Derivation of Process Networks and the use of Bernstein Expansion to compute FIFO Sizes

Sven Verdoolaege

Katholieke Universiteit Leuven, Belgium

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Outline

1. Introduction and Motivation
2. Derivation of Process Networks
   - Polytope Model
   - Dataflow Analysis
   - Parametric Counting
   - Parametric Maximization
3. Bernstein Expansion
   - Interval
   - Polytopes
   - Simplification
   - Incremental Computation
4. Conclusion

Motivation

Goal: map sequential programs to parallel hardware (MPSoC)

```
 1 for( i=0; i<2*Nrw; i++ )
     for( j=0; j<2*Ncl; j++ )
       a[i][j] = ReadImage();

   for( i=0; i<Nrw; i++ ) {
 6   // 1D DWT in vertical direction with subsampling
     for( j=0; j<2*Ncl; j++ ) {
       tmpLine = (i==Nrw-1) ? a[2*i][j] : a[2*i+2][j];
       Hf[j] = high_flt_vert( a[2*i][j], a[2*i+1][j], tmpLine );

11     tmp = (i==0) ? Hf[j] : oldHf[j];
       low_flt_vert( tmp,a[2*i][j],Hf[j],&oldHf[j],&Lf[j] );
     }

     // 1D DWT in horizontal direction with subsampling ---------
16   for( j=0; j<Ncl; j++ ) {
       tmp = (j==Ncl-1) ? Lf[2*j] : Lf[2*j+2];
       HL[i][j] = high_flt_hor( Lf[2*j], Lf[2*j+1], tmp );

       tmp = (j==0) ? HL[i][j] : HL[i][j-1];
21     LL[i][j] = low_flt_hor( tmp, Lf[2*j],  HL[i][j] );
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31   }
   }
   // The Outputs  ------------------------------------------------
   for( i=0; i<Nrw; i++)
     for( j=0; j<Ncl; j++) {
36     Sink( LL[i][j] );
       Sink( HL[i][j] );
       Sink( LH[i][j] );
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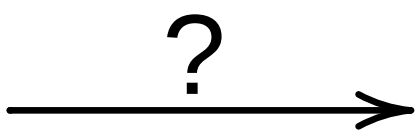
     }
```





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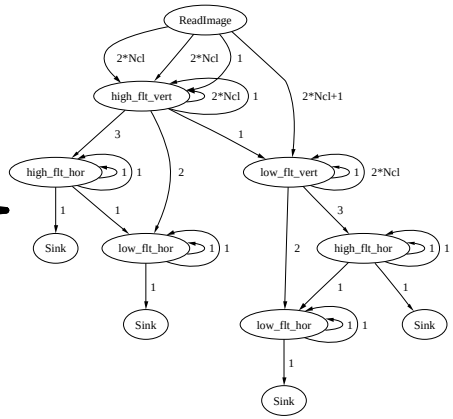

process network

Kahn Process Networks

- Parallel specification
  $\Rightarrow$ easier to map onto parallel hardware
- Consists of
  - independent processes
  - communication channels
    $\Rightarrow$ unbounded FIFOs / memories
  - synchronization: blocking read + blocking write

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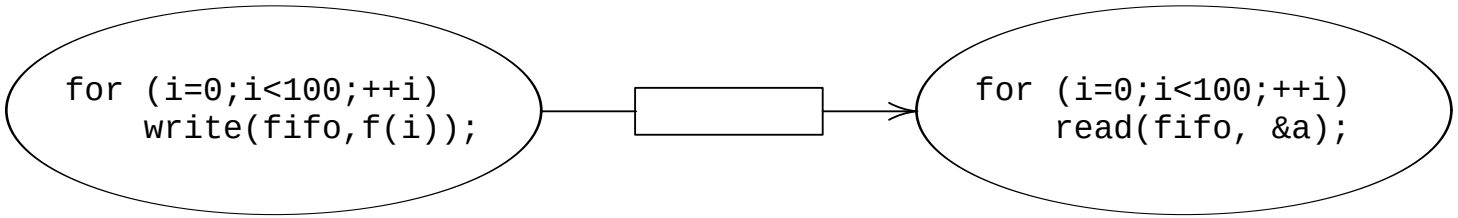

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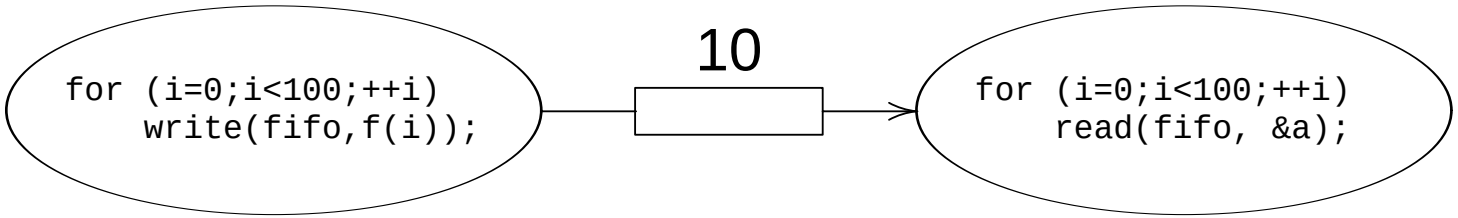

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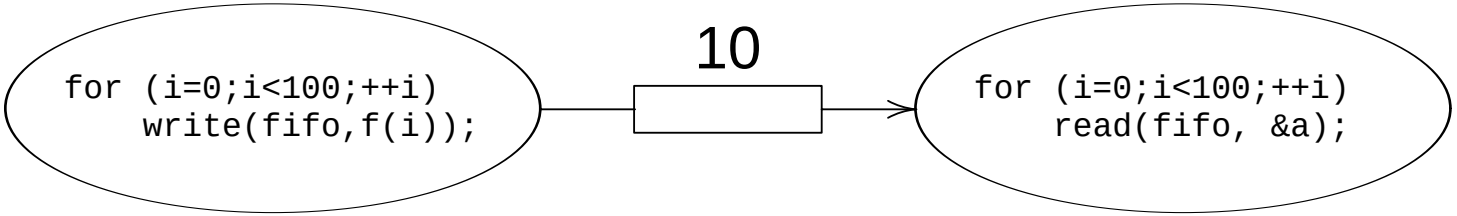

