On the notion of strong relative degree and its implications on the design of extended PD-controllers for time-delay systems*

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***Abstract*— The fundamental notion of relative degree is studied for a class of linear time-delay systems of retarded type, where the standard assumption of commensurate delays is dropped. Even though the analyzed systems do not have a non-trivial feedthrough from input to output, it is shown that the relative degree may be sensitive to infinitesimal delay perturbations. This observation is at the basis of a novel notion of relative degree, called strong relative degree, which is characterized algebraically and computationally. It plays a key role in guaranteeing well-posedness of systems controlled with extended PD controllers, for which a design procedure and numerical experiments are presented.**

I. Introduction

The relative degree is a fundamental notion in systems theory which has for instance a major influence on the mechanisms, possibilities and limitations of feedback control. It plays a role in input-output feedback linearization of nonlinear systems [5], [19], possibly combined with a decoupling [10], [12], [11], [26], [27], inversion and tracking [3], [7], [17]. In this paper, we provide several definitions and mathematical characterizations, considering the linear time-delay system described by

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the state, input and output at time t, the system matrices are assumed to be real valued and the time-delays τ_i are ordered such that

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m. \tag{2}$$

Intuitively, the relative degree of a system is equal to the number of times the output must be differentiated to reveal an explicit relation with the input. For delay systems, this creates some ambiguity as delayed inputs may appear by differentiation. In article [12], addressing a class of nonlinear systems with single delay, a classification is made into Type-I relative degree (any explicit relation with the input), Type-II relative degree (this relation reveals the input at the present time) and Type-III relative degree (this relation reveals the input at the present time only). If $m = 1$ in (1), the adopted definition in the paper corresponds to the Type-I relative degree.

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In most works addressing the relative degree of time-delay systems or control problems for which this notion is crucial, such as [5], [3], [4], [7], [8], [11], [17] and the references therein, it is assumed that there is a single delay in the model or that the delays in (1) are commensurate, i.e., they are all multiples of a common basis delay. For linear time-delays systems, this assumption gives rise to algebraic characterizations in terms of univariate matrix polynomials whose variable corresponds to the shift or delay operator. In this paper we drop the assumption of commensurate delays.

The relative degree is a structural property of the system that is known to be sensitive to small perturbations on the coefficient matrices in case the perturbations do not respect the underlying structure, inter-dependencies or sparsity pattern of the coefficient matrices induced by inherent properties of the modeled system (e.g., if they induce interconnections between inputs, outputs and different parts of the system that are physically not present). What is far less intuitive, but we demonstrate with an example, is that the relative degree of an output may be sensitive to infinitesimal perturbations of the values of the state delays in (1) and that, as a consequence, important structural information may also be encoded in the inter-dependence between the delay values. This example leads us to the main contribution of the paper, in the form of the definition and algebraic characterizations of a robustified notion of relative degree called *strong* relative degree, whose terminology is inspired by the notion of strong stability of delay equations of neutral type and delay-differential-algebraic equations (DDAEs) [15], [20], [22]. We note that also delay-independent stability along a particular direction in the delay-parameter space may be sensitive to infinitesimal perturbations to the direction [21].

It is important to mention that if the nominal delay are commensurate and perturbations are restricted so that the ratio between delays cannot change, the above sensitivity problem of the relative degree cannot occur. This directly follows from the induced interpretation of the transfer function as a rational function whose coefficients are polynomials of $z := \exp(-hs)$ with h the basis delay. For non-commensurate delays, such coefficients are rather multivariate polynomials (number of variables determined by the number of rationally independent components).

