Rational interpolation: I. Least square convergence ⋆

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Abstract

Given a positive bounded Borel measure μ on the interval $[-1, 1]$, we provide convergence results in L_2^μ-norm to a function f of its sequence of interpolating rational functions at the nodes of rational Gauss-type quadrature formulas associated with the measure μ. For this, we use the connection between rational Gauss-type quadrature formulas on $[-1, 1]$ and rational Szegő quadrature formulas associated with a positive symmetric Borel measure $\mathring{\mu}$ on the complex unit circle.

Key words: Orthogonal rational functions, rational interpolation, least square convergence.

1 Introduction

Consider a positive bounded Borel measure μ on the interval $I := [-1, 1]$, and a function $f \in L_2^\mu(I)$. Let $\{f_n\}_{n=0}^\infty$ be the sequence of polynomials of degree n that interpolate f at the zeros of the orthogonal polynomials (or, equivalently, at the nodes of the Gaussian quadrature formulas). The Erdő-Turán Theorem (see e.g. [5]) then assures the L_2^μ-convergence of the polynomials f_n to the function f. The Gauss-Radau and Gauss-Lobatto points are the nodes of the

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corresponding quadrature formulas that have one or two nodes fixed at the endpoints of I. In [6], the authors extended the Erdő-Turán Theorem to these nodes. This was done by using the Joukowski [2] Transformation $x = \frac{1}{2}(z+z^{-1})$, which maps $x \in I$ to $z \in \mathbb{T} := \{e^{\mathbf{i}\theta} : \theta \in [-\pi, \pi]\}$. In this way, the measure $d\mu$ transforms into $d\mathring{\mu}(\theta) = |d\mu(\cos\theta)|$ (see [11, Thm. 11.5]), and the Gauss-type quadrature formulas into Szegő quadrature formulas (introduced by Jones et. al. in [10]). This allows to apply known results about the convergence on interpolation by Laurent polynomials at the zeros of so-called para-orthogonal polynomials for the measure $\mathring{\mu}$.

When f has singularities outside (but possibly close to) the interval I, it is often more appropriate to not consider polynomial interpolation, but rather consider rational interpolation where the poles of the interpolating rational function simulate the singularities of f. In such a case, one needs to consider more general spaces of rational functions, were the orthogonal polynomials are replaced by orthogonal rational functions with preassigned poles, and the interpolation points are taken from rational Gauss-type quadrature formulas instead.

A theory of orthogonal rational functions on the complex unit circle $\mathbb{T}$ and the extended real line $\mathbb{R} \cup \{\infty\}$ has been studied intensively (see e.g. [1]) in many papers devoted to their applications in numerical quadrature. Of course by a Joukowski Transform one may relate poles, nodes, weights, and measures on I and $\mathbb{T}$. The aim of this paper is then to study the convergence in L_2^μ-norm to a function f of its sequence of interpolating rational functions at the nodes of rational Gauss-type quadrature formulas associated with μ. This will be done in a similar way as in [6] by using the connection between the rational Gauss-type quadrature formulas with certain rational Szegő quadrature formulas, as described in [9].

The outline of the paper is as follows. After giving the necessary theoretical background in Section 2, in Section 3 we are mainly concerned with the extension of a known $L_2^{\mathring{\mu}}$ convergence result with respect to a positive bounded Borel measure $\mathring{\mu}$ on $\mathbb{T}$ for sequences of interpolating rational functions with poles among $\{\beta_i, 1/\overline{\beta}_i\}$, with β_i not on $\mathbb{T}$, to the case of interpolating rational functions with poles among $\{\beta_i, 1/\beta_i\}$. Once the existing result is adapted to this modified situation, we are ready to study the convergence of the rational approximants to f in Section 4.

[2] As stated in [6, p. 968], the Joukowski Transformation is also called the Joukowsky or Zhukovsky Transform.

2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$ and for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Further, the positive half line will be represented by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $a \in \mathbb{C}$, then $\Re\{a\}$ refers to the real part of a, while $\Im\{a\}$ refers to the imaginary part, and the imaginary unit will be denoted by $\mathbf{i}$. The unit circle and the open unit disk are denoted respectively by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Whenever the value zero is omitted in a set $X \subseteq \overline{\mathbb{C}}$, this will be represented by X_0. Similarly, the complement of a set $Y \subset \overline{\mathbb{C}}$ with respect to a set $X \subseteq \overline{\mathbb{C}}$ will be denoted by X_Y; i.e., $X_Y = \{t \in X : t \notin Y\}$.

For any complex function f, we define the involution operation or substar conjugate by

$$f_* : \overline{\mathbb{C}} \to \overline{\mathbb{C}} : z \mapsto \overline{f(1/\overline{z})}.$$

Next, we define the super-c conjugate by

$$f^c : \overline{\mathbb{C}} \to \overline{\mathbb{C}} : z \mapsto \overline{f(\overline{z})},$$

and consequently

$$f_*^c : \overline{\mathbb{C}} \to \overline{\mathbb{C}} : z \mapsto f(1/z).$$

Note that, if $f(z)$ has a pole at $z = p$, then $f_*(z)$ (respectively $f^c(z)$ and $f_*^c(z)$) has a pole at $z = 1/\overline{p}$ (respectively $z = \overline{p}$ and $z = 1/p$).

In this paper, we will consider functions on the interval $I = [-1, 1]$ and on the complex unit circle $\mathbb{T}$. Although x and z are both complex variables, we reserve the notation x for the interval and z for the unit circle.

Let $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$ be a sequence of poles, fixed in advance, where the poles are arbitrary complex or infinite; hence, they do not have to appear in pairs of complex conjugates. We then define the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = \frac{x^k}{\prod_{j=1}^{k}(1 - x/\alpha_j)}, \quad k = 1, 2, \ldots . \tag{1}$$

These basis functions generate the nested spaces of rational functions with poles in $\mathcal{A}$ defined by $\mathcal{L}_{-1} = \{0\}$, $\mathcal{L}_0 = \mathbb{C}$ and

$$\mathcal{L}_k := \mathcal{L}\{\alpha_1, \ldots, \alpha_k\} = \operatorname{span}\{b_0, \ldots, b_k\}, \quad k = 1, 2, \ldots .$$

Further, with $\mathcal{L}$ we denote the closed linear span of all $\{b_k\}_{k=0}^{\infty}$. With the definition of the super-c conjugate we define

$$\mathcal{L}_k^c = \{f : f^c \in \mathcal{L}_k\} = \mathcal{L}\{\overline{\alpha}_1, \ldots, \overline{\alpha}_k\},$$

while the product of two spaces of rational functions $\mathcal{L}_k$ and $\tilde{\mathcal{L}}_j = \mathcal{L}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j\}$ is defined by

$$\mathcal{L}_k \cdot \tilde{\mathcal{L}}_j = \{f \cdot g : f \in \mathcal{L}_k \text{ and } g \in \tilde{\mathcal{L}}_j\} = \mathcal{L}\{\alpha_1, \ldots, \alpha_k, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_j\}.$$

Note that $\mathcal{L}_k$ and $\mathcal{L}_k^c$ are rational generalizations of the space $\mathcal{P}_k$ of polynomials of degree less than or equal to k. Indeed, if $\alpha_j = \infty$ for every $j \geqslant 1$, the expression in (1) becomes $b_k(x) = x^k$.