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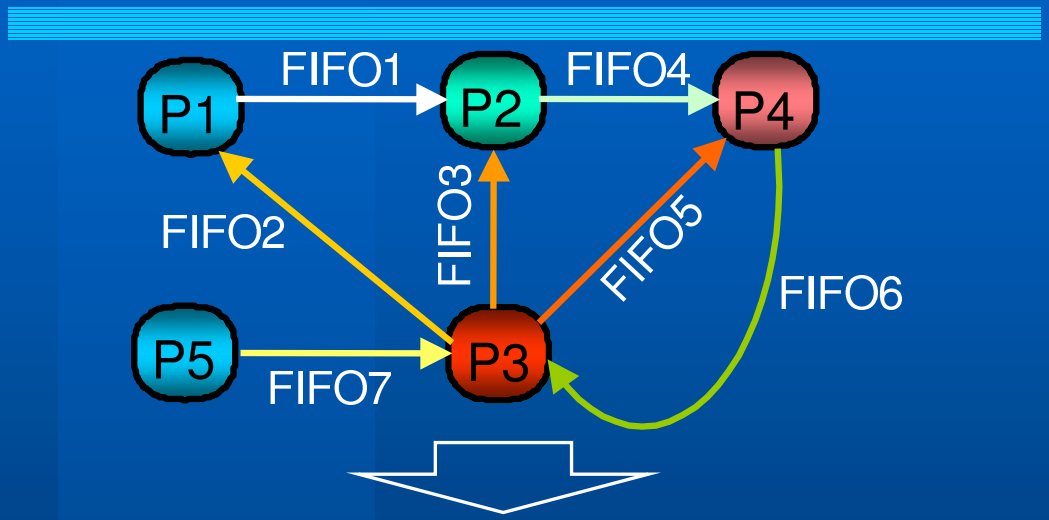
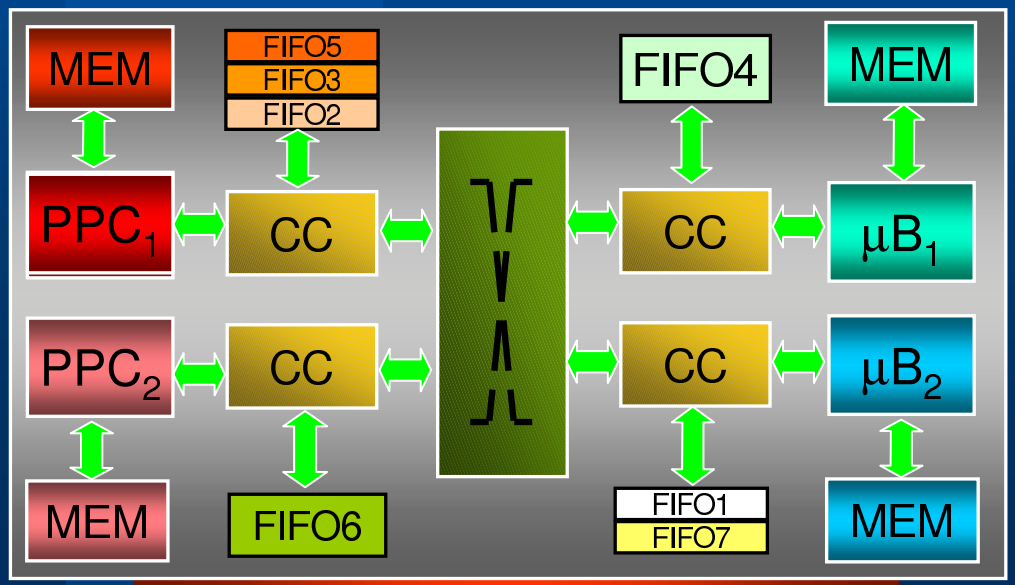
MPSoCs Programming and Mapping





- Programming model: KPN
- Example of a MPSoC platform:
  - 2 PowerPC processors
  - 2 MicroBlaze processors
  - 4 Communication memories (CM)
  - 4 Communication controllers (CC)
  - Crossbar component
- Mapping:
  - The processes are mapped onto the processors using standard GCC compiler.
  - The FIFOs are mapped onto the CMs depending on the process mapping.
  - CCs take care about the inter processor communication and synchronization

Derivation of Process Networks

Problem: what are the channels and how big are they ?

- dataflow analysis

  Given a read from an array element, what was the last write to that array element?

- counting

  Given a communication channel, how many tokens are in the channel at any given time?

- maximization (computing an upper bound)

  Given a communication channel, what is the maximal number of tokens in the channel during the execution?

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Polytope Model

```
#define N 5
for (i = 1; i <= N; ++i)
    for (j = 1; j <= i; ++j)
        a[i][j]=
```

Assumptions on sequential code:

- iterators are integers
- loops with affine bounds
- affine conditions
- affine index expressions

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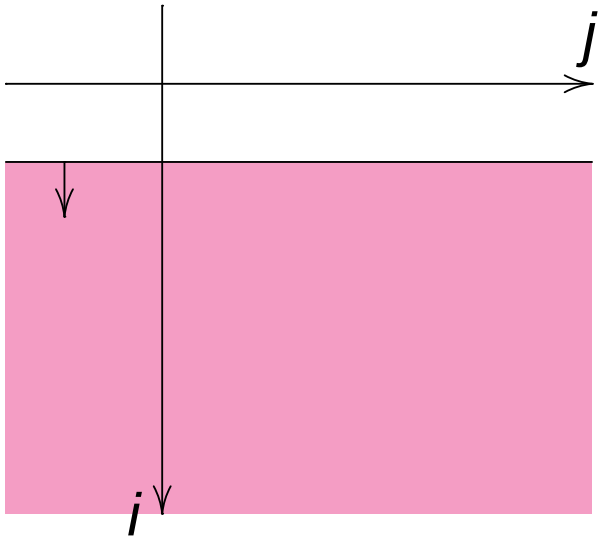

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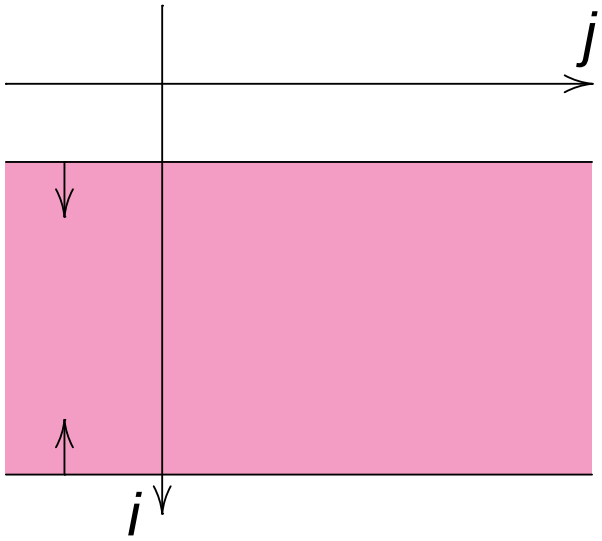

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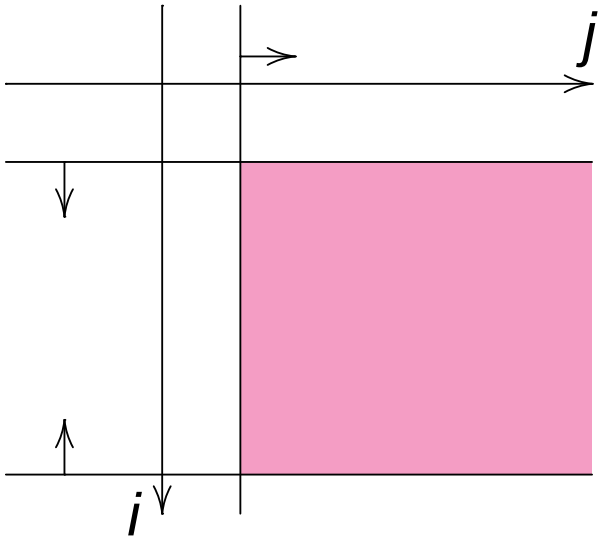

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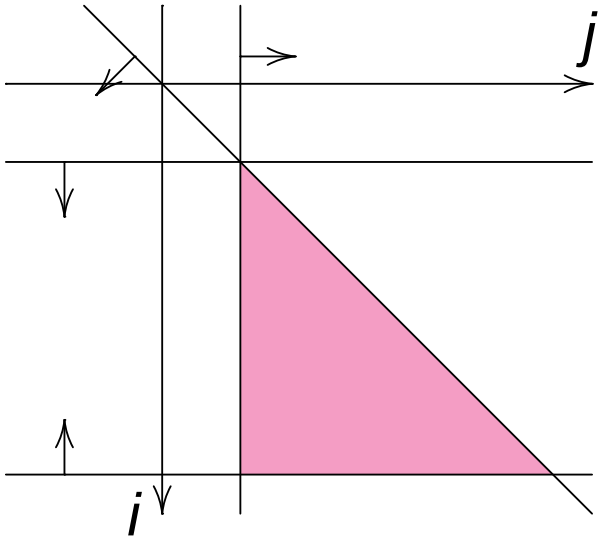

