Distributed finite-time observers over complete communication graphs

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Abstract: In this extended abstract, a distributed observer with all-to-all communication is presented to estimate the state of a linear time-invariant plant in finite-time in each observer node. The design is based on a decomposition into locally observable and unobservable substates and on properties of homogeneous systems. Each observer node can reconstruct in finite-time its locally observable substate with its measurements only. Then exploiting the coupling, a finite-time converging observer is constructed for the remaining states by adding the consensus terms. A numerical example illustrates the result.

Keywords: Distributed observers, finite-time observers, observability decomposition

1. INTRODUCTION

Driven by the fact that numerous collaborative approaches are proposed to reach efficiency in complex systems like power networks, traffic systems and robot swarms etc., the research on distributed observers has gained more attention in recent years. The idea is to take advantage of the fact that smart multi-agent systems are deployed almost everywhere with sensing, processing and communication capabilities. This leads to new opportunities for the task of state estimation but also to related fields as synchronization, which is a fundamental problem in interconnected systems, and can be represented as the problem of designing observers (Nijmeijer and Mareels, 1997).

In this context, *distributed state estimation* (Wang and Morse, 2018) has been proposed as an elementary problem. In this framework a general linear plant with multiple measurement outputs is observed by a sensor network. Each sensor acts as an observer node, with a complete model of the plant, and the aim is to reconstruct the state of the plant in each node. Since the system is not observable from a single output, the observer nodes exchange information with each other, subject to a communication graph, to obtain the estimate of the full plant state, i.e. to achieve *state-omniscience* (Park and Martins, 2017)), without falling back to a central fusion framework (as it has been shown in Silm et al. (2018b) the distributed solution outperforms the centralized one when taking into account communication lags).

In the present note, to tackle this problem we will rely on the observability decomposition of the state space. This is inspired by the fact, that each observer node has some part of the state space which is observable by its measurement, and an independent observer can be designed (the locally observable substate). This has been first exploited in Kim et al. (2016) and also other authors, for example Han et al. (2018); Millán et al. (2017). A related approach is suggested in Mitra and Sundaram (2018), but with a generalized decomposition for multi-sensors.

For the remaining part, we assume that there exists a suitable communication graph. A widely adopted approach is to use a consensus mechanism with neighboring agents. Since initially proposed for Distributed Kalman Filtering (Olfati-Saber, 2007) many methods have been presented to design the consensus gains, for example in Ugrinovskii (2011) using $\mathscr{H}_\infty$ filter technique.

One useful and alternative approach is to design distributed observers with regard to the finite-time stability property (Silm et al., 2018a). Centralized finite-time observers have been suggested before in Andrieu et al. (2008); Levant (2005); Perruquetti et al. (2008) based on homogeneity. The advantages of homogeneity based finite-time stability, is that robustness in terms of disturbances is retained while next to finite-time stability, robustness to time-delays is achieved, which is important in the context of networked observers. Additionally, in a mobile environment the communication topology can be changed at some time-instants, making it more important that time-constraints are met. In this work, following Silm et al. (2018a), we propose a distributed finite-time observer, but to include general linear plant dynamics, we additionally exploit the decoupling property on observable and unobservable subspaces, due to finite-time convergences of some part of the system.

The outline of the paper is as follows. In the preliminaries the finite-time observer design is presented for the case of a centralized observer and the observability decomposition is recapitulated. These results are used in the main part to introduce the distributed finite-time observer design, first, for

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the observable and, second, for the unobservable part. An example in the last section illustrates the result.

Notation

- For $x = [x_1 \ \dots \ x_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\alpha = [\alpha_1 \ \dots \ \alpha_n]^{\mathrm{T}} \in \mathbb{R}^n_+$, we denote by $\lceil x \rfloor^{\alpha} = [\mathrm{sign}(x_1)|x_1|^{\alpha_1} \ \dots \ \mathrm{sign}(x_n)|x_n|^{\alpha_n}]^{\mathrm{T}}$ the sign preserving element-wise exponentiation.
- $\mathrm{diag}(x)$ denotes the diagonal matrix with $x_1, \dots, x_n$ on the main diagonal and 0 otherwise.
- $1_n \in \mathbb{R}^n$ is the vector with all entries equal 1
- $E_n \in \mathbb{R}^{1\times n}$ is the unit row vector $E_n = [1 \ \dots \ 0]$
- Denote with $\mathcal{I}_n = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & \dots & 0 \end{bmatrix}$ a matrix with dimension $n \times n$.