As an illustration of the applicability of the presented results, we enlarge the scope of a control design method for linear systems with delays in states, inputs, and outputs, where the new concepts play a key-role. Recently, an increasing interest in P(I)D controllers for time-delay systems can be observed [6], [13], [18], [24], [25], [30], [31]. These

low-order controllers offer as advantages the simplicity of an implementation (compared to full order, infinite-dimensional controllers) and the need to tune only a small amount of parameters, while for particular classes of systems, even analytic design procedures are available. In case the relative degree of an output is larger than one, the range of a PD controller can be extended by also using higher order output derivatives for feedback (leading to *extended* PD controllers, following the terminology of [29]). As we shall illustrate, such an extension may significantly increases the range of plants that can be stabilized. However, there is a major pitfall since the resulting closed-loop system may be ill-posed or fragile if an infinitesimal parametric perturbation would induce a decrease of the relative degree. Therefore, the maximal order of employed output derivatives should be determined from the strong relative degree rather than the standard relative degree. Relying on minimizing the spectral abscissa of the closed-loop system as a function of the controller parameters, the design algorithm which we will present bears conceptual similarities to the algorithms for computing fixed-order controllers for systems described by DDAEs and for synthesizing non-fragile PID controllers for retarded systems, presented in [20] and [1], respectively.

The structure of the paper is as follows. In Section II we give a motivating example exhibiting a discontinuity of the relative degree with respect to delay parameters. In Section III we introduce the notion of strong relative degree and outline various properties and characterizations. In Section IV we present the design of extended PD controllers. Finally, in Section V we draw some conclusions and briefly comment on the existence and properties of a normal form [4], [16], [9], which is tightly connected with the notion of relative degree.

All technical proofs, the generalization of the presented results towards MIMO systems, and algorithmic aspects can be found in [23].

II. Motivating example

We consider the case where $m = 3$ and the matrices are spelled out as

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\kappa \\ 0 & -2 & 0 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

with $\kappa \in \mathbb{R}$ a parameter used to demonstrate several scenarios, in this section and Section IV. The transfer function is given by

$$H(s,\ \vec{\tau}) = \frac{e^{-2s\tau_2} - e^{-s(\tau_1+\tau_3)}}{s^2(s-1)} + \kappa \frac{4e^{-s(2\tau_1+\tau_2)} + 2e^{-s(\tau_1+2\tau_2)} + 2e^{-s(2\tau_1+\tau_3)} + e^{-s(\tau_1+\tau_2+\tau_3)}}{s^3(s-1)}, \tag{4}$$

with $\vec{\tau} = (\tau_1, \ldots, \tau_m)$, which can be verified using symbolic software. We conclude that the relative degree, in the sense of Type-I relative degree in [12] (see the introduction for the different types), satisfies

$$RD(\vec{\tau}) = \begin{cases} 3, & 2\tau_2 \neq \tau_1 + \tau_3, \\ 4, & 2\tau_2 = \tau_1 + \tau_3 \text{ and } \kappa \neq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5}$$

It follows that the relative degree may be sensitive to *infinitesimal* delay perturbations, as is the case at nominal delay values satisfying $2\tau_2 = \tau_1 + \tau_3$.

Remark 1: The relative degree is sensitive with respect to infinitesimal perturbations to matrices A_i, B, C as well. However, matrix perturbations inducing a change in relative degree are, as a rule, perturbations that do not respect the underlying structure of the system. For instance, replacing a zero element by a nonzero element may correspond to introducing an interconnection that is physically not present. Surprisingly, the example illustrates that important structural information is encoded in the inter-dependence between the different delay values, and motivates us to take small delay perturbations into account, in particular in applications where the delays in model (1) correspond to different physical parameters. Note in this context that if system (1) with matrices (3) is combined with controller $u(t) = K_0 y(t - r) + \sum_{i=1}^{4} K_i y^{(i)}(t-r)$ and $K_4 \neq 0$, the closed-loop system is well-posed if $2\tau_2 = \tau_1 + \tau_3$ (the model can be recast in terms of DDAEs of index one), but ill-posed if $2\tau_2 \neq \tau_1 + \tau_3$ (delay equations of advanced type).