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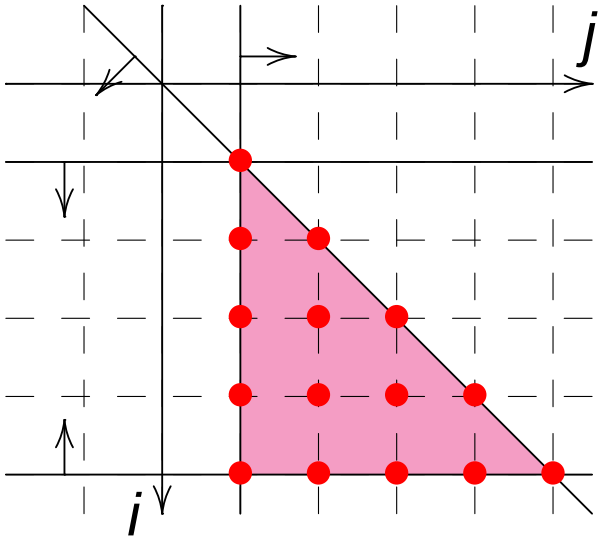

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$\mathcal{H}$-Parametric Polytopes

```
for (i = 1; i <= N; ++i)
    for (j = 1; j <= i; ++j)
        a[i][j]=
```

$$P(N) = \{ (i,j) \mid i \geq 1 \wedge i \leq N \wedge j \geq 1 \wedge j \leq i \}$$

$\Rightarrow$ a set of polytopes, one for each value of N

parametric vertices:

$$P = \text{conv}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} N \\ 1 \end{bmatrix}, \begin{bmatrix} N \\ N \end{bmatrix} \right\}$$

In general: different (active) vertices on different parts of the parameter space (chamber decomposition)

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Chamber Decomposition

$$P(\mathbf{s}) = \left\{ \mathbf{t} \in \mathbb{Q}^2 \mid -s_1 + 2s_2 + t_1 - 2t_2 \geq 0 \wedge s_1 - s_2 - t_1 + t_2 \geq 0 \wedge t_1 \geq 0 \wedge t_2 \geq 0 \right\}$$

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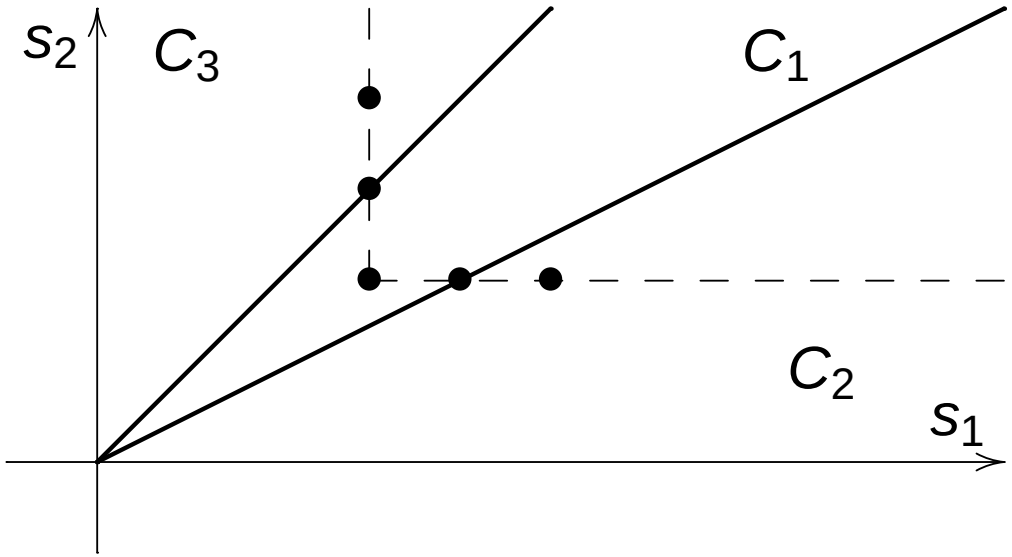

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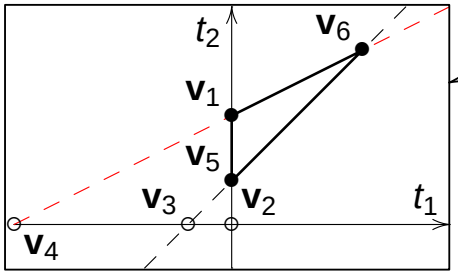

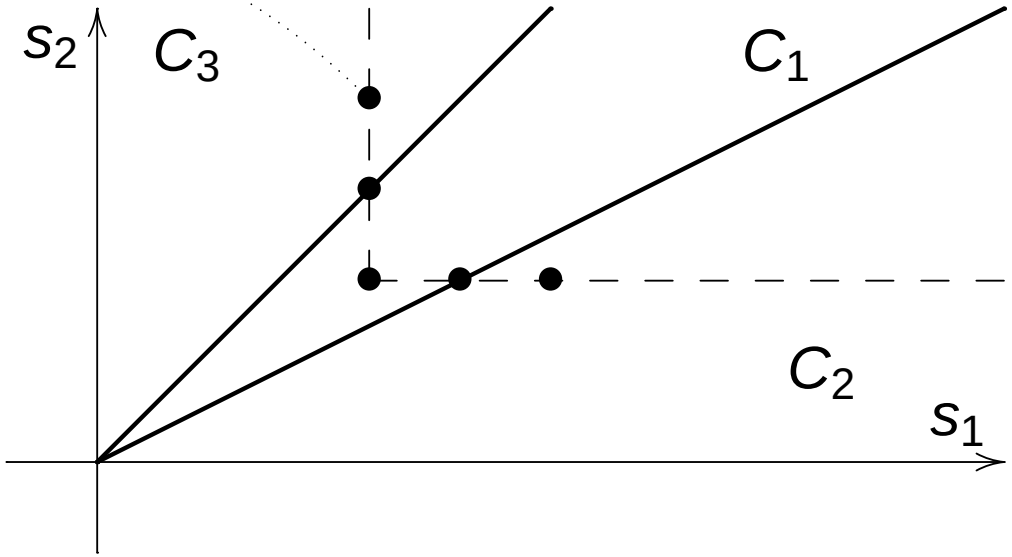

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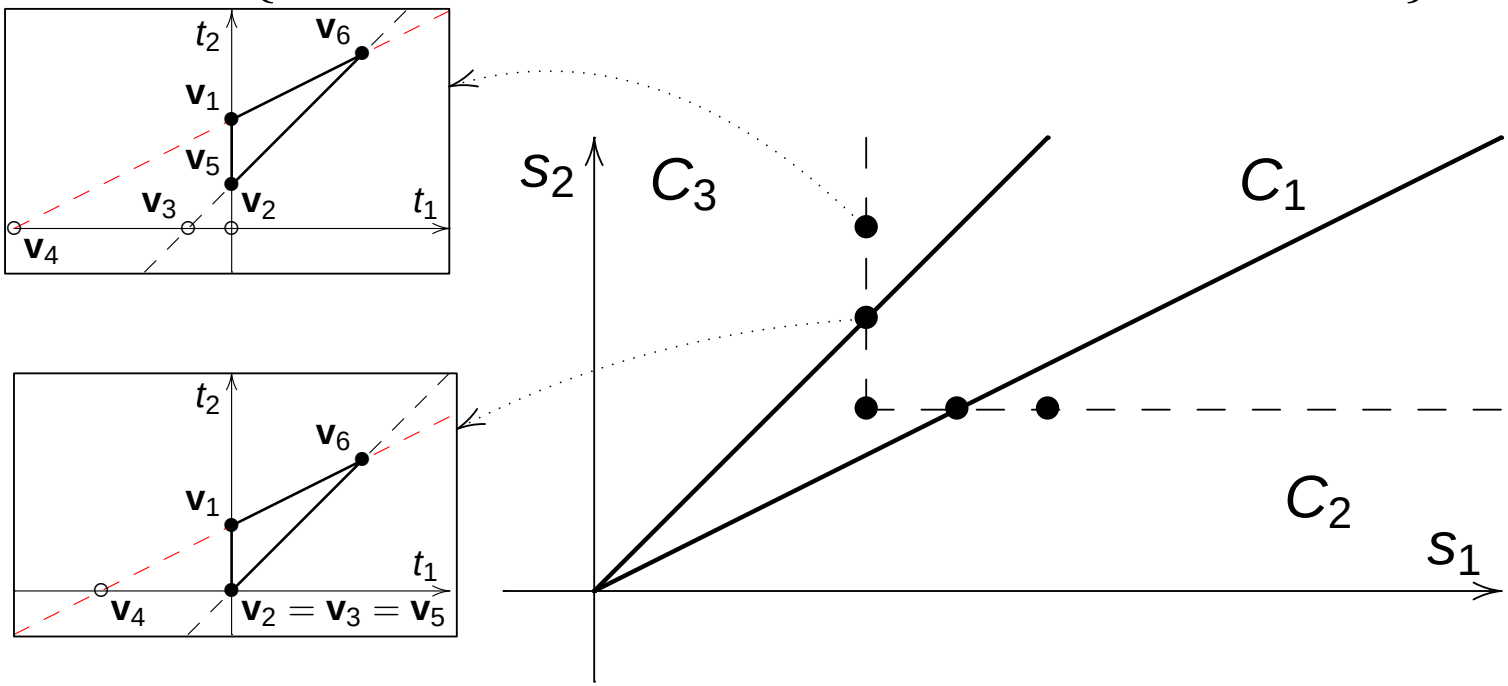

