A pseudo-spectrum based characterization of the robust strong H-infinity norm of time-delay systems with real-valued and structured uncertainties

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Abstract

This paper provides a mathematical characterization of the robust (strong) H-infinity norm of an uncertain linear time-invariant system with discrete delays in terms of the robust distance to instability of an associated characteristic matrix. The considered class of uncertainties consists of real-valued, structured, Frobenius norm bounded matrix uncertainties that act on the coefficient matrices. The robust H-infinity norm, defined as the worst-case value of the H-infinity norm over all admissible uncertainty values, is an important measure of robust performance to quantify the worst-case disturbance rejection of an uncertain dynamical system. For the considered system class, this robust H-infinity norm is however a fragile measure, as for a particular instance of the uncertainties, the H-infinity norm might be sensitive to arbitrarily small perturbations on the delays. Therefore, we introduce the robust strong H-infinity norm, inspired by the notion of strong stability of delay differential equations of neutral type, which takes into account both the uncertainties on the system matrices and infinitesimal delay perturbations. This quantity is a continuous function of both the elements of the system matrices and the delays. The main contribution of this work is the introduction of a relation between this robust strong H-infinity norm and the robust distance to instability of an associated characteristic matrix. This relation is subsequently employed in a novel algorithm for computing the robust strong H-infinity norm of uncertain time-delay systems.

Keywords: Time-delay systems; H-infinity norm; Distance to instability; Pseudo-spectrum; Uncertain systems.

1 Introduction

Literature review and contribution. For uncertain dynamical systems, the robust $\mathcal{H}_\infty$-norm is defined as the worst-case value of the $\mathcal{H}_\infty$-norm over all admissible uncertainty values. In robust control theory, this measure is used to quantify the worst-case input to output disturbance rejection of the system (Xie & de Souza Carlos 1992). For systems with discrete delays, this quantity is already examined in, among others, Li & de Souza (1996), Kokame *et al.* (1998), De Souza & Li (1999), Mahmoud (2000) and Ji *et al.* (2006), which all derive linear matrix inequalities that guarantee that the $\mathcal{H}_\infty$-norm of the closed loop system is below a certain specified bound for all admissible uncertainty values. However, this upper bound is typically an overestimate for the actual robust $\mathcal{H}_\infty$-norm of the system. Furthermore, these papers are mostly limited to systems with one delay. In contrast, we will consider dynamical systems with multiple discrete delays and derive an exact characterization of the robust $\mathcal{H}_\infty$-norm. Furthermore, we will consider systems of which the direct feed-through part may include multiple delays. However, for such systems, the robust $\mathcal{H}_\infty$-norm is *fragile* with respect to infinitesimal delay changes. This fragility problem originates from the sensitivity of the $\mathcal{H}_\infty$-norm of a particular instance of the uncertain system with respect to arbitrary small delay changes (Gumussoy & Michiels 2011). This work will therefore introduce the robust strong $\mathcal{H}_\infty$-norm, which is defined as the worst-case value of the strong $\mathcal{H}_\infty$-norm[1] over all admissible uncertainty values. This robust strong $\mathcal{H}_\infty$-norm takes both the uncertainties on the system matrices and infinitesimal perturbations on the delay values into account. The main contribution of this work is a relation between this robust strong $\mathcal{H}_\infty$-norm and the robust distance to instability of an associated characteristic matrix with both real- and complex-valued perturbations. This relation can be seen as an extension of the well known result (Hinrichsen & Pritchard 2005, Theorem 5.3.3), which relates the $\mathcal{H}_\infty$-norm of a linear time-invariant (LTI) system without delays and uncertainties to the structured distance to instability of its state matrix. To conclude this paper, we introduce a numerical algorithm for computing the robust strong $\mathcal{H}_\infty$-norm. This algorithm is inspired by recent developments in numerical linear algebra (Borgioli & Michiels 2020) that allow to efficiently calculate the distance to instability of a non-linear eigenvalue problem.

In this work we consider LTI systems with discrete delays of the following form:

$$\begin{cases} \dot{x}(t) = \sum_{k=0}^{K} A_k\, x(t-\tau_k) + \sum_{k=0}^{K} B_k\, w(t-\tau_k) \\ z(t) = \sum_{k=0}^{K} C_k\, x(t-\tau_k) + \sum_{k=0}^{K} D_k\, w(t-\tau_k). \end{cases} \tag{1.1}$$

with $x \in \mathbb{R}^n$ the state, $w \in \mathbb{R}^m$ the input and $z \in \mathbb{R}^p$ the output; A_k, B_k, C_k and D_k for $k = 0, \ldots, K$ real-valued system matrices of appropriate dimension; and $0 = \tau_0 < \tau_1 < \cdots < \tau_K < \infty$ discrete delays.

[1] as defined in Gumussoy & Michiels (2011, Definition 4.4)

The corresponding transfer function from w to z equals:

$$T(s;\vec{\tau}) = \left(\sum_{k=0}^{K} C_k e^{-s\tau_k}\right)\left(sI - \sum_{k=0}^{K} A_k e^{-s\tau_k}\right)^{-1}\left(\sum_{k=0}^{K} B_k e^{-s\tau_k}\right) + \sum_{k=0}^{K} D_k e^{-s\tau_k},$$

with $\vec{\tau} = (\tau_1, \ldots, \tau_K)$[2]. If system (1.1) is exponentially stable, its $\mathcal{H}_\infty$-norm is defined as the supremum of its frequency response, i.e. the transfer function evaluated at the imaginary axis, measured in spectral norm:

$$\|T(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}^+} \sigma_1\big(T(\jmath\omega;\vec{\tau})\big)$$

with $\sigma_1(\cdot)$ the largest singular value of its matrix argument, see Zhou & Doyle (1998, Section 4.2). The $\mathcal{H}_\infty$-norm is an important performance measure as it quantifies the worst-case disturbance rejection of the system:

$$\|T(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty} = \sup_{w\in\mathcal{L}_2^m} \frac{\|z\|_{\mathcal{L}_2^p}}{\|w\|_{\mathcal{L}_2^m}},$$

with $\|w\|_{\mathcal{L}_2^m} = \sqrt{\int_0^{+\infty} \|w(t)\|_2^2\, dt}$, $\mathcal{L}_2^m = \{w : [0,+\infty) \mapsto \mathbb{R}^m \text{ such that } \|w\|_{\mathcal{L}_2^m}^2 < +\infty\}$, $\|w(t)\|_2$ the Euclidean norm of $w(t)$ and $\|z\|_{\mathcal{L}_2^p}$ defined analogously, see Zhou & Doyle (1998, Section 4.2).

For systems of form (1.1), the $\mathcal{H}_\infty$-norm continuously depends on the elements of the system matrices at values for which the system is exponentially stable. The function $\vec{\tau} \mapsto \|T(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty}$ might however be discontinuous, even if the system remains stable. The cause for this behaviour is the potential sensitivity of the asymptotic frequency response, which for (1.1) is equal to

$$T_a(\jmath\omega;\vec{\tau}) = D_0 + \textstyle\sum_{k=1}^{K} D_k e^{-\jmath\omega\tau_k},$$

with respect to infinitesimal delay changes, see Gumussoy & Michiels (2011). The name asymptotic frequency response stems from the following property.

Property 1. (Gumussoy & Michiels 2011, Proposition 3.3) It holds that,

$$\lim_{\bar{\omega}\to\infty} \max_{\omega\geq\bar{\omega}} \sigma_1\big(T(\jmath\omega) - T_a(\jmath\omega)\big) = 0.$$

This potential discontinuity with respect to the delay values can be eliminated by considering the strong $\mathcal{H}_\infty$-norm, which is defined in Gumussoy & Michiels (2011, Definition 4.4) as the smallest upper bound for the $\mathcal{H}_\infty$-norm that is insensitive to infinitesimal delay changes:

$$|||T(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} := \limsup_{\gamma\to 0+} \left\{\|T(\cdot;\vec{\tau}_\gamma)\|_{\mathcal{H}_\infty} : \vec{\tau}_\gamma \in \mathcal{B}(\vec{\tau},\gamma) \cap (\mathbb{R}^+)^K\right\},$$

with $\mathcal{B}(\vec{\tau},\gamma)$ a ball in $\mathbb{R}^K$ with radius γ and centered at $\vec{\tau}$. The strong $\mathcal{H}_\infty$-norm of the asymptotic frequency response, $|||T_a(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty}$, is defined analogously. The following properties of the strong $\mathcal{H}_\infty$-norm will be used in the remainder of this work.

[2]Note that τ_0 is not included in $\vec{\tau}$ as it is kept fixed to 0.

Property 2. (Gumussoy & Michiels 2011, Theorem 4.5) The strong $\mathcal{H}_\infty$-norm is continuous as a function of both the elements of the system matrices and the delays at values for which the system is exponentially stable.

Property 3. (Gumussoy & Michiels 2011, Theorem 4.5 & Proposition 4.3) The strong $\mathcal{H}_\infty$-norm of (1.1) satisfies

$$|||T(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} = \max\left\{\|T(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty}, |||T_a(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty}\right\}.$$

Furthermore, it holds that

$$|||T_a(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} = \max_{\vec{\theta}\in[0,2\pi)^K} \sigma_1\left(D_0 + \textstyle\sum_{k=1}^K D_k e^{\jmath\theta_k}\right). \tag{1.2}$$

Property 4. It holds that $|||T(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} \geq \|T(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty}$ and $|||T_a(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} \geq \|T_a(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty}$.

Property 5. (Gumussoy & Michiels 2011, Proposition 4.3) If the delays are rationally independent[3], then $|||T_a(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} = \|T_a(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty}$ and as a consequence $|||T(\cdot;\vec{\tau})|||_{\mathcal{H}_\infty} = \|T(\cdot;\vec{\tau})\|_{\mathcal{H}_\infty}$.

The following example illustrates the aforementioned sensitivity of the $\mathcal{H}_\infty$-norm with respect to infinitesimal delay changes.

Example 1. Let us consider the following system

$$\begin{cases} \dot{x}(t) = -2x(t) + x(t-\tau_1) - \ \ w(t) - 0.5w(t-\tau_2) \\ z(t) = -2x(t) + x(t-\tau_2) + 5w(t) + 1.5w(t-\tau_1) - 3w(t-\tau_2). \end{cases} \tag{1.3}$$

The corresponding frequency response is

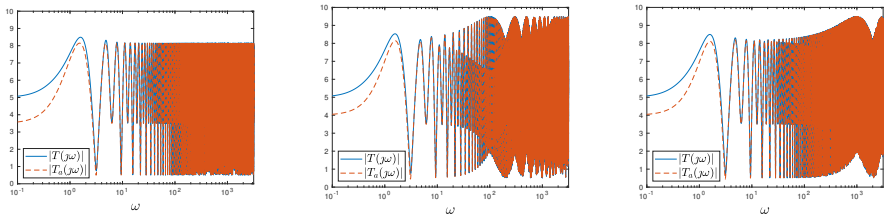
$$T(\jmath\omega;(\tau_1,\tau_2)) = \frac{(-2+e^{-\jmath\omega\tau_2})(-1-0.5e^{-\jmath\omega\tau_2})}{\jmath\omega+2-e^{-\jmath\omega\tau_1}} + 5 + 1.5e^{-\jmath\omega\tau_1} - 3e^{-\jmath\omega\tau_2}$$

and its asymptotic frequency response is equal to

$$T_a(\jmath\omega;(\tau_1,\tau_2)) = 5 + 1.5e^{-\jmath\omega\tau_1} - 3e^{-\jmath\omega\tau_2}.$$

Figure 1a plots the magnitude of both the frequency response and the asymptotic frequency response in function of ω for $\tau_1 = 1$ and $\tau_2 = 2$. For these values of delays, the magnitude of the frequency response attains a maximum of approximately 8.4907. Figure 1b shows the effect of a small change in delay parameters, $\tau_1 = 1$ and $\tau_2 = 2+\pi/100$. Now the $\mathcal{H}_\infty$-norm is equal to approximately 9.5. For an even smaller perturbation on the delays, such as $\tau_1 = 1$ and $\tau_2 = 2+\pi/1000$, we observe

[3] A collection of real numbers is rationally independent if no member can be written as a linear combination of the other elements with rational coefficients.