III. Definitions and properties

The adopted definition of relative degree is rooted in explicit expressions for output derivatives. Instrumental to these expressions are matrix polynomials $P_{k_0,\ldots,k_m}$, with $k_j \in \mathbb{Z}_{\geq 0}, j = 0, \ldots, m$, which are recursively defined through

$$P_{0,\ldots,0} := I, \tag{6}$$

$$P_{k_0,\ldots,k_m} := A_0 P_{k_0-1,k_1,\ldots,k_m} + A_1 P_{k_0,k_1-1,\ldots,k_m} + \ldots + A_m P_{k_0,k_1,\ldots,k_m-1} \tag{7}$$

and

$$P_{k_0,\ldots,k_m} := 0 \text{ if any } k_j \in \mathbb{Z}_{<0}, j = 0, \ldots, m, \tag{8}$$

and also appear in the finiteness analysis of the $\mathcal{H}_2$-norm of systems described by DDAEs [14]. For instance, for $m = 1$ and $k_0 + k_1 \leq 3$, the matrix polynomials are

$$\begin{aligned} &P_{0,0} = I, \\ &P_{1,0} = A_0, \quad P_{0,1} = A_1, \\ &P_{2,0} = A_0^2, \quad P_{1,1} = A_0A_1 + A_1A_0, \quad P_{0,2} = A_1^2, \\ &P_{3,0} = A_0^3, \quad P_{2,1} = A_0^2A_1 + A_0A_1A_0 + A_1A_0^2, \\ &P_{1,2} = A_0A_1^2 + A_1A_0A_1 + A_1^2A_0, \quad P_{0,3} = A_1^3. \end{aligned}$$

Note that $P_{k_0,\ldots,k_m}$ can be interpreted as the sum of all monomials in $(A_0,\ldots,A_m)$ that have power k_i in matrix A_i for $i=0,\ldots,m$. If we further define

$$\Omega_j := \left\{(k_0,\ldots,k_m)\in\mathbb{Z}_{\geq 0}^{m+1}:\ k_0+\cdots+k_m=j\right\}$$

for $j\in\mathbb{Z}_{\geq 0}$, then we can express

$$\begin{aligned}
\dot{y}(t) &= CA_0x(t)+\textstyle\sum_{i=1}^m CA_ix(t-\tau_i)+CBu(t)\\
&= \textstyle\sum_{\vec{k}\in\Omega_1} CP_{k_0,\ldots,k_m}x(t-k_1\tau_1\cdots-k_m\tau_m)\\
&\quad +CP_{0,\ldots,0}Bu(t)\\
\ddot{y}(t) &= CA_0^2x(t)+\textstyle\sum_{i=1}^m C(A_0A_i+A_iA_0)\\
&\quad x(t-\tau_i)+CA_0Bu(t)\\
&\quad +\textstyle\sum_{i_1=1}^m\sum_{i_2=1}^m CA_{i_1}A_{i_2}x((t-\tau_{i_1}-\tau_{i_2})\\
&\quad +CB\dot{u}(t)+\textstyle\sum_{i=1}^m CA_iBu(t-\tau_i)\\
&= \textstyle\sum_{\vec{k}\in\Omega_2} CP_{k_0,\ldots,k_m}x(t-k_1\tau_1-k_2\tau_2\cdots-k_m\tau_m)\\
&\quad +\textstyle\sum_{i=0}^1\left(\sum_{\vec{k}\in\Omega_i} CP_{k_0,\ldots,k_m}B\right.\\
&\quad \left. u^{(1-i)}(t-k_1\tau_1-k_2\tau_2\cdots-k_m\tau_m)\right),
\end{aligned}$$

which can be generalized for any $j\geq 1$ to

$$\begin{aligned}
y^{(j)}(t) &= \sum_{\vec{k}\in\Omega_j} CP_{k_0,\ldots,k_m}x(t-\vec{k}\cdot\vec{\tau}_e)\\
&\quad +\sum_{i=0}^{j-1}\left(\sum_{\vec{k}\in\Omega_i} CP_{k_0,\ldots,k_m}B\ u^{(j-1-i)}(t-\vec{k}\cdot\vec{\tau}_e)\right), \quad (9)
\end{aligned}$$

with $\vec{\tau}_e=(0,\tau_1,\ldots,\tau_m)$ and $\vec{k}=(k_0,k_1,\ldots,k_m)$.