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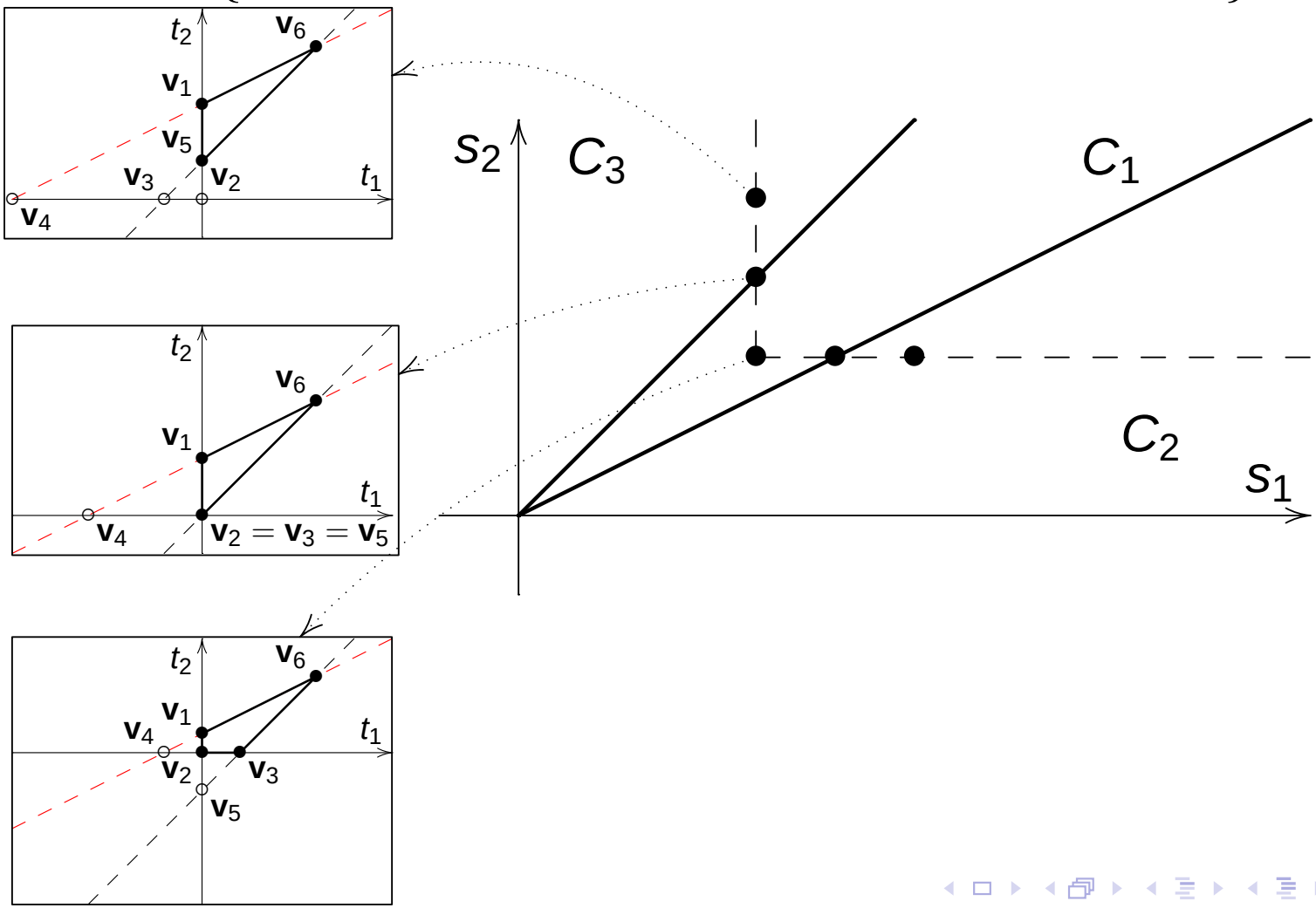

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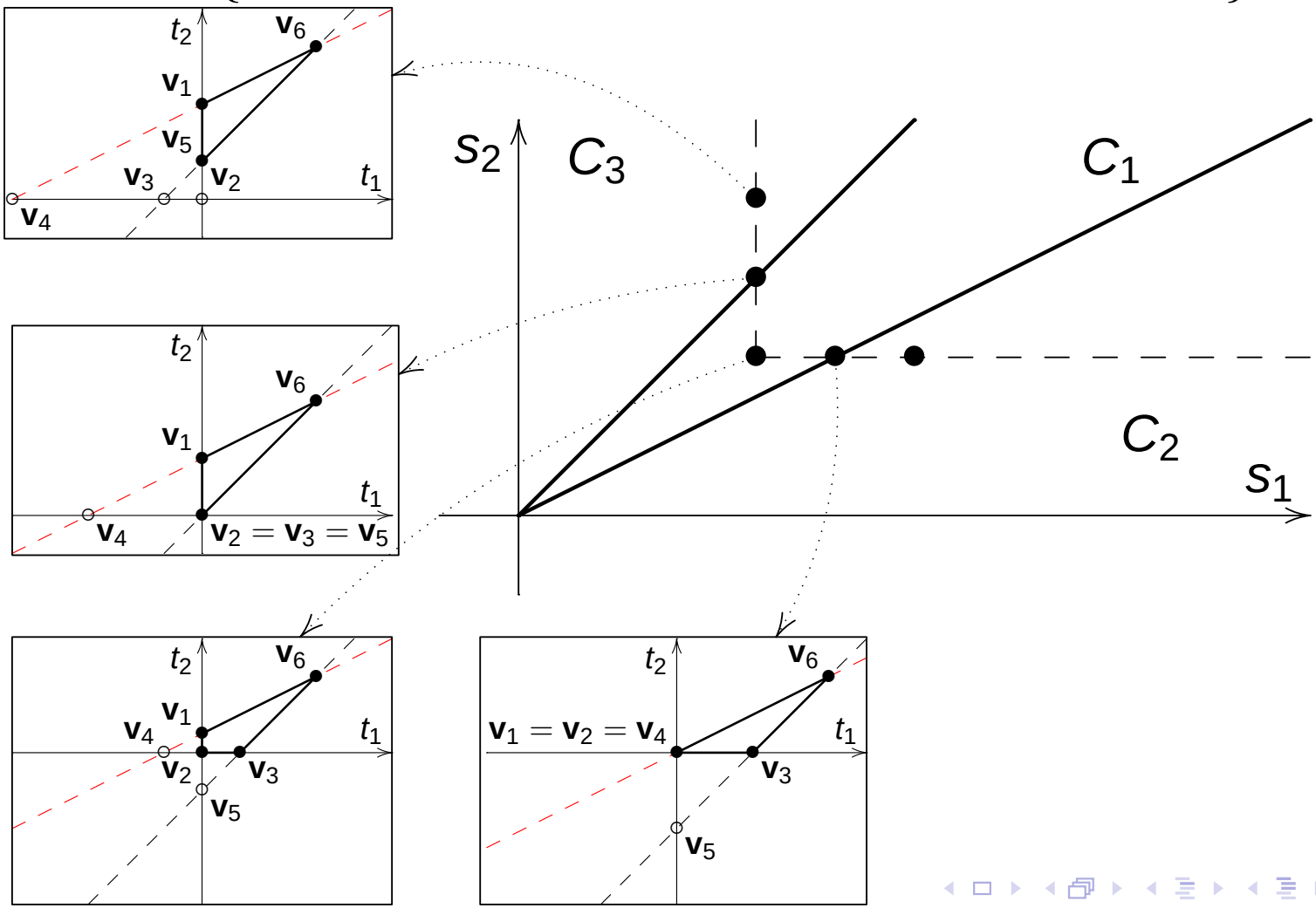