(a) (b) (c)

Figure 1: The magnitude of the frequency response (blue) and asymptotic frequency response (red) of system (1.3) in function of ω for (τ_1, τ_2) equal to $(1, 2)$ (left), $(1, 2 + \pi/100)$ (center) and $(1, 2 + \pi/1000)$ (right).

a similar effect, see Figure 1c. Furthermore, from Properties 3 and 5 it follows that for every $i \in \mathbb{N}_{>0}$

$$\begin{aligned}
\|T(\cdot;(1,2+\pi/i))\|_{\mathcal{H}_\infty} &= |||T\ (\cdot;(1,2+\pi/i))|||_{\mathcal{H}_\infty} \\
&\geq |||T_a(\cdot;(1,2+\pi/i))|||_{\mathcal{H}_\infty} \\
&= \max_{\vec{\theta}\in[0,2\pi)^2} \sigma_1\left(5 + 1.5e^{\jmath\theta_1} - 3e^{\jmath\theta_2}\right) \\
&= 9.5.
\end{aligned}$$

One can thus choose delay values arbitrarily close to $(1, 2)$ for which the $\mathcal{H}_\infty$-norm jumps to at least 9.5. From the figures it is also clear that this discontinuity is due to the asymptotic frequency response.

By considering the strong $\mathcal{H}_\infty$-norm we remove a *fragility* problem of the $\mathcal{H}_\infty$-norm, namely being potentially sensitive to infinitesimal perturbations on the delays. Note however that the strong $\mathcal{H}_\infty$-norm is still a property of the nominal model (and infinitesimal perturbations of the delays). The strong $\mathcal{H}_\infty$-norm can be computed using the method presented in Gumussoy & Michiels (2011), which extends the well-known BBBS algorithm, see Boyd *et al.* (1989) and Bruinsma & Steinbuch (1990), to systems with discrete time-delays.

Until now we assumed that we are able to obtain an exact model for the system we want to examine. In almost all applications there is however a mismatch between the mathematical model that describes the dynamical system and the real system, due to unmodelled (non-linear) behavior, model reductions and imprecise measurements of parameters. To take these deviations into account during the design process, one often adds uncertainties to the nominal model. In this work, the considered uncertainties are real-valued, as the model itself is real-valued, norm-bounded, reflecting the distance between model and reality, and structured, as often only a certain parameter or group of parameters is affected by an uncertainty. This is in contrast to most other works which consider complex-valued, unstructured uncertainties and hence lead to conservative results. In this work we

will consider L real-valued matrix uncertainties bounded in Frobenius norm:

$$\delta_l \in \mathbb{R}^{q_l \times r_l} \text{ with } \|\delta_l\|_F \leq \overline{\delta_l} \text{ for } l = 1, \ldots, L.$$

For notational convenience, we denote $\delta = (\delta_1, \ldots, \delta_L)$ as the collection of uncertainties and $\mathcal{D}$ is the set of admissible uncertainty values:

$$\mathcal{D} = \{\delta \in \mathbb{R}^{q_1 \times r_1} \times \cdots \times \mathbb{R}^{q_L \times r_L} : \|\delta_l\|_F \leq \bar{\delta}_l \text{ for } l = 1, \ldots, L\}.$$

These uncertainties influence the system matrices in the following way:

$$\tilde{R}(\delta) = R + \textstyle\sum_{l=1}^{L} \sum_{s=1}^{S_l^R} G_{l,s}^R \delta_l H_{l,s}^R$$

where $G_{l,s}^R$ and $H_{l,s}^R$ are real-valued shape matrices of appropriate dimension. Note that this definition allows a single uncertainty to affect multiple blocks in the same system matrix and even multiple system matrices as the same uncertainty may be present at multiple locations. This results in the following uncertain system:

$$\begin{cases} \dot{x}(t) = \sum_{k=0}^{K} \tilde{A}_k(\delta)\, x(t-\tau_k) + \sum_{k=0}^{K} \tilde{B}_k(\delta)\, w(t-\tau_k) \\ z(t) = \sum_{k=0}^{K} \tilde{C}_k(\delta)\, x(t-\tau_k) + \sum_{k=0}^{K} \tilde{D}_k(\delta)\, w(t-\tau_k) \end{cases} \tag{1.4}$$

with $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^m$ the input, $z \in \mathbb{R}^p$ the output, $0 = \tau_0 < \tau_1 < \cdots < \tau_K < \infty$ discrete delays, $\tilde{A}_k(\delta)$, $\tilde{B}_k(\delta)$, $\tilde{C}_k(\delta)$ and $\tilde{D}_k(\delta)$ uncertain system matrices as defined above. In the remainder of this work we make the following assumption:

Assumption 1. *System* (1.4) *is internally exponentially stable for all admissible uncertainty values, i.e. the characteristic roots of* $\lambda I - \sum_{k=0}^{K} \tilde{A}_k(\delta) e^{-\lambda \tau_k}$ *lie in the open left half-plane for all* $\delta \in \mathcal{D}$.

For uncertain systems, such as (1.4), it is desirable to quantify the worst-case behavior over all admissible values of the uncertainties. In the context of the $\mathcal{H}_\infty$-norm this led to the notion of the robust $\mathcal{H}_\infty$-norm which is defined as the worst-case value of the $\mathcal{H}_\infty$-norm over all admissible uncertainty values:

$$\|T(\cdot;\cdot,\vec{\tau})\|_{\mathcal{H}_\infty}^{\mathcal{D}} := \max_{\delta \in \mathcal{D}} \|T(\cdot;\delta,\vec{\tau})\|_{\mathcal{H}_\infty}$$

in which $T(s;\delta,\vec{\tau})$ is the transfer functions associated with a given instance of (1.4).

The robust $\mathcal{H}_\infty$-norm, which quantifies robust performance, can also be interpreted as the supremum of the worst-case gain function, which we define as follows

$$\Upsilon : \mathbb{R}^+ \ni \omega \mapsto \max_{\delta \in \mathcal{D}} \sigma_1\big(T(\jmath\omega;\delta,\vec{\tau})\big) \in \mathbb{R}^+,$$

i.e.

$$\|T(\cdot;\cdot,\vec{\tau})\|_{\mathcal{H}_\infty}^{\mathcal{D}} = \sup_{\omega \in \mathbb{R}^+} \Upsilon(\omega).$$

The potential discontinuity of the nominal $\mathcal{H}_\infty$-norm with respect to the delay values carries over to the robust $\mathcal{H}_\infty$-norm. Therefore, this paper works with the robust strong $\mathcal{H}_\infty$-norm instead, which we define as the worst-case value of the strong $\mathcal{H}_\infty$-norm

$$|||T(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} := \max_{\delta\in\mathcal{D}} |||T(\cdot;\delta,\vec{\tau})|||_{\mathcal{H}_\infty}.$$

Furthermore, by Property 3, it follows that

$$|||T(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} = \max\left\{ \max_{\delta\in\mathcal{D}} |||T_a(\cdot;\delta,\vec{\tau})|||_{\mathcal{H}_\infty}, \|T(\cdot;\delta,\vec{\tau})\|_{\mathcal{H}_\infty}^{\mathcal{D}} \right\}.$$

We thus can thus distinguish two cases. In the first case $|||T(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} = \max_{\delta\in\mathcal{D}} |||T_a(\cdot;\delta,\vec{\tau})|||_{\mathcal{H}_\infty}$, i.e. the robust strong $\mathcal{H}_\infty$-norm is equal to the worst-case value of the strong $\mathcal{H}_\infty$-norm of the asymptotic transfer function. In the remainder of this work, this last quantity will be referred to as the robust strong asymptotic $\mathcal{H}_\infty$-norm, which is defined as follows:

$$|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} := \max_{\delta\in\mathcal{D}} |||T_a(\cdot;\delta,\vec{\tau})|||_{\mathcal{H}_\infty}.$$

In the second case $|||T(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} > \max_{\delta\in\mathcal{D}} |||T_a(\cdot;\delta,\vec{\tau})|||_{\mathcal{H}_\infty}$, which means that $|||T(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} = \|T(\cdot;\cdot,\vec{\tau})\|_{\mathcal{H}_\infty}^{\mathcal{D}} = \max_{\omega\in\mathbb{R}^+} \Upsilon(\omega)$, in which the supremum operator is replaced by the maximum operator as in this case the maximum is attained at at least one finite ω. We will come back to this division into two case in Section 3.

The main contribution of this work is a characterization of the robust strong $\mathcal{H}_\infty$-norm of uncertain system (1.4) in terms of the robust distance to instability of a corresponding characteristic matrix. This result is inspired by the following result for LTI systems without uncertainties and delays.

Proposition 1.1. The $\mathcal{H}_\infty$-norm of

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{1.5}$$

with corresponding transfer function

$$T(s) = C(Is - A)^{-1}B + D,$$

is equal to the maximum of $\sigma_1(D)$ and the reciprocal of the distance to instability of

$$M(\lambda;\Delta) := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda - \begin{bmatrix} A & B & 0 \\ 0 & -I & \Delta \\ C & D & -I \end{bmatrix}, \tag{1.6}$$

which is defined as the smallest ϵ such that there exists a $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ for which characteristic matrix (1.6) has (a) characteristic root(s) in the closed right half-plane.

Proof. By introducing slack variables ζ_w and ζ_z, system (1.5) can be rewritten as a descriptor system:

$$\begin{cases} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\zeta}_w(t) \\ \dot{\zeta}_z(t) \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ 0 & -I & 0 \\ C & D & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \zeta_w(t) \\ \zeta_z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ \zeta_w(t) \\ \zeta_z(t) \end{bmatrix}. \end{cases}$$

The proposition now follows from Benner & Voigt (2014, Proposition 3.2) by noting that $\lim_{\omega\to\infty} \sigma_1(T(\jmath\omega)) = \sigma_1(D)$. Note that this result is equivalent with Hinrichsen & Pritchard (2005, Theorem 5.3.3). The reformulation as descriptor system will however be useful to get some intuition for the derivations in the remainder of this work. □

This work will extend this result by examining the relationship between the robust strong $\mathcal{H}_\infty$-norm of (1.4) and the robust distance to instability of a characteristic matrix with discrete delays and a singular leading matrix. More specifically, the remainder of this paper is organized as follows. In Section 2 we examine some properties of characteristic matrices with discrete delays and a singular leading matrix. Section 3 contains the main theoretical result of the paper as it gives the relation between the robust strong $\mathcal{H}_\infty$-norm of (1.4) and the robust distance to instability of the aforementioned characteristic matrix. This relationship is illustrated in Section 4 using four simple examples. Next, Section 5 presents a numerical algorithm to compute the robust strong $\mathcal{H}_\infty$-norm, based on the theory developed in Section 3. Finally, Section 6 gives some concluding remarks.

Preliminary results, focusing on systems without direct feed-through, i.e. $\tilde{D}_k(\delta) \equiv 0$ for $k = 0, \ldots, K$, have been presented in Appeltans & Michiels (2019). In this special case $T_a(s; \delta, \vec{\tau}) \equiv 0$ and as a consequence the robust $\mathcal{H}_\infty$-norm and the robust strong $\mathcal{H}_\infty$-norm coincide.

2 Delay eigenvalue problems with a singular leading matrix

In this section, we consider nonlinear eigenvalue problems $M(\lambda; \vec{\tau})v = 0$ where the characteristic matrix is of the following form:

$$M(\lambda; \vec{\tau}) := Q\lambda - P_0 - \textstyle\sum_{k=1}^{K} P_k e^{-\lambda\tau_k} \tag{2.1}$$

with $Q, P_0, \ldots, P_K$ square matrices of size n_M $(= n+m+p)$, $0 < \tau_1 < \cdots < \tau_K < \infty$ discrete delays and $\vec{\tau} = (\tau_1, \ldots, \tau_K)$. The leading matrix Q is singular and in the remainder of this paper U and V denote $n_M \times (n_M - \text{rank}(Q))$-dimensional matrices whose columns form a basis for the left and right nullspace of Q, respectively. The spectrum of (2.1), i.e. the set of characteristic roots, is defined as

$$\Lambda(\vec{\tau}) := \{\lambda \in \mathbb{C} : \det(M(\lambda; \vec{\tau})) = 0\}. \tag{2.2}$$

The characterization of this spectrum is not trivial due to the singular leading matrix. Therefore this section will review some related theory. In Section 2.1, the focus lies on the nominal characteristic matrix and its characteristic roots. Subsequently the effect of structured perturbations on the location of these characteristic roots is examined in Section 2.2. The derivations in Section 2.2 are restricted to matrices Q, P_0, $\ldots$, P_K with a specific structure. This structure is chosen such that there exists a connection between the robust distance to instability of this perturbed characteristic matrix and the robust strong $\mathcal{H}_\infty$-norm of (1.4).