If the relative degree of the output, $RD(\vec{\tau})$, is larger than one, it is determined by the highest value of j for which

$$\begin{aligned}
&\textstyle\sum_{\vec{k}\in\Omega_0} CP_{k_0,\ldots,k_m}B\ u^{(j-1)}(t-\vec{k}\cdot\vec{\tau}_e)=0,\\
&\qquad\qquad\vdots\\
&\textstyle\sum_{\vec{k}\in\Omega_{j-2}} CP_{k_0,\ldots,k_m}B\ u^{(1)}(t-\vec{k}\cdot\vec{\tau}_e)=0,
\end{aligned}\quad (10)$$

independently of the input signal - recall that we look for an explicit relation between output derivatives and the (possibly delayed) input, as is the case for the Type-I relative degree in [12]. In [23], various algebraic characterizations of $RD(\vec{\tau})$ are provided, based on (10) .

The sensitivity of the relative degree with respect to infinitesimal delay perturbations, apparent from (5), motivates the following robustified notion[1].

Definition 1: The strong relative degree of SISO system (1) is given by

$$sRD(\vec{\tau}) := \lim_{\epsilon\to 0+}\min\left\{RD(\vec{\nu}):\ \vec{\nu}\in\mathcal{B}(\vec{\tau},\epsilon)\right\}$$

with

$$\mathcal{B}(\vec{\tau},\epsilon) := \left\{\vec{\nu}\in\mathbb{R}_{\geq 0}^m:\ \|\vec{\nu}-\vec{\tau}\|_2<\epsilon\right\}.$$

For system (1) and (3) we have $sRD(\vec{\tau})=3$, independently of the values of the delays and parameter κ, which is apparent from the transfer function.

Expression (9) for the output derivatives is key to a simple algebraic characterization of the strong relative degree. Definition 1 explicitly takes small delay perturbations into account, and there are always rationally independent delays arbitrarily close to the nominal delays, see [23, Proposition 2.2]. For rationally independent delays $\vec{\tau}$, it is not possible that different terms in (10) share the same input delay. Indeed, assuming that $\vec{k}\cdot\vec{\tau}_e=\vec{l}\cdot\vec{\tau}_e$ with $\vec{k}\neq\vec{l}$, we would have $(\vec{k}-\vec{l})\cdot\vec{\tau}_e=0$, contradicting that the delays $\vec{\tau}$ are rationally independent. Hence, (10) can only be zero for any sufficiently smooth input signal if all coefficients are zero. In this way we have sketched the proof of the following result.

Theorem 1: The output has strong relative degree (sRD) equal to r if and only if

$$\begin{aligned}
&CP_{k_0,\ldots,k_m}B=0,\\
&\forall(k_0,\ldots,k_m)\in\mathbb{Z}_{\geq 0}^{m+1}:\ k_0+\cdots+k_m<r-1
\end{aligned}\quad (11)$$

and there exist some $(k_0,\ldots,k_m)$ satisfying $k_0+\cdots+k_m=r-1$ such that $CP_{k_0,\ldots,k_m}B\neq 0$.
The output has strong relative degree $sRD=+\infty$ if (11) holds for all $r\geq 1$.

