Delay compensation using predictive and periodic feedback

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Abstract Time-delay systems form a class of infinite-dimensional systems, inducing severe challenges from a control perspective. In the chapter, we first review basic properties of time-delay systems and shed light on these limitations by means of the design of static output feedback controllers. For input delay systems, we then make a survey of two complementary control techniques that allow to render the closed-loop system in some sense finite-dimensional. The first one is based on including predictors to compensate delays in the control loop. The second, much less known control-induced reduction technique, consists of using periodic output feedback. We show that an appropriate design of the time-varying gains renders the monodromy operator of the closed-loop system finite-dimensional. Moreover, this matrix can be made nilpotent or even zero, leading to fixed-time stability. We conclude the chapter with a discussion on the related problem of observation in a guaranteed, prescribed time.

1 Introduction

We feel privileged to contribute to this book volume on *Nonlinear and Constrained Control*, and to have participated in the wonderful workshop at its origin in Brussels. The chapter presents a *self-contained* mini-tutorial on time-delay systems. On the one hand, it may be a-typical, in the sense that all systems addressed are linear time-invariant, without constraints on inputs and states, seemingly "orthogonal" to the scope of the book. On the other hand, it is well known that time-delays in control loops and interconnections impose major limitations on the achievable robustness,

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performance and even on ensuring stability. As such, the presence of delays arguably imposes major constraints on a control design problem. In addition, as we shall see, the frequency domain analysis gives rise to nonlinear eigenvalue problems and non-rational transfer matrices, necessitating dedicated design tools and an integration of analytical and computational approaches.

Throughout the chapter, the focus is on revealing distinctive properties of time-delay systems and the main ideas behind analysis and control design methods, by means of examples *as simple as possible*. This also explains the restriction to linear systems with only one discrete and constant delay, whose dynamics are already very rich. Sections 2-3 are mainly devoted to a presentation of basic results, whereas Section 4 touches very recent research results of the authors. As a distinctive flavor of the chapter, not only detrimental effects of delays are addressed, but beneficial effects and opportunities of using delay elements in control schemes are also outlined.

The developments are guided by means of the stabilization of systems described by the model

$$\dot{x}\,(t) = Ax\,(t) + Bu(t-\tau), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state-variable at time t, $u(t) \in \mathbb{R}^m$ is the input at time t, and $\tau > 0$ is an input delay, under the assumption that the pair (A, B) is controllable. We will start with the stabilization using instantaneous state feedback,

$$u(t) = -Kx(t), \tag{2}$$

and subsequently move on to more powerful yet slightly more complex generalizations of (2).

The structure of the chapter is as follows. In Section 2 we discuss fundamental properties of time-delay systems, starting with a discussion of the initial value problem. In Sections 3 and 4 we outline the principles behind prediction based feedback and periodic feedback as a means to achieve dimension reduction, by compensating the delay and rendering the monodromy operator finite-dimensional, respectively. In Section 5 we formulate some concluding remarks and discuss the generalization of the presented results to nonlinear time-delay systems.

2 Basic properties of time-delay systems

In this section we review representations in time and frequency domain, based on [21, 8], and shed light on the effects of delays in control loops.

2.1 Functional differential equation

The equations governing the closed-loop dynamics of (1) and (2) from $t \geq \tau$ on constitute a special instance of

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \quad x(t) \in \mathbb{R}^n, \tag{3}$$

which is an example of a *functional differential equation* (FDE) of *retarded type*. The term *FDE* stems from the property that the right-hand side can be interpreted as a functional evaluated at a piece of trajectory. The adjective *retarded* expresses that the right-hand side does not explicitly depend on past instances of $\dot{x}$ [14, 15].

As a first difference with an ordinary differential equation, the initial condition of (3) at $t = 0$ is a function ϕ mapping the interval $[-\tau,\ 0]$ into $\mathbb{R}^n$. For all $\phi \in C([-\tau,\ 0], \mathbb{R}^n)$, a forward solution $x(\phi)$ exists and is uniquely defined.

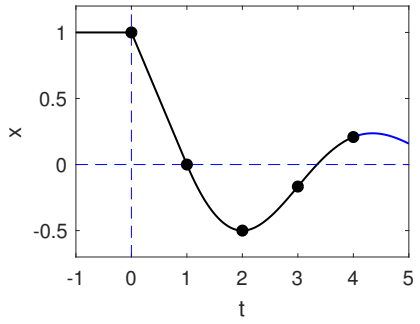


Fig. 1 Solution of (4) for $\tau = 1,\ k = 1$ and initial condition $\phi \equiv 1$.

A solution of the scalar equation

$$\dot{x}(t) = -kx(t-\tau) \tag{4}$$

is shown in Figure 1. The discontinuity in the derivative at $t = 0$ stems from $A_0\phi(0) + A_1\phi(-\tau) \neq \lim_{\theta\to 0^-} \dot{\phi}(\theta)$. Due to the smoothing property of an integrator, however, at $t = n\tau,\ n \in \mathbb{N}$, the discontinuity will only be present in the $(n+1)$−th derivative. This illustrates a second property of functional differential equations of retarded type: solutions become smoother as time evolves. As a third major difference with ODEs, backward continuation of solutions is not always possible [23, 10]. However, note that for the considered linear time-invariant systems, exponential solutions corresponding to characteristic roots (see Section 2.3) can be continued backwards.

Remark 1 Specifically for system (1), we will consider as initial condition at $t = 0$ a pair $(x_0, u_0) \in X := \mathbb{R}^n \times \mathcal{PC}([-\tau,\ 0], \mathbb{R}^m)$, with $\mathcal{PC}([-\tau,\ 0], \mathbb{R}^m)$ the space of piecewise-continuous functions from $[-\tau,\ 0]$ to $\mathbb{R}^m$, and allow piecewise-continuous input signals. This choice is motivated by the use of a switched feedback law in Section 4.1.

