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LINEAR ALGEBRA ASPECTS OF THE CHRISTOFFEL TRANSFORM FOR ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract

Let μ be a Borel measure on the unit circle and let $\mathcal{L}$ be a bilinear form defined on the space of polynomials by $\mathcal{L}(p,q) = \int p(z)q_*(z)d\mu(z)$, with $q_*(z) = \overline{q(1/\overline{z})}$. Let ρ_n be the corresponding Verblunsky coefficients. The GGT (Gragg-Golub-Teplyaev) matrix is an upper Hessenberg matrix that corresponds to the multiplication with the complex variable with respect to the basis of orthogonal polynomials. The Christoffel transform is a spectral transform that corresponds to a new bilinear form $\mathcal{L}_c(p,q) = \mathcal{L}(|z-\alpha|p(z), |z-\alpha|q(z))$, $\alpha \in \mathbb{C}$. Computing the Verblunsky coefficients for $\mathcal{L}_c$, given the Verblunsky coefficients for $\mathcal{L}$ corresponds to a QR decomposition of the GGT matrix. However, using the unitarity of the GGT matrix and its decomposition into elementary orthogonal transformations, it is possible to compute the first n Verblunsky coefficients for the Christoffel transform in $O(n)$ computations, just using as input the original Verblunsky coefficients. The algorithm is very stable, requiring only elementary orthogonal transforms, and never forming the GGT matrix explicitly.

If time allows, we shall say something about the Geronimus transform which corresponds to the bilinear form $\mathcal{L}_g(p,q) = \mathcal{L}(\frac{p(z)}{|z-\alpha|}, \frac{q(z)}{|z-\alpha|}) + m\delta_\alpha(p,q) + \overline{m}\delta_{1/\overline{\alpha}}(p,q)$ with $\delta_\alpha(p,q) = p(\alpha)q_*(\alpha)$ and $|\alpha| > 1$. This algorithm is fast as well, but more involved and its numerical stability is not guaranteed.

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