Example 1: For system (1) with matrices (3) we get

$$\begin{array}{l}
CP_{0,0,0,0}B=CB=0,\\
CP_{1,0,0,0}B=CA_0B=0,\ \ CA_1B=0,\ \ CA_2B=0,\ \ CA_3B=0,\\
CP_{2,0,0,0}B=CA_0^2B=0,\ \ CA_1^2B=0,\ \ CA_2^2B=1,\ CA_3^2B=0,\\
CP_{1,1,0,0}B=C(A_0A_1+A_1A_0)B=0,\ \ C(A_0A_2+A_2A_0)B=0,\\
C(A_0A_3+A_3A_0)B=0,\ \ C(A_1A_2+A_2A_1)B=0,\\
C(A_1A_3+A_3A_1)B=-1,\ \ C(A_2A_3+A_3A_2)B=0,
\end{array}$$

hence, (11) is satisfied for $r=3$ but not for $r=4$, which confirms $sRD=3$. Note that the drop in the (standard) relative degree whenever $2\tau_2=\tau_1+\tau_3$ is due to the relation

$$CP_{0,0,2,0}B+CP_{0,1,0,1}B=0.$$

According to Theorem 1, determining whether the strong relative degree is equal to infinity, corresponds to verifying an infinite number of equalities, which can, however, be reduced to checking a finite number of equalities using a generalization of the Cayley-Hamilton theorem to multi-powers (6)-(8), as spelled out in [14, Theorem 5]. A combination with Theorem 1 leads us to the following proposition.

Proposition 1: The output has strong relative degree $sRD=+\infty$ if and only if $CP_{k_0,\ldots,k_m}B=0,\ \ \forall(k_0,\cdots,k_m):\ k_0+\ldots+k_m<(m+1)n$, with n the state dimension.

We conclude the section by listing some properties of the strong relative degree.

Theorem 2: SISO system (1) has the following properties:

1) $\mathrm{sRD}\leq\mathrm{RD}(\vec{\tau})$;
2) sRD is independent of the delays;
3) $\mathrm{sRD}=\mathrm{RD}(\vec{\tau})$ for rationally independent $\vec{\tau}$;
4) the output has $\mathrm{sRD}=r$ if and only if $C\left(A_0+\sum_{i=1}^m A_iz_i\right)^iB\equiv 0$ for $i\in\{0,\ldots,r-2\}$ and $C\left(A_0+\sum_{i=1}^m A_iz_i\right)^{r-1}B\not\equiv 0$, where $\equiv 0$ stands for all values of $(z_1,\ldots,z_m)\in\mathbb{C}^m$;
5) the output has $\mathrm{sRD}=r$ if and only if

$$\begin{aligned}
&\lim_{\omega\to+\infty}\max_{\vec{z}\in\mathcal{D}_m}\omega^i\hat{G}(\imath\omega,z_1,\ldots,z_m)=0,\\
&i\in\{0,\ldots,r-1\},\\
&\lim_{\omega\to+\infty}\max_{\vec{z}\in\mathcal{D}_m}\omega^r\left|\hat{G}(\imath\omega,z_1,\ldots,z_m)\right|>0,
\end{aligned}$$

[1]The name is inspired by the notion of strong stability for neutral delay equations. The notion is rather related to fragility (sensitivity with respect to *infinitesimal* parametric perturbations) than robustness.

with

$$\hat{G}(s, z_1, \ldots, z_m) := C \left(sI - A_0 - \sum_{i=1}^{m} A_i z_i \right)^{-1} B$$

and $\mathcal{D}_m =: \{(z_1, \ldots, z_m) \in \mathbb{C}^m : \ |z_i| = 1, \ i = 1, \ldots, m\}$.

The first three assertions stem from the properties that the set of rationally dependent delays is dense in $\mathbb{R}^m_{\geq 0}$ and that the sRD does not depend on the delay values. The last two assertions follow from the third one and the fact that for any $j \geq 0$, the set $\{P_{k_0,\ldots,k_m} : \ k_0 + k_1 + \cdots + k_m = j\}$ can be identified as the set of coefficients of the multivariate matrix polynomial $(A_0 + A_1 z_1 + \cdots + A_m z_m)^j$.