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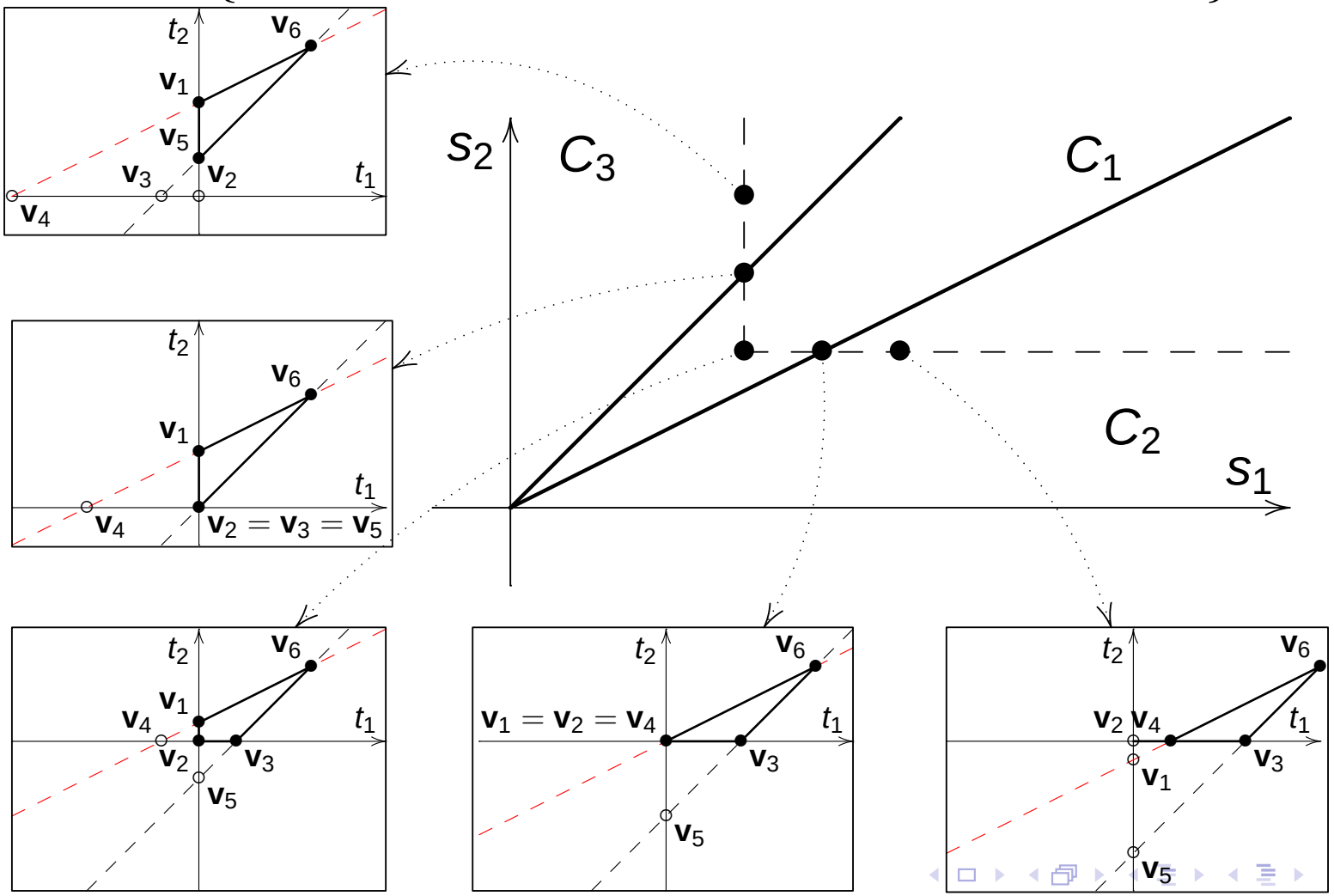

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Example

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int N, M;
#pragma parameter N 10 100
#pragma parameter M 10 100

int main(void)
{
    int i, j, k;
    int a[N][N];

    for (i = 0; i < N; ++i)
        for (j = 0; j < N; ++j)
            a[i][j] = Read();
    for (j = 0; j < N; ++j) {
        for (k = 0; k < N; ++k)
            if (M < N)
                Switch(a[j][k], a[M][k], &a[j][k], &a[M][k]);
        for (i = j+1; i < N; ++i)
           a[i][j] = Imp(a[i][j], a[j][j]);
        for (k = j+1; k < N; ++k)
            for (i = j+1; i < N; ++i)
                 a[i][k] = MulSubs(a[i][k],a[i][j],a[j][k]);
    }
    for (i = 0; i < N; ++i)
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Step 1: last write through given write access

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Context: $10 \leq M \leq 100 \wedge 10 \leq N \leq 100$

Iteration domain:

$$S = \{(j,k,i) \mid 0 \leq j < N \wedge j+1 \leq k < N \wedge j+1 \leq i < N\}$$

$$\text{lexmax}\{(j_w,k_w,i_w) \in S \mid (j_w,k_w,i_w) \prec (j_r,k_r,i_r) \wedge (j_r,k_r,i_r) \in S \wedge (k_w,i_w) = (k_r,i_r)\}$$

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Lexicographic maximum of parametric polytope

Variables: write iterators

Parameters: read iterators and structural parameters

Dataflow Analysis

Step 1: last write through given write access

Iteration domain:

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Example

```
int N, M;
#pragma parameter N 10 100
#pragma parameter M 10 100

int main(void)
{
    int i, j, k;
    int a[N][N];

    for (i = 0; i < N; ++i)
        for (j = 0; j < N; ++j)
            a[i][j] = Read();
    for (j = 0; j < N; ++j) {
        for (k = 0; k < N; ++k)
            if (M < N)
                Switch(a[j][k], a[M][k], &a[j][k], &a[M][k]);
        for (i = j+1; i < N; ++i)
           a[i][j] = Imp(a[i][j], a[j][j]);
        for (k = j+1; k < N; ++k)
            for (i = j+1; i < N; ++i)
                 a[i][k] = MulSubs(a[i][k],a[i][j],a[j][k]);
    }
    for (i = 0; i < N; ++i)
        for (j = 0; j < N; ++j)
            a[i][j] = Pass(a[i][j]);
}
```

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    int i, j, k;
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            a[i][j] = Read();
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            if (M < N)
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    }
    for (i = 0; i < N; ++i)
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```

