A Lyapunov approach to stability analysis of partial synchronization in delay-coupled networks

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Abstract: Networks of interconnected dynamical systems may exhibit a so-called partial synchronization phenomenon, which refers to synchronous behaviors of some but not all of the systems. The patterns of partial synchronization are often characterized by partial synchronization manifolds, which are linear invariant subspace of the state space of the network dynamics. Here, we propose a Lyapunov-Krasovskii approach to analyze the stability of partial synchronization manifolds in delay-coupled networks. First, the synchronization error dynamics are isolated from the network dynamics in a systematic way. Second, we use a parameter-dependent Lyapunov-Krasovskii functional to assess the local stability of the manifold, by employing techniques originally developed for linear parameter-varying (LPV) time-delay systems. The stability conditions are formulated in the form of linear matrix inequalities (LMIs) which can be solved by several available tools.

Keywords: Partial synchronization, linear parameter-varying systems, time-delay systems, linear matrix inequalities

1. INTRODUCTION

In recent decades, synchronization of networks with interconnected dynamical systems has received increasing attention. Synchronization of networked systems has been observed widely in various fields, ranging from nature (Buck and Buck (1976), Lewis et al. (2014)) to engineering (Nijmeijer and Rodriguez-Angeles (2003), K. Y. Pettersen (2006), Ploeg et al. (2014)). In nature, synchronization often happens spontaneously, while in engineering, it is generally a designed phenomenon.

Sometimes, networks may show a form of incomplete synchronization, called *partial synchronization* or *cluster synchronization*, which refers to the situation where only some but not all the systems in the networks synchronize, Pogromsky et al. (2002), Belykh et al. (2008), Dahms et al. (2012). Partial synchronization is often observed in complex systems; for example, synchronous firing of neurons in parts of the human brain, Gray (1994). Besides, it should be pointed out that complete (full) synchronization is not always desirable: synchronization of excessive amount of neurons can cause brain disorders like epilepsy and Parkinson's disease, Bennett and Zukin (2004). Therefore, partial synchronization has become an important topic in complex systems like biological networks (especially neural networks), power grids, communication networks, *etc.*, see Sorrentino et al. (2016) and references therein. It is also noticed that in some networks, there may exist time delays in and between the systems, which can impact the existence and stability of partial synchronization. Some research has been done to study partial synchronization of networked systems with delayed coupling, see, *e.g.* Dahms et al. (2012), Orosz (2012), Steur et al. (2016), Ryono and Oguchi (2015). In Dahms et al. (2012), a master stability function (Pecora and Carroll (1998)) based method is used for characterization and stability analysis of partial synchronization of such networks. In Orosz (2012), the network dynamics are decomposed around cluster states for stability analysis of delay-coupled networks of identical systems. In Ryono and Oguchi (2015), an LMI-based condition for stability of partial synchronization is presented for delay-coupled systems with diffusive couplings which are invasive (couplings that do not vanish when systems are synchronized).

In this paper, we focus on partial synchronization in networks of systems interconnected via linear diffusive time-delay couplings. The systems may have one or more types of input-output dynamics. More precisely, it is allowed for some (but not all) systems to have different dynamics.

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The couplings can be *invasive* (coupling terms remain when systems are synchronized) or *non-invasive* (coupling terms vanish when systems are synchronized). Here, we use partial synchronization manifolds to describe patterns of partial synchronization, which are linear invariant subspaces of the state space of the systems in networks (Steur et al. (2016)). To analyze the stability of such manifolds, firstly, the synchronization errors dynamics (differences between the states of the system within each cluster) are isolated from the network dynamics and then linearized around the zero equilibrium solution; secondly, such dynamics are viewed as a linear parameter-varying (LPV) system. In this way, by assessing the stability of the LPV time delay system, we can derive sufficient condition for the local stability of partial synchronization manifolds. Several papers have been devoted to the stability analysis of LPV time-delay systems, see Wu and Grigoriadis (2001), Zhang et al. (2002), Briat (2014). The stability of such systems is often assessed by using the Lyapunov-Krasovskii theorem with a quadratic Lyapunov-Krasovskii functional candidate. The condition for the derivative of the functional along the solutions can be expressed in linear matrix inequalities (LMIs). By choosing appropriate Lyapunov functionals and techniques for the LMIs, the conservatism can be reduced (see Zhang et al. (2002), Briat (2008)). Although similar LMIs conditions deduced from parameter-independent Lyapunov functionals have been used for stability analysis of full synchronization (see Li and Chen (2004),Li et al. (2008) and Oguchi et al. (2008)), few works exploit this framework for partial synchronization case. Therefore, we will use here such method based on a delay- and parameter-dependent Lyapunov functional for stability analysis of partial synchronization manifolds. The structure of this paper is as follows. Section 2 introduces some basic concepts on partial synchronization, including the definition of partial synchronization manifolds and their existence conditions. Section 3 shows the separation of the synchronization error dynamics from the network dynamics. Section 4 shows its local stability condition inferred from a Lyapunov-Krasovskii functional. Section 5 presents an example where the method is applied to a network of Hindmarsh-Rose neuron models. Finally, Section 6 provides the conclusions.