Consider the integral

$$\mathfrak{I}_\mu(f) := \int_{-1}^{1} f(x) d\mu(x),$$

where μ is a positive bounded Borel measure with infinite support on the interval I (in short, a measure on I). To approximate $\mathfrak{I}_\mu(f)$, where f is a possibly complex function that can have singularities (possibly close to, but) outside the interval, rational interpolatory quadrature formulas are often preferred. By taking a set of appropriate nodes, an nth positive rational interpolatory quadrature is obtained by integrating an interpolating rational function in $\mathcal{L}_{n-1}$, and is of the form

$$\mathfrak{I}_n(f) := \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k}), \quad \{x_{n,k}\}_{k=1}^n \subset I, \quad x_{n,j} \neq x_{n,k} \text{ if } j \neq k, \quad \{\lambda_{n,k}\}_{k=1}^n \subset \mathbb{R}_0^+,$$

so that the equality $\mathfrak{I}_\mu(f) = \mathfrak{I}_n(f)$ holds for at least every $f \in \mathcal{L}_{n-1}$. Suppose $\mathcal{L}_{n-1} \cdot \tilde{\mathcal{L}}_j$, with $\tilde{\mathcal{L}}_j \subseteq \left(\mathcal{L}\{\alpha_n\} \cdot \mathcal{L}_{n-1}^c\right)$, is the largest space of rational functions for which this equality holds, then $\mathcal{L}_{n-1} \cdot \tilde{\mathcal{L}}_j$ is called the domain of validity of the quadrature formula. For reasons of notational simplicity, in the remainder we will write x_k and λ_k, meaning $x_{m,k}$ and $\lambda_{m,k}$ for a certain index m. At any time, the index m should be clear from the context.

Next, consider the inner product defined by

$$\langle f\,, g \rangle_\mu = \mathfrak{I}_\mu(f g^c), \quad f, g \in \mathcal{L}, \tag{2}$$

and let $\|f\|_{\mu,2} := \sqrt{\langle f\,, f \rangle_\mu}$. Orthogonalizing the basis functions $\{b_0, b_1, \ldots\}$ with respect to this inner product, we obtain a sequence of orthogonal rational functions (ORFs) $\{\varphi_0, \varphi_1, \ldots\}$, with $\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, so that $\varphi_k \perp_\mu \mathcal{L}_{k-1}$; i.e., $\langle \varphi_k\,, \varphi_j \rangle_\mu = d_k \delta_{k,j}$, $d_k \in \mathbb{R}_0^+$ and $k, j = 0, 1, \ldots$, where $\delta_{k,j}$ is the Kronecker Delta.

Whenever $\alpha_n \in \overline{\mathbb{R}}_I$, the zeros x_k of $\varphi_n(x)$ are all distinct and in the open interval $(-1, 1)$, and hence, can be chosen as nodes for the quadrature formula $\mathfrak{I}_n(f)$. In this way we obtain the n-point rational Gaussian quadrature formula, which is unique and has maximal domain of validity; i.e., the approximation is exact for every function $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$. It is well known that the weights λ_k in the rational Gaussian quadrature are all positive (see e.g. [8, Thm.

2.3.5]). In [9, Sect. 2] it is proved, however, that the n-point rational Gaussian quadrature formula with domain of validity $\mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$ does not exist whenever $\alpha_n \notin \overline{\mathbb{R}}$.

Other special kinds of positive rational interpolatory quadrature formulas are the so-called rational Gauss-Radau quadrature formulas and the rational Gauss-Lobatto quadrature formula (see also [9]). An n-point rational Gauss-Radau quadrature formula is obtained by fixing one of the nodes in the quadrature formula to either 1 or -1, and choosing the $n-1$ remaining nodes in the open interval $(-1,1)$ in such a way that the quadrature has domain of validity $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$. In contrast with the n-point rational Gaussian quadrature formula, the n-point rational Gauss-Radau quadrature formulas always exist, even when the last pole $\alpha_n \notin \overline{\mathbb{R}}$. On the other hand, the $(n+1)$-point rational Gauss-Lobatto quadrature formula is obtained by fixing two nodes in the quadrature formula to 1 and -1, and choosing the $n-1$ remaining nodes in the open interval $(-1,1)$ in such a way that the quadrature has domain of validity $\mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$. As in the case of the n-point rational Gaussian quadrature formula, the $(n+1)$-point rational Gauss-Lobatto quadrature formula only exists whenever $\alpha_n \in \overline{\mathbb{R}}_I$.

In the remainder we will refer to the nodes in an n-point rational Gauss-type (Gaussian, Gauss-Radau and Gauss-Lobatto) quadrature formula as nth rational Gauss-type-nodes (or, more specific, Gaussian-nodes, Gauss-Radau-nodes and Gauss-Lobatto-nodes).

Another sequence of basis functions will be used for the unit circle. Given a sequence of complex numbers $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, we define the Blaschke products for $\mathcal{B}$ as

$$B_0(z) \equiv 1, \quad B_k(z) = \prod_{j=1}^{k} \frac{z-\beta_j}{1-\overline{\beta}_j z},\ k = 1, 2, \ldots\ . \tag{3}$$

These Blaschke products generate the nested spaces of rational functions $\mathring{\mathcal{L}}_{-1} = \{0\}$, $\mathring{\mathcal{L}}_0 = \mathbb{C}$ and

$$\mathring{\mathcal{L}}_k := \mathring{\mathcal{L}}\{\beta_1, \ldots, \beta_k\} = \operatorname{span}\{B_0, \ldots, B_k\},\ \ k = 1, 2, \ldots\ .$$

Similarly as before, we denote with $\mathring{\mathcal{L}}$ the closed linear span of all $\{B_k\}_{k=0}^{\infty}$. With the definition of the substar conjugate and the super-c conjugate we can define

$$\begin{aligned} &\mathring{\mathcal{L}}_{k*} = \{f : f_* \in \mathring{\mathcal{L}}_k\} = \mathring{\mathcal{L}}\{1/\overline{\beta}_1, \ldots, 1/\overline{\beta}_k\} \\ &\mathring{\mathcal{L}}_k^c = \{f : f^c \in \mathring{\mathcal{L}}_k\} = \mathring{\mathcal{L}}\{\overline{\beta}_1, \ldots, \overline{\beta}_k\} \\ \text{and} \quad &\mathring{\mathcal{L}}_{k*}^c = \{f : f_*^c \in \mathring{\mathcal{L}}_k\} = \mathring{\mathcal{L}}\{1/\beta_1, \ldots, 1/\beta_k\}, \end{aligned}$$

while the product of two spaces of rational functions $\mathring{\mathcal{L}}_k$ and $\mathring{\tilde{\mathcal{L}}}_j = \mathcal{L}\{\tilde{\beta}_1, \ldots, \tilde{\beta}_j\}$ is defined by

$$\mathring{\mathcal{L}}_k \cdot \mathring{\tilde{\mathcal{L}}}_j = \{f \cdot g : f \in \mathring{\mathcal{L}}_k \text{ and } g \in \mathring{\tilde{\mathcal{L}}}_j\} = \mathring{\mathcal{L}}\{\beta_1, \ldots, \beta_k, \tilde{\beta}_1, \ldots, \tilde{\beta}_j\}.$$

Note that $\mathring{\mathcal{L}}_k$ and $\mathring{\mathcal{L}}_k^c$ are rational generalizations of $\mathcal{P}_k$ too. Indeed, if all $\beta_j = 0$ (or equivalently, $1/\overline{\beta}_j = \infty$ for every $j \geqslant 1$), the expression in (3) becomes $B_k(z) = B_k^c(z) = z^k$.