2.1 Spectral properties

First, we define the concepts of regularity and index for the following matrix pencil:

$$N(\lambda) := Q\lambda - P. \tag{2.3}$$

Matrix pencil (2.3) is regular when the polynomial, $\mathbb{C} \ni \lambda \mapsto \det(N(\lambda))$, does not vanish identically. If regular, characteristic matrix (2.3) can be brought in Weierstrass-Kronecker canonical form by invertible matrices W and T (Gantmacher 1959, Chapter XII):

$$W\left(Q\lambda - P\right)T = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}\lambda - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}$$

with J and N in Jordan form and N a nilpotent matrix. The index of matrix pencil (2.3) is defined as zero if N is void and is defined as the smallest integer ν such that $N^\nu = 0$ for non-void N.

We now return to our original characteristic matrix (2.1). As in Fridman (2002, Section 2.1), characteristic matrix (2.1) is said to be regular if the corresponding matrix pencil $Q\lambda - P_0$ is regular and (if regular) its index is defined as the index of $Q\lambda - P_0$. Based on these notions of regularity and index, we define well-posedness in the remainder of this work in the following way.

Definition 1. Characteristic matrix (2.1) is called well-posed when it is regular and has at most index 1.

The following lemma allows us to easily verify this well-posedness condition.

Lemma 2.1. *Characteristic matrix* (2.1) *is well-posed if and only if* $U^H P_0 V$ *is non-singular.*

Proof. The lemma follows immediately from Definition 1 and Bunse-Gerstner *et al.* (1999, Lemma 2). □

Next, we restrict our attention to well-posed characteristic matrices and examine some properties of their spectrum as defined in (2.2). More specifically, the spectral abscissa of $M(\lambda;\vec{\tau})$ is defined as follows,

$$\alpha(\vec{\tau}) := \sup\left\{\Re(\lambda) : \lambda \in \Lambda(\vec{\tau})\right\}.$$

From Michiels (2011, Section 3.2), it follows that the function $\vec{\tau} \mapsto \alpha(\vec{\tau})$ is potentially discontinuous. This behavior is similar to the sensitivity of the $\mathcal{H}_\infty$-norm with respect to infinitesimal delay changes, as discussed in Section 1, and the sensitivity of the stability of the null solution of neutral delay differential equations with respect to infinitesimal delay changes, as discussed in Hale & Verduyn Lunel (2002). To resolve this fragility problem, we will consider the strong spectral abscissa, which is defined in Michiels (2011, Proposition 3) as

$$\alpha_s(\vec{\tau}) := \limsup_{\gamma \to 0+} \{\alpha(\vec{\tau}_\gamma) : \vec{\tau}_\gamma \in \mathcal{B}(\vec{\tau}, \gamma) \cap (\mathbb{R}^+)^K\},$$

and which is continuous with respect to both $\vec{\tau}$ and the elements of the matrices Q, P_0,..., P_K. In the remainder of this work, we will use the following properties of the strong spectral abscissa.

Property 6. (Michiels 2011, Propostition 3) For well-posed characteristic matrices, the strong spectral abscissa satisfies

$$\alpha_s(\vec{\tau}) = \max\left\{\alpha_{D,s}(\vec{\tau}), \alpha(\vec{\tau})\right\}$$

with $\alpha_{D,s}(\vec{\tau})$ the strong spectral abscissa of the difference equation associated with (2.5):

$$M_D(\lambda; \vec{\tau}) = U^T P_0 V + \sum_{k=1}^{K} U^T P_k V e^{-\lambda \tau_k}.$$

Furthermore, $\alpha_{D,s}(\vec{\tau})$ is equal to the zero crossing of the monotonically decreasing function

$$\mathbb{R} \ni \varsigma \mapsto \max_{\vec{\theta} \in [0, 2\pi)^K} \rho\left(\sum_{k=1}^{K} \left(U^H P_0 V\right)^{-1} \left(U^H P_k V\right) e^{-\varsigma \tau_k + \jmath \theta_k}\right) - 1, \tag{2.4}$$

if such a crossing exists and otherwise $\alpha_{D,s}(\vec{\tau})$ equals $-\infty$, in which $\rho(\cdot)$ denotes the spectral radius of its matrix argument.

Property 7. If $\alpha_{D,s}(\vec{\tau}) \neq -\infty$ then for each $\gamma > 0$ there exist $\vec{\tau}_\gamma \in \mathcal{B}(\vec{\tau}, \gamma) \cap (\mathbb{R}^+)^K$ such that $\Lambda(\vec{\tau}_\gamma)$, i.e. the spectrum of (2.1) with the delay values replaced by $\vec{\tau}_\gamma$, contains a chain of characteristic roots $\{\lambda_i\}_{i \in \mathbb{N}}$ that satisfies

$$\lim_{i \to \infty} \Re(\lambda_i) = \alpha_{D,s}(\vec{\tau}) \qquad \lim_{i \to \infty} |\Im(\lambda_i)| = +\infty.$$

If $\alpha_{D,s}(\vec{\tau}) < 0$, there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for any $\vec{\tau}_{\epsilon_1} \in \mathcal{B}(\vec{\tau}, \epsilon_1) \cap (\mathbb{R}^+)^K$ the number of characteristic roots of $M(\lambda; \vec{\tau}_{\epsilon_1})$ that lie to the right of $-\epsilon_2$ is finite.

Proof. For each $\vec{\tau} \in (\mathbb{R}^+)^K$ and $\gamma > 0$ there exist $\vec{\tau}_\gamma \in \mathcal{B}(\vec{\tau}, \gamma) \cap (\mathbb{R}^+)^K$ such that the elements of $\vec{\tau}_\gamma$ are rationally independent. The first assertion then follows from Michiels (2011, Proposition 2.3 and Lemma 1). Furthermore, if $\alpha_{D,s}(\vec{\tau}) < 0$ then by the definition of $\alpha_{D,s}(\vec{\tau})$ there exists $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\alpha_D(\vec{\tau}_{\epsilon_1}) \leq -\epsilon_2 < 0$ for all $\vec{\tau}_{\epsilon_1} \in \mathcal{B}(\vec{\tau}, \epsilon_1) \cap (\mathbb{R}^+)^K$. The second assertion then follows from Michiels (2011, Lemma 2). □

To conclude this subsection, we introduce the following definition of strong stability based on the notion of well-posedness and the strong spectral abscissa.

Definition 2. A characteristic matrix $M(\lambda;\vec{\tau})$ is strongly stable if it is well-posed and its strong spectral abscissa is strictly negative.

The stability definition given above stems from the relation between (2.1) and the following dynamical system of delay differential algebraic equations:

$$Q\,x(t) = P_0\,x(t) + \textstyle\sum_{k=1}^{K} P_k\,x(t-\tau_k).$$

More specifically, in Michiels (2011, Section 2 and Theorem 1) it is shown that this dynamical system has a unique and well-defined solution for appropriate initial conditions and that its null solution is exponential stable, even when considering infinitesimal delay changes, if and only if $M(\lambda;\vec{\tau})$ fulfills the definition of strong stability given above.

2.2 Robust distance to instability

This subsection examines the effect of adding perturbations to a strongly stable characteristic matrix. Inspired by Proposition 1.1 and its proof, we focus on perturbed characteristic matrices of the following form:

$$M(\lambda;\delta,\Delta,\vec{\tau}) := Q\lambda - \tilde{P}_0(\delta,\Delta) - \textstyle\sum_{k=1}^{K} \tilde{P}_k(\delta)e^{-s\tau_k} \tag{2.5}$$

with

$$Q = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \tilde{P}_0(\delta,\Delta) = \begin{bmatrix} \tilde{A}_0(\delta) & \tilde{B}_0(\delta) & 0 \\ 0 & -I & \Delta \\ \tilde{C}_0(\delta) & \tilde{D}_0(\delta) & -I \end{bmatrix} \text{ and } \tilde{P}_k(\delta) = \begin{bmatrix} \tilde{A}_k(\delta) & \tilde{B}_k(\delta) & 0 \\ 0 & 0 & 0 \\ \tilde{C}_k(\delta) & \tilde{D}_k(\delta) & 0 \end{bmatrix}$$

where $\tilde{A}_k(\delta)$, $\tilde{B}_k(\delta)$, $\tilde{C}_k(\delta)$, $\tilde{D}_k(\delta)$ and $\tau_1,\ldots,\tau_K$ are as defined in Section 1 and $\Delta \in \mathbb{C}^{m\times p}$. Observe that the perturbed characteristic matrix (2.5) shares some similarities with (1.6), but there are two main differences: firstly (2.5) contains delay terms and secondly (2.5) has both real and complex-valued perturbations. The real-valued perturbations, δ, originate from the uncertainties in model (1.4) and are therefore confined to the set $\mathcal{D}$. The complex-valued perturbation, Δ, on the other hand plays a similar role as in Proposition 1.1: we are interested in the smallest ϵ such that there exists a $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ for which $M(\lambda;\delta,\Delta,\vec{\tau})$ is not strongly stable for at least one $\delta \in \mathcal{D}$. This critical ϵ will be called the *robust* distance to instability, where the specifier "robust" is used to indicate that we take the worst-case value over all $\delta \in \mathcal{D}$. In Section 3 it will be shown that there exists a relation between the robust distance to instability of (2.5) and the robust strong $\mathcal{H}_\infty$-norm of system (1.4). The remainder of this subsection characterizes therefore this robust distance to instability in greater detail.

From the definition of strong stability (see Definition 2), it follows that as ϵ increases there are two ways in which a loss of strong stability can occur. Firstly, as ϵ increases there might exist $\delta \in \mathcal{D}$ and $\Delta \in \mathbb{C}^{m\times P}$ with $\|\Delta\|_2 \leq \epsilon$ such

that $M(\lambda;\delta,\Delta,\vec{\tau})$ is non well-posed. The robust distance to non well-posedness is defined as the smallest ϵ for which this occurs:

$$\mathrm{d_{NWP}}(\mathcal{D}) := \begin{cases} +\infty, \text{ if } M(\lambda;\delta,\Delta,\vec{\tau}) \text{ is well-posed for all } \Delta \in \mathbb{C}^{m\times p} \text{ and } \delta \in \mathcal{D} \\ \min\limits_{\substack{\delta\in\mathcal{D} \\ \Delta\in\mathbb{C}^{m\times p}}} \{\|\Delta\|_2 : M(\lambda;\delta,\Delta,\vec{\tau}) \text{ is not well-posed}\}, \text{ otherwise.} \end{cases} \tag{2.6}$$

Secondly, an instance of (2.5) is not strongly stable if its strong spectral abscissa is non-negative. Therefore we study the strong pseudo-spectral abscissa of (2.5), which for $\epsilon \in \big[0, \mathrm{d_{NWP}}(\mathcal{D})\big)$ is defined as the maximal strong spectral abscissa over all instances of $\delta \in \mathcal{D}$ and $\Delta \in \mathbb{C}^{m\times P}$ with $\|\Delta\|_2 \leq \epsilon$:

$$\alpha_s^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) := \max_{\delta\in\mathcal{D}} \max_{\|\Delta\|_2\leq\epsilon} \alpha_s(\delta,\Delta,\vec{\tau})$$

with $\alpha_s(\delta,\Delta,\vec{\tau})$ the strong spectral abscissa of $M(\lambda;\delta,\Delta,\tau)$. It follows from Property 6 that:

$$\alpha_s^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) = \max\{\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}), \alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})\}, \tag{2.7}$$

with

$$\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) := \max_{\delta\in\mathcal{D}} \max_{\|\Delta\|_2\leq\epsilon} \alpha_{D,s}(\delta,\Delta,\vec{\tau})$$

and

$$\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) := \max_{\delta\in\mathcal{D}} \max_{\|\Delta\|_2\leq\epsilon} \alpha(\delta,\Delta,\vec{\tau}).$$