2.2 Abstract ordinary differential equation

The state of system (3) at time t is naturally defined as the minimal information needed to continue the solution, which, once again, boils down to a function segment $x_t(\phi)$ where $x_t(\phi)(\theta) = x(t+\theta),\ \theta \in [-\tau,\ 0]$. In Figure 1 the function x_t is shown in blue for $t = 5$. In fact, the state can be observed in a time-window of length τ that slides along a solution, see Figure 2. As a consequence, (3) can be reformulated as an abstract ordinary differential equation over the infinite-dimensional space $C([-\tau,\ 0], \mathbb{R}^n)$. This governing equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \mathcal{A}\ z(t),\ z(t) \in C([-\tau,\ 0], \mathbb{R}^n) \tag{5}$$

where the operator $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \left\{ \phi \in C([-\tau,\ 0],\ \mathbb{R}^n) :\ \dot{\phi} \in C([-\tau,\ 0], \mathbb{R}^n),\ \dot{\phi}(0) = A_0\phi(0) + A_1\phi(-\tau) \right\}, \\ \mathcal{A}\ \phi &= \tfrac{\mathrm{d}\phi}{\mathrm{d}\theta}. \end{aligned} \tag{6}$$

The relation between solutions of (3) and (5) is given by $z(t)(\theta) = x(t+\theta),\ \theta \in [-\tau,\ 0]$, see [23, Chapter 1].

Remark 2 While forward solutions of (3) are uniquely defined for any initial condition $\phi \in C([-\tau,\ 0], \mathbb{R}^n)$, the domain of $\mathcal{A}$ requires the function to be continuously differentiable. Hence, taking into account the smoothing property previously mentioned, the description (5) only holds for $t \geq \tau$. For more details related to the connections between solutions of (3) and (5), we refer to [11].

Note that all system information is concentrated in the non-local boundary condition describing the domain of $\mathcal{A}$. The representation (5) is closely related to a description by an advection PDE with a non-local boundary condition [17]. We note that for systems with exogenous inputs, a decoupling of the current time value and the history is natural, see [4] where $\mathbb{R}^n \times \mathcal{L}_2([-\tau,\ 0], \mathbb{R}^n)$ is used as the state space.

The reformulation of (3) into the standard form (5) allows us to define stability notions and to generalize the stability theory for ordinary differential equations in a straightforward way, with the main change that the state space is $C([-\tau,\ 0], \mathbb{R}^n)$. For example, the null solution of (3) is exponentially stable if and only if there exist constants $C > 0$ and $\gamma > 0$ such that

$$\|x_t(\phi)\|_s \leq C\mathrm{e}^{-\gamma t}\|\phi\|_s,\ \ \forall \phi \in C([-\tau,\ 0], \mathbb{R}^n),$$

where $\|\cdot\|_s$ is the supremum norm, $\|\phi\|_s = \sup_{\theta \in [-\tau,\ 0]} \|\phi(\theta)\|_2$. Since the system is linear, asymptotic and exponential stability are equivalent.

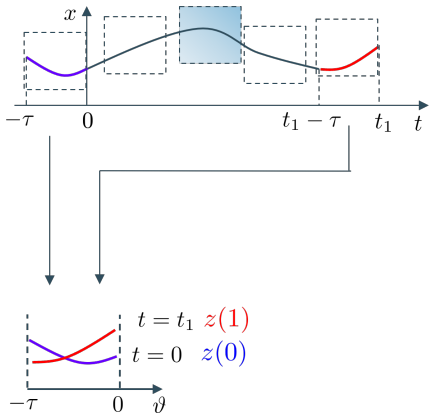


Fig. 2 Idea behind the reformulation as an ordinary differential equation on $C([-\tau,\ 0],\mathbb{R}^n)$.

2.3 Two equivalent eigenvalue problems

The substitution of an exponential solution in (3) leads us to the *nonlinear eigenvalue problem*

$$\left(\lambda I - A_0 - A_1 \mathrm{e}^{-\lambda\tau}\right) v = 0,\ \lambda \in \mathbb{C},\ v \in \mathbb{C}^n,\ v \neq 0. \tag{7}$$

The solutions of the equation $\det(\lambda I - A_0 - A_1\mathrm{e}^{-\lambda\tau}) = 0$ are called characteristic roots. Similarly, formulation (5) leads us to the equivalent *infinite-dimensional linear eigenvalue problem*

$$(\lambda I - \mathcal{A})\, u = 0,\ \lambda \in \mathbb{C},\ u \in C([-\tau,\ 0],\mathbb{C}^n),\ u \not\equiv 0. \tag{8}$$

The combination of these two representations lays at the basis of most methods for computing characteristic roots, see [20, 1]. On the one hand, discretizing (8), i.e., approximating $\mathcal{A}$ with a matrix, and solving the resulting standard eigenvalue problems enables retrieving global information, for example, estimates of *all* characteristic roots in a given compact set or in a given right half plane. On the other hand, the (finitely many) nonlinear equations (7) allow to make *local corrections* on characteristic root approximations up to the desired accuracy using Newton type methods. Popular discretization techniques to obtain a matrix approximation of $\mathcal{A}$ include (pseudo)spectral collocation [3] and the so-called *semi-discretization* technique [13]. Alternatively, the solution operator is discretized using a linear multi-step method in the stability routines of the software package DDE-BIFTOOL [7].

Linear time-delay systems satisfy spectrum determined growth properties of solutions. For instance, the zero solution of (3) is asymptotically stable if and only if all characteristic roots are located in the open left half plane.

In Figure 3 (left), the rightmost characteristic roots of (4) are depicted for $k\tau = 1$. Note that, since the characteristic equation can be written as $\lambda\tau + k\tau\mathrm{e}^{-\lambda\tau} = 0$, k and

τ can be combined into one parameter. In Figure 3 (right), we show the real parts of the characteristic roots as a function of $k\tau$. The plots illustrate some important spectral properties of retarded type FDEs. First, even though there are in general infinitely many characteristic roots, the number of them in *any* right half plane is always finite. Second, the individual characteristic roots, as well as the *spectral abscissa*, i.e., the supremum of the real parts of all characteristic roots, continuously depend on parameters. Related to this, a loss or gain of stability is always associated with characteristic roots crossing the imaginary axis. Figure 3 (right) also illustrates the transition to a delay-free system as $k\tau \to 0+$. For more details, see [23].