Consider now the integral

$$\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) := \int_{-\pi}^{\pi} \mathring{f}(z) d\mathring{\mu}(\theta), \ z = e^{\mathbf{i}\theta},$$

where $\mathring{\mu}$ is a positive bounded Borel measure with infinite support on the complex unit circle $\mathbb{T}$ [3] (in short, a measure on $\mathbb{T}$), and $\mathring{f}$ is a (possibly complex) function bounded on $\mathbb{T}$. The so-called n-point rational Szegő quadrature formulas to approximate $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f})$ are then of the form

$$\mathring{\mathfrak{I}}_n(f) := \sum_{k=1}^{n} \mathring{\lambda}_{n,k} \mathring{f}(z_{n,k}), \ \ \{z_{n,k}\}_{k=1}^n \subset \mathbb{T}, \ \ z_{n,j} \neq z_{n,k} \text{ if } j \neq k, \ \ \{\mathring{\lambda}_{n,k}\}_{k=1}^n \subset \mathbb{R}_0^+, \tag{4}$$

so that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) = \mathring{\mathfrak{I}}_n(\mathring{f})$ for every $\mathring{f} \in \mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$. Rational Szegő quadrature formulas on $\mathbb{T}$ are the equivalent of rational Gaussian quadrature formulas on I in the sense that they have maximal domain of validity (i.e., it is not possible to construct a quadrature formula of the form (4) that is exact either for every $\mathring{f} \in \mathring{\mathcal{L}}_n \cdot \mathring{\mathcal{L}}_{(n-1)*}$ or for every $\mathring{f} \in \mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{n*}$). From now on we will write z_k and $\mathring{\lambda}_k$, meaning $z_{m,k}$ and $\mathring{\lambda}_{m,k}$ for a certain index m, where the index m should again be clear at any time from the context.

Let $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$ denote an nth ORF with respect to the inner product

$$\left\langle \mathring{f}, \mathring{g} \right\rangle_{\mathring{\mu}} = \mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}\mathring{g}_*), \ \ \mathring{f}, \mathring{g} \in \mathring{\mathcal{L}},$$

and let $\left\| \mathring{f} \right\|_{\mathring{\mu},2} := \sqrt{\left\langle \mathring{f}, \mathring{f} \right\rangle_{\mathring{\mu}}}$. We then define a para-orthogonal rational function

$$\mathring{Q}_{n,\tau}(z) = \phi_n(z) + \tau B_n(z) \phi_{n*}(z), \ \ \tau \in \mathbb{T}. \tag{5}$$

The zeros z_k of $\mathring{Q}_{n,\tau}(z)$ are all distinct and on the unit circle $\mathbb{T}$, and hence, can be chosen as nodes for an n-point rational Szegő quadrature formula $\mathring{\mathfrak{I}}_n(\mathring{f})$. It is well known that in this case the weights $\mathring{\lambda}_k$ are all positive too. Conversely, it is proved in [1, Chapt. 5.3] that the nodes in an n-point rational Szegő quadrature formula with domain of validity $\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$ are the zeros of a

[3] The measure $\mathring{\mu}$ on $\mathbb{T}$ induces a measure on $[-\pi, \pi]$ for which we shall use the same notation $\mathring{\mu}$.

para-orthogonal rational function of the form (5). Consequently, an n-point rational Szegő quadrature formula always exists, but it is not unique due to the presence of the parameter τ in (5).

In the remainder we will refer to the nodes in the n-point rational Szegő quadrature formulas as nth rational Szegő-nodes.

We denote the Joukowski Transformation $x = \frac{1}{2}(z+z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}_I$ and the unit circle $\mathbb{T}$ onto the interval I. When $z = e^{\mathbf{i}\theta}$, then $x = J(z) = \cos\theta$. In everything that follows, we will assume that x and z are related by this transformation. The inverse mapping is denoted by $z = J^{inv}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}_I$. With the sequence $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$ we associate a sequence $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, so that $\beta_k = J^{inv}(\alpha_k)$ for every $k > 0$. A connection can then be made between rational quadrature formulas on the unit circle and the interval I. If μ is a measure on I, we obtain a measure on $\mathbb{T}$ by setting

$$\mathring{\mu}(E) = \mu\left(\{\cos\theta, \theta \in E \cap [0,\pi)\}\right) + \mu\left(\{\cos\theta, \theta \in E \cap [-\pi,0)\}\right), \qquad (6)$$

for every $E \subseteq [-\pi,\pi)$. Clearly, this measure $\mathring{\mu}$ is then symmetric (i.e.; $d\mathring{\mu}(-\theta) = -d\mathring{\mu}(\theta)$), so that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}^c_*) = \mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f})$ for every function $\mathring{f}$ on $\mathbb{T}$.

Note that by the Joukowski Transformation, the function f transforms into the function $\mathring{f} = (f \circ J)$, so that $\mathring{f}^c_*(z) = \mathring{f}(z)$ and

$$\mathfrak{I}_\mu(f) = \frac{1}{2}\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}). \qquad (7)$$

Further, from [8, Lem. 3.2.1] it follows that by the Joukowski Transformation, every function $f \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ transforms into a function $\mathring{f} \in \mathring{\mathcal{L}}^c_k \cdot \mathring{\mathcal{L}}_{k*}$, so that $\mathring{f} \notin \mathring{\mathcal{L}}^c_k \cdot \mathring{\mathcal{L}}_{(k-1)*}$ and $\mathring{f} \notin \mathring{\mathcal{L}}^c_{k-1} \cdot \mathring{\mathcal{L}}_{k*}$.

The following connection between the rational Gauss-type quadratures on I and rational Szegő quadratures on $\mathbb{T}$ has been proved in [9, Sect. 3].

Theorem 1. *Let the positive measures μ on I and $\mathring{\mu}$ on $\mathbb{T}$ be related by* (6).

(1) Assume that $\alpha_n \in \overline{\mathbb{R}}_I$, and suppose that the set of n distinct nodes $\{x_k\}_{k=1}^n \subset (-1,1)$ forms a set of nth rational Gaussian-nodes such that $\mathfrak{I}_\mu(f) = \mathfrak{I}_n(f)$ for every $f \in \mathcal{L}_n \cdot \mathcal{L}^c_{n-1}$. Set

$$z_k = \overline{z}_{n+k} = J^{inv}(x_k), \quad k = 1, \ldots, n.$$

Then the set of $2n$ distinct nodes $\{z_k\}_{k=1}^{2n} \subset \mathbb{T}$ forms a set of $2n$th rational Szegő-nodes such that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) = \mathring{\mathfrak{I}}_n(\mathring{f})$ for every $\mathring{f} \in \left(\mathring{\mathcal{L}}_n \cdot \mathring{\mathcal{L}}^c_{n-1}\right) \cdot \left(\mathring{\mathcal{L}}_{n*} \cdot \mathring{\mathcal{L}}^c_{(n-1)*}\right)$.

(2) *Suppose that the set of n distinct nodes $\{x_k\}_{k=1}^{n} \subset I$ forms a set of nth rational Gauss-Radau-nodes such that $\mathfrak{I}_\mu(f) = \mathfrak{I}_n(f)$ for every $f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$. Assume that the nodes are ordered in such a way that $\{x_k\}_{k=1}^{n-1} \subset (-1,1)$ and $x_n \in \{-1,1\}$. Set*

$$z_k = \overline{z}_{n-1+k} = J^{inv}(x_k), \quad k = 1, \ldots, n-1, \quad \text{and} \quad z_{2n-1} = x_n.$$

Then the set of $(2n-1)$ distinct nodes $\{z_k\}_{k=1}^{2n-1} \subset \mathbb{T}$ forms a set of $(2n-1)$th rational Szegő-nodes such that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) = \mathring{\mathfrak{I}}_n(\mathring{f})$ for every $\mathring{f} \in \left(\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{n-1}^c\right) \cdot \left(\mathring{\mathcal{L}}_{(n-1)*} \cdot \mathring{\mathcal{L}}_{(n-1)*}^c\right)$.