2. PARTIAL SYNCHRONIZATION MANIFOLDS

In this section, we introduce some basic concepts regarding partial synchronization of delay-coupled networks, adopting the setting and results from Steur et al. (2016). First, the definition of partial synchronization manifolds is presented. Second, existence conditions for such manifolds are provided.

Here, we focus on networks of systems which interact via linear time-delay couplings. The networks are represented by directed graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where

- $\mathcal{V}$ is a finite set of nodes with cardinality $|\mathcal{V}| = N \in \mathbb{Z}_+$(*i.e.*, the number of nodes);
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the *ordered* set of edges, where the edge (i, j) points from node i to node j;
- $A = (a_{i,j}) \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix, where $a_{i,j} > 0$ represents the weight of edge (i, j) when $(i, j) \in \mathcal{E}$, and $a_{i,j} = 0$ when $(i, j) \notin \mathcal{E}$.

The networks we consider are simple and strongly connected. A graph $\mathcal{G}$ is simple if it contains neither self-loops nor multiple edges. Self loops are edges connecting a node to itself, and multiple edges are two or more edges connecting an ordered pair of nodes, Gibbons (1985). A graph $\mathcal{G}$ is strongly connected if and only if, for any two nodes $i, j \in \mathcal{V}$, there exist a directed path from i to j and a directed path from j to i, Bollobas (1998).

Every node in the networks hosts a time-invariant dynamical system of the following form

$$\begin{cases} \dot{x}_i(t) = f_i(x_i(t)) + B_i u_i(t) \\ y_i(t) = C_i x_i(t) \end{cases} \tag{1}$$

where $i \in \mathcal{V}$, state $x_i(t) \in \mathbb{R}^n$, sufficiently smooth function $f_i : \mathbb{R}^n \to \mathbb{R}^n$, input(s) $u_i(t) \in \mathbb{R}^m$, output(s) $y_i(t) \in \mathbb{R}^m$, input matrices $B_i \in \mathbb{R}^{n\times m}$ and output matrices $C_i \in \mathbb{R}^{m\times n}$, $i = 1, \ldots, N$. Here, f_i, B_i and C_i can vary for different nodes, that is, in this setting, it is allowed for some systems to have different input-output dynamics.

Assumption 1. Systems (1) are strictly $\mathcal{C}^1$-semipassive.

The definition of semipassive is given below.

Definition 2. Pogromsky and Nijmeijer (2001) Consider a system of the following form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and sufficiently smooth functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. This system is strictly $\mathcal{C}^r$-semipassive if there exist a nonnegative storage function $V(x(t)) \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}_+), r \geq 1$ and a scalar function $S : \mathbb{R}^n \to \mathbb{R}$ positive outside some ball $\mathcal{B} = \{x \in \mathbb{R}^n |\ |x| < R\}$ such that

$$\dot{V}(x(t)) \leq y^\top(t)u(t) - S(x(t)). \tag{3}$$

The systems (1) interact via either one of the following two types of diffusive couplings:

$$u_i(t) = k \sum_{j\in\mathcal{N}_i} a_{i,j}[y_j(t-\tau) - y_i(t)], \tag{4}$$

or

$$u_i(t) = k \sum_{j\in\mathcal{N}_i} a_{i,j}[y_j(t-\tau) - y_i(t-\tau)], \tag{5}$$

where $\mathcal{N}_i$ is the neighboring set of node i, defined as $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i, j) \in \mathcal{E}\}$, and τ, k are, respectively, the time-delay and coupling strength. To distinguish between these two types of couplings, we adopt the terminologies "invasive" and "non-invasive", which are commonly used in literature (see Schöll et al. (2009), Jüngling et al. (2011), Ünal and Michiels (2013), Steur et al. (2014), *etc.*). Coupling (4) is called *invasive* coupling since the coupling does not vanish when all the nodes synchronize; while coupling (5) is called *non-invasive* coupling since the coupling vanishes when all the nodes synchronize, Steur et al. (2014).

For the coupled systems (1), (4) or (1), (5), a solution is a partial synchronous solution if there exist $i, j \in \mathcal{V}$ with $i \neq j$ such that

$$x_i(t) = x_j(t), \quad \forall t \geq t_0, \tag{6}$$

whenever $x_i(t) = x_j(t)$ for $t \in [t_0 - \tau,\ t_0]$.

For every set of nodes satisfying condition (6), they are grouped into one cluster. To describe the clustering of the

nodes, a natural way is to use the concept of partition which is a set of nonempty, disjoint subsets of $\mathcal{V}$ while the union of these subsets is $\mathcal{V}$. The subsets are referred as *parts* of the partition, and one part represents one cluster of the nodes. The number of parts is denoted by κ. Here, a partition $\mathcal{P}$ is parameterized by a $N \times N$ permutation matrix Π associated with an equivalence relation $\sim$ such that $i \sim j$ if the ij^{th} entry of Π is equal to 1. It is easy to prove $\kappa = \dim \ker(I_N - \Pi)$.