Remark 1. The quantity $\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$ can alternatively be characterized using the pseudo-spectrum of (2.5) which is defined as

$$\Lambda^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) := \bigcup_{\delta\in\mathcal{D}} \bigcup_{\substack{\Delta\in\mathbb{C}^{m\times p} \\ \|\Delta\|_2\leq\epsilon}} \big\{\lambda\in\mathbb{C} : \det\big(M(\lambda;\delta,\Delta,\vec{\tau})\big) = 0\big\}. \tag{2.8}$$

The pseudo-spectral abscissa, $\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$, is then equal to the supremum of the real parts of the points in this pseudo-spectrum:

$$\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) = \sup\{\Re(\lambda) : \lambda \in \Lambda^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})\}.$$

$\diamond$

Remark 2. Both $\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$ and $\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$ continuously depend on ϵ for $\epsilon \in \big[0, \mathrm{d_{NWP}}(\mathcal{D})\big)$. This follows from the fact that both $\alpha_{D,s}(\delta,\Delta,\vec{\tau})$ and $\alpha(\delta,\Delta,\vec{\tau})$ continuously depend on the elements of Δ for $\|\Delta\|_2 \leq \mathrm{d_{NWP}}(\mathcal{D})$. As a consequence, also $\alpha_s^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$ continuously depends on ϵ for $\epsilon \in \big[0, \mathrm{d_{NWP}}(\mathcal{D})\big)$. $\diamond$

Based on (2.7), we now introduce two new distance measures. Firstly, the robust distance to a characteristic root chain crossing is defined as

$$\mathrm{d_{CHAIN}}(\mathcal{D}) := \begin{cases} +\infty, \text{ if } \alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) < 0 \quad \forall \epsilon \in \big[0, \mathrm{d_{NWP}}(\mathcal{D})\big) \\ \min\{\varepsilon \in \big[0, \mathrm{d_{NWP}}(\mathcal{D})\big) : \alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) \geq 0\}, \text{ otherwise.} \end{cases} \tag{2.9}$$

Secondly, the robust distance to a finite root crossing is defined as

$$\mathrm{d_{FIN}}(\mathcal{D}) := \begin{cases} +\infty, \text{ if } \alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau}) < 0 \\ \qquad\qquad \forall \epsilon \in \left[0, \min\{\mathrm{d_{NWP}}(\mathcal{D}), \mathrm{d_{CHAIN}}(\mathcal{D})\}\right) \\ \min\{\epsilon \in \left[0, \min\{\mathrm{d_{NWP}}(\mathcal{D}), \mathrm{d_{CHAIN}}(\mathcal{D})\}\right) : \\ \qquad\qquad \alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau}) \geq 0\}, \text{ otherwise.} \end{cases} \tag{2.10}$$

Remark 3. Because both $\alpha^{\mathrm{ps}}_{D,s}(\mathcal{D}, \epsilon, \vec{\tau})$ and $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ continuously depend on ϵ for $\epsilon \in \left[0, \mathrm{d_{NWP}}(\mathcal{D})\right)$, both $\mathrm{d_{CHAIN}}(\mathcal{D})$ and $\mathrm{d_{FIN}}(\mathcal{D})$, when finite, are characterized by critical $\epsilon^\star_{CHAIN}$ and $\epsilon^\star_{FIN}$ for which $\alpha^{\mathrm{ps}}_{D,s}(\mathcal{D}, \epsilon^\star_{CHAIN}, \vec{\tau}) = 0$ and $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon^\star_{FIN}, \vec{\tau}) = 0$, respectively. ◇

The names of these two distances can be understood using Property 7. If the function $\epsilon \mapsto \alpha^{\mathrm{ps}}_{D,s}(\mathcal{D}, \epsilon, \vec{\tau})$ has a zero crossing in the interval $\left[0, \mathrm{d_{NWP}}(\mathcal{D})\right)$, this zero crossing corresponds to the smallest ϵ such that for each $\gamma > 0$ there exist $\delta \in \mathcal{D}$, $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ and $\vec{\tau}_\gamma \in \mathcal{B}(\vec{\tau}, \gamma) \cap (\mathbb{R}^+)^K$ for which the spectrum of $M(\lambda; \delta, \Delta, \vec{\tau}_\gamma)$ contains a chain of characteristic roots, as introduced in Property 7, which has the imaginary axis as vertical asymptote. At the same time, for $\epsilon < \min\{\mathrm{d_{CHAIN}}(\mathcal{D}), \mathrm{d_{NWP}}(\mathcal{D})\}$ the number of characteristic roots of any admissible instance of $M(\lambda; \delta, \Delta, \tau)$ in the closed right half-plane is finite and the chains of characteristic roots, as described in Property 7, are bounded away from the imaginary axis, even for infinitesimal delay perturbations. A zero crossing of $\epsilon \mapsto \alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ in the interval $\left[0, \min\{\mathrm{d_{NWP}}(\mathcal{D}), \mathrm{d_{CHAIN}}(\mathcal{D})\}\right)$ thus corresponds to the smallest ϵ for which there exist $\delta \in \mathcal{D}$ and $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ such that $M(\lambda; \delta, \Delta, \vec{\tau})$ has (finitely many) characteristic roots on the imaginary axis.

Remark 4. Based on the alternative definition of $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ in terms of the pseudo-spectrum $\Lambda^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$, we can introduce the following equivalent definition for the robust distance to a finite root crossing:

$$\mathrm{d_{FIN}}(\mathcal{D}) := \begin{cases} +\infty, \text{ if } \alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau}) < 0 \\ \qquad\qquad \forall \epsilon \in [0, \min\{\mathrm{d_{NWP}}(\mathcal{D}), \mathrm{d_{CHAIN}}(\mathcal{D})\}) \\ \min\{\epsilon \in [0, \min\{\mathrm{d_{NWP}}(\mathcal{D}), \mathrm{d_{CHAIN}}(\mathcal{D})\}) : \\ \qquad\qquad \exists \omega \in \mathbb{R} \text{ such that } \jmath\omega \in \Lambda^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})\}, \text{ otherwise,} \end{cases} \tag{2.11}$$

i.e. the robust distance to a finite root crossing is equal to the smallest ϵ such that the pseudo-spectrum touches the imaginary axis. ◇

These three distance measures will be illustrated in greater detail in Section 4. To conclude this section we define the robust distance to instability as follows.

Definition 3. The robust distance to instability is defined as the minimum of the three distance measures introduced above:

$$\mathrm{d_{INS}}(\mathcal{D}) := \min\left\{\mathrm{d_{NWP}}(\mathcal{D}),\, \mathrm{d_{CHAIN}}(\mathcal{D}),\, \mathrm{d_{FIN}}(\mathcal{D})\right\}. \tag{2.12}$$

3 Relation between the robust strong $\mathcal{H}_\infty$-norm and the robust distance to instability

This section establishes the relation between the robust strong $\mathcal{H}_\infty$-norm of (1.4) and the robust distance to instability ($\mathrm{d_{INS}}$) of characteristic matrix (2.5). This section is structured as follows. Section 3.1 reviews some preliminary results. Section 3.2 focuses on the relation between the robust strong asymptotic $\mathcal{H}_\infty$-norm $\left(|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}\right)$ and the robust distances to non well-posedness ($\mathrm{d_{NWP}}$) and a characteristic root chain crossing ($\mathrm{d_{CHAIN}}$). Next, Section 3.3 investigates the link between the worst-case gain function, $\omega \mapsto \Upsilon(\omega)$, and the robust distance to a finite root crossing ($\mathrm{d_{FIN}}$). Finally, Section 3.4 combines these results in the central theorem of this work.

3.1 Preliminary results

We start with some technical lemmas.

Lemma 3.1. *For a matrix $R \in \mathbb{C}^{p\times m}$ it holds that*

$$\sigma_1(R)^{-1} = \min_{\Delta\in\mathbb{C}^{m\times p}} \{\|\Delta\|_2 : \det(I_m - \Delta R) = 0\}$$

and

$$\sigma_1(R)^{-1}vu^H \in \operatorname{argmin}_{\Delta\in\mathbb{C}^{m\times p}}\{\|\Delta\|_2 : \det(I_m - \Delta R) = 0\}.$$

in which u and v are normalized left and right singular vectors of R associated with $\sigma_1(R)$, respectively.

Proof. The lemma follows from a similar argument as in Packard & Doyle (1993, Section 3 for the case $S = 0$ and $F = 1$). □

Lemma 3.2. *The robust distance to instability of (2.5) is strictly positive if and only if the characteristic roots of $I\lambda - \tilde{A}_0(\delta) - \sum_{k=1}^{K}\tilde{A}_k(\delta)e^{-\lambda\tau_k}$ lie in the open left half-plane for all $\delta \in \mathcal{D}$.*

Proof. Because

$$U^H\tilde{P}_0(\delta, \mathbf{0}_{m\times p})V = \begin{bmatrix} -I & 0 \\ \tilde{D}_0(\delta) & -I \end{bmatrix}$$

is invertible for all $\delta \in \mathcal{D}$, it follows from Lemma 2.1 that the robust distance to non well-posedness is strictly positive. Furthermore, from (2.4) it is easy to verify that

$$\alpha^{\mathrm{ps}}_{D,s}(\mathcal{D}, 0, \vec{\tau}) = -\infty$$

and thus also the robust distance to a characteristic root chain crossing is is strictly positive. Finally, it holds that

$$\alpha^{\mathrm{ps}}(\mathcal{D}, 0, \vec{\tau}) = \max_{\lambda\in\mathbb{C}}\left\{\Re(\lambda) : \exists\delta \in \mathcal{D} \text{ such that } \det\left(I\lambda - \sum_{k=0}^{K}\tilde{A}_k(\delta)e^{-\lambda\tau_k}\right) = 0\right\},$$

from which the lemma follows. □

By Assumption 1 this means that all characteristic matrices examined in the remainder of this paper have a strictly positive robust distance to instability.

Lemma 3.3. *Consider $\delta^\star \in \mathcal{D}$. Assume $\epsilon > 0$ and the complex number s is not a characteristic root of $I\lambda - \tilde{A}_0(\delta^\star) - \sum_{k=1}^{K} \tilde{A}_k(\delta^\star)e^{-\lambda\tau_k}$. Then there exists a $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ such that s is a characteristic root of $M(\lambda; \delta^\star, \Delta, \vec{\tau})$ if and only if $\sigma_1\big(T(s; \delta^\star, \vec{\tau})\big) \geq \epsilon^{-1}$.*

Proof. By definition, a complex number s is a characteristic root of $M(\lambda; \delta^\star, \Delta, \vec{\tau})$ if and only if

$$\det\big(M(s; \delta^\star, \Delta, \vec{\tau})\big) = \det\left(\left[\begin{array}{c:cc} Is - \sum\limits_{k=0}^{K} \tilde{A}_k(\delta^\star)e^{-s\tau_k} & -\sum\limits_{k=0}^{K} \tilde{B}_k(\delta^\star)e^{-s\tau_k} & 0 \\ \hdashline 0 & I & -\Delta \\ -\sum\limits_{k=0}^{K} \tilde{C}_k(\delta^\star)e^{-s\tau_k} & -\sum\limits_{k=0}^{K} \tilde{D}_k(\delta^\star)e^{-s\tau_k} & I \end{array}\right]\right) = 0.$$

Because $sI - \tilde{A}_0(\delta^\star) - \sum_{k=1}^{K} \tilde{A}_k(\delta^\star)e^{-s\tau}$ is invertible, it follows from Schur's determinant formula for block partitioned matrices, see Schur (1917) and Zhang (2005, Equation 0.3.2), that this last equation is equivalent with

$$\det\left(\begin{bmatrix} I & -\Delta \\ -T(s; \delta^\star, \vec{\tau}) & I \end{bmatrix}\right) = 0.$$

By applying Schur's determinant formula once again, we find that this is equivalent with

$$\det\big(I - \Delta T(s; \delta^\star, \vec{\tau})\big) = 0.$$

From Lemma 3.1 it follows that there exists a $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ such that this condition is met if and only if

$$\sigma_1\left(T(s; \delta^\star, \vec{\tau})\right) \geq \epsilon^{-1}.$$

□

3.2 Link between the robust strong asymptotic $\mathcal{H}_\infty$-norm and the robust distances to non well-posedness and a characteristic root chain crossing

We start with a characterization of the robust distance to non well-posedness, $\mathrm{d}_{\mathrm{NWP}}$, in terms of the uncertain direct feed-through term associated with $\tau_0 = 0$ of system (1.4).

