tag{16}
$$

Define the estimation error $e_z(t)=z(t)-\hat{z}(t)$ and $e_x(t)=x(t)-\hat{x}(t)$. Combining (14) and (16) yields

$$
e_z(t)=A_ce_z(t)-L_{A_c}(t)\bar{C}e_z(t-r).
$$

Since $W=M^{-1}(A_c,r)$, it follows from (10) that

$$
\begin{aligned}
\Delta(r,n_1\tau)=&\mathrm{e}^{2A_cr}\Big(I-W\int_{n_1\tau+r}^{n_1\tau+2r}\mathrm{e}^{A_c^{\mathrm{T}}(s-n_1\tau-r)}\bar{C}^{\mathrm{T}} \\
&\times\bar{C}\mathrm{e}^{A_c(s-n_1\tau-r)}H_r(s-n_1\tau)\mathrm{d}s\Big) \\
=&\mathrm{e}^{2A_cr}(I-WM(A_c,r))=0,
\end{aligned}
$$

which implies that $e_z(t)=0,\forall t\geq n_1\tau+2r$ according to Proposition 1. Thus we have

$$
\begin{aligned}
e_x(t)&=x(t)-R_3(d)\xi(t)+(R_3(d)C_2(d)-R_4(d))\hat{\eta}_2(t) \\
&=R_3(d)\eta_1(t)+R_4(d)\eta_2(t)+R_3(d)C_2(d)\hat{\eta}_2(t) \\
&\quad -R_4(d)\hat{\eta}_2(t)-R_3(d)(\eta_1(t)+C_2(d)\eta_2(t)) \\
&=(R_4(d)-R_3(d)C_2(d))(\eta_2(t)-\hat{\eta}_2(t)) \\
&=(R_4(d)-R_3(d)C_2(d))T_L(d)e_z(t) \\
&=0,\forall t\geq(n_1+n_2)\tau+2r.
\end{aligned}
$$

The proof is completed. ■

Remark 2: Because of the generalized coordinate transformation $R(d)x(t)$, the initial condition of the transformed system becomes $R(d)\varphi(s), s\in[-k_0\tau+\deg(R(d))\tau,\deg(R(d))\tau]$. Similarly, the generalized coordinate transformation $T(d)\eta_2(t)$ leads to a corresponding change of the initial condition. For simplicity, we directly define $n_1=\max\{\deg(F(d)),\deg(\tilde{A}_{21}(d)),\deg(\tilde{B}_2(d)-F(d)D_2(d)),\deg(D_2(d))\}$. Thus, the time-varying gain $L_{A_c}(t)$ will start to be injected at $n_1\tau$, ensuring consistency with the initial time of the transformed system.

Remark 3: It can be observed that the dimension of observer (15) is $(p-l)l^*$, which may be greater than the

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dimension n of the original system. This is a result of the generalized coordinate transformation.

Remark 4: Unlike high-gain observers, the method of periodic delayed feedback does not introduce singularities, and the convergence time is unaffected by the initial conditions, which will be illustrated in the numerical example.

IV. Numerical Example

Consider the linearized model of Mach number in a wind tunnel [21]

$$\begin{cases} \dot{x}_1(t) = -ax_1(t) + kax_2(t-\tau) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = -\omega^2 x_2(t) - 2\zeta\omega x_3 + \omega^2 u(t) \end{cases}$$

where a, k, ω, ζ are constants. Therefore, the coefficient matrices of the system are

$$A(d) = \begin{bmatrix} -a & kad & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -2\zeta\omega \end{bmatrix}, B(d) = \begin{bmatrix} 0 \\ 0 \\ \omega^2 \end{bmatrix}.$$

Let the output matrix and the unknown input matrix be

$$C(d) = \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \end{bmatrix}, E(d) = \begin{bmatrix} 1 \\ 1 \\ d \end{bmatrix}.$$