IV. Extended PD control

In order to sketch the idea behind the extension, consider as an example a system whose output has strong relative degree equal to 3. Then static output feedback can be directly generalized to

$$u(t) = K_1 y(t) + K_2 \dot{y}(t) + K_3 \ddot{y}(t), \tag{12}$$

since the output derivatives in (12) can be explicitly expressed as a function of (delayed) states only. It is also possible to use, in addition, output derivative $\dddot{y}(t)$ for feedback, which can be expressed in terms of (delayed) states and the input, but more care is needed in the design, due to the resulting implicit expression for the input and neutral dynamics, as we shall see.

Letting the highest derivative of the output, used for feedback, depend on the strong rather than the standard relative degree assures that the closed-loop system remains well-posed in the presence of small deviations of the delays. From an application of view it is also important that model mismatch affecting the system matrices does not alter the strong relative degrees (see Remark 1).

In Section IV-A we present the general setting for the extended PD control along with stability conditions and a procedure to compute the gain values. In Section IV-B we illustrate the methodology on a numerical example.

A. Computing strongly stable feedback gains

We start from model (1) and assume $sRD = R$. We assume $R < \infty$ and consider an extended PD controller of the form

$$u(t) = \sum_{j=1}^{R} K_j \ y^{(j)}(t - \tau_f), \tag{13}$$

where $\tau_f \geq 0$ models the delay in the feedback control loop.

If $K_R = 0$, the closed-loop system is of retarded type. Otherwise, it may correspond to a DDAE of neutral type [20] with associated delay-difference equation

$$u(t) = K_R \left(\sum_{\vec{k} \in \Omega_{R-1}} C P_{k_0,\ldots,k_m} B u(t - \vec{k} \cdot \vec{\tau}_e - \tau_f) \right), \tag{14}$$

where we used (9). For neutral type systems, it is well known that exponential stability may be lost when applying infinitesimal delay perturbations, which led in [15] to the notion of strong stability. Inspired by the analysis in [22], we define strong stability in the context of the paper as follows.

Definition 2: Closed-loop system (1) and (13) is strongly exponentially stable if the null solution is exponentially stable and the exponential stability is robust against small perturbations of delays $(\tau_1, \ldots, \tau_m, \tau_f)$.

In order to arrive at a mathematical characterization, let us define matrices $\mathcal{M}_i, \ i = 1, \ldots, m+1$, depending on the gains values, such that

$$\det \left(I - K_R C \left(A_0 + A_1 z_1 + \cdots + A_m z_m \right)^{R_i - 1} z_0 B \right) = \det \left(I - \textstyle\sum_{i=0}^{m} \mathcal{M}_i z_i \right)$$

holds for all $(z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. Such matrices can always be constructed using the lifting procedure described in [22, Section 4.2].

Theorem 3: Closed-loop system (1), with $sRD = R$, and (13) is strongly exponentially stable if and only if its null solution is exponentially stable and

$$\left(\max_{\vec{\theta} \in [0, \ 2\pi]^{m+1}} r_\sigma \left(\sum_{i=0}^{m} \mathcal{M}_i e^{\imath \theta_i} \right) \right) < 1, \tag{15}$$

with $\imath = \sqrt{-1}$ the imaginary unit.

Proof: The characteristic function of delay-difference equation (14) can be expressed as

$$\det \left(I - K_R C \left(A_0 + A_1 e^{-s\tau_1} + \cdots + A_m e^{-s\tau_m} \right)^{R_i - 1} e^{-s\tau_f} B \right) = \det \left(I - \textstyle\sum_{i=1}^{m} \mathcal{M}_i e^{-s\tau_i} - \mathcal{M}_0 e^{-s\tau_f} \right).$$

The assertion now follows from the theory developed in [15], [20]. ∎

Expressing the employed output derivatives explicitly as a function of delayed state variables and possibly inputs, enables the design approach of [20] to compute the gain values. This spectral approach is based on *numerically minimizing the so-called robust spectral abscissa* [20, Section 3.2] of the closed-loop system as a function of the controller parameters, since a negative value of the robust spectral abscissa is a necessary and sufficient condition for strong exponential stability. The algorithm has been integrated in the software TDS-CONTROL [2]. In this way, the additional constraint (15) for robustness of stability against small delay perturbations is taken into account.