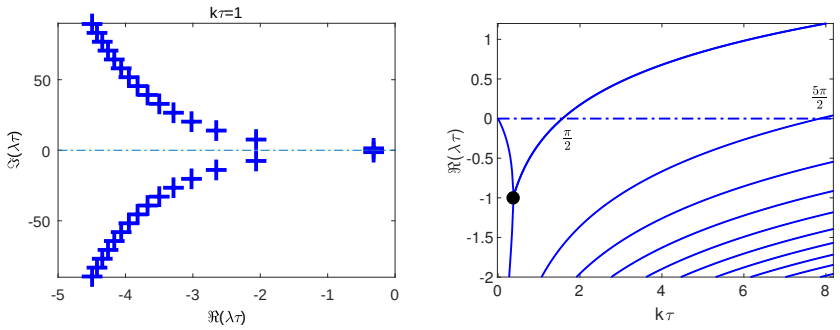


Fig. 3 (left) Rightmost characteristic roots of (4) for $k\tau = 1$. (right) Real parts of rightmost characteristic roots as a function of $k\tau$.

2.4 Control

It is well known that time-delays in control loop may lead to a significant degradation of performance and robustness, and even to instability [26, 28, 9]. Let us return to example (4), where k is interpreted as a controller parameter. As indicated in Figure 3, the system loses stability if τ reaches the value $\pi/(2k)$, while it can be shown that stability is not recovered for larger delay. The maximum achievable exponential decay rate of the solutions, which corresponds to the minimum of the spectral abscissa, is given by $-1/\tau$ (see the right pane of Figure 3), hence, large delays can only be tolerated at the price of a degradation of the rate of convergence. It should be noted here that the limitations induced by delays are even more stringent if the uncontrolled system is exponentially unstable, which is not the case for (4).

The analysis in Sections 2.1-2.3 gives a hint why control is difficult in the presence of delays: the system is inherently infinite-dimensional. As a consequence, most control design problems which involve determining a finite number of parameters can be interpreted as reduced-order control design problems or as control design

problems for under-actuated systems, both of which are known to be challenging problems.

Most standard control design techniques lead to controllers whose dimension is larger or equal to the dimension of the system. For infinite-dimensional time-delay system, such controllers might have a disadvantage of being complicated and hard to implement. To see this, for a system with delay τ in the state x but no delay in the input, the generalization of static state feedback (2) is given by $u(t) = -\int_{-\tau}^{0} x(t+\theta)\mathrm{d}\mu(\theta)$, where μ is a function of bounded variation, since the state corresponds to the function x_t introduced earlier. However, in the context of large-scale systems it is known that reduced-order controllers often perform relatively well compared to full order controllers, while they are much easier to implement.

Recently new methods for the design of controllers with a prescribed order (dimension) and/or structure have been proposed for linear time-invariant systems, starting from [20]. These methods rely on a direct optimization of cost functions such as the spectral abscissa, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of appropriately defined transfer matrices, as a function of the available controller parameters. While $\mathcal{H}_2$-criteria can be addressed within a derivative based optimization framework, $\mathcal{H}_\infty$-criteria and the spectral abscissa require targeted methods for *non-smooth optimization problems*. To illustrate the need for such methods, we consider again Figure 3 (right): minimizing the spectral abscissa for a given value of τ as a function of the controller gain k leads to an optimum (indicated with the black circle), where the objective function is not differentiable, even not locally Lipschitz continuous. For these parameters a double non-semisimple rightmost characteristic root occurs. We note that the tendency towards multiple roots in optima has also lead to analytic design methods based on direct assignment of multiple roots, see, e.g., [19].

An implementation of state-of-the-art routines for the design of structured controllers has been done in the open-source software package TDS-CONTROL [1], whose manual has been conceived as a survey on spectral properties and spectrum based control techniques for time-delay systems.

We conclude the section with another example.

Example 1 With the matrices specified as

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & -0.2 & -0.07 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.1 \end{bmatrix}, \quad \tau = 5, \tag{9}$$

model (1) is unstable for $u(t) \equiv 0$, with rightmost characteristic roots $0.0217654 \pm 0.195684\mathrm{i}$. Optimizing the spectral abscissa for feedback law (2) using TDS-CONTROL leads us to the minimum -0.187685, reached for

$$K = \begin{bmatrix} 0.420480 & 0.0336486 & 0.443844 \end{bmatrix}. \tag{10}$$

In Figure 4 the (rightmost) characteristic roots of the open and closed-loop are shown. In this example, the global minimum of the spectral abscissa is characterized by a defective real eigenvalue of multiplicity four.

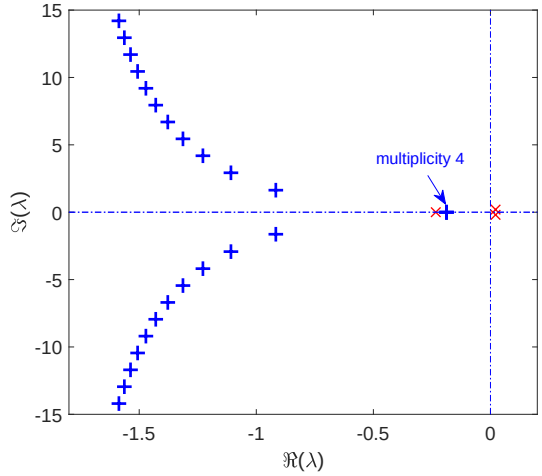


Fig. 4 Spectrum of the open-loop ('x') and closed-loop system ('+'), formed by (1) and (2). The matrices and delay are described by (9) and (10).