It is easy to verify that Assumptions 1 and 2 hold. According to Eqs. (6), (7) and (11), the corresponding matrices can be calculated directly as follows:

$$R(d) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -d & 0 & 1 \end{bmatrix}, R(d)B(d) = \begin{bmatrix} 0 \\ 0 \\ \omega^2 \end{bmatrix},$$

$$S(d) = \begin{bmatrix} 0 & 1 \\ 1 & -(1+d) \end{bmatrix}, S(d)C(d)R^{-1}(d) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

$$\tilde{A}(d) = \begin{bmatrix} -a + kad & kad & 0 \\ a + (1-ak)d & -kad & 1 \\ (a - akd - 2\zeta\omega)d - \omega^2 & -kad^2 - \omega^2 & -2\zeta\omega \end{bmatrix},$$

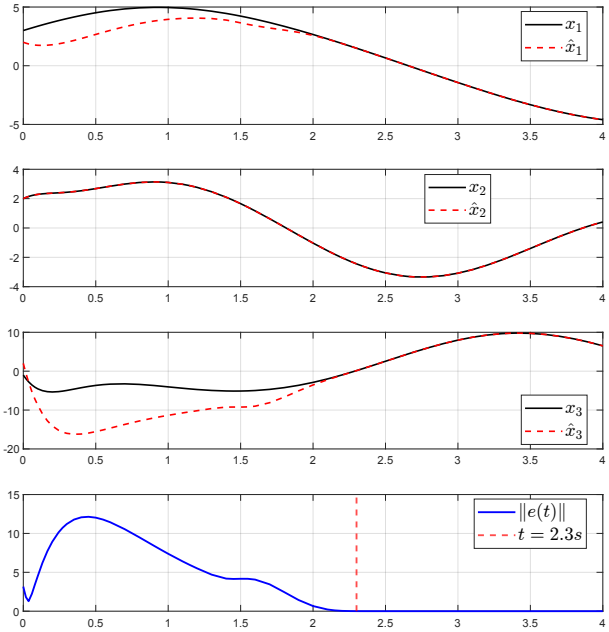
$$T(d) = \begin{bmatrix} -1 & 0 \\ -2\zeta\omega & -1 \end{bmatrix}, F(d) = \begin{bmatrix} -(a + 2\zeta\omega + d) \\ -(ad + 2a\zeta\omega) \end{bmatrix},$$

where $\tilde{A}(d) = R(d)A(d)R^{-1}(d)$. Then it can be obtained that $n_1 = 2, n_2 = 1$. Construct the $\mathbf{S}_p^{(k)}(r)$ function $H_r(t)$ in (1) with $a_1 = 0.5$, $a_2 = 0.5$ and $k = 1$. The constant gain L can be chosen as $L = [5, 6]^{\mathrm{T}}$ such that the poles of A_c are configured at -2 and -3. We select initial conditions of the observer as $\hat{z}_0 = [0, 0]^{\mathrm{T}}$. In the subsequent simulations, we will investigate the effects of time delay r and the initial conditions x_0 of the model on the convergence time of the error.

