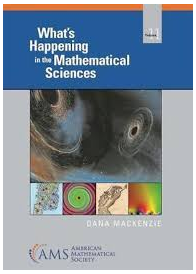
What's happening in the mathematical sciences, vol. 11 by *Dana Mackenzie*, AMS, 2019 (vi+136 p.), isbn: 9781470441630.

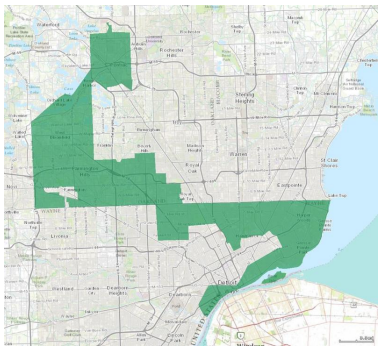




Dana Mackenzie

I have reviewed in this newsletter volumes 7 (Sept. 2010) and 8 (March 2012) of this wonderful series *What's happening in the mathematical sciences*. I skipped volumes 9 and 10, but I think it is useful to bring this series once more to your attention. Dana Mackenzie brings again nine texts, richly illustrates, in which he reports about interesting mathematical achievements obtained in recent years, in a way that is accessible for a broad public. I restrict myself to inform you about some of them.

One topic is *gerrymandering*. This word is less familiar to us, but it is important for elections in the US where a system of 'the winner takes it all' is in vigor. It refers to US vice president Gerry (1812) and salamander. He signed a bill in which the boundaries of a voting district looked very odd like a salamander. Districts are redrawn after the updated sensus data. The idea was that by carefully designing the boundaries of the districts, this could bring an advantage in the election process. Suppose in a whole region the majority of votes goes to the Democrats, but subdividing the region into districts, one may do that in such a way that there are large districts where the republicans have a slight overweight, while the bulk of the democratic voters are packed in a few smaller districts, then the republicans can still win the most representatives in Congress for the whole state.

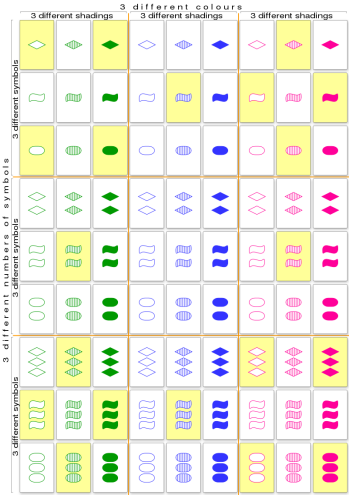


Credit: Wikipedia

Michigan's 14th congressional district 2020

A candidate winning with 50.01% is as good as one with 99.99% of the votes. Mathematicians were asked to produce a criterion by which this manipulative design can be condemned in court, but that is not so easy. One may think of minimizing the length of the perimeter, but then natural geographical restrictions are neglected and they are sensitive to small wiggles. Moreover, people do not live equidistributed. Then it has been tried to define redistricting such that it would not give a significant increase in efficiency of a vote for one party over the other. Whatever proposal that has been made, it remains a problem to find a fair criterion. The redistricting commission is often composed of the party that is in charge. Making that commission independent of the parties, would at least mend the cause of all this instead of fighting the symptoms.

Other contributions describe a PDE model for the dynamics of human metabolism, from which significant conclusion can be drawn concerning the effects of diets and weight loss. With the LIGO, VIRGO, LISA and other detectors, data are collected about gravitational waves. It turns out to be a gigantic numerical challenge to actually extract useful information about colliding black holes or neutron stars from the collected data which are, despite all precautions still noisy.