(3) *Assume that $\alpha_n \in \overline{\mathbb{R}}_I$, and suppose that the set of $(n+1)$ distinct nodes $\{x_k\}_{k=1}^{n+1} \subset I$ forms a set of $(n+1)$th rational Gauss-Lobatto-nodes such that $\mathfrak{I}_\mu(f) = \mathfrak{I}_n(f)$ for every $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$. Assume that the nodes are ordered in such a way that $\{x_k\}_{k=1}^{n-1} \subset (-1,1)$ and $x_n = -x_{n+1} \in \{-1,1\}$. Set*

$$z_k = \overline{z}_{n-1+k} = J^{inv}(x_k), \quad k = 1, \ldots, n-1, \quad \text{and} \quad z_{2n-1} = -z_{2n} = x_n.$$

Then the set of $2n$ distinct nodes $\{z_k\}_{k=1}^{2n} \subset \mathbb{T}$ forms a set of $2n$th rational Szegő-nodes such that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) = \mathring{\mathfrak{I}}_n(\mathring{f})$ for every $\mathring{f} \in \left(\mathring{\mathcal{L}}_n \cdot \mathring{\mathcal{L}}_{n-1}^c\right) \cdot \left(\mathring{\mathcal{L}}_{n*} \cdot \mathring{\mathcal{L}}_{(n-1)*}^c\right)$.

3 The space of rational functions $\mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$

In [2,3] the $L_2^{\mathring{\mu}}(\mathbb{T})$ convergence has been studied for sequences of interpolating rational functions in nested subspaces of $\mathring{\mathcal{L}} + \mathring{\mathcal{L}}_*$ to a function $\mathring{f}$ defined on $\mathbb{T}$. The subspaces under consideration were of the form $\mathring{\mathcal{L}}_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$, with $p(n)$ and $q(n)$ non-negative integers such that $p(n) + q(n) = n$ and

$$\lim_{n\to\infty} \frac{q(n)}{n} = r \in (0,1),$$

and so that $\mathring{\mathcal{L}}_{p(n-1)} \cdot \mathring{\mathcal{L}}_{q(n-1)*} \subset \mathring{\mathcal{L}}_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$ for every $n > 0$. In the next section we will study the $L_2^\mu(I)$ convergence when considering sequences of interpolating rational functions of increasing degree to a function f defined on I. This will be done by passing from the interval to the unit circle by means of the Joukowski Transformation $x = J(z)$. However, from the previous section we deduce that the corresponding interpolating rational functions for the function $\mathring{f} = (f \circ J)$ on $\mathbb{T}$ are in nested subspaces of $\mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$, that are of the form $\mathring{\mathcal{L}}_{p(n)}^c \cdot \mathring{\mathcal{L}}_{q(n)*}$. Hence, these subspaces differ from those considered in the references, unless the numbers β_k, $k = 1, 2, \ldots$, are all real. The aim of this section is to extend [3, Thm. 4.1], for the case of nested subspaces of

$\mathring{\mathcal{L}}+\mathring{\mathcal{L}}_*$, to the case of nested subspaces of $\mathring{\mathcal{L}}^c+\mathring{\mathcal{L}}_*$. The main result is collected in Theorem 6, which is a consequence of the following four lemmas.

Lemma 2. *Consider the spaces of rational functions* $\mathring{\mathcal{S}}_n := \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$ *and* $\hat{\mathring{\mathcal{L}}}_n := \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)}$. *Then for every function* f *and* g *in* $\mathring{\mathcal{S}}_n$ *it holds that* $fg_* \in \hat{\mathring{\mathcal{L}}}_n \cdot \hat{\mathring{\mathcal{L}}}_{n*}$. *Conversely, for every function* f *and* g *in* $\hat{\mathring{\mathcal{L}}}_n$ *it holds that* $fg_* \in \mathring{\mathcal{S}}_n \cdot \mathring{\mathcal{S}}_{n*}$.

Proof. From the definition of $\mathring{\mathcal{S}}_n$ and $\hat{\mathring{\mathcal{L}}}_n$ it follows that

$$\begin{aligned}\mathring{\mathcal{S}}_n \cdot \mathring{\mathcal{S}}_{n*} &= \left(\mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}\right) \cdot \left(\mathring{\mathcal{L}}^c_{p(n)*} \cdot \mathring{\mathcal{L}}_{q(n)}\right) \\ &= \left(\mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)}\right) \cdot \left(\mathring{\mathcal{L}}^c_{p(n)*} \cdot \mathring{\mathcal{L}}_{q(n)*}\right) = \hat{\mathring{\mathcal{L}}}_n \cdot \hat{\mathring{\mathcal{L}}}_{n*}.\end{aligned}$$

This ends the proof. ∎

Lemma 3. *Let* $\mathring{\lambda}$ *denote the Lebesgue measure on* $\mathbb{T}$. *Then the space* $\mathring{\mathcal{L}}^c+\mathring{\mathcal{L}}_*$ *is dense in* $L^{\mathring{\lambda}}_p(\mathbb{T})$, $1 \leqslant p < \infty$, *and in the class* $C(\mathbb{T})$ *of continuous* 2π*-periodic functions on* $\mathbb{T}$ *(with respect to the Chebyshev norm) iff* $\sum_{k=1}^{\infty}(1-|\beta_k|) = \infty$.

Proof. The statement has been proved in [1, Thm. 7.1.2] for the space $\mathring{\mathcal{L}}+\mathring{\mathcal{L}}_*$ with poles among $\{\beta_k, 1/\overline{\beta}_k\}_{k=0}^{\infty}$. It is easily verified that the proof in [1] remains valid when considering the space $\mathring{\mathcal{L}}^c+\mathring{\mathcal{L}}_*$ with poles among $\{\beta_k, 1/\beta_k\}_{k=0}^{\infty}$ instead. ∎

Lemma 4. *Suppose* $\mathring{\mu}$ *is a measure on* $\mathbb{T}$, *and consider the nested spaces of rational functions* $\mathring{\mathcal{S}}_n := \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$, $n = 0, 1, \ldots$. *Then,*

(1) for $n = 1, 2, \ldots$, *there exist* n *distinct points* $\{z_k\}_{k=1}^n$ *on* $\mathbb{T}$ *and* n *positive numbers* $\mathring{\lambda}_k$, *so that* $\mathring{\mathfrak{I}}_n(\mathring{f}) := \sum_{k=1}^n \mathring{\lambda}_k \mathring{f}(z_k) = \mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f})$ *for every function* $\mathring{f}$ *of the form* $\mathring{f} = fg_*$, *with* $f, g \in \mathring{\mathcal{S}}_{n-1}$.
(2) if $\sum_{k=1}^{\infty}(1-|\beta_k|^2) = \infty$, *then* $\lim_{n\to\infty} \mathring{\mathfrak{I}}_n(\mathring{f}) = \mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f})$ *for all* $\mathring{\mu}$*-integrable functions* $\mathring{f}$.

Proof. The first statement has been proved in [4, Thm. 1] for functions $\mathring{f}$ of the form $\mathring{f} = \tilde{f}\tilde{g}_*$, with $\tilde{f}, \tilde{g} \in \mathring{\mathcal{L}}_{n-1}$. Clearly, what holds true for the nested space of rational functions $\mathring{\mathcal{L}}_n$, $n = 0, 1, \ldots$, also holds true for the nested space of rational functions $\hat{\mathring{\mathcal{L}}}_n := \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)}$, $n = 0, 1, \ldots$. As a result of Lemma 2, we now have that for every f and g in $\mathring{\mathcal{S}}_{n-1}$ there exist $\tilde{f}$ and $\tilde{g}$ in $\hat{\mathring{\mathcal{L}}}_{n-1}$, so that $fg_* = \tilde{f}\tilde{g}_*$. This proves the first statement.