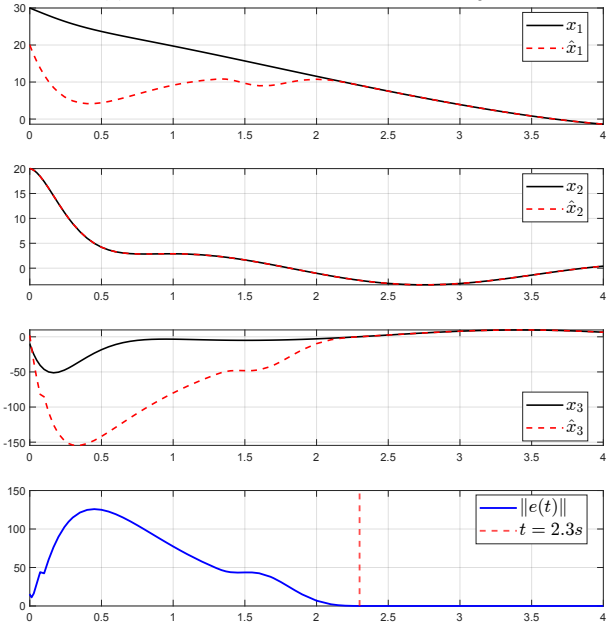
As in [21] the parameters are set to $k = -0.0117\mathrm{deg}^{-1}$, $h = 1.964$s, $\zeta = 0.8$, $\omega = 6.0$rads/s and $a = 1/h$. Set the known input $u(t)$, the unknown input $w(t)$ and the delay τ as $u(t) = 2\sin(2t)$, $w(t) = 6\cos(t)$ and $\tau = 0.1$s.

First, we set initial condition x_0 of the model as $x_1(0) = 3, x_2(s) = 2, \forall s \in [-\tau, 0], x_3(0) = -1$ and $r = 1$s to investigate the effects of different initial values on the convergence of the error $e(t)$. For the observer designed as (15), the estimation $\hat{x}(t)$ and the error $e(t)$ corresponding to the initial values of $x(0) = x_0$, $x(0) = 10x_0$ are recorded in Fig. 1. From Fig. 1, it can be observed that the estimation error $e(t)$ converges to zero after $3\tau + 2r = 2.3$s, and the convergence time is independent of the initial conditions, which coincides with Remark 4.

Next, we set $r = 0.6$s with the same initial condition x_0. In this case the estimation $\hat{x}(t)$ and the error $e(t)$ are recorded in Fig. 2. From Fig. 2, it can be observed that the estimation error $e(t)$ converges to zero after $3\tau + 2r = 1.5$s, where the convergence time is precisely what we anticipated.



(a) Estimation with initial condition x_0.



(b) Estimation with initial condition $10x_0$.

Fig. 1: Estimation with different initial conditions and $r = 1$s.

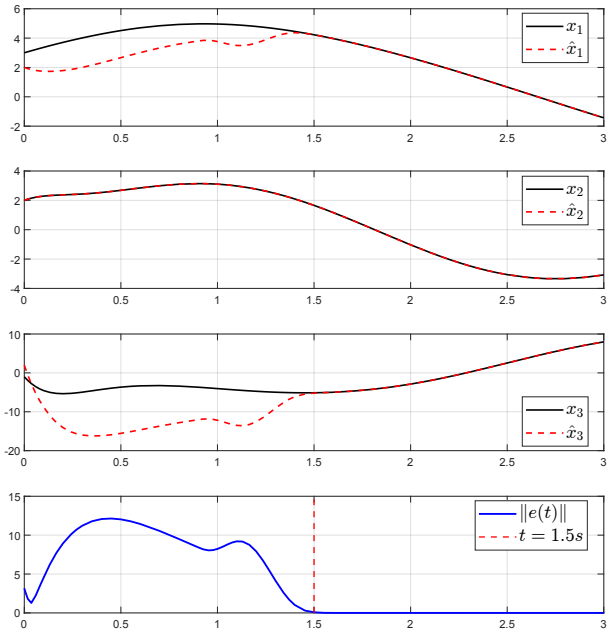


Fig. 2: Estimation with initial condition x_0 and $r = 0.6$s.

V. Conclusion

This work addressed the issue of prescribed-time observer design for a large class of time-delay systems with commensurate delays and unknown inputs. Based on ring theory and periodic delayed feedback methods, a Luenberger observer is designed such that the state observation of the system can be achieved within a finite time. At last, a numerical example is provided to validate the effectiveness of the method. Furthermore, since the generalized coordinate transformation is not restricted to systems with commensurate delays, it is expected that an extension to systems with non-commensurate delays is possible, relying on the multivariate polynomial ring framework.

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