Proposition 3.1. It holds that

$$\mathrm{d}_{\mathrm{NWP}}(\mathcal{D})^{-1} = \max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big).$$

Proof. It follows from Lemma 2.1 that a characteristic matrix of form (2.5) is not well-posed if and only if

$$U^H \tilde{P}_0(\delta, \Delta) V = \begin{bmatrix} -I & \Delta \\ \tilde{D}_0(\delta) & -I \end{bmatrix}$$

is singular. As a consequence of the Schur's determinant formula for block partitioned matrices, this condition can be rewritten as: characteristic matrix (2.5) is not well-posed if and only if $I - \Delta\tilde{D}_0(\delta)$ is singular. The robust distance to non well-posedness is thus equal to

$$\mathrm{d}_{\mathrm{NWP}}(\mathcal{D}) = \begin{cases} +\infty, \text{ if } \det\big(I - \Delta\tilde{D}_0(\delta)\big) \neq 0 \text{ for all } \Delta \in \mathbb{C}^{m\times p} \text{ and } \delta \in \mathcal{D} \\ \min\limits_{\substack{\delta\in\mathcal{D} \\ \Delta\in\mathbb{C}^{m\times p}}} \big\{\|\Delta\|_2 : \det\big(I - \Delta\tilde{D}_0(\delta)\big) = 0\big\}, \text{ otherwise} \end{cases}$$

Using Lemma 3.1, one finds

$$\begin{aligned} \mathrm{d}_{\mathrm{NWP}}(\mathcal{D}) &= \begin{cases} +\infty, \text{ if } \sigma_1\big(\tilde{D}_0(\delta)\big) = 0 \text{ for all } \delta \in \mathcal{D} \\ \min\limits_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big)^{-1}, \text{ otherwise} \end{cases} \\ &= \begin{cases} +\infty, \text{ if } \sigma_1\big(\tilde{D}_0(\delta)\big) = 0 \text{ for all } \delta \in \mathcal{D} \\ \Big(\max\limits_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big)\Big)^{-1}, \text{ otherwise} \end{cases} \end{aligned}$$

which completes the proof. □

The following lemma gives a lower bound for the robust strong asymptotic $\mathcal{H}_\infty$-norm.

Lemma 3.4. *It holds that*

$$\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) \leq |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}.$$

Proof. Consider the matrix-valued function $\mathbb{C} \ni s \mapsto D_0 + \sum_{k=1}^K D_k e^{-s} \in \mathbb{C}^{p\times m}$, of which every entry is an analytic and bounded function of s for s in the closed right half-plane, $\overline{\mathbb{C}}^+ = \{s \in \mathbb{C} : \Re(s) \geq 0\}$. From Boyd & Desoer (1984, Theorem 2.1 and Equation 2.3) it follows that the function $s \mapsto \sigma_1(D_0 + \sum_{k=1}^K D_k e^{-s})$ attains its maximum over $\overline{\mathbb{C}}^+$ on the imaginary axis, $\jmath\mathbb{R}$. And thus

$$\begin{aligned} \sigma_1(D_0) = \lim_{\Re(s)\to\infty} \sigma_1\big(D_0 + \textstyle\sum_{k=1}^K D_k e^{-s}\big) &\leq \max_{s\in\overline{\mathbb{C}}^+} \sigma_1\big(D_0 + \textstyle\sum_{k=1}^K D_k e^{-s}\big) \\ &= \max_{\omega\in\mathbb{R}} \sigma_1\big(D_0 + \textstyle\sum_{k=1}^K D_k e^{-\jmath\omega}\big) \\ &\leq \max_{\vec{\theta}\in[0,2\pi)^K} \sigma_1\big(D_0 + \textstyle\sum_{k=1}^K D_k e^{\jmath\theta_k}\big), \end{aligned}$$

from which the lemma follows. □

Next we show that the robust distance to a characteristic root chain crossing, $\mathrm{d}_{\mathrm{CHAIN}}$, is finite if and only if the above mentioned lower bound is not attained.

Lemma 3.5. *The robust distance to a characteristic root chain crossing of* (2.5) *is finite if and only if*

$$\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) < |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}.$$

Proof. First we prove sufficiency. It follows from the definition of the robust strong asymptotic $\mathcal{H}_\infty$-norm and (1.2) that $\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) < |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}$ implies that there exist $\vec{\theta}^\star \in [0,2\pi)^K$ and $\delta^\star \in \mathcal{D}$ such that

$$\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) < \sigma_1\big(\tilde{D}_0(\delta^\star) + \textstyle\sum_{k=1}^K \tilde{D}_k(\delta^\star)e^{\jmath\theta_k^\star}\big).$$

By Lemma 3.1 and Proposition 3.1, this means that there exists a $\Delta^\star$ with

$$\|\Delta^\star\|_2 = \sigma_1\big(\tilde{D}_0(\delta^\star)+\textstyle\sum_{k=1}^K \tilde{D}_k(\delta^\star)e^{\jmath\theta_k^\star}\big)^{-1} < \Big(\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big)\Big)^{-1} = \mathrm{d}_{\mathrm{NWP}}(\mathcal{D}),$$

such that

$$\det\Big(I - \Delta^\star\Big(\tilde{D}_0(\delta^\star) + \textstyle\sum_{k=1}^K \tilde{D}_k(\delta^\star)e^{\jmath\theta_k^\star}\Big)\Big) = 0.$$

Because $I - \Delta^\star\tilde{D}_0(\delta^\star)$ is invertible, this last equation is equivalent to

$$\det\Big(I - \big(I - \Delta^\star\tilde{D}_0(\delta^\star)\big)^{-1}\Delta^\star\Big(\textstyle\sum_{k=1}^K \tilde{D}_k(\delta^\star)e^{\jmath\theta_k^\star}\Big)\Big) = 0. \tag{3.1}$$

Furthermore, using the Shur-Banachiewicz inversion formula (Zhang 2005, Equation 0.8.1) one finds that for characteristic matrices of form (2.5),

$$\begin{aligned}\rho\Big(\textstyle\sum_{k=1}^K \big(U^H\tilde{P}_0(\delta^\star,\Delta^\star)V\big)^{-1}\big(U^H\tilde{P}_k(\delta^\star)V\big)e^{\jmath\theta_k^\star}\Big)& \\ = \rho\Big((I - \Delta^\star\tilde{D}_0(\delta^\star))^{-1}\Delta^\star\Big(\textstyle\sum_{k=1}^K \tilde{D}_k(\delta^\star)e^{\jmath\theta_k^\star}\Big)\Big).&\end{aligned} \tag{3.2}$$

Equation (3.1) thus implies that $\rho\Big(\sum_{k=1}^K \big(U^H\tilde{P}_0(\delta^\star,\Delta^\star)V\big)^{-1}\big(U^H\tilde{P}_k(\delta^\star)V\big)e^{\jmath\theta_k^\star}\Big) \geq 1$. This inequality has as a consequence that $\alpha_{D,s}(\delta^\star,\Delta^\star,\vec{\tau}) \geq 0$, and hence $\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\|\Delta^\star\|_2,\vec{\tau}) \geq 0$. This means that $\mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D}) \leq \|\Delta^\star\|_2 < \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})$.

To prove necessity, we use an argument by contradiction and assume $\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) \geq |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}$. This implies that there do not exist values $\delta^\star \in \mathcal{D}$, $\vec{\theta}^\star \in [0,2\pi)^K$ and $\Delta^\star \in \mathbb{C}^{m\times p}$ with $\|\Delta^\star\|_2 < \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})$ such that (3.1) holds. By noting that $\det(e^{\jmath\vartheta}R) = 0$ if and only if $\det(R) = 0$, this implies that the matrix

$$\textstyle\sum_{k=1}^K \big(U^H\tilde{P}_0(\delta,\Delta)V\big)^{-1}\big(U^H\tilde{P}_k(\delta)V\big)e^{\jmath\theta_k}$$

has no eigenvalue with unit modulus for all $\delta \in \mathcal{D}$, $\vec{\theta} \in [0,2\pi)^K$ and $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 < \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})$. Because 0 is the only eigenvalue of $\sum_{k=1}^K \big(U^H\tilde{P}_0(\delta,\mathbf{0}_{m\times p})V\big)^{-1}\big(U^H\tilde{P}_k(\delta)V\big)e^{\jmath\theta_k}$ and by the continuity of (3.2) with respect to the elements of Δ for $\|\Delta\|_2 < \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})$ this implies that

$$\rho\Big(\textstyle\sum_{k=1}^K \big(U^H\tilde{P}_0(\delta,\Delta)V\big)^{-1}\big(U^H\tilde{P}_k(\delta)V\big)e^{\jmath\theta_k}\Big) < 1$$

for all $\delta \in \mathcal{D}$, $\vec{\theta} \in [0,2\pi)^K$ and $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 < \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})$. It follows that $\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) < 0$ for all $\epsilon \in [0,\mathrm{d}_{\mathrm{NWP}}(\mathcal{D}))$ and hence $\mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D}) = +\infty$. □

Lemma 3.6. *For $\epsilon \in \big[0, \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})\big)$, the inequality $\epsilon \geq \big(|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}\big)^{-1}$ is a necessary and sufficient condition for $\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) \geq 0$.*

Proof. Sufficiency follows from the fact that $\epsilon \geq \big(|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}\big)^{-1}$ implies that there exist $\delta \in \mathcal{D}$, $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ and $\vec{\theta} \in [0,2\pi)^K$ such that (3.1) holds and a similar argument as in the sufficiency part of Lemma 3.5. We use an argument by contradiction to prove necessity. Assume $\epsilon < \big(|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}\big)^{-1}$. By Lemma 3.1 this means that there do not exist $\delta \in \mathcal{D}$, $\|\Delta\|_2 \leq \epsilon$ and $\theta \in [0,2\pi)^K$ such that $\det\big(I - \Delta\big(\tilde{D}_0(\delta) + \sum_{k=1}^K \tilde{D}_k(\delta)e^{\jmath\theta_k}\big)\big) = 0$. The lemma now follows from a similar argument as the necessity part of the proof of Lemma 3.5. □

By combining these results we arrive at an expression for the robust strong asymptotic $\mathcal{H}_\infty$-norm in terms of the robust distances to non well-posedness and a characteristic root chain crossing.

Proposition 3.2. The robust strong asymptotic $\mathcal{H}_\infty$-norm of (1.4) is equal to the reciprocal of the minimum of the robust distance to non well-posedness and the robust distance to a characteristic root chain crossing, i.e.

$$|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} = \min\{\mathrm{d}_{\mathrm{NWP}}(\mathcal{D}), \mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})\}^{-1}.$$

Proof. First we consider the case where $\mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})$ is finite and hence $\mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D}) < \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})$. It follows from Lemma 3.6 that $\epsilon \geq \big(|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}\big)^{-1}$ is a necessary and sufficient condition for $\alpha_{D,s}^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau}) \geq 0$. The smallest ϵ such that this first inequality holds, is equal to $\big(|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}}^{\mathcal{D}}\big)^{-1}$ and thus

$$|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}}^{\mathcal{D}} = \mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})^{-1}.$$

Next we consider the case where $\mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})$ is not finite. In this case it follows from Lemma 3.5 that $\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) \geq |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}$. But by Lemma 3.4 we have $\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) \leq |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}$. Thus in this case

$$|||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}} = \max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}_0(\delta)\big) = \mathrm{d}_{\mathrm{NWP}}(\mathcal{D})^{-1}.$$

□

3.3 Link between the worst-case gain function and the robust distance to a finite root crossing

The previous subsection established a relation between the robust strong asymptotic $\mathcal{H}_\infty$-norm and the robust distances to non well-posedness and a characteristic root chain crossing. In this subsection the link, between the worst-case gain function $\omega \mapsto \Upsilon(\omega)$ and the robust distance to a finite root crossing, $\mathrm{d}_{\mathrm{FIN}}$, is examined.