3 Dimension reduction by prediction based feedback

In the special case of systems with input or output delays, these delays can be compensated using a predictor. The basic idea, sketched in Figure 5 for model (1), consists of a decomposition of the system, where the new input v is related to u via the relation $v(t) = u(t-\tau)$ for $t \geq \tau$. Since we know the input up to the current time t, we know input v of the ODE part up to time $t+\tau$, so if we can measure $x(t)$, we can predict $x(t+\tau)$. From the variation-of-constants formula the prediction $x(t+\tau)$, denoted by $\hat{x}(t)$, reads as

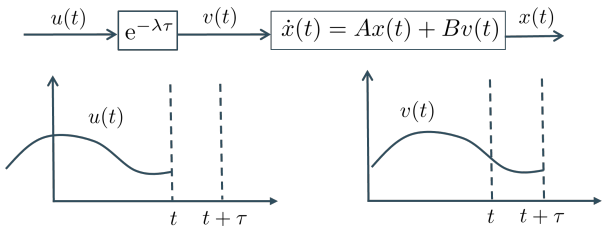


Fig. 5 Principle behind state prediction.

$$\begin{aligned} \hat{x}(t) &= e^{A\tau}x(t) + \int_t^{t+\tau} e^{A(t+\tau-s)}Bv(s)\,ds \\ &= e^{A\tau}x(t) + \int_t^{t+\tau} e^{A(t+\tau-s)}Bu(s-\tau)\,ds. \end{aligned} \tag{11}$$

Using the predicted state for feedback, $u(t) = -K\hat{x}(t)$, the characteristic equation of the closed-loop system becomes

$$\det(\lambda I - A + BK) = 0, \tag{12}$$

hence, the delay in the control loop has been compensated by the prediction.

A design based on shaping the roots of (12) is denoted by *finite spectrum assignment*. Prediction based feedback is also known under the term *infinite-dimensional backstepping* [17], in particular if the design is accompanied by the construction of a Lyapunov function. A core operation in the recursive backstepping procedure namely consists of turning a stabilizing control law for a (sub)system into a stabilizing control law for an enlarged system obtained by adding an integrator to the input. The analogy is complete by replacing the integrator by the delay operator. Closely related, in the LTI case the approach is known as the *Artstein reduction* technique [2]. This term stems from the transformation

$$z(t) = x(t) + \int_{t-\tau}^{t} \mathrm{e}^{A(t-\tau-s)} Bu(s)\,\mathrm{d}s,$$

which reduces (1) to the target system

$$\dot{z}(t) = Az(t) + \mathrm{e}^{-A\tau} Bu(t).$$

Remark 3 If the feedback gain can be chosen arbitrarily small, the second term in (11) can be dropped as it is dominated by the first term, resulting in so-called *truncated predictor feedback* [33]. Another scheme that does not require the convolution term is based on using the target closed-loop dynamics for the prediction, leading to the control law $u(t) = -K\mathrm{e}^{(A-BK)\tau}x(t)$. This approach, known as *pseudo-predictor feedback*, is applicable provided an additional stability condition is satisfied [32].

The predictor (11) is static in nature, in the sense that in an implementation, the predicted quantity needs to be computed online. Also dynamic prediction schemes have been proposed in the literature, starting from Smith predictor [29]. Smith's original scheme assumes a plant model described by the SISO transfer function.

$$P(\lambda) = P_0(\lambda)\mathrm{e}^{-\lambda\tau}, \tag{13}$$

with $P_0(\lambda)$ a stable rational function. A feedback controller of the form

$$C(\lambda) = \frac{C_0(\lambda)}{1 + C_0(\lambda)P_0(\lambda) - C_0(\lambda)P_0(\lambda)\mathrm{e}^{-\lambda\tau}}$$

is then proposed, which induces the closed-loop transfer function (from external reference r to output y)

$$H_{cl}(\lambda) = \frac{C_0(\lambda)P_0(\lambda)}{1 + C_0(\lambda)P_0(\lambda)}\mathrm{e}^{-\lambda\tau}.$$

Since the exponential function does not appear in the denominator, the controller C_0 can be designed for the target system described by the undelayed part $P_0(\lambda)$ only. In Figure 6 the block diagram of the feedback interconnection of plant P with

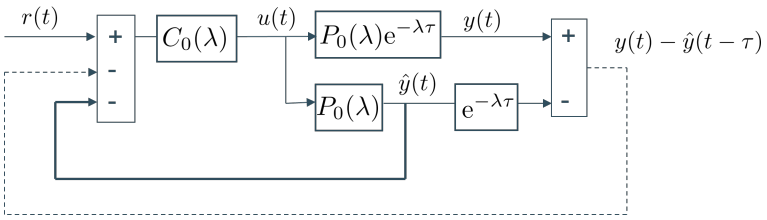


Fig. 6 Rearranged block diagram of the Smith predictor. Relevant signals in the time domain are also indicated.

controller C is rearranged, clarifying that the Smith predictor is (also) based on using a prediction of the output, $\hat{y}$, for feedback.

In order to connect the Smith predictor with the other predictors introduced in this section, and clarify the notions of "static" and "dynamic" predictor, we consider the special case where $P(\lambda)$ is the transfer function of system (1) with output $y(t) = Fx(t)$, and $C_0(s)$ corresponds to a constant gain Γ. Then, in accordance with Figure 6, the action of $C(s)$ can be described by

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \\ \hat{y}(t) = F\hat{x}(t), \\ u(t) = -\Gamma\hat{y}(t) - \Gamma\left(y(t) - \hat{y}(t-\tau)\right), \end{cases} \tag{14}$$

where the first term in the expression for $u(t)$ describes the feedback of the predicted output, while the second term describes the adjustment in response to an observed prediction error, unavoidably with delay. This predictor is dynamic as $\hat{x}$ is now generated by an observer desynchronized (shifted) in time, whereas in static predictors, $\hat{x}(t)$ is directly computed from (11).

Delay compensation allows to drastically improve the performance of the controlled system, yet the limitations of time-delays manifest themselves in often poor robustness. In particular, if the plant is unstable and/or the delays are large, the prediction requires a very accurate model and perturbations to the system need to be small. For static predictors, a point of attention are high-frequency instabilities related to the implementation of (11). Note that a discretization of the integral reveals a direct (static) connection between the current and past inputs. For some unstable open loop systems, the discretization may induce instability independent of its resolution [6, 25]. What concerns dynamic predictors, the following example sheds light on the sensitivity of the Smith predictor with respect to modeling errors.