Dataflow Analysis

Step 1: last write through given write access
Step 2: combination of results over all write accesses

```
for (j = 0; j < N; ++j)
  for (k = j+1; k < N; ++k)
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```

Final result:

$$(j_w, k_w, i_w) = \begin{cases} (j_r - 1, k_r, i_r) & \text{if } N \leq M \\ (j_r - 1, k_r, i_r) & \text{if } M < N \wedge M < i_r \\ (j_r - 1, k_r, i_r) & \text{if } i_r < M < N \end{cases}$$

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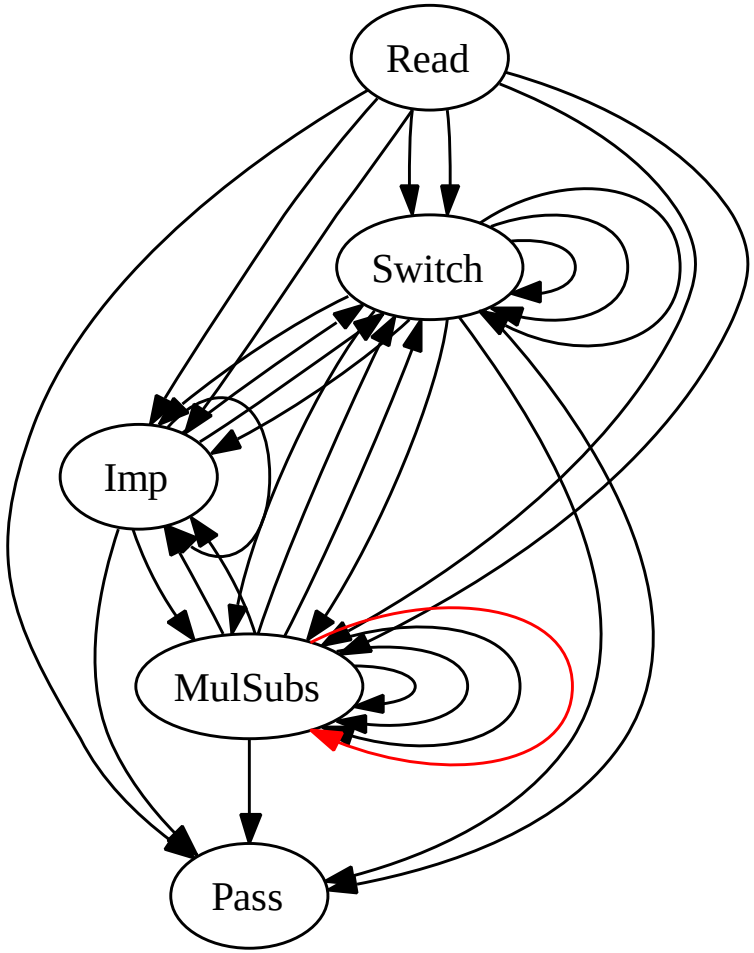
Dataflow Analysis



Derivation of Process Networks

Problem: what are the channels and how big are they ?

- dataflow analysis

  Given a read from an array element, what was the last write to that array element?

- counting

  Given a communication channel, how many tokens are in the channel at any given time?

- maximization (computing an upper bound)

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Parametric Counting

Given a communication channel, how many tokens are in the channel at any given time?

Iteration domain:

$$S = \{(j,k,i) \mid 0 \leq j < N \wedge j+1 \leq k < N \wedge j+1 \leq i < N\}$$

Dependence relation (partial result from dataflow analysis):

$$R = R_1 \cup R_2 \cup R_3$$

$$R_2 = \{((j_r - 1, k_r, i_r), (j_r, k_r, i_r)) \in S \times S \mid M < N \wedge M < i_r\}$$

Assume communication channel is a FIFO:
Number of elements in FIFO at (j,k,i) is number of writes before (j,k,i) minus number of reads before (j,k,i)

$$\#\{(j_w, k_w, i_w) \in \operatorname{dom} R \mid (j_w, k_w, i_w) \prec (j,k,i)\}$$
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Parametric Counting: Order Predicate

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Ordering of iterations

- fixed within the same process
- unknown at compile-time between different processes
  $\Rightarrow$ combine all processes into a single process
  $\Rightarrow$ perform counting in the combined process

Process Merging

Given: iteration domains of dimension $d_i \leq D$

1. Place in a common iteration space of dimension D
   (no change of order within an iteration domain)
   Heuristic: identify dimensions related through dependence relation
2. Add constant offset to all iteration domains
   (without violating any dependence)
   Heuristic: put dependent iterations “close” together

Process Merging: Common Iteration Space

```
for (i = 0; i <= 5; ++i)
    a[i] = ...
for (i = 0; i <= 5; ++i)
    for (j = 0; j <= 5; ++j)
        b[i][j] = a[i];
```

$$R = \{(i, i', j') \mid 0 \leq i \leq 5 \wedge 0 \leq i' \leq 5 \wedge 0 \leq j' \leq 5 \wedge i = i'\}$$

$\Rightarrow$ identify i and i'

$\Rightarrow$, not

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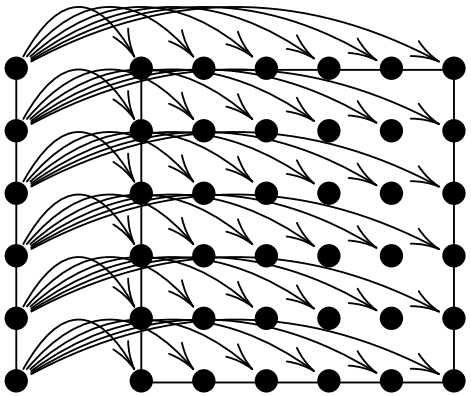
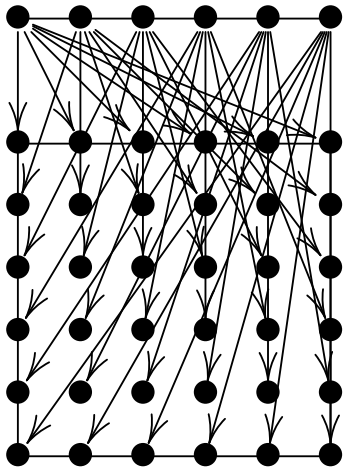
$\Rightarrow$  , not 

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Process Merging: Example

Original QR code and code equivalent to merged QR

```
for (j = 0; j < N; ++j)
  for (i = j; i < N; ++i)
    R[j][i] = Zero();
for (k = 0; k < K; ++k)
  for (j = 0; j < N; ++j)
    X[k][j] = ReadMatrix();
for (k = 0; k < K; ++k)
  for (j = 0; j < N; ++j) {
    Vectorize(R[j][j], X[k][j],
              &R[j][j], &X[k][j], &t);
  for (i = j+1; i < N; ++i)
    Rotate(R[j][i], X[k][i], t,
           &R[j][i], &X[k][i], &t);
  }
for (j = 0; j < N; ++j)
  for (i = j; i < N; ++i)
    WriteMatrix(R[j][i]);
```