Next, from Lemma 2 it also follows that for every $\tilde{f}$ and $\tilde{g}$ in $\hat{\mathring{\mathcal{L}}}_{n-1}$, there exist f and g in $\mathring{\mathcal{S}}_{n-1}$, so that $\tilde{f}\tilde{g}_* = fg_*$. Consequently, $\mathring{\mathfrak{I}}_n(\mathring{f})$ is an n-point rational

Szegő quadrature formula. The second statement now follows from [4, Thm. 5 and Cor. 6]. This ends the proof. ∎

Lemma 5. *Let $\mathring{\mu}$ be a measure on $\mathbb{T}$, and consider a function $\mathring{f}$, bounded on $\mathbb{T}$, for which the Riemann–Stieltjes integral $I_{\mathring{\mu}}(\mathring{f})$ exists. Let $\mathring{\mathcal{T}}_n^{[P]}$, $n = 0, 1, \ldots$, represent the classes of (complex) trigonometric functions of degree n at most, and set $\mathring{\mathcal{T}}^{[P]} = \bigcup_{n=0}^{\infty} \mathring{\mathcal{T}}_n^{[P]}$. Further, assume $\sum_{k=1}^{\infty}(1 - |\beta_k|) = \infty$, and let the classes $\mathring{\mathcal{T}}_n^{[R]}$, $n = 0, 1, \ldots$, of (complex) rational trigonometric functions be defined by*

$$\mathring{\mathcal{T}}_n^{[R]} = \left\{ R_n(\theta) = \frac{P_n(\theta)}{\prod_j^n (1 - \beta_j e^{\mathbf{i}\theta})(e^{\mathbf{i}\theta} - \beta_j)} : P_n \in \mathring{\mathcal{T}}_n^{[P]} \right\},$$

and $\mathring{\mathcal{T}}^{[R]} = \bigcup_{n=0}^{\infty} \mathring{\mathcal{T}}_n^{[R]}$. Then for any $\epsilon > 0$ there exists a rational trigonometric function $R \in \mathring{\mathcal{T}}^{[R]}$, so that

$$\left\| \mathring{f} - R \right\|_{\mathring{\mu},2}^2 < 2(M + \epsilon)\epsilon,$$

where

$$M = \sup_{\theta \in [-\pi, \pi]} \left| \Re\{\mathring{f}(e^{\mathbf{i}\theta})\} \right| + \sup_{\theta \in [-\pi, \pi]} \left| \Im\{\mathring{f}(e^{\mathbf{i}\theta})\} \right|. \tag{8}$$

Proof. Let ϵ' be an arbitrary positive number. From [11, Thm. 1.5.4] and the proof of [3, Thm. 4.1] it follows that there exists a trigonometric polynomial $P \in \mathring{\mathcal{T}}^{[P]}$, so that

$$\mathring{\mathfrak{I}}_{\mathring{\mu}}(|\mathring{f} - P|) < \epsilon' \quad \text{and} \quad |\mathring{f} - P| < 2(M + \epsilon'),$$

where M is given by (8). Under the condition $\sum_{k=1}^{\infty}(1 - |\beta_k|) = \infty$, it follows from Lemma 3 that $\mathring{\mathcal{T}}^{[R]}$ is dense in $C(\mathbb{T})$, while P is in $C(\mathbb{T})$. Consequently, there exist sequences $\{R_n\}_{n=0}^{\infty}$ in $\mathring{\mathcal{T}}^{[R]}$ such that

$$\lim_{n \to \infty} R_n = P, \quad \text{uniformly in } [-\pi, \pi].$$

Thus, for every $\epsilon'' > 0$ we can find an integer k so that for all $n \geqslant k$:

$$|P - R_n| < 2\epsilon'', \quad \forall \theta \in [-\pi, \pi].$$

Because the convergence $R_n \to P$ is uniform in $[-\pi, \pi]$, we also have

$$\lim_{n \to \infty} \mathring{\mathfrak{I}}_{\mathring{\mu}}(R_n) = \mathring{\mathfrak{I}}_{\mathring{\mu}}(P).$$

Thus, for every $\epsilon'' > 0$ we can find an integer l so that for all $n \geqslant l$:

$$\mathring{\mathfrak{I}}_{\mathring{\mu}}(|P - R_n|) < \epsilon''.$$

Setting $R = R_N \in \mathring{\mathcal{T}}^{[R]}$ for a certain $N > \max\{k, l\}$, we have that

$$|\mathring{f} - R| \leqslant |\mathring{f} - P| + |P - R| < 2(M + \epsilon' + \epsilon''),$$

and

$$\mathring{\mathfrak{I}}_{\mathring{\mu}}(|\mathring{f} - R|) \leqslant \mathring{\mathfrak{I}}_{\mathring{\mu}}(|\mathring{f} - P|) + \mathring{\mathfrak{I}}_{\mathring{\mu}}(|P - R|) < \epsilon' + \epsilon''.$$

Setting $\epsilon = (\epsilon' + \epsilon'')$, we obtain in this way that

$$\left\|\mathring{f} - R\right\|^2_{\mathring{\mu},2} = \mathring{\mathfrak{I}}_{\mathring{\mu}}(|\mathring{f} - R|^2) < 2(M + \epsilon) \cdot \mathring{\mathfrak{I}}_{\mathring{\mu}}(|\mathring{f} - R|) < 2(M + \epsilon)\epsilon.$$

This concludes the proof. ∎

We are now in a position to prove the following main result on $L^{\mathring{\mu}}_2(\mathbb{T})$ convergence that will be needed in the next section to study the $L^{\mu}_2(I)$ convergence.

Theorem 6. *Suppose $\mathring{\mu}$ is a measure on $\mathbb{T}$, and consider the nested spaces of rational functions $\mathring{\mathcal{S}}_n := \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)*}$ and $\hat{\mathring{\mathcal{L}}}_n := \mathring{\mathcal{L}}^c_{p(n)} \cdot \mathring{\mathcal{L}}_{q(n)}$. Assume $\sum_{j=1}^{\infty}(1-|\beta_j|) = \infty$, and suppose $\{z_k\}_{k=1}^n$, $n = 1, 2, \ldots$, are sets of nth rational Szegő-nodes on $\mathbb{T}$ so that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) = \mathring{\mathfrak{I}}_n(\mathring{f})$ for every $\mathring{f} \in \hat{\mathring{\mathcal{L}}}_{n-1} \cdot \hat{\mathring{\mathcal{L}}}_{(n-1)*}$. Then for any function $\mathring{f}$ bounded on $\mathbb{T}$, for which the Riemann–Stieltjes integral $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f})$ exists, the sequence of interpolating rational functions $\{S_{n-1}\}_{n=1}^{\infty}$, with $S_{n-1} \in \mathring{\mathcal{S}}_{n-1}$, at the points $\{z_k\}_{k=1}^n$ converge to $\mathring{f}$ in $L^{\mathring{\mu}}_2$-norm; i.e;*

$$\lim_{n\to\infty} \left\|\mathring{f} - S_{n-1}\right\|_{\mathring{\mu},2} = 0.$$

Proof. The proof is the same as in [3, Thm. 4.1] when replacing Theorem 2.1, Theorem 2.2 and Lemma 2.5 in [3] with respectively Lemma 4, Lemma 5 and Lemma 2 above. ∎

4 Rational interpolation and $L^{\mu}_2(I)$ convergence

In this section we are mainly concerned with the $L^{\mu}_2(I)$ convergence when considering sequences of interpolating rational functions of increasing degree to a function f defined on I, the interpolation points being rational Gauss-type-nodes. This will be done by passing from the interval to the unit circle by means of the Joukowski Transformation $x = J(z)$, and using the result for the $L^{\mathring{\mu}}_2(\mathbb{T})$ convergence in Theorem 6 when considering sequences of interpolating rational functions of increasing degree to a function $\mathring{f}$ defined on $\mathbb{T}$. We will start with the case in which one of the interpolation points is in the endpoints $\{-1, 1\}$ and the remaining interpolation points are in the open interval $(-1, 1)$, since this is the easiest case. We then have the following theorem.