Let $\mathcal{C}([-\tau, 0], \mathbb{R}^{Nn})$ be the space of continuous functions that map the interval $[-\tau, 0] \subset \mathbb{R}$ into $\mathbb{R}^{Nn}$. That is, $\mathcal{C}([-\tau, 0], \mathbb{R}^{Nn})$ is the state space of the delay-coupled systems. Denote $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^{Nn})$ as the state of the network, the condition (6) can be expressed as $x_t \in \mathcal{M}(\Pi), \forall t \geq t_0$, where

$$\begin{aligned}\mathcal{M}(\Pi) := \{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^{Nn}) \,|\, \phi(\theta) = \text{col}(\phi_1(\theta),\\ \ldots, \phi_N(\theta)), \phi_i(\theta) \in \mathbb{R}^n,\, i = 1, \ldots, N,\\ \phi(\theta) \in \ker(I_{Nn} - \Pi \otimes I_n)\, \forall \theta \in [-\tau, 0]\}\end{aligned}$$

is the set of partially synchronous states induced by the permutation matrix Π.

Definition 3. Steur et al. (2016) The set $\mathcal{M}(\Pi)$ with permutation matrix Π for which $1 < \kappa < N$ is a partial synchronization manifold for the coupled systems (1), (4), or (1), (5), if and only if it is positively invariant under the dynamics (1), (4), or (1), (5), respectively.

If the set $\mathcal{M}(\Pi)$ is a partial synchronization manifold, the partition $\mathcal{P}$ associated with Π is called *viable*.

Given a partition $\mathcal{P}$, the nodes can be relabelled by clusters such that the first κ_1 nodes belong to cluster 1, the second κ_2 belong to cluster 2 and so on, where $\kappa_i, i = 1, \ldots, \kappa$ are the sizes of clusters of $\mathcal{P}$. This can also be done by using another permutation matrix R, which is called *reordering matrix*, which satisfies

$$R^\top \Pi R = \begin{pmatrix} \Pi_C(\kappa_1) & & & 0 \\ & \Pi_C(\kappa_2) & & \\ & & \ddots & \\ 0 & & & \Pi_C(\kappa_\kappa) \end{pmatrix}, \quad \sum_{i=1}^{\kappa} \kappa_i = N, \tag{7}$$

where $\Pi_C(\kappa_i), i = 1, \ldots, \kappa$ are $\kappa_i \times \kappa_i$-dimensional cyclic permutation matrices. Using R, the *reordered* adjacency matrix can be constructed.

$$R^\top A R = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1\kappa} \\ A_{21} & A_{22} & \cdots & A_{2\kappa} \\ \vdots & \ddots & \ddots & \vdots \\ A_{\kappa 1} & A_{\kappa 2} & \cdots & A_{\kappa\kappa} \end{pmatrix}, \quad A_{ij} \in \mathbb{R}^{\kappa_i \times \kappa_j}. \tag{8}$$

With the reordered adjacency matrix defined, the existence condition of partial synchronization manifolds can be formulated below.

Theorem 4. Su et al. (2018) Given an adjacency matrix A and a permutation matrix Π of the same dimension. Assume system (1) is left-invertible (the system input-output map is injective), then the following statements are equivalent:

1) $\mathcal{M}(\Pi)$ is a partial synchronization manifold for (1) and (4), respectively (1) and (5);
2) all blocks, respectively all off-diagonal blocks, of the reordered adjacency matrix (8), partitioned in blocks of size $k_i \times k_j$ have constant row-sums and, in addition, $\mathcal{F}, \mathcal{B}$ and $\mathcal{C}$ defined by

$$\mathcal{F} := \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \\ \vdots \\ f_N(\cdot) \end{bmatrix}, \mathcal{B} := \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}, \mathcal{C} := \begin{bmatrix} C_1^\top \\ C_2^\top \\ \vdots \\ C_N^\top \end{bmatrix}$$

satisfy the conditions $\mathcal{F} = (\Pi \otimes I_n)\mathcal{F}$, $\mathcal{B} = (\Pi \otimes I_n)\mathcal{B}$ and $\mathcal{C} = (\Pi \otimes I_n)\mathcal{C}$.

Here, the conditions $\mathcal{F} = (\Pi \otimes I_n)\mathcal{F}$, $\mathcal{B} = (\Pi \otimes I_n)\mathcal{B}$ and $\mathcal{C} = (\Pi \otimes I_n)\mathcal{C}$ indicate all the nodes in the same cluster host systems with the same dynamics. Note that this theorem is an extension of Theorem 3 and 4 in Steur et al. (2016) where only networks of identical systems are considered.

3. DYNAMICS DECOMPOSITION OF PARTIALLY SYNCHRONIZED NETWORK

In this section, we show how to separate the synchronization error dynamics from the network dynamics, using the procedure presented in Su et al. (2018). The synchronization error dynamics are linearized and will be used for analyzing the local stability of the partial synchronization manifolds.