```
for (a = 0; a <= K-1 a++)
  for (b = 0; b <= N-1; b++)
    for (c = b; c <= 2*N-2; c++) {
      if (a == 0 && c <= N-1)
        R[b][c] = Zero();
      if (b == 0 && N-1 <= c)
        X[a][c-N+1] = ReadMatrix();
      if (c == N-1)
        Vectorize(R[b][b], X[a][b],
                  &R[b][b], &X[a][b], &t);
      if (b <= c-N)
        Rotate(R[b][c-N+1],X[a][c-N+1],t,
               &R[b][c-N+1],&X[a][c-N+1],&t);
      if (a == K-1 & b <= c-N+1)
        WriteMatrix(R[b][c-N+1]);
    }
```

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Number of integer points in parametric polytope

Variables: write/read iterators

Parameters: “inspection” iterators and structural parameters

$$\begin{cases} i - Nk - Ni + N^2 & \text{if } i \leq k \leq i+1 \wedge i \geq M+2 \\ -k + i^2 - 2Ni + 2i + N^2 - N + 1 & \text{if } k \geq i+2 \wedge i \geq M+2 \end{cases}$$

$$(\text{on } \operatorname{ran} R_2 \cap \{(j,k,i) \mid i = j+1\})$$

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Maximal number of tokens:

$$\max\Big(\max_{(j,k,i)\in T_1} i - Nk - Ni + N^2, \max_{(j,k,i)\in T_2} -k + i^2 - 2Ni + 2i + N^2 - N + 1\Big)$$

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Upper bound on polynomial over parametric polytope

Variables: “inspection” iterators
Parameters: structural parameters

We would like maximum over integer points
Bernstein expansion gives upper bound over rational points

Maximization

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Bernstein expansion gives upper bound over rational points

Maximization

$$\max\left(\max_{(j,k,i)\in T_1} i-Nk-Ni+N^2,\ \max_{(j,k,i)\in T_2} -k+i^2-2Ni+2i+N^2-N+1\right)$$

$$T_1 = \{(j,k,i) \mid i = j+1 \wedge i \leq k \leq i+1 \wedge i \geq M+2 \wedge k \leq N-1\}$$
$$T_2 = \{(j,k,i) \mid i = j+1 \wedge k \geq i+2 \wedge i \geq M+2 \wedge k \leq N-1\}$$

Upper bound on polynomial over parametric polytope

Variables: “inspection” iterators
Parameters: structural parameters

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Derivation of Process Networks

Problem: what are the channels and how big are they ?

- dataflow analysis
  Lexicographic maximum of parametric polytope (`PIP`)
  $\Rightarrow$ piecewise quasi-affine map
- counting
  Number of elements in parametric polytope (`barvinok`)
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  Given a communication channel, what is the maximal number of tokens in the channel during the execution?

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  Upper bound of parametric polynomial over parametric polytope (`bernstein`)

Bernstein Expansion over an Interval

Write $p(x) \in \mathbb{Q}[x]$ in terms of Bernstein base polynomials

$$p(x) = \sum_{i=0}^{d} a_i x^i = \sum_{i=0}^{d} a_i x^i (x + (1-x))^{d-i} = \sum_{k=0}^{d} b_k^d B_k^d(x)$$

with

$$B_k^d(x) = \binom{d}{k} x^k (1-x)^{d-k}$$

Bernstein coefficients

$$b_k^d = \sum_{i=0}^{k} \frac{\binom{k}{i}}{\binom{d}{i}} a_i$$

Example:

$$p(x) = x^3 - 5x^2 + 2x + 4 = 4B_0^3(x) + \frac{14}{3} B_1^3(x) + \frac{11}{3} B_2^3(x) + 2B_3^3(x)$$

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Properties of Bernstein base polynomials:

- $1 = (x + (1-x))^d = \sum_{k=0}^{d} B_k^d(x)$
- on interval $[0,1]$: $B_k^d(x) \geq 0$

$\Rightarrow p(x)$ is convex combination of b_k^d on interval $[0,1]$

$$\min_{0 \leq i \leq d} b_i^d \leq p(x) \leq \max_{0 \leq i \leq d} b_i^d \qquad \text{for } x \in [0,1]$$

bound is sharp if min or max of b_i^d is b_0^d or b_d^d

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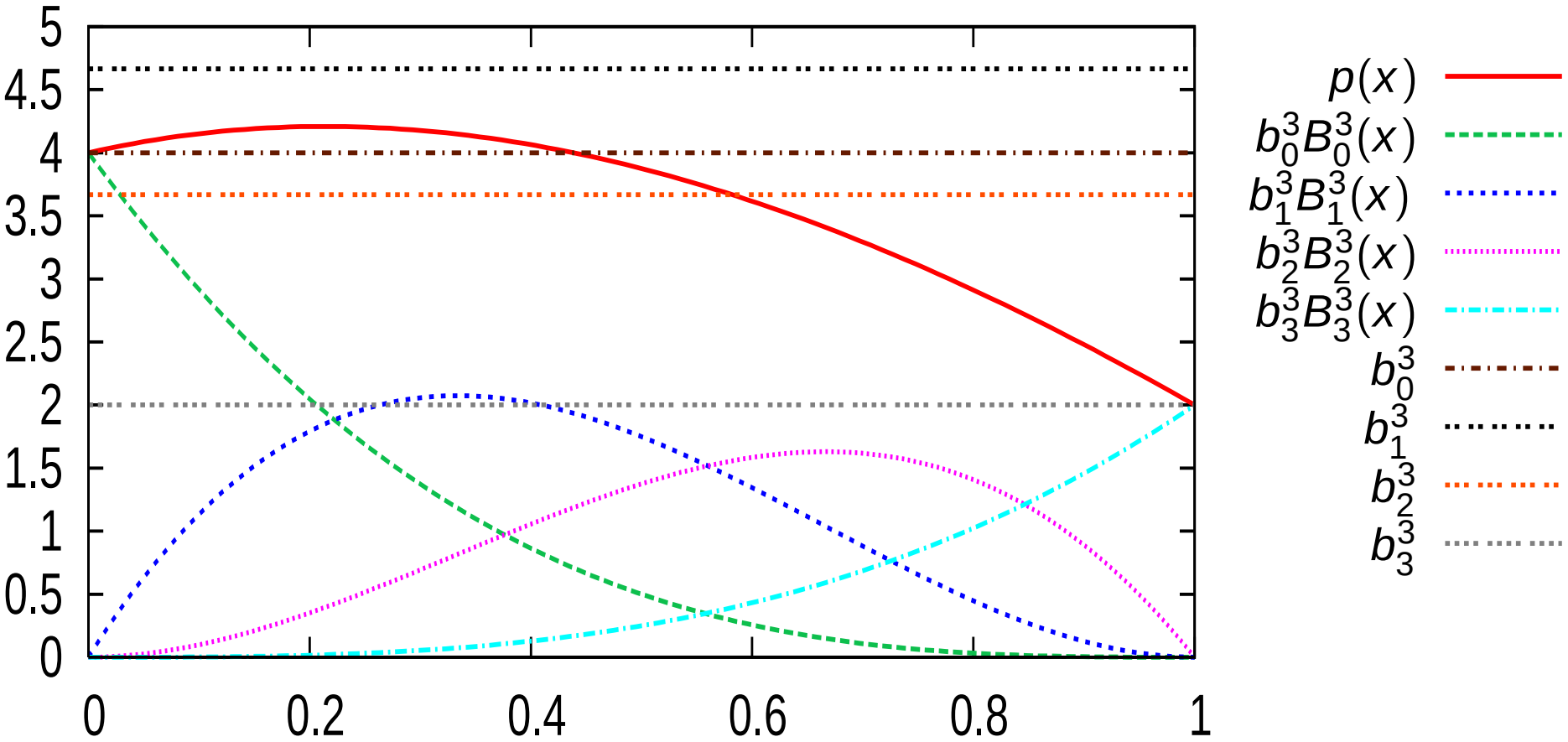

Bernstein Expansion over an Interval: Extensions

- Parametric boxes (Clauss and Tchoupaeva 2004)
  Product Bernstein base polynomials over unit cube $[0,1]^n$
  Linear transformation of parametric boxes
  $[l_1(\mathbf{q}), u_1(\mathbf{q})] \times [l_2(x_1, \mathbf{q}), u_2(x_1, \mathbf{q})] \times \cdots \times$
  $[l_n(x_1, \ldots, x_{n-1}, \mathbf{q}), u_n(x_1, \ldots, x_{n-1}, \mathbf{q})]$

$$y_i = \frac{x_i - a_i(x_1, \ldots, x_{i-1}, \mathbf{q})}{b_i(x_1, \ldots, x_{i-1}, \mathbf{q}) - a_i(x_1, \ldots, x_{i-1}, \mathbf{q})}$$

  - – only works on boxes
  - \+ bounds can be arbitrary polynomials
  - – requires detection and handling of roots of $b_i - a_i$
- General parametric polytopes
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Write

$$x = \alpha_0\, 0 + \alpha_1\, 1$$

with $\alpha_i \geq 0$ and $\alpha_0 + \alpha_1 = 1$
$\Rightarrow (\alpha_0, \alpha_1)$ barycentric coordinates on $[0, 1]$

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$\mathcal{V}$-parametric polytopes

$P : D \rightarrow \mathbb{Q}^n$:

$$\mathbf{q} \mapsto P(\mathbf{q}) = \left\{ \mathbf{x} \mid \exists \alpha_i \in \mathbb{Q} : \mathbf{x} = \sum_i \alpha_i \mathbf{v}_i(\mathbf{q}), \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$

$D \subset \mathbb{Q}^r$: parameter domain
$\mathbf{v}_i(\mathbf{q}) \in \mathbb{Q}[\mathbf{q}]$ arbitrary polynomials in parameters
$\mathbf{v}_i(\mathbf{q})$ are generators of the polytope

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polytope

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$$p(\boldsymbol{\alpha}) = p(\textstyle\sum_i \alpha_i \mathbf{v}_i)$$