Example 2 We investigate the effect of a delay mismatch between plant and model. More precisely, we assume that the plant is correctly described by (13) but that the

Smith predictor is designed for assumed delay $\tau+\delta$, with $\delta \geq -\tau$ the delay mismatch. Setting

$$P_0(\lambda) = \frac{B_1(\lambda)}{A_1(\lambda)}, \quad C_0(\lambda) = \frac{B_2(\lambda)}{A_2(\lambda)},$$

the closed-loop characteristic equation is given by

$$A_1(\lambda)A_2(\lambda) + B_1(\lambda)B_2(\lambda) + B_1(\lambda)B_2(\lambda)(1 - \mathrm{e}^{-\lambda\delta})\mathrm{e}^{-\lambda\tau} = 0,$$

see [22]. As expected, the exponential function reappears for $\delta \neq 0$, inducing infinitely many characteristic roots. We now set

$$P_0(\lambda) = \frac{1}{s+1}, \quad C_0(\lambda) = \frac{s}{2} + 2,$$

leading to the characteristic equation

$$\frac{3}{2}\lambda + 3 + \left(\frac{\lambda}{2} + 2\right)\mathrm{e}^{-\lambda\tau} + \left(\frac{-\lambda}{2} - 2\right)\mathrm{e}^{-\lambda(\tau+\delta)} = 0. \tag{15}$$

In Figure 7, the stability chart of (15) in the (τ, δ)-plane is depicted, which separate regions in which the number of characteristic roots in the right half plane is constant. These curves are computed with TDS-CONTROL. The stability region lies in the center of the figure.

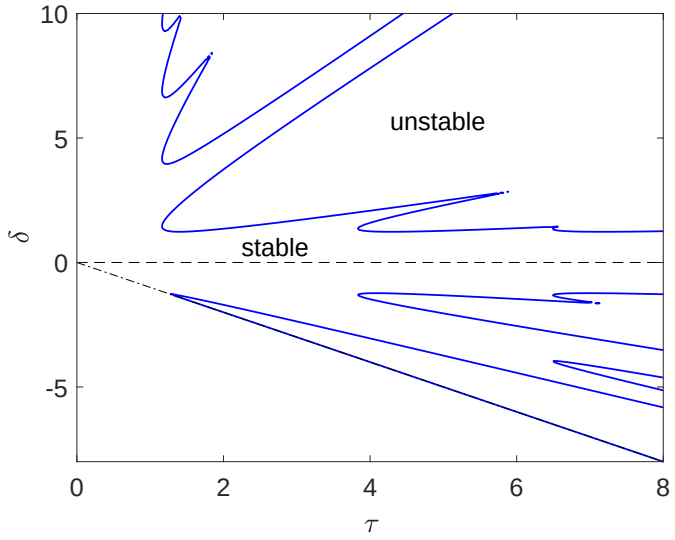


Fig. 7 Stability chart of (15) in the (τ, δ)-parameter space.

It can be observed that for increasing plant delay τ, the absolute stability margin in δ varies around a nonzero constant. As a consequence, as τ increases, the relative margin tends to zero for maintaining the most elementary property of stability, despite the asymptotically stable open loop dynamics.

4 Dimension reduction by periodic feedback

We outline a complementary control approach to induce a form of dimension reduction of the closed-loop system, grounded in periodic feedback. We first address periodically switched state feedback, followed by smooth periodic state feedback, thereby pointing to the capability to achieve fixed-time stability. We conclude the section by giving perspectives on observation in a prescribed time.

4.1 Act-and-wait control

We consider system (1) and modify the static state feedback control law (2) by switching it on and off periodically. More precisely, we employ a linear periodic feedback controller of the form

$$u(t) = -g_\tau(t)Kx(t), \tag{16}$$

with $g_\tau : \mathbb{R} \to \mathbb{R}$ the function with period 2τ defined by

$$g_\tau(t) = \begin{cases} 0, \, t \in (0, \ \tau), \\ 1, \, t \in [\tau, \ 2\tau], \end{cases} \tag{17}$$

which corresponds to a realization of the *act-and-wait control* strategy. This strategy was introduced in [12, 31], albeit for systems with delay in the measured output, and later on it was extended to the time-delayed control of nonlinear systems [16, 27].

To shed light on the control mechanism, we first note that in the interval $[0, \ \tau]$, the closed-loop dynamics are determined by the initial conditions. In the interval $[\tau, \ 2\tau]$, system (1)-(16) works in open loop mode, hence, we can express

$$x(t) = \mathrm{e}^{A(t-\tau)}x(\tau), \ t \in [\tau, \ 2\tau].$$

In the interval $[2\tau, \ 3\tau]$ the dynamics are described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BKx(t-\tau) \\ &= Ax(t) - BK\mathrm{e}^{A(t-2\tau)}x(\tau) \end{aligned}$$

and using the variable of constants formula we arrive at

$$x(3\tau) = \Delta \, x(\tau),$$

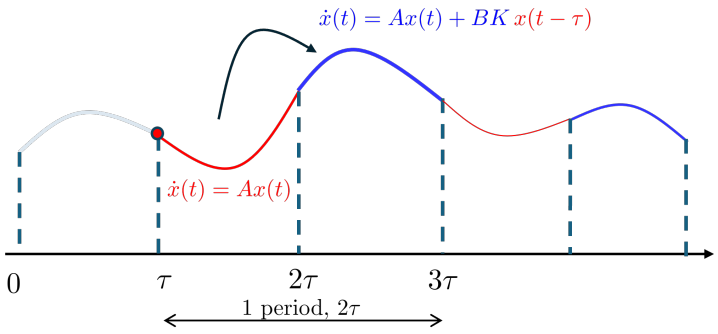


Fig. 8 Principle behind act-and-wait control.