Theorem 7. *Let μ be a measure on I, and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by* (6)*. Suppose R_{n-1} is the unique rational function in $\mathcal{L}_{n-1}$ that interpolates f at the n distinct nodes $\{x_k\}_{k=1}^{n} \subset I$, with $\{x_k\}_{k=1}^{n-1} \subset (-1,1)$ and $x_n \in \{-1,1\}$. Set*

$$z_k = \overline{z}_{n-1+k} = J^{inv}(x_k), \quad k = 1, \ldots, n-1, \quad \text{and} \quad z_{2n-1} = x_n, \tag{9}$$

and assume that S_{2n-2} is the unique rational function in $\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$ that interpolates the function $\mathring{f} = (f \circ J)$ at the $(2n-1)$ distinct nodes $\{z_k\}_{k=1}^{2n-1} \subset \mathbb{T}$. For any function f bounded on I it then holds that

$$\|f - R_{n-1}\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - S_{2n-2} \right\|_{\mathring{\mu},2}.$$

Proof. First, note that for f bounded on I, the interpolating rational function R_{n-1} always exists and is unique in $\mathcal{L}_{n-1}$. Clearly, the corresponding function $\mathring{f}$ is bounded too on $\mathbb{T}$, so that the interpolating rational function S_{2n-2} again always exists and is unique in $\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$. Thus, let $\mathring{R}_{n-1} = (R_{n-1} \circ J)$. Then it follows from (7) that

$$\|f - R_{n-1}\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - \mathring{R}_{n-1} \right\|_{\mathring{\mu},2}.$$

Moreover, $\mathring{R}_{n-1}$ is a rational function in $\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$ that interpolates $\mathring{f}$ at the nodes $\{z_k\}_{k=1}^{2n-1} \subset \mathbb{T}$. Due to the uniqueness, it follows that $\mathring{R}_{n-1}(z) = S_{2n-2}(z)$, which ends the proof. ∎

Next, consider the case in which the interpolation points are nth rational Gauss-Radau-nodes. The following theorem then gives a convergence result in the L^{μ}_2-norm $\|f - R_{n-1}\|_{\mu,2}$ for n tending to infinity.

Theorem 8. *Let μ be a measure on I, and assume that $\sum_{j=1}^{\infty}(1-|J^{inv}(\alpha_j)|) = \infty$. Then for any function f bounded on I, for which the Riemann–Stieltjes integral $\mathfrak{I}_{\mu}(f)$ exists, the sequence of interpolating rational functions $\{R_{n-1}\}_{n=1}^{\infty}$, with $R_{n-1} \in \mathcal{L}_{n-1}$, at the nth rational Gaus-Radau-nodes $\{x_k\}_{k=1}^{n}$, $n = 1, 2, \ldots$, converge to f in L^{μ}_2-norm; i.e,*

$$\lim_{n\to\infty} \|f - R_{n-1}\|_{\mu,2} = 0.$$

Proof. Suppose that, for every $n > 0$, the nth rational Gaus-Radau-nodes are ordered in such a way that $\{x_k\}_{k=1}^{n-1} \subset (-1,1)$ and $x_n \in \{-1,1\}$, and let S_{2n-2} denote the unique rational function in $\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$ that interpolates the function $\mathring{f} = (f \circ J)$ at the $(2n-1)$ distinct nodes $\{z_k\}_{k=1}^{2n-1} \subset \mathbb{T}$, given by (9). From Theorem 7 it then follows that

$$\lim_{n\to\infty} \|f - R_{n-1}\|_{\mu,2} = \frac{1}{\sqrt{2}} \lim_{n\to\infty} \left\| \mathring{f} - S_{2n-2} \right\|_{\mathring{\mu},2}.$$

Further, from Theorem 1(2) we deduce that the corresponding sets of $(2n-1)$ distinct nodes $\{z_k\}_{k=1}^{2n-1} \subset \mathbb{T}$ are $(2n-1)$th rational Szegő-nodes on $\mathbb{T}$ so that $\mathring{\mathfrak{I}}_{\mathring{\mu}}(\mathring{f}) = \mathring{\mathfrak{I}}_n(\mathring{f})$ for every $\mathring{f} \in \left(\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}^c_{n-1}\right) \cdot \left(\mathring{\mathcal{L}}_{(n-1)*} \cdot \mathring{\mathcal{L}}^c_{(n-1)*}\right)$. Hence, we can use Theorem 6 to find that

$$\lim_{n\to\infty} \left\| \mathring{f} - S_{2n-2} \right\|_{\mathring{\mu},2} = 0.$$

This concludes the proof. ∎

Next, consider the case in which all the interpolation points are inside the open interval $(-1,1)$. We then have the following theorem.

Theorem 9. *Let μ be a positive measure on I, and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (6). Suppose R_{n-1} is the unique rational function in $\mathcal{L}_{n-1}$ that interpolates f at the nodes $\{x_k\}_{k=1}^n \subset (-1,1)$. Set*

$$z_k = \overline{z}_{n+k} = J^{inv}(x_k), \quad k = 1, \ldots, n, \tag{10}$$

and assume that S_{2n-1} and $\tilde{S}_{2n-1}$ are the unique rational functions in respectively $\mathring{\mathcal{L}}^c_n \cdot \mathring{\mathcal{L}}_{(n-1)*}$ and $\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{n*}$ that interpolate $\mathring{f} = (f \circ J)$ at the $2n$ distinct nodes $\{z_k\}_{k=1}^{2n} \subset \mathbb{T} \setminus \{-1,1\}$. For any function f bounded on I it then holds that

$$\|f - R_{n-1}\|_{\mu,2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - S_{2n-1} \right\|_{\mathring{\mu},2} = \frac{1}{\sqrt{2}} \left\| \mathring{f} - \tilde{S}_{2n-1} \right\|_{\mathring{\mu},2}.$$

Proof. The proof is similar to the proof of Theorem 7, except that we now have that $\mathring{R}_{n-1} \in \mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$, while $S_{2n-1} \in \mathring{\mathcal{L}}^c_n \cdot \mathring{\mathcal{L}}_{(n-1)*}$ and $\tilde{S}_{2n-1} \in \mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{n*}$. However, since $\left(\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}\right) \subset \left(\mathring{\mathcal{L}}^c_n \cdot \mathring{\mathcal{L}}_{(n-1)*}\right)$ and $\left(\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}\right) \subset \left(\mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{n*}\right)$, and due to the uniqueness, we can again conclude that $\mathring{R}_{n-1}(z) = S_{2n-1}(z) = \tilde{S}_{2n-1}(z)$. ∎

Consider now the case in which the interpolation points are nth rational Gaussian-nodes. Recall from Section 2 that the nth rational Gaussian-nodes can only exist if $\alpha_n \in \overline{\mathbb{R}}_I$. For this reason, chose some arbitrary pole $\alpha = J(\beta) \in \overline{\mathbb{R}}_I$, and define the auxiliary subspaces

$$\breve{\mathcal{L}}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_{n-1}, \alpha\} \qquad \text{and} \qquad \breve{\mathring{\mathcal{L}}}_n = \mathring{\mathcal{L}}\{\beta_1, \ldots, \beta_{n-1}, \beta\}. \tag{11}$$

Note that this auxiliary subspaces are nested subspaces of $\mathcal{L}$ and $\mathring{\mathcal{L}}$ respectively. Moreover, for every $n > 0$, the zeros of a rational function $\breve{\varphi}_n \in \breve{\mathcal{L}}_n$ that is orthogonal to $\mathcal{L}_{n-1}$ with respect to the inner product (2) are then all distinct and in the open interval $(-1,1)$, so that they can be chosen as nth rational Gaussian-nodes. The following theorem then gives a convergence result in the L^μ_2-norm $\|f - R_{n-1}\|_{\mu,2}$ for n tending to infinity.