For simplicity, we assume the systems have been pre-ordered by clusters according to a viable partition $\mathcal{P}$ associated with Π as follows

$$\begin{array}{cc} x_{1,1},\ x_{1,2}, \ldots, x_{1,\kappa_1} & \text{cluster 1,} \\ x_{2,1},\ x_{2,2}, \ldots, x_{2,\kappa_2} & \text{cluster 2,} \\ \vdots & \vdots \\ x_{\kappa,1},\ x_{\kappa,2}, \ldots, x_{\kappa,\kappa_\kappa} & \text{cluster } \kappa, \end{array}$$

where κ_i is the number of nodes in cluster i with $\sum_{i=1}^{\kappa} \kappa_i = N$. Here, $x_{1,1}, \ldots, x_{\kappa,1}$ are referred as the reference systems of each cluster. Now, we denote the synchronization errors by

$$E_i = \begin{bmatrix} e_{i,2} \\ \vdots \\ e_{i,\kappa_i} \end{bmatrix} = \begin{bmatrix} x_{i,2} - x_{i,1} \\ \vdots \\ x_{i,\kappa_i} - x_{i,1} \end{bmatrix}, i = 1, \ldots, \kappa. \tag{9}$$

We also denote $R_{i,j}$ as the value of the row sums of the ij^{th} block of the adjacency matrix for $i, j \in \{1, \ldots, \kappa\}$ (in case of non-invasive coupling, $i \neq j$). Note that these blocks has constant row sums, since $\mathcal{P}$ is viable. In case of non-invasive coupling, we define $R_{ij} = 0$ when $i = j$. Recall that the nodes in each cluster host the same dynamical system, we denote the dynamics of the nodes in cluster i by $g_i, \tilde{B}_i, \tilde{C}_i$, $i = 1, 2, \ldots, \kappa$, where $g_1 = f_1 = f_2 = \cdots = f_{\kappa_1}$, $g_2 = f_{\kappa_1+1} = f_{\kappa_1+1} = \cdots = f_{\kappa_1+\kappa_2}$, $\tilde{B}_1 = B_1 = B_2 = \cdots = B_{\kappa_1}$, $\tilde{C}_1 = C_1 = C_2 = \cdots = C_{\kappa_1}$, and so on.

3.1 Networks with invasive coupling

When denoting the row sums of the adjacency matrix, corresponding to each cluster by

$$R_i = \sum_{j=1}^{\kappa} R_{i,j}, \quad i = 1, \ldots, \kappa$$

the linearized error dynamics (around the zero equilibrium solution $E_i = 0, i = 1, \cdots, N$) can be expressed as

$$\begin{bmatrix} \dot{E}_1(t) \\ \vdots \\ \dot{E}_\kappa(t) \end{bmatrix} = A_{I0} \begin{bmatrix} E_1(t) \\ \vdots \\ E_\kappa(t) \end{bmatrix} + A_{I1} \begin{bmatrix} E_1(t-\tau) \\ \vdots \\ E_\kappa(t-\tau) \end{bmatrix}, \quad (10)$$

with

$$A_{I0} = \begin{bmatrix} I_{\kappa_1-1} \otimes (\frac{\partial g_1}{\partial x}(x_{1,1}(t)) - kR_1\tilde{B}_1\tilde{C}_1) & & 0 \\ & \ddots & \\ 0 & & I_{\kappa_\kappa-1} \otimes (\frac{\partial g_\kappa}{\partial x}(x_{\kappa,1}(t)) - kR_\kappa\tilde{B}_\kappa\tilde{C}_\kappa) \end{bmatrix} \quad (11)$$

$$A_{I1} = k\tilde{\mathcal{B}}(A_{\text{red}} \otimes I_m)\tilde{\mathcal{C}}, \quad (12)$$

$$\tilde{\mathcal{B}} = \text{diag}(I_{\kappa_1-1} \otimes \tilde{B}_1, \ldots, I_{\kappa_\kappa-1} \otimes \tilde{B}_\kappa),$$

$$\tilde{\mathcal{C}} = \text{diag}(I_{\kappa_1-1} \otimes \tilde{C}_1, \ldots, I_{\kappa_\kappa-1} \otimes \tilde{C}_\kappa),$$

and

$$A_{\text{red}} = T_1 A T_1^T - T_2 A T_1^T, \quad (13)$$

where $T_1, T_2 \in \mathbb{R}^{(N-\kappa)\times N}$ are defined as

$$T_1 = \text{diag}(T_{11}, \ldots, T_{1\kappa}), T_2 = \text{diag}(T_{21}, \ldots, T_{2\kappa}),$$

with

$$T_{1i} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & 1 & 0 \\ 0 & & \cdots & 0 \; 1 \end{bmatrix}, T_{2i} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(\kappa_i-1)\times\kappa_i}.$$

For more information on the derivation and a detailed interpretation of $T_1 A T_1^\top$ and $T_2 A T_1^\top$, refer to Su et al. (2018).