with Δ the monodromy matrix,

$$\Delta := \mathrm{e}^{A\tau}\left[\mathrm{e}^{A\tau} - \int_0^\tau \mathrm{e}^{-As} BK \mathrm{e}^{As}\mathrm{d}s\right]. \tag{18}$$

Furthermore, from the periodicity of the control law we get

$$x((2k+1)\tau) = \Delta\, x((2k-1)\tau),\ \forall k \geq 1. \tag{19}$$

Thus, by an appropriate combination of open-loop and closed-loop phases, sketched in Figure 8, the infinite-dimensional monodromy operator of the closed-loop system, which maps the state of the system onto the corresponding state one period ahead, only has n nonzero eigenvalues, which are captured by the spectrum of the n-by-n matrix Δ. In fact, exponential stability is achieved if the spectrum of Δ is confined to the open unit disc in the complex plane. A blow-up of a solution between the time-instants considered in (19) cannot occur since in the open and closed-loop phase of a cycle, the solution is described by an autonomous and non-autonomous ODE, respectively, as clarified by the coloring in Figure 8. In addition, such a segment scales linearly with the initial point, $x((2k-1)\tau)$.

The finite spectrum of the monodromy operator offers potential for direct spectrum assignment since the number of degrees of freedom in the feedback (16) is larger or equal to n. Furthermore, whereas the tuning of (2) by minimizing the spectral abscissa of the closed-loop system gives rise to a non-smooth optimization problem, a smooth formulation of the spectrum assignment problem of Δ is possible, in terms of traces of matrices. Note that the trace of a square matrix, denoted by $\mathrm{tr}\,(\cdot)$, is defined as the sum of its eigenvalues or, equivalently, as the sum of its diagonal elements. The spectrum assignment problem of the monodromy matrix Δ can namely be formulated as

$$\begin{cases} \operatorname{tr}\left(\Delta(K)\right) = \lambda_1 + \ldots + \lambda_n, \\ \operatorname{tr}\left(\Delta^2(K)\right) = \lambda_1^2 + \ldots + \lambda_n^2, \\ \qquad\qquad \vdots \\ \operatorname{tr}\left(\Delta^n(K)\right) = \lambda_1^n + \ldots + \lambda_n^n, \end{cases} \tag{20}$$

with $\{\lambda_1, \ldots, \lambda_n\}$ the desired eigenvalues of Δ. Equations (20) constitute a system of polynomial equations in the elements of the controller gain K, which directly follows from interpreting the trace in the left-hand sides in terms of sums of diagonal elements. As shown in [24], where formulation (20) is proposed, we can express

$$\frac{\partial \operatorname{tr}\left(\Delta^k\right)}{\partial K_{ij}} = k \operatorname{tr}\left(\Delta^{k-1} \frac{\partial \Delta}{\partial K_{ij}}\right)$$

and

$$\frac{\partial \Delta(K)}{\partial K_{ij}} = -\int_0^\tau \mathrm{e}^{A(\tau-s)} B e_i e_j^{\mathrm{T}} \mathrm{e}^{As} \mathrm{d}s,$$

enabling a Newton type algorithm for finding solutions of (20).

Furthermore, there is even potential for fixed-time stability, if all eigenvalues can be set to zero. The closed-loop system (1) and (16) is said to be fixed-time stable with convergence time T if it is stable and

$$x(t;\ x_0, u_0) = 0, \quad \forall t \geq T,\ \forall (x_0, u_0) \in X,$$

where $x(\cdot;\ x_0, u_0)$ is the solution with initial condition (x_0, u_0) and X is defined in Remark 1. The following result holds.

Proposition 1 *System (1) and (16) is fixed-time stable if and only if matrix Δ is nilpotent. Furthermore, if Δ is nilpotent with index $\nu \in \{1, \ldots, n\}$, i.e.*

$$\Delta^{\nu-1} \neq 0, \quad \Delta^\nu = 0,$$

then the convergence time is given by $T = (1 + 2\nu)\tau$.

Proof. In the interval $[0,\ \tau]$, the dynamics are largely determined by the initial conditions. Since $\Delta^\nu = 0$ and the periodic is 2τ, it holds for any solution that $x(t) = 0$ at $t = (1 + 2\nu)\tau$. In the interval $[(1 + 2\nu)\tau,\ (2 + 2\nu)\tau]$ the system operates in open loop mode, where its dynamics are described by $\dot{x}(t) = Ax(t)$. As a consequence, we have $x(t) = 0$ for all t in this interval of length τ, which on its turn implies $x(t) = 0$ for all $t \geq (2 + 2\nu)\tau$. □

Example 3 We consider a quadruple integrator with unit input delay,

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = x_3(t),\ \dot{x}_3(t) = x_4(t),\ \ \dot{x}_4(t) = u(t-1). \tag{21}$$

A solution of equations (20) with $\lambda_i = 0,\ i = 1, \ldots, 4$, is given by

$$K = \begin{bmatrix} 0.118 & 0.605 & 1.47 & 2.21 \end{bmatrix}, \tag{22}$$

for which matrix Δ is nilpotent with index $\nu = 4$. As a consequence, the convergence time satisfies $T = (1 + 2\nu)\tau = 9$, which is consistent with the solution of the closed-loop system visualized in Figure 9.

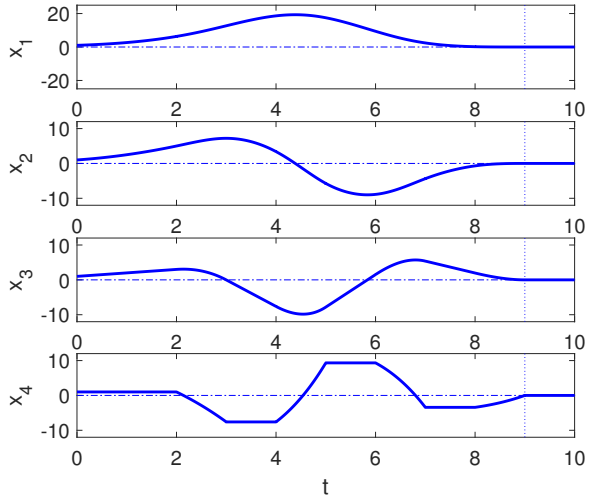


Fig. 9 Solution of (1) and (16), for model (21), gain (22) and initial conditions $x_0 = [1\ 1\ 1\ 1]^{\mathrm{T}}$ and $u_0 = 0$.