$$p(\boldsymbol{\alpha}) = \sum_{\substack{k_1,k_2,\ldots,k_n \geq 0 \\ k_1+k_2+\cdots+k_n=d}} b^d_{\mathbf{k}} B^d_{\mathbf{k}}(\boldsymbol{\alpha})$$

Bernstein base polynomials

$$B^d_{\mathbf{k}}(\boldsymbol{\alpha}) = \binom{d}{\mathbf{k}} \boldsymbol{\alpha}^{\mathbf{k}}$$

bounds

$$\min_{\substack{\mathbf{k} \geq \mathbf{0} \\ \sum k_i = d}} b^d_{\mathbf{k}} \leq p(\mathbf{x}) \leq \max_{\substack{\mathbf{k} \geq \mathbf{0} \\ \sum k_i = d}} b^d_{\mathbf{k}}$$

interval

$$\{ \alpha_0\, 0 + \alpha_1\, 1 \mid \alpha_i \geq 0, \textstyle\sum_i \alpha_i = 1 \}$$

univariate polynomial

$$p(x) \in \mathbb{Q}[x]$$

$$p(\alpha_0, \alpha_1) = p(\alpha_0\, 0 + \alpha_1\, 1)$$

$$p(\alpha_0, \alpha_1) = \textstyle\sum_{k=0}^{d} b^d_k B^d_k(\alpha_0, \alpha_1)$$

$$B^d_k(\alpha_0, \alpha_1) = \binom{d}{k} \alpha_1^k \alpha_0^{d-k}$$

$$\min_{0 \leq i \leq d} b^d_i \leq p(x) \leq \max_{0 \leq i \leq d} b^d_i$$

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polytope

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parametric polytope

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Parametric Example

$$p(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_1 + x_2$$

$$P = \text{conv}\{(0,0), (N,0), (N,N)\}$$

To compute:

$$\max_{(x_1,x_2)\in P} p(x_1, x_2)$$

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$\mathcal{H}$-parametric polytopes

$$P : \mathbb{Q}^r \rightarrow \mathbb{Q}^n : \mathbf{q} \mapsto P(\mathbf{q}) = \{\mathbf{x} \in \mathbb{Q}^d \mid A\mathbf{x} \geq D\mathbf{q} + \mathbf{d}\}$$

Conversion to $\mathcal{V}$-parametric polytopes:

- Compute parametric vertices
- Compute chamber decomposition

Example:

$$T_1 = \{(j,k,i) \mid i = j+1 \wedge i \leq k \leq i+1 \wedge i \geq M+2 \wedge k \leq N-1\}$$

$$\begin{cases} \text{conv}\left\{ \begin{pmatrix} M+1 \\ N-1 \\ M+2 \end{pmatrix}, \begin{pmatrix} M+1 \\ M+2 \\ M+2 \end{pmatrix}, \begin{pmatrix} N-2 \\ N-2 \\ N-2 \end{pmatrix} \right\} & \text{if } N-4 \leq M \leq N-3 \\ \text{conv}\left\{ \begin{pmatrix} M+1 \\ M+3 \\ M+2 \end{pmatrix} \begin{pmatrix} N-3 \\ N-1 \\ N-2 \end{pmatrix} \begin{pmatrix} M+1 \\ M+2 \\ M+2 \end{pmatrix} \begin{pmatrix} N-2 \\ N-1 \\ N-1 \end{pmatrix} \right\} & \text{if } M \leq N-5 \end{cases}$$

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FIFO Size Example

$$\max\left(\max_{(j,k,i)\in T_1} i-Nk-Ni+N^2,\ \max_{(j,k,i)\in T_2} -k+i^2-2Ni+2i+N^2-N+1\right)$$

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Simplification

Some polynomials $p_i(\mathbf{q})$ in max-expression may be redundant
$\Rightarrow$ drop $p_i(\mathbf{q})$ if $p_i(\mathbf{q}) - p_j(\mathbf{q}) \leq 0$ $(i \neq j)$ on their domain

Easy ways of determining sign of $d(\mathbf{q}) = p_i(\mathbf{q}) - p_j(\mathbf{q})$

- $d(\mathbf{q}) = D$, a constant
- $d(\mathbf{q})$ is degree-1 polynomial (add constraint to domain)
- domain is bounded $\Rightarrow$ apply Bernstein expansion
  $(M+1)N^2 - 4N - 2(M+1) = (N^2 + 2N - 1) =$
  $3+M = 2N^2 - 7N - 2MN$ on $10 \leq M \leq N = 5 \leq 95$
  Bernstein coefficients: 116, 8756, 17813, 199, 27, 1274
  univariate $\Rightarrow$ Cauchy's bounds
- lower bound on parameter
  $\Rightarrow$ check coefficients of Taylor expansion about lower bound

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Context: $10 \leq M \leq 100 \wedge 10 \leq N \leq 100$

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Simplification

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Incremental Computation

- Parametric boxes
  $[l_1(\mathbf{q}), u_1(\mathbf{q})] \times [l_2(x_1, \mathbf{q}), u_2(x_1, \mathbf{q})] \times \cdots \times$
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  $\Rightarrow$ symbolic Bernstein expansion on last coordinate

  $$\min_{0 \leq i \leq d} b_i^d(\hat{\mathbf{x}}, \mathbf{q}) \leq p(\hat{\mathbf{x}}, \mathbf{q})(x_n) \leq \max_{0 \leq i \leq d} b_i^d(\hat{\mathbf{x}}, \mathbf{q})$$

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Incremental Computation

- $\mathcal{H}$-parametric polytopes
  - Consider all but one variable to be parameters
  - Compute parametric vertices (of interval)
  - Compute chamber decomposition
  - Compute Bernstein coefficients
  - Apply recursively on each coefficient over each chamber

Incremental Computation: Comparison

Number of Bernstein coefficients
= number of multinomial coefficients $\binom{d}{k_1,k_2,\ldots,k_n}$
= $\binom{d+n-1}{d} > \frac{n^d}{d!}$
d: degree of polynomial; n: number of vertices

Incremental computation
$\Rightarrow$ 2 vertices (for each level, polynomial and chamber)
$\Rightarrow d + 1$ Bernstein coefficients

But: multiple levels and polynomials + maybe more chambers
But: may be more accurate

Also:
redundant coefficients may be removed at intermediate levels

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  - Bounds on polynomial over parametric polytopes
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- Bounds on *quasi*-polynomials
- Further simplifications

`bernstein` library included in `barvinok`, available from `http://freshmeat.net/projects/barvinok/`

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