Lemma 3.7. *Under Assumption 1, the robust distance to a finite root crossing is finite if and only if*

$$||T(\cdot;\cdot,\vec{\tau})||_{\mathcal{H}_\infty}^{\mathcal{D}} > |||T_a(\cdot;\cdot,\vec{\tau})|||_{\mathcal{H}_\infty}^{\mathcal{D}}.$$

Furthermore, if this condition is fulfilled it holds that

$$\mathrm{d}_{\mathrm{FIN}}(\mathcal{D}) = \Big(\max_{\omega \in \mathbb{R}} \Upsilon(\omega) \Big)^{-1}.$$

Proof. We first prove the necessity of the above mentioned condition. If the robust distance to a finite root crossing is finite, it follows from Remark 3 that there exists a $\epsilon^\star < \min\{\mathrm{d}_{\mathrm{NWP}}(\mathcal{D}), \mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})\}$ such that $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon^\star, \vec{\tau}) = 0$. This means that there exist $\delta^\star \in \mathcal{D}$, $\Delta^\star$ with $\|\Delta^\star\|_2 = \epsilon^\star$ and an $\omega \in \mathbb{R}$ such that $\jmath\omega$ is a characteristic root of $M(\lambda; \delta^\star, \Delta^\star, \vec{\tau})$. By Lemma 3.3, this means that

$$\sigma_1(T(\jmath\omega; \delta^\star, \vec{\tau})) \geq (\epsilon^\star)^{-1} > \left(\min\{\mathrm{d}_{\mathrm{NWP}}(\mathcal{D}), \mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})\}\right)^{-1}.$$

Using the definition of $\|T(\cdot;\cdot,\vec{\tau})\|^{\mathcal{D}}_{\mathcal{H}_\infty}$ and Proposition 3.2 one finds that

$$\|T(\cdot;\cdot,\vec{\tau})\|^{\mathcal{D}}_{\mathcal{H}_\infty} \geq \sigma_1(T(\jmath\omega; \delta^\star, \vec{\tau})) \geq (\epsilon^\star)^{-1} > |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}.$$

Next, we prove sufficiency. If $||T(\cdot;\cdot,\vec{\tau})||^{\mathcal{D}}_{\mathcal{H}_\infty} > |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$, then there exists $\delta^\star \in \mathcal{D}$ and $\omega^\star \in \mathbb{R}$ such that $\sigma_1(T(\jmath\omega^\star; \delta^\star, \vec{\tau})) > |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$. From Lemma 3.3 and Proposition 3.2 it follows that there exists a $\Delta^\star$ with

$$\|\Delta^\star\|_2 = \sigma_1(T(\jmath\omega^\star; \delta^\star, \vec{\tau}))^{-1} < \left(|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}\right)^{-1} = \min\left\{\mathrm{d}_{\mathrm{NWP}}(\mathcal{D}), \mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})\right\}$$

such that characteristic matrix $M(\lambda; \delta^\star, \Delta^\star, \vec{\tau})$ has a characteristic root at $\jmath\omega^\star$.

Finally, we prove the equality $\mathrm{d}_{\mathrm{FIN}}(\mathcal{D}) = \left(\max_{\omega\in\mathbb{R}} \Upsilon(\omega)\right)^{-1}$, when $\mathrm{d}_{\mathrm{FIN}}(\mathcal{D})$ is finite. It follows from Lemma 3.3 that $\jmath\omega \in \Lambda^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ if and only if $\epsilon \geq \left(\max_{\delta\in\mathcal{D}} \sigma_1\left(T(\jmath\omega; \delta, \vec{\tau})\right)\right)^{-1}$. Using the alternative definition of $\mathrm{d}_{\mathrm{FIN}}$ in (2.11), we obtain

$$\mathrm{d}_{\mathrm{FIN}}(\mathcal{D}) = \min_{\omega\in\mathbb{R}^+} \Big(\max_{\delta\in\mathcal{D}} \sigma_1(T(\jmath\omega; \delta^\star, \vec{\tau})) \Big)^{-1} = \Big(\max_{\omega\in\mathbb{R}^+} \Upsilon(\omega) \Big)^{-1}.$$

□

3.4 Main theoretical result

This subsection combines the results of the two previous subsections to characterize the robust strong $\mathcal{H}_\infty$-norm in terms of the robust distance to instability.

Theorem 3.1. *The robust strong $\mathcal{H}_\infty$-norm of an internally exponentially stable system of form* (1.4) *is equal to the reciprocal of the robust distance to instability of characteristic matrix* (2.5)*, i.e.*

$$|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = \mathrm{d}_{\mathrm{INS}}(\mathcal{D})^{-1}$$

Proof. As mentioned in Section 1, we have to consider two cases: either $|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ or $|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} > |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ in which case $|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = \max_{\omega\in\mathbb{R}^+} \Upsilon(\omega)$. In the former case, the theorem follows from Proposition 3.2. In the latter case, the theorem follows from Lemma 3.7. □

4 Illustration

In Section 2.2 we encountered three ways in which a perturbed characteristic matrix of form (2.5) can loose strong stability. Thereafter in Section 3, we established the relation between the robust distance to instability of this characteristic matrix and the robust strong $\mathcal{H}_\infty$-norm of an uncertain system of form (1.4). This section presents 4 examples to illustrate these distance measures and their relation with the robust strong $\mathcal{H}_\infty$-norm.

4.1 Robust distance to non well-posedness

Example 2. Consider the following uncertain system:

$$\begin{cases} \dot{x}(t) = -(10+\delta_1)\,x(t) + (2+3\delta_1)\,x(t-1) + (4-\delta_1+\delta_2)\,w(t) \\ z(t) = (-8-3\delta_1+2\delta_2)\,x(t) + 4\,w(t) \end{cases} \tag{4.1}$$

where δ_1 and δ_2 are confined to $|\delta_1| \leq 0.2$ and $|\delta_2| \leq 0.1$. Because $T_a(\jmath\omega; \delta, \vec{\tau}) = 4$, the robust strong asymptotic $\mathcal{H}_\infty$-norm, $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$, equals 4. Next, Figure 2 shows the worst-case gain function $\Upsilon(\omega)$ for $\omega \in [10^{-2}, 10^3]$. The dotted line indicates $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$. It can be verified that $\Upsilon(\omega)$ lies below $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ for all finite ω. It thus holds that

$$|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = \max_{\delta\in\mathcal{D}} \sigma_1(\tilde{D}_0(\delta)).$$

Next we consider the associated perturbed characteristic matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda - \begin{bmatrix} -(10+\delta_1) & 4-\delta_1+\delta_2 & 0 \\ 0 & -1 & \Delta \\ -8-3\delta_1+2\delta_2 & 4 & -1 \end{bmatrix} - \begin{bmatrix} 2+3\delta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{-\lambda}. \tag{4.2}$$

From the theory developed in Section 3, it follows that $\mathrm{d}_{\mathrm{INS}}(\mathcal{D}) = \mathrm{d}_{\mathrm{NWP}}(\mathcal{D}) = 0.25$. Next, we examine the pseudo-spectrum, $\Lambda^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$, of (4.2) for ϵ close to 0.25. Figure 3 shows the part of $\Lambda^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ in the region $[-8, 0.5] \times \jmath[-40, 40]$ for $\epsilon = 0.249$. For this value of ϵ, the pseudo-spectrum consists of islands around the nominal characteristic roots, i.e. the characteristic roots of (4.2) for δ and Δ equal to zero. Furthermore, these islands lie completely in the open left half-plane. However, if we increase ϵ to 0.251 this no longer holds, as shown in Figure 4. Now the pseudo-spectrum includes almost the complete right half-plane and the corresponding (strong) pseudo-spectral abscissa is equal to $+\infty$. The function $\epsilon \mapsto \alpha_s(\mathcal{D}, \epsilon, \vec{\tau})$ is thus not continuous in the point $\epsilon = \mathrm{d}_{\mathrm{NWP}}(\mathcal{D}) = 0.25$.

4.2 Distance to characteristic root chain crossing

Example 3. Next we consider the following uncertain system:

$$\begin{cases} \dot{x}(t) = (-3+\delta_2)\,x(t) + (-1+3\delta_1)\,x(t-1) + 4\,w(t) \\ z(t) = (-2-3\delta_1+2\delta_2)\,x(t) + (3+\delta_1)\,w(t) + (1+\delta_1+\delta_2)\,w(t-1) \end{cases} \tag{4.3}$$

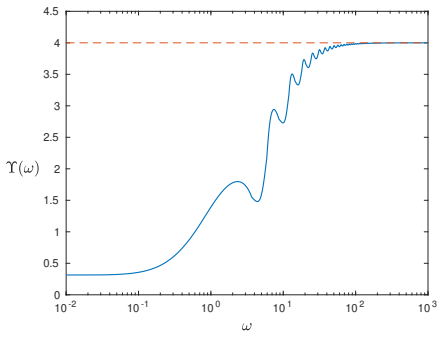


Figure 2: The worst-case gain function $\Upsilon(\omega)$ associated with (4.1). The dashed line indicates $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$.

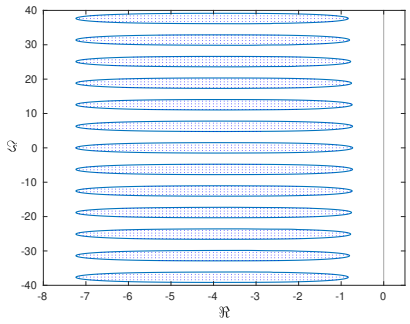


Figure 3: The part of the pseudo-spectrum of (4.2) for $\epsilon = 0.249$ in the region $[-8,\, 0.5] \times \jmath[-40,\, 40]$.

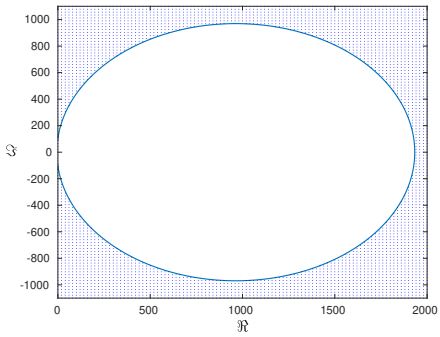


Figure 4: The part of the pseudo-spectrum of (4.2) for $\epsilon = 0.251$ in the region $[0,\, 2000] \times \jmath[-1100,\, 1100]$.

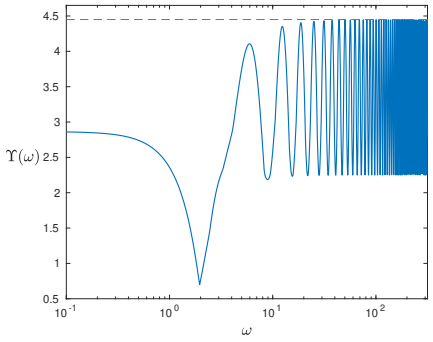


Figure 5: The worst-case gain function $\Upsilon(\omega)$ associated with system (4.3). The dashed line indicates $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$.

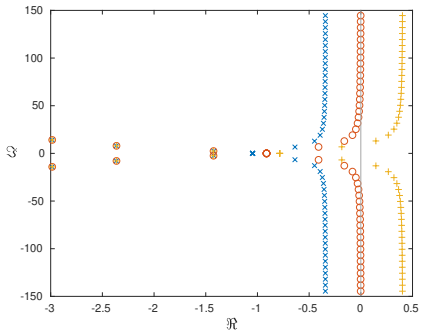


Figure 6: The spectrum of (4.4) for $\delta_1 = 0.1$, $\delta_2 = 0.25$ and Δ equal to to 0.2 (blue x), 1/4.45 (red o) and 0.25 (yellow +).

with $|\delta_1| \leq 0.1$ and $|\delta_2| \leq 0.25$. First, we examine the robust strong asymptotic $\mathcal{H}_\infty$-norm:

$$|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = \max_{\substack{|\delta_1|\leq 0.1 \\ |\delta_2|\leq 0.25}} \max_{\theta\in[0,2\pi)} \left|3+\delta_1+(1+\delta_1+\delta_2)e^{j\theta}\right| = 4.45.$$

Next, Figure 5 shows the worst-case gain function $\Upsilon(\omega)$ for $\omega \in [10^{-1}, 10^{2.5}]$. Also in this example, $\Upsilon(\omega)$ lies below $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ for all finite ω and thus $|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ is again equal to $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$. However, $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ is now strictly larger than $\max_{\delta\in\mathcal{D}} \sigma_1\big(\tilde{D}(\delta)\big) = \max_{|\delta_1|\leq 0.1} |3+\delta_1| = 3.1$.