3.2 Networks with non-invasive coupling

Defining a series of vectors $\tilde{R}_i \in \mathbb{R}^{\kappa_i-1}$ with their elements being the row sums of the rows in A corresponding to $x_{i,2}, \ldots, x_{i,\kappa_i}$ for $i = 1, \ldots, \kappa$, the linearized error dynamics of networks with non-invasive couplings can be expressed as

$$\begin{bmatrix} \dot{E}_1(t) \\ \vdots \\ \dot{E}_\kappa(t) \end{bmatrix} = A_{N0} \begin{bmatrix} E_1(t) \\ \vdots \\ E_\kappa(t) \end{bmatrix} + A_{N1} \begin{bmatrix} E_1(t-\tau) \\ \vdots \\ E_\kappa(t-\tau) \end{bmatrix}. \quad (14)$$

with

$$A_{N0} = \begin{bmatrix} I_{\kappa_1-1} \otimes \frac{\partial g_1}{\partial x}(x_{1,1}(t)) & & 0 \\ & \ddots & \\ 0 & & I_{\kappa_\kappa-1} \otimes \frac{\partial g_\kappa}{\partial x}(x_{\kappa,1}(t)) \end{bmatrix} \quad (15)$$

$$A_{N1} = -k\tilde{\mathcal{B}}(L_{\text{red}} \otimes I_m)\tilde{\mathcal{C}}, \quad (16)$$

$$L_{\text{red}} = \text{diag}\left(\tilde{R}_1, \ldots, \tilde{R}_\kappa\right) - A_{\text{red}}. \quad (17)$$

4. STABILITY ANALYSIS OF PARTIAL SYNCHRONIZATION MANIFOLDS

After comparing the structure of (10) with that of (14), we notice that they can be re-written in a unified form,

$$\dot{\mathcal{X}}(t) = \mathcal{A}(\rho(t))\mathcal{X}(t) + \mathcal{A}_d\mathcal{X}(t-\tau), \quad (18)$$

where

$$\mathcal{X}(t) = \left[\, E_1^\top(t) \; \cdots \; E_\kappa^\top(t) \,\right]^\top$$

$$\mathcal{A}(\rho(t)) = \begin{cases} A_{I0} & \text{invasive coupling,} \\ A_{N0} & \text{non-invasive coupling,} \end{cases}$$

$$\rho(t) = [x_{1,1}(t), \ldots, x_{\kappa,1}(t)],$$

$$\mathcal{A}_d = \begin{cases} A_{I1} & \text{invasive coupling,} \\ A_{N1} & \text{non-invasive coupling.} \end{cases}$$

In this way, we can interpret the error dynamic system as a linear parameter-varying (LPV) time-delay system, whose dynamics depend on exogeneous non-stationary parameters $\rho(t)$. Then the synchronization problem of the delay-coupled networks can be reformulated as the stability analysis problem of LPV time-delay systems. Due to the time-varying natures of LPV systems, the existing results based on the frequency-domain method or eigenvalue-based analysis techniques cannot be applied for the analysis of this class of systems. Alternatively, we resort to the Lyapunov method to analyze the stability of underlying systems. In particular, some sufficient delay-dependent conditions to check the partial synchronization problems can be obtained in what follows.

Recall that under Assumption 1, the systems (1) are strictly $\mathcal{C}^1$-semipassive. That is, systems (1) have radially unbounded storage functions $V_1(x_1)$, $V_2(x_2)$, $\cdots$, $V_N(x_N)$, satisfying (3). Then, networks of such systems interconnected via the invasive coupling (4) have ultimately bounded solutions (Steur and Nijmeijer (2011)). Furthermore, there exists a radially unbounded nonnegative function $\mathcal{V}(x_1, x_2, \cdots, x_N) = \sum_{i=1}^N \vartheta_i V_i(x_i)$, where ϑ_i is the i-th entry of the vector $\vartheta \in \mathbb{R}^N$ with all entries positive and $\vartheta^\top L = 0$, and a constant $\eta^* \geq 0$ such that $\dot{\mathcal{V}}(x_1, x_2, \cdots, x_N) < 0$, for any $\eta \geq \eta^*$ and all possible $(x_1, x_2, \cdots, x_N)$ subject to $\mathcal{V}(x_1, x_2, \cdots, x_N) \geq \eta$. More precisely, the set $\{x \in \mathbb{R}^{Nn} | \mathcal{V}(x_1, x_2, \cdots, x_N) \leq \eta\}$ is a compact forward invariant set in the context of dynamics (1) coupled via (4), and all solutions converge to this compact set in finite time. In the non-invasive case, similar results hold, but in general conditions on the coupling strength and delay are needed.