2. PRELIMINARIES

2.1 Finite-Time Observers

First we revise the design of a centralized finite-time observer for a linear time-invariant system

$$\dot{x} = Ax, \quad y = Cx, \tag{1}$$

with state $x \in \mathbb{R}^n$, dynamics matrix $A \in \mathbb{R}^{n\times n}$ and output matrix $C \in \mathbb{R}^{1\times n}$. It is known that if the rank of the observability matrix $\mathscr{O}(A,C)$ is n, then the system is observable and using a Luenberger observer an asymptotically stable system can be designed.

Additionally, there exists a state transformation $x = T_c v, T_c \in \mathbb{R}^{n\times n}$ describing the system as an integrator chain plus additional output feedback (observer canonical form (Kailath, 1980))

$$\dot{v} = {T_c}^{-1} A T_c v = \mathcal{I}_n v - \underbrace{[a_1 \ \dots \ a_v]^{\mathrm{T}}}_{a} y, \tag{2}$$

with $y = CT_c v = v_1$ and where $a \in \mathbb{R}^{n\times 1}$ consists of the coefficients of the characteristic polynomial of A.

Following the works of Bhat and Bernstein (2005) and Perruquetti et al. (2008), a nonlinear observer for the system (2) of the form

$$\dot{z} = \mathcal{I}_n z - a y + \mathrm{diag}\,(L) \lceil (1_n (z_1 - y) \rfloor^{\gamma} \tag{3}$$

with the observer gain $L \in \mathbb{R}^n$ will lead to a homogeneous error system of degree $-1/n < \nu \leq 0$ for the error $\hat{e} = z - v$, if the entries of $\gamma \in \mathbb{R}^n$ are selected as $\gamma_s = 1 + s\nu$. This leads to the following theorem:

Theorem 1. For the system (2) with observer (3) the estimation error will be globally finite-time stable provided that

(1) L is selected, so that $\mathcal{I}_n + \mathrm{diag}(L) E_n$ is Hurwitz

(2) $-\dfrac{\exp(1)}{2} \dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(P) \| P \,\mathrm{diag}(L) \|} < \nu < 0\,,$

where the positive definite matrices $Q, P \in \mathbb{R}^{n\times n}$ solve the Lyapunov equation

$$(\mathcal{I}_n + \mathrm{diag}(L) E_n)^{\mathrm{T}} P + P(\mathcal{I}_n + \mathrm{diag}(L) E_n) + Q = 0$$

and $\lambda_{\min}(Q), \lambda_{\max}(P)$ are the smallest and largest eigenvalues of Q and P, respectively.

Proof. The lower bound of ν in condition (2) is a sufficient condition that the origin of the corresponding error system is asymptotically stable, provided it is asymptotically stable for $\nu = 0$ (See Silm et al. (2018a) for details). The for a homogenous system with negative degree this implies global finite-time stability (Nakamura et al., 2002).

2.2 Observability Decomposition

Consider a system with $N > 1$ outputs

$$\dot{\tilde{x}} = \widetilde{A}\tilde{x}, \quad y_i = \widetilde{C}_i \tilde{x}, \quad i = 1, \dots, N \tag{4}$$

with dynamics matrix $\widetilde{A} \in \mathbb{R}^{n\times n}$ and output matrices $\widetilde{C}_i \in \mathbb{R}^{1\times n}$. For each output y_i the observable subspace has dimension $r_i = \mathrm{rank}\, \mathcal{O}(\widetilde{A}, \widetilde{C}_i)$ and the unobservable subspace may be unstable, hence the system is not detectable from any single output y_i.