Theorem 10. *Let μ be a measure on I, and assume that $\sum_{j=1}^{\infty}(1-|J^{inv}(\alpha_j)|) = \infty$. Then for any function f bounded on I, for which the Riemann–Stieltjes integral $J_\mu(f)$ exists, the sequence of interpolating rational functions $\{R_{n-1}\}_{n=1}^{\infty}$, with $R_{n-1} \in \mathcal{L}_{n-1}$, at the nth rational Gaussian-nodes $\{x_k\}_{k=1}^{n}$ (obtained by means of the auxiliary subspace $\breve{\mathcal{L}}_n$ from above), $n = 1, 2, \ldots$, converge to f in L_2^μ-norm; i.e,*

$$\lim_{n\to\infty} \|f - R_{n-1}\|_{\mu,2} = 0.$$

Proof. The proof is similar to the proof of Theorem 8, when using (10), Theorem 9 and Theorem 1(1) instead of respectively (9), Theorem 7 and Theorem 1(2). ∎

Finally, let us consider the case in which two of the interpolation points are in the endpoints -1 and 1, and the remaining interpolation points are in the open interval $(-1, 1)$. We then have the following theorem.

Theorem 11. *Let μ be a positive measure on I, and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (6). Suppose R_n is the unique rational function in $\mathcal{L}_n$ that interpolates f at the $n+1$ distinct nodes $\{x_k\}_{k=1}^{n+1} \subset I$, with $\{x_k\}_{k=1}^{n-1} \subset (-1, 1)$ and $x_n = -x_{n+1} = 1$. Set*

$$z_k = \overline{z}_{n-1+k} = J^{inv}(x_k), \quad k = 1, \ldots, n-1, \quad \text{and} \quad z_{2n-1} = -z_{2n} = x_n, \quad (12)$$

and assume that S_{2n-1} and $\tilde{S}_{2n-1}$ are the unique rational functions in respectively $\mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{(n-1)*}$ and $\mathring{\mathcal{L}}_{n-1}^c \cdot \mathring{\mathcal{L}}_{n*}$ that interpolate $\mathring{f} = (f \circ J)$ at the $2n$ distinct nodes $\{z_k\}_{k=1}^{2n} \subset \mathbb{T}$. For any function f bounded on I it then holds that

$$\|f - R_n\|_{\mu,2} \leqslant \frac{1}{\sqrt{2}} \left\|\mathring{f} - S_{2n-1}\right\|_{\mathring{\mu},2} \quad \text{and} \quad \|f - R_n\|_{\mu,2} \leqslant \frac{1}{\sqrt{2}} \left\|\mathring{f} - \tilde{S}_{2n-1}\right\|_{\mathring{\mu},2}.$$

Proof. The first part of the proof is similar to the first part of the proof of Theorem 7, except that we now have that $\mathring{R}_n \in \mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{n*}$, while $S_{2n-1} \in \mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{(n-1)*}$ and $\tilde{S}_{2n-1} \in \mathring{\mathcal{L}}_{n-1}^c \cdot \mathring{\mathcal{L}}_{n*}$. Note, however, that $\left(\mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{(n-1)*}\right) \subset \left(\mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{n*}\right)$ and $\left(\mathring{\mathcal{L}}_{n-1}^c \cdot \mathring{\mathcal{L}}_{n*}\right) \subset \left(\mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{n*}\right)$. So, let us define

$$S_{2n}(z) = \frac{1}{2}\{S_{2n-1}(z) + \tilde{S}_{2n-1}(z)\} \in \mathring{\mathcal{L}}_n^c \cdot \mathring{\mathcal{L}}_{n*}.$$

Clearly, we then have that $S_{2n}(z_k) = \mathring{f}(z_k)$ for $k = 1, \ldots, 2n$, and $S_{2n}(z^{-1}) = S_{2n}(z)$ for every $z \in \overline{\mathbb{C}}$. Consequently, there exists a function $F_n \in \mathcal{L}_n$ so that $S_{2n} = (F_n \circ J)$ and $F_n(x_k) = f(x_k)$ for $k = 1, \ldots, n+1$. Since the interpolating rational function $R_n \in \mathcal{L}_n$ is unique, it follows that $F_n(x) = R_n(x)$, and hence,

$\mathring{R}_n(z) = S_{2n}(z)$. Consequently,

$$\left\|\mathring{f} - \mathring{R}_n\right\|_{\mathring{\mu},2} = \left\|\mathring{f} - S_{2n}\right\|_{\mathring{\mu},2} = \left\|\frac{1}{2}(\mathring{f} + \mathring{f}) - \frac{1}{2}\{S_{2n-1} + \tilde{S}_{2n-1}\}\right\|_{\mathring{\mu},2}$$
$$\leqslant \frac{1}{2}\left\{\left\|\mathring{f} - S_{2n-1}\right\|_{\mathring{\mu},2} + \left\|\mathring{f} - \tilde{S}_{2n-1}\right\|_{\mathring{\mu},2}\right\}.$$

Finally, note that $S^c_{(2n-1)*}(z) = \tilde{S}_{2n-1}$ due to the fact that $\left(\mathring{\mathcal{L}}^c_n \cdot \mathring{\mathcal{L}}_{(n-1)*}\right)^c_* = \mathring{\mathcal{L}}^c_{n-1} \cdot \mathring{\mathcal{L}}_{n*}$. Consequently, since the measure $\mathring{\mu}$ is symmetric, it holds that $\left\|\mathring{f} - S_{2n-1}\right\|_{\mathring{\mu},2} = \left\|\mathring{f} - \tilde{S}_{2n-1}\right\|_{\mathring{\mu},2}$. This ends the proof. ∎

Consider now the case in which the interpolation points are $(n+1)$th rational Gauss-Lobatto-nodes. Recall from Section 2 that the $(n+1)$-point rational Gauss-Lobatto quadrature formula can only exist if $\alpha_n \in \overline{\mathbb{R}}_I$, so that we again will need to consider the auxiliary subspace $\breve{\mathcal{L}}_n$ and $\breve{\mathring{\mathcal{L}}}_n$, given by (11). The following theorem then gives a convergence result in the L^μ_2-norm $\|f - R_n\|_{\mu,2}$ for n tending to infinity.

Theorem 12. *Let μ be a measure on I, and assume that $\sum_{j=1}^{\infty}(1-|J^{inv}(\alpha_j)|) = \infty$. Then for any function f bounded on I, for which the Riemann–Stieltjes integral $J_\mu(f)$ exists, the sequence of interpolating rational functions $\{R_n\}_{n=1}^{\infty}$, with $R_n \in \breve{\mathcal{L}}_n$, at the $(n+1)$th rational Gauss-Lobatto-nodes $\{x_k\}_{k=1}^{n+1}$, $n = 1, 2, \ldots$, converge to f in L^μ_2-norm; i.e,*

$$\lim_{n\to\infty} \|f - R_n\|_{\mu,2} = 0.$$

Proof. The proof is similar to the proof of Theorem 8, when using (12), Theorem 11 and Theorem 1(3) instead of respectively (9), Theorem 7 and Theorem 1(2). ∎

Remark 13. In order to ensure the existence of an n-point rational Gaussian quadrature formula (respectively, $(n+1)$-point rational Gauss-Lobatto quadrature formula) for every $n > 0$ in Theorems 10 and 12, we had to introduce auxiliary subspaces $\breve{\mathcal{L}}_n$. However, it is clear that the convergence in L^μ_2-norm does not depend on

(1) the choice for the fixed pole $\alpha \in \overline{\mathbb{R}}_I$ in definition (11). So, we may as well replace the pole α in the definition of $\breve{\mathcal{L}}_n$ with poles $\tilde{\alpha}_n \in \overline{\mathbb{R}}_I$, which depend on n.
(2) the order in which the poles in $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\}$ appear. Consequently, suppose $\{\alpha_{k_1}, \alpha_{k_2}, \ldots\}$ is the subset of real poles in $\mathcal{A}$. In the case of rational Gauss-Lobatto nodes we then may as well redefine $\breve{\mathcal{L}}_n$ as follows for $n \geqslant k_1$:

$$\breve{\mathcal{L}}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_{k_i-1}, \alpha_{k_i+1}, \ldots, \alpha_n, \alpha_{k_i}\}, \quad k_i \leqslant n \leqslant k_{i+1}-1 \text{ and } i = 1, 2, \ldots .$$

Note that in this way, $\breve{\mathcal{L}}_n = \mathcal{L}_n$ for every $n \geqslant k_1$.