Next, we examine the associated perturbed characteristic matrix:

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -3+\delta_2 & 4 & 0 \\ 0 & -1 & \Delta \\ -2-3\delta_1+2\delta_2 & 3+\delta_1 & -1 \end{bmatrix} - \begin{bmatrix} -1+3\delta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1+\delta_1+\delta_2 & 0 \end{bmatrix} e^{-\lambda}. \tag{4.4}$$

It follows from Section 3 that the robust distance to instability is equal to the robust distance to a characteristic root chain crossing, which on its turn is equal to $\big(|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}\big)^{-1} = 1/4.45$. This is illustrated in Figure 6, which for several ϵ shows the spectrum of a realization of (4.4) with $\delta \in \mathcal{D}$ and $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \epsilon$ chosen such that $\alpha_s(\delta,\Delta,\vec{\tau}) = \alpha_s^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$. We see that as ϵ increases a chain of characteristic roots crosses the imaginary axis.

Example 4. As third example we consider the following uncertain system:

$$\begin{cases} \dot{x}(t) = (-2+\delta_1)\,x(t) + (1+\delta_2)\,x(t-1) - w(t) + (-0.5+\delta_1)\,w(t-2) \\ z(t) = (-2+2\delta_2)\,x(t) + x(t-2) + (5+4\delta_1)\,w(t) + 1.5\,w(t-1) + (-3+\delta_1)\,w(t-2) \end{cases} \tag{4.5}$$

where δ_1 and δ_2 are confined to $|\delta_1| \leq 0.2$ and $|\delta_2| \leq 0.3$. We start again with the robust strong asymptotic $\mathcal{H}_\infty$-norm. In this case

$$|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = \max_{|\delta_1|\leq 0.2} \max_{\vec{\theta}\in[0,2\pi)^2} |5 + 4\delta_1 + 1.5e^{\jmath\theta_1} + (-3+\delta_1)e^{\jmath\theta_2}| = 10.1.$$

In Figure 7, we again plot the worst-case gain function $\Upsilon(\omega)$ and indicate $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ with a dashed line. Also in this example, $\Upsilon(\omega)$ lies below $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ for all ω. In contrast to the two previous examples, it holds that $\limsup_{\omega\to+\infty} \Upsilon(\omega) < |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$. This means that $\|T_a(\cdot;\cdot,\vec{\tau})\|^{\mathcal{D}}_{\mathcal{H}_\infty} \neq |||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ and that, as in Example 1, the asymptotic behavior of the transfer function is sensitive to infinitesimal perturbations on the delays, as illustrated in Figure 8.

Next, we examine its associated perturbed characteristic matrix:

$$\begin{bmatrix}\lambda & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} - \begin{bmatrix}-2+\delta_1 & -1 & 0\\ 0 & -1 & \Delta\\ -2+2\delta_2 & 5+4\delta_1 & -1\end{bmatrix} - \begin{bmatrix}1+\delta_2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1.5 & 0\end{bmatrix} e^{-\lambda} - \begin{bmatrix}0 & -0.5+\delta_1 & 0\\ 0 & 0 & 0\\ 1 & -3+\delta_1 & 0\end{bmatrix} e^{-2\lambda}. \tag{4.6}$$

Using the theory of Section 3 we conclude that $\mathrm{d}_{\mathrm{INS}}(\mathcal{D}) = \mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D}) = 1/10.1$. In contrast to the previous example, the spectrum of (4.6) is bounded away from the imaginary axis for all $\delta \in \mathcal{D}$ and $\|\Delta\|_2 \leq \mathrm{d}_{\mathrm{INS}}(\mathcal{D})$. However, for each $\gamma > 0$ there exist $\delta \in \mathcal{D}$, $\Delta \in \mathbb{C}^{m\times p}$ with $\|\Delta\|_2 \leq \mathrm{d}_{\mathrm{INS}}(\mathcal{D})$ and $\vec{\tau}_\gamma \in \mathcal{B}((1,2),\gamma) \cap (\mathbb{R}^+)^K$ such that the spectrum of (4.6) with the delay values $(1,2)$ replaced by $\vec{\tau}_\gamma$ contains a chain of characteristic roots with the imaginary axis as vertical asymptote. This is illustrated in Figure 9, which shows the spectrum of (4.6) for $\Delta = 1/10.1$, $\delta_1 = 0.2$ and $\delta_2 = 0$ for the nominal delays $\vec{\tau} = (1,2)$ and a small perturbation on the delays $\vec{\tau}_\gamma = (1, 2+\pi/100)$, where δ and Δ are chosen such that $\alpha_s(\delta,\Delta,\vec{\tau}) = \alpha_s^{\mathrm{ps}}\big(\mathcal{D}, \mathrm{d}_{\mathrm{INS}}(\mathcal{D}), \vec{\tau}\big)$.

4.3 distance to a finite root crossing

Example 5. In this final example we consider the following uncertain system

$$\begin{cases} \dot{x}(t) = (-7+3\delta_1)\,x(t) + (-5+2\delta_2)\,x(t-1) + 4\,w(t)\\ z(t) = (2-2\delta_2)\,x(t) + (1+\delta_1)\,w(t) + w(t-1)\end{cases} \tag{4.7}$$

with $|\delta_1| \leq 0.15$ and $|\delta_2| \leq 0.2$. For this example the robust strong asymptotic $\mathcal{H}_\infty$-norm is equal to

$$|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} = \max_{|\delta_1|\leq 0.15} \max_{\theta\in[0,2\pi)} \left|(1+\delta_1) + 1e^{\jmath\theta}\right| = 2.15.$$

Figure 10 shows the worst-case gain function $\Upsilon(\omega)$ for $\omega \in [10^{-1}, 10^{2.5}]$ which attains a maximum of approximately 6.012 at $\omega \approx 2.734$. In this case $|||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ is thus equal to $\max_{\omega\in\mathbb{R}^+} \Upsilon(\omega)$.

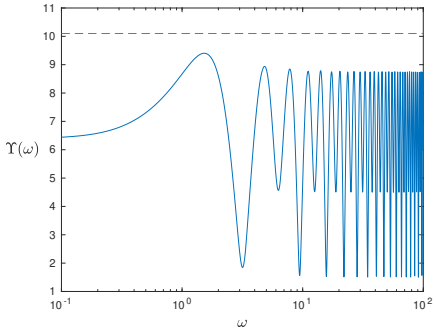


Figure 7: The worst-case gain function $\Upsilon(\omega)$ associated with system (4.5). The dashed line indicates $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$.

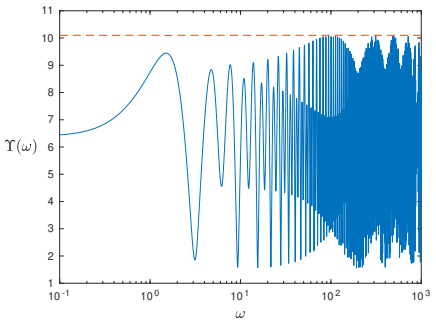
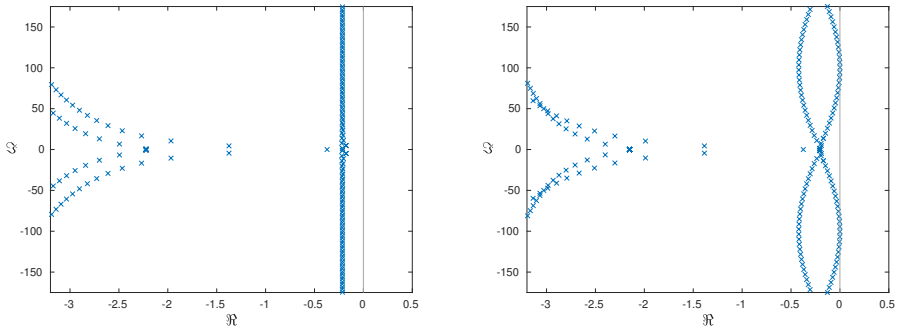


Figure 8: The worst-case gain function $\Upsilon(\omega)$ associated with system (4.5) where the delays $(1,2)$ are replaced by $(1,2+\frac{\pi}{100})$. The dashed line indicates $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$.



(a) (b)

Figure 9: The spectrum of (4.6) for $\delta_1 = 0.2$, $\delta_2 = 0$, $\Delta = 1/10.1$ and $\vec{\tau} = (1,2)$ (left) and $\vec{\tau} = (1, 2+\pi/100)$ (right)

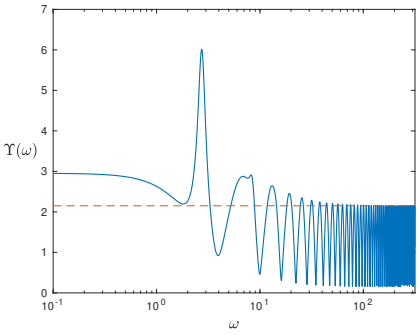


Figure 10: The worst-case gain function $\Upsilon(\omega)$ associated with system (4.7). The dashed line indicate $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$.

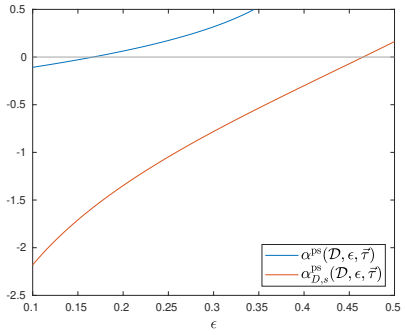


Figure 11: The functions $\epsilon \mapsto \alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$ (blue) and $\epsilon \mapsto \alpha^{\mathrm{ps}}_{D,s}(\mathcal{D},\epsilon,\vec{\tau})$ (red) for perturbed characteristic matrix (4.8).

Next we examine the corresponding characteristic matrix:

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} (-7+3\delta_1) & 4 & 0 \\ 0 & -1 & \Delta \\ (2-2\delta_2) & (1+\delta_1) & -1 \end{bmatrix} - \begin{bmatrix} (-5+2\delta_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} e^{-\lambda}. \quad (4.8)$$

It follows from Lemma 3.7 that $\mathrm{d}_{\mathrm{INS}}(\mathcal{D}) = \mathrm{d}_{\mathrm{FIN}}(\mathcal{D}) = \left(\max_{\omega\in\mathbb{R}^+} \Upsilon(\omega)\right)^{-1} \approx 0.1663$. This is illustrated in Figure 11 which shows $\alpha^{\mathrm{ps}}(\mathcal{D},\epsilon,\vec{\tau})$ and $\alpha^{\mathrm{ps}}_{D,s}(\mathcal{D},\epsilon,\vec{\tau})$ in function of ϵ. From this figure it is clear that $\mathrm{d}_{\mathrm{FIN}}(\mathcal{D})$ is indeed smaller than $\mathrm{d}_{\mathrm{CHAIN}}(\mathcal{D})$. To conclude this example, we return to the alternative definition of the robust distance to a finite root crossing in (2.11). Figure 12, shows the pseudo-spectrum of (4.6) for ϵ smaller than, equal to, and larger than the robust distance to instability. We observe that for $\epsilon = \mathrm{d}_{\mathrm{INS}}(\mathcal{D})$ the pseudo-spectrum touches the imaginary axis in approximately $\pm\jmath 2.734$.

5 Numerical algorithm for computing the robust strong $\mathcal{H}_\infty$-norm

This section introduces a high-level description of a numerical algorithm to compute the robust strong $\mathcal{H}_\infty$-norm of system (1.4). This algorithm exploits the relation of the robust strong $\mathcal{H}_\infty$-norm with the robust distance to instability of characteristic matrix (2.5).