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The 81 different SET cards

A cap set is a subset of the n-dimensional affine space $\mathbb{Z}_3^n$ over a field with three elements where no triple of points is collinear. The cap set problem is to find the largest possible cap set as a function of n. For $n = 4$, this problem matches perfectly with the card game SET in which a card has 4 'attributes' (number, colour, shape, and fill) and each of these can take 3 values. So it has $81 = 3^4$ unique cards. The game is to recognize a threesome in which all attributes are different or the same (called a SET). A cap set is the largest possible number of cards that you can select that will not contain a SET. For $n = 4$ this is 20. Exact cap set sizes are known for small $n \in \mathbb{Z}_{\geq 0}$ (1, 2, 4, 9, 20, 45, 112, ...), upper and lower bounds for general n are known, narrowing the gap is an ongoing research topic.

Another problem where mathematical modelling helps is how to distribute docking stations for citybikes. That is a standard mixed integer linear programming problem. If bikes accumulate at some stations, there is an extra cost of redistributing the bikes at the end of the day. But if it can be combined with an app that suggests the user to choose for a nearby station where the chance is higher to find a bike, this can improve a lot on the distribution cost and the number of bikes. Similarly an Uber taxi could possibly be shared by two customers to reduce the cost and the number of taxis. Perhaps taxis and bikes can be combined to improve the commuting problem of a city.

Expanders are sparse but strongly connected graphs. This is what one wants when modelling a brain: connect neurons fast to each other with few axions. The adjacency matrix of such graphs have a large spectral gap between the largest eigenvalue and the other ones. These graphs are also used to design of error correcting codes, recommendation systems, and knot theory.

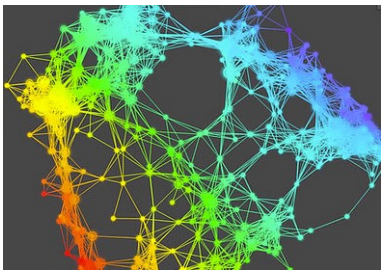
One essay gives an introduction to Quantum Computing and operations on qubits. All possible logical functions of two bits can be built with only NOT and NOR gates. The problem now is to find such a similar set of two 'golden gates' for two qubits.

After Wiles proved Fermat's Last Theorem (FLT), mathematicians tried to solve generalizations like the Asymptotic version (AFLT) in more general number fields. Here it is explained how elliptic curves and modular forms are connected and how this contributes in reducing the search for solutions of the equation from an infinite set to a finite one. It all starts by solving a simple equation $x + y = 1$ where x and y are rational numbers whose numerators and denominators are only divisible by prime numbers from a finite set.

Let me finish with another chapter on data analysis, which is another hot topic today. This is a remarkable story about the unexpected role played by topology in this field. Zeolites are porous materials that are used as adsorbents. The atoms of such material can be plotted as a well structured point cloud with empty holes. Assume now that we have an arbitrary set of data. Can we detect its topological structure? Can we find holes, and data that are closely connected? To detect such structures one has to draw balls around each point and consider points connected if their environments intersect. The radius of these balls can vary from very small, where there is no overlap anywhere, and as the radius increases, more and more points and components become connected, holes are formed, but as the radius increases more, everything will be connected, but the big 'holes' will be most persistent. This needs some introduction to persistent homology. The idea is to see which topological invariants are most persistent under increasing distortion of the space. Invariants are for example the Betti numbers giving the numbers of connected components (β_0), the number of tunnels (β_1), and the number of cavities (β_2). There are other ways of topological data analysis, using successive filters, but the result is similar. One arrives mathematically at a 'simplicial complex'.



zeolite



data

Credit: Wikipedia

This kind of analysis knows many applications. We can think of searching for zeolites with extreme porosity. It was used on clinical data to identify defects in a variety of genes that can be linked with the different types of diabetes, which defined more appropriate treatments. It was detected which brain cells, not in the visual cortex, were active when rats are maze running. Given a two-dimensional structure of a molecule, it is a challenge to find out how it will fold and what its physical properties are. Classical methods of computational chemistry keeps track of the thousands of atoms, but the data reduction via the topological analysis leads to a much simpler solution.

Adhemar Bultheel