We decompose now the system into observable and unobservable substates. Choose a transformation $\tilde{x} = T_i x_i$, where $T_i = [T_{o,i} \ T_{u,i}]$ is an orthogonal matrix and $T_{u,i}$ is a matrix whose columns form a basis of $\ker \mathcal{O}(\widetilde{A}, \widetilde{C}_i) \in \mathbb{R}^{n\times(n-r_i)}$, gives

$$A_i = {T_i}^{\mathrm{T}} \widetilde{A} T_i, \quad C_i = \widetilde{C}_i T_i, \tag{5}$$

where A_i and C_i are decomposed as

$$A_i = \begin{bmatrix} A_{o,i} & 0 \\ A_{r,i} & A_{u,i} \end{bmatrix}, \quad C_i = [C_{o,i} \ 0]\,. \tag{6}$$

The system in each new coordinate $x_i = \left[x_{o,i}^{\mathrm{T}} \ x_{u,i}^{\mathrm{T}}\right]^{\mathrm{T}}$ is then

$$\dot{x}_i = A_i x_i, \quad y_i = C_i x_i, \tag{7}$$

where the observable substate and the output

$$\dot{x}_{o,i} = A_{o,i} x_{o,i}, \quad y_i = C_{o,i} x_{o,i} \tag{8}$$

are independent of the substate $x_{u,i}$.

3. DISTRIBUTED OBSERVER

A finite-time distributed observer is designed now for the system (4). By design the observable subsystem (8) can be transformed with a transformation matrix $T_{c,i} \in \mathbb{R}^{r_i \times r_i}$ into observer canonical form and consequently a finite-time observer as in Theorem 1 can be applied to estimate the observable substate $v_{o,i}$ obtained by $x_{o,i} = T_{c,i} v_{o,i}$, whose dynamics has the form

$$\dot{v}_{o,i} = \mathcal{I}_{r_i} v_{o,i} - \underbrace{[a_{1,i} \ \dots \ a_{v,i}]^{\mathrm{T}}}_{a_{o,i}} y, \tag{9}$$

where $a_{o,i} \in \mathbb{R}^{r_i \times 1}$ consists of the coefficients of the characteristic polynomial of $A_{o,i}$. Then the observers

$$\dot{z}_{o,i} = \mathcal{I}_{r_i} z_{o,i} - a_{o,i} y_i + \mathrm{diag}\,(L_{o,i}) \lceil (1_{r_i} (z_{o,i,1} - y_i) \rfloor^{\gamma_{o,i}}, \tag{10}$$

with observer gain $L_{o,i} \in \mathbb{R}^{r_i}$ and $\gamma_{o,i} \in \mathbb{R}^{r_i}, \gamma_{o,i,s} = 1 + s\nu_{o,i}$, will lead to finite-time stable systems for the errors $e_{o,i} = z_{o,i} - v_{o,i}$:

Corollary 2. For the multi-output system (4), each observer node can estimate the observable substate $x_{o,i}$ with observers (10) in finite-time if

(1) $L_{o,i}$ is selected, so that $\mathcal{I}_{r_i} + \mathrm{diag}(L_{o,i}) E_{r_i}$ is Hurwitz

(2) $-\dfrac{\exp(1)}{2} \dfrac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i) \| P_i \,\mathrm{diag}(L_{o,i}) \|} < \nu_{o,i} < 0\,,$

where the positive definite matrices $Q_i, T_{c,i} \in \mathbb{R}^{r_i \times r_i}$ solve the Lyapunov equation

$$(\mathcal{I}_{r_i} + \mathrm{diag}(L_{o,i}) E_{r_i})^{\mathrm{T}} P_i + P_i (\mathcal{I}_{r_i} + \mathrm{diag}(L_{o,i}) E_{r_i}) + Q_i = 0.$$

Proof. This claim is a direct consequence of Theorem 1.

3.1 Unobservable subspace

As the unobservable part is not similar to a chain of integrators, let us introduce a hypothesis:

Assumption 3. The unobservable substates can be expressed as

$$x_{u,i} = \frac{1}{N}\sum_{j=1}^{N} Q_{i,j} x_{o,j}. \tag{11}$$

For the system (4) with transformation (5) this holds, if the pair $\mathcal{O}(\widetilde{A}, \left[\widetilde{C}_1^{\mathrm{T}} \ \dots \ \widetilde{C}_N^{\mathrm{T}}\right]^{\mathrm{T}})$ is observable. Indeed, the matrices $Q_{i,j}$ can be obtained from

$$T_{u,i}^{\mathrm{T}}\tilde{x} = \frac{1}{N}\sum_{j=1}^{N} Q_{i,j} T_{o,j}^{\mathrm{T}}\tilde{x} \tag{12}$$

as solutions of

$$[T_{o,1} \ \dots \ T_{o,N}] \begin{bmatrix} Q_{i,1}^{\mathrm{T}} \\ \vdots \\ Q_{i,N}^{\mathrm{T}} \end{bmatrix} = N T_{u,i}, \tag{13}$$

which has always a solution since $\operatorname{rank}([T_{o,1} \ \dots \ T_{o,N}]) = n$.