Based on the discussion above, we can conclude that the parameters $x_{i,1}(t)$, $i = 1, 2, \cdots, \kappa$ range between some extremal values $\underline{\rho}_i$ and $\bar{\rho}_i$, i.e., $x_{i,1}(t) \in [\underline{\rho}_i, \bar{\rho}_i]$. The boundedness of $x_i(t)$ also leads to the variation rates $\bar{x}_i(t)$ can be confined as $\dot{x}_{i,1}(t) \in [\underline{v}_i, \bar{v}_i]$, $i = 1, 2, \cdots, \kappa$, where $\underline{v}_i$ and $\bar{v}_i$ are, respectively, the lower and upper bounds on $\dot{x}_{i,1}(t)$. The latter can be readily derived from $\dot{\mathcal{X}}(t) = \mathcal{A}(\rho(t))\mathcal{X}(t) + \mathcal{A}_d\mathcal{X}(t-\tau)$. Specifically, if $\|\mathcal{X}(t)\| \leq \rho_o$ for all $t \geq 0$, it follows that $\|\dot{\mathcal{X}}(t)\| \leq \max_{\{\|\mathcal{X}(t)\|\leq\rho_o, \|\mathcal{X}(t-\tau)\|\leq\rho_o\}} \|\mathcal{A}(\rho(t))\mathcal{X}(t) + \mathcal{A}_d\mathcal{X}(t-\tau)\|$ holds. However, as $\|\mathcal{A}(\rho(t))\|$ and $\|\mathcal{A}_d\|$ also depend on the coupling parameter k, the bounds of $\dot{x}_{i,1}(t)$ relate to k. So, it is reasonable to assume that, given a fixed pair of coupling parameters (k, τ), $(\rho(t), \dot{\rho}(t)) \in \Gamma_\rho \times \Gamma_v$, that is, the parameter vector $\rho(t)$ is constrained into a hypercube Γ_ρ, and the variation rate of $\rho(t)$ also belongs to a hypercube Γ_v. Hence, parameter-dependent analysis conditions for all $(\rho(t), \dot{\rho}(t)) \in \Gamma_\rho \times \Gamma_v$, as we now provide, are sufficient for the stability of the partial synchronization manifold.

Theorem 5. The LPV time-delay system (18) is asymptotically stable if there exist positive definite symmetric matrices $P(\rho), Q_j, Z_j \in \mathbb{R}^{(Nn-\kappa n)\times(Nn-\kappa n)}$, and matrices $G_j \in \mathbb{R}^{(Nn-\kappa n)\times(Nn-\kappa n)}$, $j = 1, 2, 3$, such that the following LMI holds for all possible $(\rho, \dot{\rho}) \in \Gamma_\rho \times \Gamma_v$,

$$\begin{bmatrix} \Phi & \frac{\tau}{3}G_1 & \frac{\tau}{3}G_2 & \frac{\tau}{3}G_3 \\ * & -\frac{\tau}{3}Z_1 & \mathbf{0} & \mathbf{0} \\ * & * & -\frac{\tau}{3}Z_2 & \mathbf{0} \\ * & * & * & -\frac{\tau}{3}Z_3 \end{bmatrix} < 0, \tag{19}$$

where (letting $\mathrm{He}(A) := (A + A^\top)$)

$$\begin{cases} \Phi := \Phi_1 + \mathrm{He}\Big(\Lambda_0 P(\rho)\bar{\mathcal{A}} + \sum_{j=1}^{3} G_j\Lambda_j\Big) + \bar{\mathcal{A}}^\top\Big(\frac{\tau}{3}\sum_{j=1}^{3} Z_j\Big)\bar{\mathcal{A}}, \\ \Phi_1 := \mathrm{diag}\Big(Q_1 + \frac{\partial P(\rho)}{\partial \rho}\dot{\rho}, -Q_1 + Q_2, \\ \qquad\qquad - Q_2 + Q_3, -Q_3\Big), \\ \bar{\mathcal{A}} := [\,\mathcal{A}(\rho)\ \mathbf{0}\ \mathbf{0}\ \mathcal{A}_d\,], \\ \Lambda_0 := [\,\mathbf{I}\ \ \mathbf{0}\ \ \mathbf{0}\ \mathbf{0}\,]^\top, \\ \Lambda_1 := [\,\mathbf{I}\ -\mathbf{I}\ \mathbf{0}\ \mathbf{0}\,], \\ \Lambda_2 := [\,\mathbf{0}\ \mathbf{I}\ -\mathbf{I}\ \mathbf{0}\,], \\ \Lambda_3 := [\,\mathbf{0}\ \mathbf{0}\ \mathbf{I}\ -\mathbf{I}\,]. \end{cases} \tag{20}$$

Proof. By constructing the following parameter-dependent Lyapunov-Krasovskii functional,

$$V(t,\rho) = \mathcal{X}^\top(t)P(\rho)\mathcal{X}(t) + V_1(t) + V_2(t) + V_3(t) \tag{21}$$

where

$$\begin{aligned} V_j(t) := &\int_{t-j\frac{\tau}{3}}^{t-(j-1)\frac{\tau}{3}} \mathcal{X}^\top(s)Q_j\mathcal{X}(s)\mathrm{d}s \\ &+\frac{\tau}{3}\int_{-j\frac{\tau}{3}}^{-(j-1)\frac{\tau}{3}}\int_{t+\theta}^{t} \dot{\mathcal{X}}^\top(s)Z_j\dot{\mathcal{X}}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{22}$$

together with the free-weighting matrix techniques (He et al. (2004)), we can readily arrive at the conclusion.