B. Numerical example

As an example we take the system described by

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} x\left(t - \tfrac{1}{2}\right) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} x(t),$$

whose strong relative degree is equal to two. For zero input, the system is unstable with a strictly positive spectral

feedback	K_1	K_2	K_3	(robust)
				spectral abscissa
no				0.8636
y	-0.6347			0.2303
$y,\ \dot{y}$	-0.6141	-0.3865		-0.3882
$y,\ \dot{y},\ \ddot{y}$	-0.6688	-0.4185	-0.061306	-0.5599

TABLE I

THE (ROBUST) SPECTRAL ABSCISSA OF THE OPEN LOOP SYSTEM AND AFTER A MINIMIZATION AS A FUNCTION OF THE EXTENDED PD CONTROLLER GAINS.

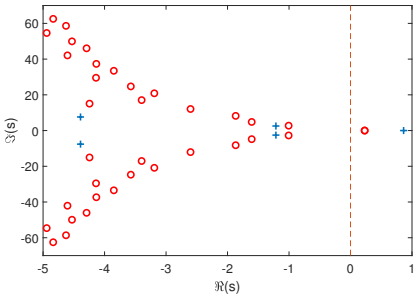


Fig. 1. Rightmost characteristic roots of the uncontrolled system ('+') and after minimizing the spectral abscissa using static output feedback ('o').

abscissa. In contrast to the example with matrices (3), it has infinitely many characteristic roots without control. We can express

$$\begin{array}{rcl} \dot{y}(t) & = & \left[\begin{array}{ccc} 1 & 1 & 2 \end{array}\right] x(t) + \left[\begin{array}{ccc} -1 & -4 & 4 \end{array}\right] x\left(t-\frac{1}{2}\right), \\ \ddot{y}(t) & = & \left[\begin{array}{ccc} 0 & 1 & -5 \end{array}\right] x(t) + \left[\begin{array}{ccc} 8 & -2 & -6 \end{array}\right] x\left(t-\frac{1}{2}\right) \\ & & + \left[\begin{array}{ccc} 1 & -2 & 8 \end{array}\right] x(t-1) + 2u(t) + 4u\left(t-\frac{1}{2}\right) \end{array}$$

and consider a feedback control law of the form

$$u(t) = K_1 y(t-1) + K_2 \dot{y}(t-1) + K_3 \ddot{y}(t-1).$$

The results of minimizing the robust spectral abscissa are shown in Table I, where we gradually increase the number of gains considered as free parameters (the others are set to zero), while the open and closed-loop spectra are visualized in Figures 1-2. In the last considered case, the variable related to gain K_3 was rescaled by a factor 100 before the optimization was done, to counter an observed significantly higher sensitivity of the robust spectral abscissa with respect to this gain. The choice of the scaling parameter is a hyper parameter in the overall process.

Using output feedback only, achieving exponential stability is not possible, in contrast to the other scenarios. When using the output and its first derivative, the closed-loop system is of retarded type, such that the robust spectral abscissa equals the spectral abscissa. When using the second-order derivative of the output, the closed-loop system is of neutral type, which is consistent with the characteristic shape of the spectrum shown in Figure 2.