Remark 4. If the communication graph is not complete, we would require that $x_{u,i} = \frac{1}{N}\sum_{j\in\mathcal{V}_i}^{N} Q_{i,j} x_{o,j}$, where $\mathcal{V}_i$ denotes the set of neighboring nodes of i. This is fulfilled in special cases depending on the graph topology and the output matrices of the observer nodes.

Let $z_{u,i}$ denote an estimate for the unobservable substate $x_{u,i}$. To evaluate each $z_{u,i}$, we use the estimates of the observable substates $z_{o,j}$ of all observer nodes in the form of a consensus term

$$\Psi_i = \frac{1}{N}\sum_{j=1}^{N} (z_{u,i} - Q_{i,j} P_j z_{o,j}). \tag{14}$$

with the transformation matrices $Q_{i,j}$ and P_j defined above. Then the observers for the unobservable part are selected as

$$\begin{aligned} \dot{z}_{u,i} = {} & A_{r,i} z_{o,i} + A_{u,i} z_{u,i} + K_i \Psi_i \\ & + \operatorname{diag}(L_{u,i}) \lceil 1_{n-r_i} E_{(n-r_i)} \Psi_i \rfloor^{\gamma_{u,i}}, \end{aligned} \tag{15}$$

with $K_i \in \mathbb{R}^{(n-r_i)\times(n-r_i)}$, $L_{u,i} \in \mathbb{R}^{n-r_i}$ and $\gamma_{u,i} \in \mathbb{R}^{n-r_i}$ to be selected.

Error system Let $e_{u,i} = z_{u,i} - x_{u,i}$ denote the error in the unobservable substate of observer node i. By adding and subtracting (11) the consensus term can be written in terms of the errors as

$$\begin{aligned} \Psi_i &= z_{u,i} - x_{u,i} + \frac{1}{N}\sum_{i=1}^{N} (Q_{i,j} P_j x_{o,j} - Q_{i,j} P_j z_{o,j}) \\ &= e_{u,i} + \frac{1}{N}\sum_{j=1}^{N} Q_{i,j} P_j e_{o,j}. \end{aligned} \tag{16}$$

With (7) and (15) this yields the error dynamics of the unobservable part as

$$\begin{aligned} \dot{e}_{u,i} = {} & A_{r,i} P_i e_{o,i} + A_{u,i} e_{u,i} + K_i \Psi_i \\ & + \operatorname{diag}(L_{u,i}) \lceil 1_{n-r_i} E_{(n-r_i)} \Psi_i \rfloor^{\gamma_{u,i}}. \end{aligned} \tag{17}$$

Due to finite-time stability, $e_{o,j}(t) = 0$ for all $t \in [t_1, \infty]$ for some $t_1 < \infty$. Then (17) simplifies to the following decoupled error dynamics:

$$\dot{e}_{u,i}(t) = (A_{u,i} + K_i) e_{u,i}(t) + \operatorname{diag}(L_{u,i}) \lceil 1_{n-r_i} e_{u,i,1}(t) \rfloor^{\gamma_i}, \tag{18}$$

for $t > t_1$, demonstrating the decoupling effect of finite-time stability.

We exploit this to design the finite-time observer for the unobservable states, since by taking $K_i = \mathcal{J}_{n-r_i} - A_{u,i}$, with appropriate weights $\gamma_{u,i}$ the systems will be homogeneous with negative degree. In order to make them finite-time stable, we choose $L_{u,i}$ and $\nu_{u,i}$ as in Theorem 1 for each i.

Theorem 5. Under Assumption 3, there exist estimates $\tilde{z}_i$ for each node $i = 1, \dots, N$ in system (4) such that they converge to $\tilde{x}$ in finite time.