Remark 6. It is noted that the delay-dependent conditions in Theorem 5 are semi-infinite inequalities due to their parametric dependence. To cast the parameter-dependent conditions into a finite-dimensional optimization problem, the Lyapunov matrix $P(\rho)$ in (21) can be approximated by a finite set of basis functions (Wu and Grigoriadis (2001)). The basis functions $\{h_l(\rho)\}_{l=1}^{n_h}$ for $P(\rho)$ should guarantee that

$$P(\rho) = \sum_{l=1}^{n_h} h_l(\rho)P_l,\ P_l = P_l^\top. \tag{23}$$

To further eliminate the dependence on the parameter vector ρ, a finite gridding $\{\rho_r\}_{r=1}^{L}$ of the parameter space can be introduced to generate finite-dimensional convex optimization conditions (Zhang et al. (2002)). We note that sufficient conditions in terms of finite LMIs can be obtained using Polya's relaxation or sum-of-squares techniques (Oliveira and Peres (2007)).

Remark 7. It should be pointed out that, by setting $Z_j = 0$ in (22), we can also derive the delay-independent condition for the stability of system (18). In that case, the only change is that the inequality (19) becomes

$$\tilde{\Phi} < 0, \tag{24}$$

with

$$\tilde{\Phi} := \Phi_1 + \mathrm{He}\big(\Lambda_0 P(\rho)\bar{\mathcal{A}}\big). \tag{25}$$

5. NUMERICAL EXAMPLE

Consider a network of four Hindmarsh-Rose neurons described by the following dynamics:

$$\begin{aligned} \dot{x}_{i,1}(t) &= c - dx_{i,3}^2(t) - x_{i,1}(t) \\ \dot{x}_{i,2}(t) &= r_i(s(x_{i,3}(t) + v_0) - x_{i,2}(t)) \\ \dot{x}_{i,3}(t) &= -ax_{i,3}^3(t) + bx_{i,3}^2(t) + x_{i,1}(t) - x_{i,2}(t) + E_m + u(t) \\ y_i(t) &= x_{i,3}(t) \end{aligned} \tag{26}$$

where the parameters a, b, c, d, r_i, s, v_0, E_m are constants, and $x_{i,1}(\cdot)$, $x_{i,2}(\cdot)$, $x_{i,3}(\cdot)$ and $u_i(\cdot)$ are the recovery variable, the adaptation variable, the membrane potential, and the external current of the i-th, $i = 1, 2, 3, 4$, Hindmarsh and Rose (1984). It has been proved in Steur et al. (2009) that this Hindmarsh-Rose model is strictly semipassive. The values of these parameters are $a = 1$, $b = 3$, $c = 1$, $d = 5$, $r_1 = r_3 = 0.004$, $r_2 = r_4 = 0.005$, $s = 4$, $v_0 = 1.618$ and $E_m = 3.25$, and the structure of the network is described by the adjacency matrix below

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}. \tag{27}$$

Here, we assume that the systems are coupled via (4). It is easy to identify a viable partition $\{\{1, 3\}, \{2, 4\}\}$, and the corresponding linearized synchronization error dynamics

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{A}_d\mathcal{X}(t-\tau), \tag{28}$$

where

$$\begin{aligned} &\mathcal{A} := \mathrm{diag}(A_1, A_2), \\ &A_1 := \begin{bmatrix} -1 & 0 & -2dx_{1,3}(t) \\ 0 & -r_1 & r_1 s \\ 1 & -1 & -3ax_{1,3}^2(t) + 2bx_{1,3}(t) - 3k \end{bmatrix}, \\ &A_2 := \begin{bmatrix} -1 & 0 & -2dx_{2,3}(t) \\ 0 & -r_2 & r_2 s \\ 1 & -1 & -3ax_{2,3}^2(t) + 2bx_{2,3}(t) - 3k \end{bmatrix}, \\ &\mathcal{A}_d := kA_{\mathrm{red}} \otimes BC,\ B = C^\top = [\,0\ 0\ 1\,]^\top, \\ &A_{\mathrm{red}} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

For this network of Hindmarsh-Rose neurons, given a pair of (k, τ), the stability analysis of the partial synchronization manifolds is done at two steps: 1) estimate the parameter ($x_{1,3}(t)$ and $x_{2,3}(t)$) space and the corresponding variation-rate ($\dot{x}_{1,3}(t)$ and $\dot{x}_{2,3}(t)$) space by numerical simulation of the network; 2) check the feasibility of the LMIs in Theorem 5. Specifically, to solve the stability analysis problem, we pick five basis functions in expansion (23) as follows:

$$\begin{aligned} &h_1(\rho) = 1,\ h_2(\rho) = x_{1,3}(t),\ h_3(\rho) = x_{1,3}^2(t), \\ &h_4(\rho) = x_{2,3}(t),\ h_5(\rho) = x_{2,3}^2(t). \end{aligned} \tag{29}$$

To this end, by solving the condition (19), it has been checked that the Hindmarsh-Rose neurons with networks described by (27) are partially synchronous with $x_{1,3}(t) = x_{3,3}(t)$ and $x_{2,3}(t) = x_{4,3}(t)$ when $k = 1.05$ and $\tau = 10$. Meantime, by testing the feasible solution of the condition (19) with different values of k and τ, it has also been shown that for $k \in [0, 5]$, when $k \geq 1.05$, the partial synchronization manifold is locally stable even for a quite large value of time-delay τ.