For static state feedback (2), the limitations of the delay in the input are reflected in a spectrum consisting of infinitely many characteristic roots, to be shaped by tuning a finite number of controller parameters. More subtle, for controller (16) the limitations are reflected in the nonlinearity of equations (20). These equations are namely not always solvable, while gaining insight in the existence and number of solutions is hard (see [24] for technical conditions and case-studies). In what follows we show that by using a time-varying gain in the on-phases, the issue of solvability is overcome. Moreover, the whole monodromy matrix can then be assigned rather than only its spectrum.

4.2 Smooth periodic feedback

By assuming the controller gain time-varying during the on-phases (while respecting the periodicity with period 2τ), the control law (16) is modified to

$$u(t) = \begin{cases} 0, & t \in (0,\ \tau) \cup (2\tau,\ 3\tau) \cup \cdots \\ -K(t-\tau)x(t), & t \in [\tau,\ 2\tau] \cup\ [3\tau,\ 4\tau] \cup \cdots \end{cases} \tag{23}$$

and the monodromy operator becomes

$$\Delta = \mathrm{e}^{A\tau}\left[\mathrm{e}^{A\tau} - \int_0^\tau \mathrm{e}^{-As}BK(s)\mathrm{e}^{As}\mathrm{d}s\right]. \tag{24}$$

We prescribe the structure of the (2τ)-periodic gain such that in the interval $[0,\ \tau]$ it satisfies

$$K(t) = R_h(t)B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}t}Q\mathrm{e}^{-At}, \tag{25}$$

with the symmetric nonzero shape function $R_h : [0,\ \tau] \to \mathbb{R}_+^{m\times m}$ and matrix $Q \in \mathbb{R}^{n\times n}$ to be determined later on. Then we can express Δ as

$$\Delta = \mathrm{e}^{A\tau}\left(\mathrm{e}^{A\tau} - \mathcal{P}_c\ Q\right),$$

where

$$\mathcal{P}_c := \int_0^\tau \mathrm{e}^{-As}BR_h(s)B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}s}\,\mathrm{d}s$$

is a scaled controllability Gramian. The controllability of the pair (A, B) implies that $\mathcal{P}_c$ is invertible. For a desired monodromy matrix Δ_{des} and a given choice of function R_h, matrix Q can be determined as

$$Q = \mathcal{P}_c^{-1}\left(\mathrm{e}^{A\tau} - \mathrm{e}^{-A\tau}\Delta_{\mathrm{des}}\right),$$

since this choice induces $\Delta = \Delta_{\mathrm{des}}$.

The flexibility offered by the shape function R_h allows us for instance to smoothen the transitions between the off- and on-phases of the control. For example, if we choose it as

$$R_h(t) = \left((\mathrm{e}^t - 1)(1 - \mathrm{e}^{t-\tau})\right)^{r+1} I,$$

with $r \in \mathbb{N}$, then the function $\mathbb{R}_+ \ni t \to K(t)$ is r times continuously differentiable, while with the choice

$$R_h(t) = \mathrm{e}^{-\frac{1}{t(\tau-t)}}\ I, \tag{26}$$

it is smooth, in the sense of being infinitely many times differentiable.

We now come back to fixed-time stabilization problem. With the choice $\Delta_{\mathrm{des}} = 0$ in the procedure sketched above, the control law (23)-(25) achieves fixed-time stability with convergence time $T = 3\tau$, following from Proposition 1. Thus, with periodic feedback we can achieve that

1. all solutions convergence to zero in time 3τ;
2. the gains are bounded and any degree of smoothness can be ensured,
3. while no theoretical restrictions are put on the value of the delay and the spectrum of matrix A.

For more details on the approach and for a combination with predictors we refer to [34]. Note that the notion of fixed-time stability is stronger than the notion of finite-time stability, where the convergence time may depend on the initial conditions. Note also that using alternative methods for finite-time or fixed-time stability, such as

approaches based on time-varying high-gain feedback [30], sliding mode control or approaches inferred from homogeneity [18], the control laws either have a singularity when the convergence time is approached, or they are non-locally Lipschitz in the neighborhood of the origin.

Example 4 We reconsider system (1) with the matrices and delay specified by (9). We select R_h according to (26) and use control law (23), tuned such that $\Delta_{\text{des}} = 0$. In Figure 10 the solution of the closed-loop system is shown for initial condition $u_0 = 0$ and $x_0 = [1\ 1\ 1]^{\mathrm{T}}$. As can be observed, the convergence time is $T = 3\tau$=15.

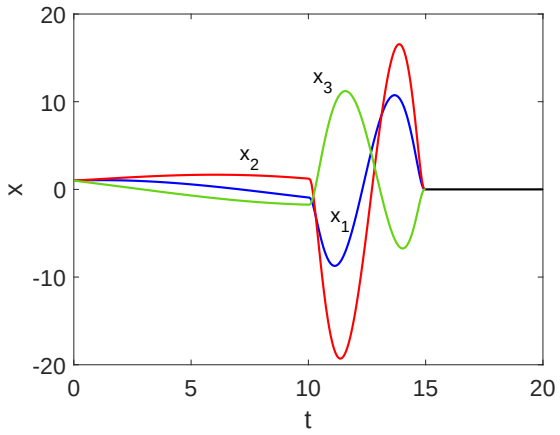


Fig. 10 Solution of (1) and (23) for data (9). By the design of the feedback law the monodromy matrix is rendered zero.