It should be mentioned that by considering the strong relative degree in the design, the resulting controllers are *non-fragile* in the sense that closed-stability cannot be destroyed by infinitesimal parametric perturbations, i.e. all

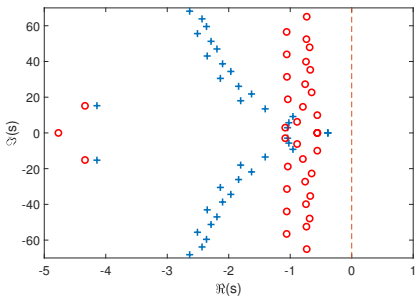


Fig. 2. Rightmost characteristic after minimizing the (robust) spectral abscissa using static feedback of $y,\ \dot{y}$ ('+'), and using $y,\ \dot{y}$ and $\ddot{y}$ ('o').

stability margins are nonzero. Computing and optimizing stability margins, for which the pseusospectral framework is amenable, is beyond the scope of this work.

V. CONCLUDING REMARKS

The potential sensitivity of the relative degree of (1) with respect to infinitesimal perturbations of the nominal delays led to the introduction of the strong relative degree, which turned out to be independent of the delays. Several algebraic characterizations were provided. It was argued that in extensions of P(I)D controllers, relying on higher-order derivatives for feedback, it is important to let the highest order be determined by the strong relative degree rather than the relative degree.

The notion of relative degree is often associated with a transformation to a normal form. For MIMO time-delay systems of the form (1), a brief discussion on this association is made in [23, Section 6] and the references therein. Existence conditions for a normal form turn out to be quite restrictive, not only because of invertibility conditions on the input-output map and the transformation. The motivating example (1) and (3) namely demonstrates another potential difficulty. When the delays are specified as $(\tau_1, \tau_2, \tau_3) = (1, 3, 6)$, the invariant zeros are described by equation

$$s\left(1 - e^{-s}\right) + \kappa\left(4e^{s} + 2e^{-s} + 2e^{-2s} + e^{-4s}\right) = 0, \quad (16)$$

where we divided by the common factor e^{-6s}. We make the following observations.

- For $\kappa = 0$, the zeros of (16) can be interpreted as characteristic roots of neutral equation $\dot{z}(t) = \dot{z}(t-1)$, or equivalently, as characteristic roots of the DDAE $\dot{z}_1(t) = 0,\ z_2(t) = z_1(t) + z_2(t-1)$.
- For $\kappa \neq 0$, the zeros correspond to a delay-differential equation of *advanced type*, due to the positive power of s.

In Figure 3, we display the invariant zeros for $\kappa = -0.03$, highlighting the typical spectrum distribution for advanced equations, and for $\kappa = 0$. We conclude that even though (1) is a delay-differential equation of retarded type, the isolation of zero dynamics, which is inherent to a description in normal form, may require DDAE models or even equations of advanced type, which are however a-causal [28].

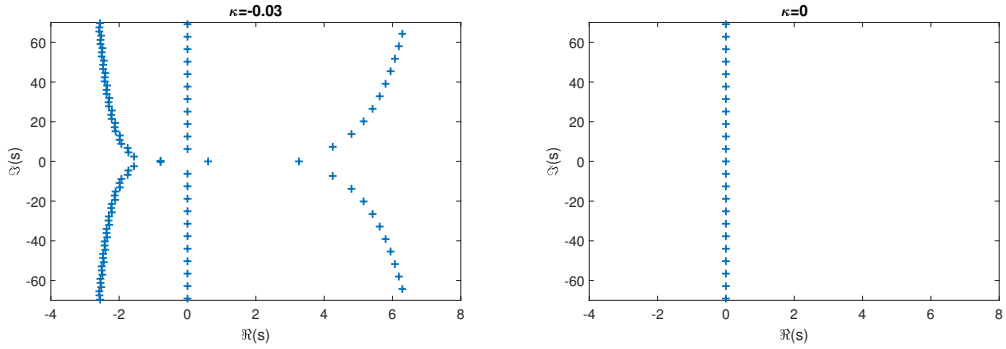


Fig. 3. Invariant zeros of system (1) with matrices (3) and delays $(\tau_1, \tau_2, \tau_3) = (1, 3, 6)$, for $\kappa = -0.03$ (left pane) $\kappa = 0$ (right pane).

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