Remark 8. Using Theorem 5, it has been validated that when k is large enough ($k \geq 1.05$), the Hindmarsh-Rose

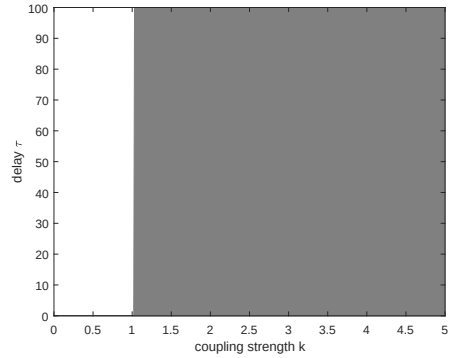

Fig. 1. Partial synchronization region (grey-colored area) in (k, τ) parameter space computed via Theorem 5

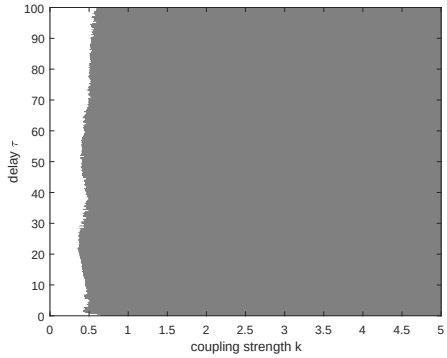

Fig. 2. Partial synchronization region (grey-colored area) in (k, τ) parameter space characterized via the system state trajectories

neurons are partially synchronized ($x_{1,3}(t) = x_{3,3}(t)$ and $x_{2,3}(t) = x_{4,3}(t)$) with a large range of τ. In fact, it is verified that (24) is satisfied provided that $k \geq 1.05$, that is, the partial synchronization manifold is locally stable independent of time delay. This case can also be explained via the concept of convergent dynamics (Steur and Nijmeijer (2011)). In particular, bearing in mind that the input matrix $B = [\,0\ 0\ 1\,]^\top$ and output matrix $C = B^\top$ in this example, it can be easily checked that $CB = 1 > 0$. Then, the system can be equivalently transformed into the following form:

$$\begin{cases} \dot{\bar{\mathcal{X}}}(t) = g_1(\bar{\mathcal{X}}(t), \bar{y}(t)), \\ \dot{\bar{y}}(t) = g_2(\bar{y}(t), \bar{\mathcal{X}}(t)) - kb\bar{D}\bar{y}(t) + kb\bar{A}\bar{y}(t-\tau), \end{cases} \tag{30}$$

where $\bar{A} := T_1^\top A T_1$, $\bar{D} := \mathrm{diag}(\tilde{R}_1, \tilde{R}_2)$, $b = CB = 1$, in addition, the $\bar{\mathcal{X}}(t)$ subsystem with $\bar{y}(t)$ as input is convergent. For this family of delay-coupled systems, it has been shown in Ryono and Oguchi (2015) that the coupled networks can realize the delay-independent partial synchronization when k is larger than some positive number k_0. However, it is difficult to compute the exact value of this threshold k_0 by the synchronization conditions in Ryono and Oguchi (2015). Resorting to Theorem 5 in this paper, we can estimate the value of k_0.

Furthermore, by using Theorem 5, we have calculated the synchronization region for $k \in [0, 5]$ and $\tau \in [0, 100]$, see Fig. 1, where the grey-colored area characterizes the occurrence of the partial synchronization ($\{\{1, 3\}, \{2, 4\}\}$). For comparisons, the synchronization region detected by comparing the state trajectories of the same network is also depicted in Fig. 2. The trajectories are calculated by simulating the network in MATLAB. It can be seen from the two figures that the partial synchronization region obtained by Theorem 5 is smaller than the one obtained by simulation. This is not unexpected since the Lyapunov method generally comes with conservatism. Besides, the systems considered are treated as a LPV time delay system. On one hand, it is an over-approximation when regarding the synchronization error dynamics as a LPV time-delay system, on the other hand, the properties of LPV functions and time-delay also increase the conservatism of the Lyapunov method.

To further demonstrate our method, let us re-consider the delay-coupled Hindmarsh-Rose neurons shown above, but with the input matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top$ and output matrix $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $CB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ does not satisfy the strict condition $CB > 0$. Typically, we cannot apply the analysis conditions proposed in Ryono and Oguchi (2015) for the synchronization validity of the underlying networks of nonlinear systems. Fortunately, by utilizing the partial synchronization analysis criterion in Theorem 5 with $k = 1.02$ and $\tau = 0.04$, one obtains the feasible solution for the stability analysis of the invasive coupled networks.

6. CONCLUSIONS

In this paper, the local stability of partial synchronization manifolds is addressed by utilizing the Lyapunov method. First, the synchronization error dynamics are isolated from the network dynamics, whose stability equates the stability of partial synchronization manifolds. Second, it is shown that the linearized synchronization error dynamics can be over-approximated by a LPV time delay system. By choosing a parameter- and delay-dependent Lyapunov-Krasovskii functional, the stability conditions are formulated as LMIs, which can be solved efficiently. A numerical example is also presented, where we apply this method to a network consisting of Hindmarsh-Rose neuron models. It has been shown that our method can apply to a boarder range of systems compared to some existing methods.

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