In a preliminary step, we have to check if Assumption 1 holds, i.e. uncertain system (1.4) is internally exponentially stable for all admissible uncertainties, using the method presented in Borgioli & Michiels (2020). Next we compute $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$

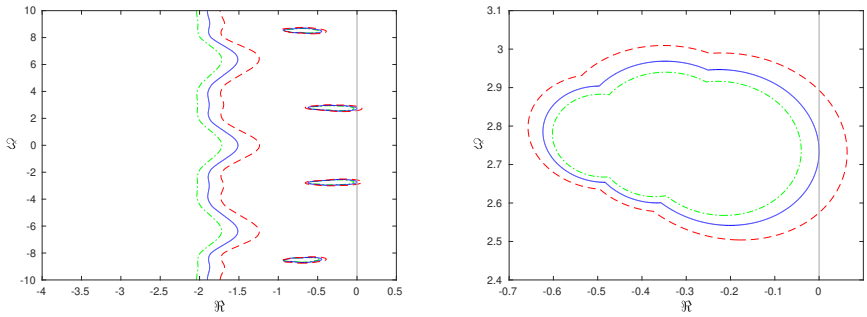


Figure 12: The pseudo-spectrum $\Lambda^{\text{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ of characteristic matrix (4.8) for ϵ equal to 1/7 (green dot dash line), $\text{d}_{\text{INS}}(\mathcal{D})$ (blue full line) and 1/5 (red dashed line).

by solving the following (constrained) optimization problem:

$$\begin{array}{ll} \underset{\delta,\vec{\theta}}{\text{maximise}} & \sigma_1\left(\tilde{D}_0(\delta) + \sum_{k=1}^{K} \tilde{D}_k(\delta) e^{\jmath\theta_k}\right) \\ \text{subject to} & \delta \in \mathcal{D} \\ & \vec{\theta} \in [0, 2\pi)^K. \end{array} \tag{5.1}$$

Appendix A.1 briefly explains how to solve this optimization problem using the projected gradient flow method. Subsequently, we compute $\text{d}_{\text{FIN}}(\mathcal{D})$ by finding the zero-crossing of the function

$$\left[0, \left(|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}\right)^{-1}\right) \ni \epsilon \mapsto \alpha^{\text{ps}}(\mathcal{D}, \epsilon, \vec{\tau}).$$

To find this zero-crossing we use the Newton-bisection method, which combines the robustness of the bisection method with the fast (local) convergence of the Newton method, see Press *et al.* (1996) for a reference implementation. This root finding method requires the evaluation of both $\alpha^{\text{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ and $\frac{d}{d\epsilon}\alpha^{\text{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ for given ϵ. The quantity $\alpha^{\text{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ is found by solving the following optimization problem:

$$\begin{array}{ll} \underset{\delta,\Delta}{\text{maximise}} & \Re\big(\lambda_{RM}(\delta, \Delta)\big) \\ \text{subject to} & \delta \in \mathcal{D} \\ & \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 \leq \epsilon, \end{array} \tag{5.2}$$

with $\lambda_{RM}(\delta, \Delta)$ the right-most characteristic root of $M(\lambda; \delta, \Delta, \vec{\tau})$. Appendix A.2 describes how this optimization problem can be solved using the projected gradient flow method and gives an expression for the derivative of $\alpha^{\text{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ with respect to ϵ.

If both $|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}$ and $\text{d}_{\text{FIN}}(\mathcal{D})$ are computed, the robust strong $\mathcal{H}_\infty$-norm follows from Theorem 3.1:

$$\begin{aligned} |||T(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty} &= \text{d}_{\text{INS}}(\mathcal{D})^{-1} = \min\left\{\text{d}_{\text{NWP}}(\mathcal{D}), \text{d}_{\text{CHAIN}}(\mathcal{D}), \text{d}_{\text{FIN}}(\mathcal{D})\right\}^{-1} \\ &= \max\left\{|||T_a(\cdot;\cdot,\vec{\tau})|||^{\mathcal{D}}_{\mathcal{H}_\infty}, \text{d}_{\text{FIN}}(\mathcal{D})^{-1}\right\}. \end{aligned}$$

Remark 5. The robust distances to non well-posedness ($\mathrm{d_{NWP}}$) and a characteristic root chain crossing ($\mathrm{d_{CHAIN}}$) can also be computed separately, although by themselves they are not necessary to find the robust strong (asymptotic) $\mathcal{H}_\infty$-norm. The following expressions for these distance measures for general Q, $\tilde{P}_0(\delta, \Delta)$ and $\tilde{P}_k(\delta)$ follow from Section 2. The robust distance to non well-posedness is equal to the smallest ϵ for which the function

$$\mathbb{R}^+ \ni \epsilon \mapsto \min_{\delta \in \mathcal{D}} \min_{\substack{\Delta \in \mathbb{C}^{m \times p} \\ \|\Delta\|_2 = \epsilon}} \left\{ \sigma_{\min} \left(U^H \tilde{P}_0(\delta, \Delta) V \right) \right\},$$

equals zero, with $\sigma_{\min}(\cdot)$ the smallest singular value of its matrix argument and recall that U and V denote $n_M \times (n_M - \mathrm{rank}(Q))$-dimensional matrices whose columns form a basis for the left and right nullspace of Q, respectively. The robust distance to a characteristic root chain crossing is equal to the zero-crossing of the function

$$[0, \mathrm{d_{NWP}}) \ni \epsilon \mapsto \max_{\delta \in \mathcal{D}} \max_{\substack{\Delta \in \mathbb{C}^{m \times p} \\ \|\Delta\|_2 \leq \epsilon}} \max_{\vec{\theta} \in [0, 2\pi)^K} \rho \left(\sum_{k=1}^{K} \left(U^H \tilde{P}_0(\delta, \Delta) V \right)^{-1} U^H \tilde{P}_k(\delta) V e^{\jmath \theta_k} \right) - 1. \quad (5.3)$$

$\diamond$

6 Conclusion

In Section 3 we examined the relation between the robust (strong) $\mathcal{H}_\infty$-norm of a time-delay system with structured uncertainties on the system matrices and the robust distance to instability of a perturbed characteristic matrix of form (2.5). Based on this relation, Section 5 introduced a novel numerical algorithm to compute this robust strong $\mathcal{H}_\infty$-norm.

The presented theory and algorithm can however be generalized to systems with uncertainties on the delays. Because the robust strong asymptotic $\mathcal{H}_\infty$-norm, the robust distance to non well-posedness and the robust distance to a characteristic root chain crossing are independent of the delay values, the proofs in Section 3.2 can be reused without modification. The results from Section 3.3 trivially generalize to the uncertain delay case by extending the definitions of the worst-case gain function and the robust distance to a finite root crossing to include the uncertainties on the delays. Also the algorithm presented in Section 5 only needs a minor modification: one has to include the uncertainties on the delays in the computation of $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$, in a similar manner as done in Borgioli & Michiels (2020).

Finally, a reference implementation of the algorithm described in Section 5 is available from `http://twr.cs.kuleuven.be/research/software/delay-control/rb_hinf/`. The presented algorithm has also been validated on some test problems: Examples 3, 4 and 5 in Section 3.4 and the open loop systems of the benchmark

problems described in Gumussoy & Michiels (2011, Section 7.3)[4] to which real-valued, structured uncertainties were added. These benchmark problems are available from the same location.

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A Projected gradient flow method

The projected gradient flow method is a continuous variant of the well-known steepest ascend/descend method for solving constrained optimization problems. It looks for a flow in the space of admissible variables along which the objective function monotonically increases/decreases. The flow is defined in such a way that the (local) optima of the objective function appear as attractive stationary points. These optimizers are found by choosing an initial point and discretizing the resulting path till convergence to a stationary point using Euler's forward method.

There already exists an extensive literature (Borgioli & Michiels 2020, Guglielmi & Lubich 2013, Guglielmi *et al.* 2014) on how to use the projected gradient flow method for computing extremal points of pseudo-spectra. We will therefore restrict ourselves to the resulting flows for the optimization problems encountered in Section 5. For more details we refer to the aforementioned papers.

A.1 Robust strong asymptotic $\mathcal{H}_\infty$-norm

This subsection briefly describes how to use the projected gradient flow method for optimization problem (5.1). In this case we are looking for a path $[0, +\infty) \ni \mathsf{t} \mapsto \big(\delta(\mathsf{t}), \theta(\mathsf{t})\big) \in \mathcal{D} \times [0, 2\pi)^K$ such that the function

$$\mathsf{t} \mapsto \sigma_1 \left(\tilde{D}_0(\delta(\mathsf{t})) + \sum_{k=1}^{K} \tilde{D}_k(\delta(\mathsf{t})) e^{\jmath \theta_k(\mathsf{t})} \right)$$

is monotonically increasing and that the (local) optimizers of (5.1) appear as attractive stationary points. It can be verified that the following path fulfills these requirements:

$$\begin{cases} \delta_l(\mathtt{t}) = \bar{\delta}_l \delta_l^n(\mathtt{t}) & l = 1, \ldots, L \\ \theta_k(\mathtt{t}) = \vartheta_k(\mathtt{t}) \bmod 2\pi & k = 1, \ldots K \end{cases}$$

with mod the modulo operator[5], which assures that $\theta_k(\mathtt{t})$ remains bounded to the half-open interval $[0, 2\pi)$ for all $\mathtt{t}$, and

$$\begin{cases} \dot{\vartheta}_k(\mathtt{t}) = -\Im\left(u(\mathtt{t})^H \tilde{D}_k(\delta(\mathtt{t}))v(\mathtt{t})e^{\jmath\theta_k(\mathtt{t})}\right) \\ \Xi_l(\mathtt{t}) = \bar{\delta}_l \sum_{k=0}^{K} \sum_{s=1}^{S_l^{D_k}} {G_{l,s}^{D_k}}^T \Re\left(u(\mathtt{t})v(\mathtt{t})^H e^{-\jmath\theta_k(\mathtt{t})}\right) {H_{l,s}^{D_k}}^T \\ \dot{\delta}_l^n(\mathtt{t}) = \begin{cases} \Xi_l(\mathtt{t}) - \left\langle \delta_l^n(\mathtt{t}), \Xi_l(\mathtt{t}) \right\rangle_F \delta_l^n(\mathtt{t}) & \text{if } \|\delta_l^n(\mathtt{t})\|_F = 1 \text{ and } \left\langle \delta_l^n(\mathtt{t}), \Xi_l(\mathtt{t}) \right\rangle_F > 0 \\ \Xi_l(\mathtt{t}) & \text{otherwise} \end{cases} \end{cases} \tag{A.1}$$

where $u(\mathtt{t})$ and $v(\mathtt{t})$ are the left and right singular vectors of unit norm associated with the largest singular value of $\tilde{D}_0(\delta(\mathtt{t})) + \sum_{k=1}^{K} \tilde{D}_k(\delta(\mathtt{t}))e^{\jmath\theta_k(\mathtt{t})}$ and $\langle A, B \rangle_F = \sum_{i,j} A_{i,j} B_{i,j}$, the Frobenius inner product of two matrices A and B. The right-hand side of (A.1) can be seen as the projection of the gradient of $\sigma_1\left(\tilde{D}_0(\delta) + \sum_{k=1}^{K} \tilde{D}_k(\delta)e^{\jmath\theta_k}\right)$ with respect to the elements of δ_l and θ_k onto the search space. The projection ensures that the constraints are fulfilled for all $\mathtt{t}$.

A.2 Pseudo-spectral abscissa

The pseudo-spectral abscissa, $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$, can be computed by solving (5.2) in a similar manner as done in Appeltans & Michiels (2019, Section 4.1). Note however that in the case considered in this paper the maximum of (5.2) might not be attained, as for $\Delta \neq 0$ the characteristic roots of $M(\lambda; \delta, \Delta, \vec{\tau})$ may have similar properties as those of a neutral delay eigenvalue problem, see Michiels & Niculescu (2014, Chapters 1.2 and 1.3). To guarantee that the maximum of (5.2) is defined, we add an additional constraint to the optimization problem:

$$\lambda_{RM}(\delta, \Delta) \in \{\lambda \in \mathbb{C} : \Im(\lambda) \in [-\beta, \beta]\}$$

with β sufficiently large. This may lead to an underestimate for $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ for a given ϵ. However for $\epsilon \in [0, \min\{\mathrm{d_{NWP}}(\mathcal{D}), \mathrm{d_{CHAIN}}(\mathcal{D})\})$, the transition to $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau}) \geq 0$ is caused by a (finite) characteristic root crossing the imaginary axis, and thus if β is sufficiently large this additional constraint does not influence the result of the overall root finding procedure to compute $\mathrm{d_{FIN}}(\mathcal{D})$.

Finally, the derivative of $\alpha^{\mathrm{ps}}(\mathcal{D}, \epsilon, \vec{\tau})$ with respect to ϵ follows from Appeltans & Michiels (2019, Equation 10).

[5]The modulo operator is defined as: $a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b$ in which the function $\lfloor \cdot \rfloor$ outputs the greatest integer smaller than or equal to its argument.