We conclude this section with the following density result.

Theorem 14. *Let μ be a measure on I. Then $\mathcal{L}$ is dense in $L_2^\mu(I)$ if $\sum_{j=1}^{\infty}(1-|J^{inv}(\alpha_j)|) = \infty$.*

Proof. Note that for every $f \in L_2^\mu(I)$ it holds that the function $\mathring{f} = (f \circ J) = \mathring{f}_*^c \in L_2^{\mathring{\mu}}(\mathbb{T})$. Further, with $\beta_j = J^{inv}(\alpha_j) \in \mathbb{D}$ for every $j > 0$ it follows from Lemma 3 that $\mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$ is dense in the class $C(\mathbb{T})$ of 2π-periodic functions on $\mathbb{T}$ if $\sum_{j=1}^{\infty}(1-|\beta_j|) = \infty$. On the other hand, the class $C(\mathbb{T})$ is also dense in $L_2^{\mathring{\mu}}(\mathbb{T})$ (see e.g. [7]). Thus for $\mathring{f} \in L_2^{\mathring{\mu}}(\mathbb{T})$ and any $\epsilon > 0$, there exists a function $\mathring{h} \in C(\mathbb{T})$ such that

$$\left\|\mathring{f} - \mathring{h}\right\|_{\mathring{\mu},2} < \epsilon/\sqrt{2}.$$

Consider now the function $h \in C(I)$ (i.e.; the class of continuous functions on I) such that $(h \circ J)(z) = [\mathring{h}(z) + \mathring{h}_*^c(z)]/2$. Then it holds that

$$\begin{aligned}\|f-h\|_{\mu,2} &= \frac{1}{\sqrt{2}}\left\|\frac{\mathring{f}+\mathring{f}_*^c}{2} - \frac{\mathring{h}+\mathring{h}_*^c}{2}\right\|_{\mathring{\mu},2} \\ &\leqslant \frac{1}{2\sqrt{2}}\left\{\left\|\mathring{f}-\mathring{h}\right\|_{\mathring{\mu},2} + \left\|\mathring{f}_*^c - \mathring{h}_*^c\right\|_{\mathring{\mu},2}\right\} = \frac{1}{\sqrt{2}}\left\|\mathring{f}-\mathring{h}\right\|_{\mathring{\mu},2} < \epsilon/2.\end{aligned}$$

Furthermore, there exists a function $\mathring{R} \in \mathring{\mathcal{L}}^c + \mathring{\mathcal{L}}_*$ so that

$$\left\|\mathring{h} - \mathring{R}\right\|_{\mathring{\mu},2} < \epsilon/\sqrt{2}.$$

Thus, let $R \in \mathcal{L}$ be such that $(R \circ J)(z) = [\mathring{R}(z) + \mathring{R}_*^c(z)]/2$, then

$$\begin{aligned}\|h-R\|_{\mu,2} &= \frac{1}{\sqrt{2}}\left\|\frac{\mathring{h}+\mathring{h}_*^c}{2} - \frac{\mathring{R}+\mathring{R}_*^c}{2}\right\|_{\mathring{\mu},2} \\ &\leqslant \frac{1}{2\sqrt{2}}\left\{\left\|\mathring{h}-\mathring{R}\right\|_{\mathring{\mu},2} + \left\|\mathring{h}_*^c - \mathring{R}_*^c\right\|_{\mathring{\mu},2}\right\} = \frac{1}{\sqrt{2}}\left\|\mathring{h}-\mathring{R}\right\|_{\mathring{\mu},2} < \epsilon/2.\end{aligned}$$

As a result,

$$\|f-R\|_{\mu,2} \leqslant \|f-h\|_{\mu,2} + \|h-R\|_{\mu,2} < \epsilon.$$

This proves the theorem. ∎

As a consequence of the previous theorem, we have the following. (The proof is exactly the same as the one of [2, Cor. 4.2], and hence, we omit it.)

Corollary 15. *Let P_n denote the nth partial sum of the expansion of a function $f \in L_2^\mu(I)$ with respect to any orthonormal basis of $\mathcal{L}$ and let $\sum_{j=1}^{\infty}(1-|J^{inv}(\alpha_j)|) = \infty$. Then $\lim_{n\to\infty}\|f-P_n\|_{\mu,2} = 0$.*

5 Conclusion

Given a positive bounded Borel measure μ on the interval $I = [-1, 1]$, we provided convergence results in L_2^μ-norm to a function f of its sequence of rational interpolating functions at the nodes of rational Gauss-type quadrature formulas associated with the measure μ. Similarly as in [6], this is done by using the connection between rational Gauss-type quadrature formulas on the interval I and certain rational Szegő quadrature formulas on the complex unit circle.

References

[1] A. Bultheel, P. González Vera, E. Hendriksen, and O. Njåstad. *Orthogonal Rational Functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 1999.

[2] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Orthogonal rational functions and interpolatory product rules on the unit circle. II. Quadrature and convergence", *Analysis* 18:185–200, 1998.

[3] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Orthogonal rational functions and interpolatory product rules on the unit circle. III. Convergence of general sequences", *Analysis* 20:99–120, 2000.

[4] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. "Quadrature formulas on the unit circle based on rational functions", *Journal of Computational and Applied Mathematics* 50(1–3):159–170, 1994.

[5] E. W. Cheney. "Introduction to Approximation Theory", Chelsea, New York, 1966.

[6] R. Cruz-Barroso, P. González-Vera, and F. Perdomo-Pío. "Orthogonality, interpolation and quadratures on the unit circle and the interval $[-1, 1]$", *Journal of Computational and Applied Mathematics* 235(4):966–981, 2010.

[7] P. J. Davis. *Interpolation and Approximation*, Blaisdell, New York, 1963. Reprint: Dover, 1975.

[8] K. Deckers. *Orthogonal Rational Functions: Quadrature, Recurrence and Rational Krylov*, Phd thesis, Department of Computer Science, Katholieke Universiteit Leuven, Leuven, Belgium, February 2009. 270 + lxiv pages.

[9] K. Deckers, A. Bultheel, R. Cruz-Barroso, and F. Perdomo-Pío. "Positive rational interpolatory quadrature formulas on the unit circle and the interval", *Applied Numerical Mathematics* 60(12):1286–1299, 2010.

[10] W. B. Jones, O. Njåstad, and W. J. Thron. "Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle", *Bull. London Math. Soc.* 21:113–152, 1989.

[11] G. Szegő. *Orthogonal Polynomials*, vol. 33 of Amer. Math. Soc. Colloq. Publ., 4th ed., American Mathematical Society, Providence, Rhode Island, 1975. First edition 1939.