Proof. We provide a sketch of the proof here. The estimates are obtained by using:

- state transformations $\tilde{x} = T_i x_i$ into the observability decomposition (7) and $x_{o,i} = T_{c,i} v_i$ into observer canonical form (9)
- observers (10) for $z_{o,i}$ according to Corollary 2
- observers (15) with consensus term (14) for $z_{u,i}$, where

(1) $K_i = \mathcal{J}_{n-r_i} - A_i$
(2) $L_{u,i}$ is selected, so that $\mathcal{J}_{n-r_i} + \operatorname{diag}(L_{u,i}) E_{(n-r_i)}$ is Hurwitz
(3) $-\dfrac{\exp(1)}{2} \dfrac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)\|P_i \operatorname{diag}(L_{u,i})\|} < \nu_{u,i} < 0\,,$

where $Q_i, T_{c,i} \in \mathbb{R}^{(n-r_i)\times(n-r_i)}$ are positive definite solutions of the Lyapunov equation

$$(\mathcal{J}_{(n-r_i)} + \operatorname{diag}(L_{u,i}) E_{n-r_i})^{\mathrm{T}} P_i + P_i (\mathcal{J}_{r_i} + \operatorname{diag}(L_{u,i}) E_{n-r_i}) + Q_i = 0.$$

To complete the proof, it is necessary to consider the boundedness of the trajectories for $e_{o,i} \neq 0$, which follows from input-to-state stability of the estimation error $e_{u,i}$ dynamics obtained from its homogeneity Bernuau et al. (2013), and to define $\tilde{z}_i = T_i \begin{bmatrix} T_{c,i} z_{o,i} \\ z_{u,i} \end{bmatrix}$. □

4. EXAMPLE

Let us consider an example with a system of order $n = 6$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

and $N = 3$ observer nodes

$$C_1 = [1 \ 2 \ 3 \ 0 \ 0 \ 0], \tag{20}$$
$$C_2 = [0 \ 0 \ 2 \ 1 \ 0 \ 0], \tag{21}$$
$$C_3 = [0 \ 2 \ 1 \ 0 \ 1 \ 0]. \tag{22}$$

which is not observable from any single output ($r_1 = r_2 = r_3 = 4$).

For simplicity we chose the same observer gains for each observable and unobservable subsystem

$$L_{o,i} = \begin{bmatrix} -4 \\ -6 \\ -4 \\ -1 \end{bmatrix}, \quad L_{u,i} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}. \tag{23}$$

The degrees of homogeneity where selected as $\nu_{o,i} = \nu_{u,i} = -0.02$. In Fig. 1 the error norm for each substate of the observer nodes in logarithmic scaling are shown, demonstrating the finite-time convergence of the observer.

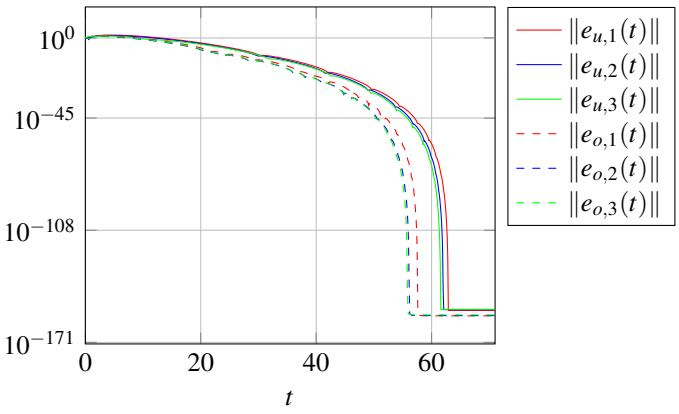

Fig. 1. Error for each observable and unobservale substate in logarithmic scaling

5. CONCLUSION

A distributed finite-time observer design was presented provided that the plant is observable from the combination of all outputs and the communication graph is complete. The proposed method can also be viable for general graph topologies, but this depends strongly on the output matrices of the observer nodes. For brevity, uncertainties like external disturbances and sensor noise were not taken into account in this extended abstract, but robustness with respect to them follows from the input-to-state property of homogeneous systems with negative degree (Bernuau et al., 2013).

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