From an implementation perspective, the open loop phases resulting from control law (23) and (25), are a main limitation if the plant is unstable. However, if there is no input delay, asymptotic stability can be achieved by a stabilizing feedback transformation. Then the introduction of artificial input delay and a control law of the form (23)-(25) allows us to turn asymptotic stability into fixed-time stability. In the next Section 4.3, the dual problem of prescribed-time observation is addressed in more detail.

4.3 Towards observation in a prescribed time

We consider system (1) with $\tau = 0$ and assume that not the full state but only output $y(t) = Cx(t)$ is available for measurement. We extend the standard Luenberger's observer with an additional term in the dynamic equation for the observer's state $\hat{x}$, in the following way:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L_0(\hat{y}(t) - y(t)) - \tilde{L}(t)(\hat{y}(t-h) - y(t-h)), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \tag{27}$$

with $h > 0$ a design parameter, L_0 a matrix of appropriate dimensions and $\tilde{L}$ a $(2h)$-periodic function satisfying

$$\tilde{L}(t) = \begin{cases} 0, & t \in (0,\ h), \\ L(t), & t \in [h,\ 2h]. \end{cases}$$

The observer error $e(t) = \hat{x}(t) - x(t)$ satisfies

$$\dot{e}(t) = (A + L_0C)e(t) - \tilde{L}(t)Ce(t-h).$$

Notice that this equation has a similar form as the closed-loop equation of (20) and (23). Hence, using the procedure in Section 4.2, function L can be designed to achieve fixed-time stability of the error system with convergence time equal to $T = 2h$ (rather than $3h$ as the initial conditions for the plant only consist of $x(0)$). However, there are two important differences:

- the delay h is now a design parameter, hence the adopted approach allows to obtain a correct state estimate in *any* prescribed time $T > 0$;
- during the "open loop" phases, the dynamics are governed by $\dot{e}(t) = (A + L_0C)e(t)$, where gain L_0 can be chosen to be stabilizing.

Hence, observer (27) combines the classical asymptotic observer with a prescribed-time stabilization mechanism. In case of a large perturbation, the last term in the observer equation ensures that within the prescribed time, synchrony between system and observer states is restored, while the penultimate term assures that during the off-phases synchrony is maintained despite small perturbations.

Remark 4 If the input delay τ in (1) is nonzero, any convergence time of the estimator larger than τ can be realized since we then have $T = \tau + 2h$. More precisely, in the interval $[0,\ \tau]$ the dynamics are determined by the initial conditions of system and observer. For $t \geq \tau$, the above analysis is applicable provided that the gain $\tilde{L}$ is shifted with time τ. Recall that $h > 0$ can be chosen arbitrarily as it is part of the observer.

A particular feature of the fixed-time observation mechanism using smooth delayed periodic feedback is that the assignable convergence time $2h$ is not merely an upper bound, but reached from almost all initial conditions [35]. This uniformity is not only a distinctive theoretical property but it may also be valuable from an application perspective. Obviously, if the prescribed convergence time is reduced towards zero, this leads to a more aggressive control action. In the context of using formal methods to analyze complex control systems, a major problem is to certify whether a system or its trajectories satisfy design specifications. In our setting, such a specification could for instance be that the system or a subsystem has converged within a prescribed period or that a correct estimate has been obtained within prescribed time T. With the adopted approach, we ensure that we realize the objective

in T seconds rather then at most T seconds (i.e., we realize what is required but no more than that) whatever the initial condition, which could be an indication of more efficiently using the controls. The presented techniques also offer potential for hierarchical control and reduction techniques, where the convergence of subsystems' states or state estimates is obtained in a sequential way.

5 Concluding remarks and perspectives

A high-level overview of properties of time-delay systems was presented, illustrated by means of simple examples. It was highlighted that time-delay systems are a class of infinite-dimensional systems, inducing challenges from an analysis and control point of view. It was also outlined how for systems with input/output delays, a dimension reduction can be achieved using two complementary techniques, relying on predictor and periodic delayed feedback, respectively. In particular, solutions to the prescribed time stabilization and estimation problems were presented, in the form of explicit smooth feedback laws. As said, a restriction was made towards linear time-invariant systems. However, many challenges in analysis and control could be related to the nonlinearity of the associated eigenvalue problem.

We conclude the chapter by discussing the generalization of the presented results towards nonlinear time-delay systems. A rather direct generalization of the representations and fundamental properties discussed in Sections 2.1-2.2 is possible. A difference, however, is that for nonlinear systems, solutions may not be forward complete. The principle behind prediction based feedback, sketched in Figure 5, remains valid but the variation-of-constants formula, at the basis of the explicit expression (11), cannot be used anymore. Instead, to compute $\hat{x}(t)$ in a static prediction scheme, an online simulation (time-stepping) of the system is needed at each time-step, which bares similarities with a model predictive control approach. Extending dynamic predictor-controller schemes like (14), which generalize observer based controllers, can be done in a more direct way. Finally, the principle behind act-and-wait control, sketched in Figure 8, also remains valid for nonlinear time-invariant systems with input delay. The relevant dynamics of the monodromy operator can once again be reduced to these induced by an operator on $\mathbb{R}^n$ that generalizes matrix Δ. However, this operator is nonlinear and, similarly to the static prediction, explicit expression like (18) and (24) do not exist in most cases. For both addressed dimension reduction techniques, the nonlinearity of the solution operator, which translates along trajectories, and the lack of explicit expressions render the analysis and design much more challenging, even though the conceptual generalization is easy.

We further note that adding nonlinearity to the controller creates opportunities. A slight disadvantage of control laws (23) and (25), when designed for stabilization in a prescribed time, is that fixed-time stability is lost in the presence of model mismatch, because the targeted monodromy operator matrix will be only approximately equal to the zero matrix. However, relying on a transformation that induces a nonlinear

non-locally Lipschitz control law, the achieved fixed-time stability for the nominal model can be preserved in the presence of small perturbations [5].

Acknowledgments

This work was supported by the project C14/22/092 of the KU Leuven Internal Funds, the project G092721N of the Research Foundation-Flanders (FWO - Vlaanderen), and the National Natural Science Foundation of China for Distinguished Young Scholars under Grant 62125303.

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