On the relations between stability optimization of linear time-delay systems and multiple rightmost characteristic roots

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Abstract

Several recent results on spectrum-based analysis and control of linear time-invariant (LTI) time-delay system concern the characterization and exploitation of situations where the so-called multiplicity-induced dominancy property holds, that is, the higher multiplicity of a characteristic roots implies that it is a rightmost root. This direction of research is inspired by observed multiple roots after minimizing the spectral abscissa as a function of controller parameters. However, unlike the relation between multiple roots and rightmost roots, barely theoretical results about the relation of the former with minimizers of the spectral abscissa are available. Consequently, in the first part of the paper the characterization of rightmost roots in such mimimizers is briefly revisited for all second-order systems with input delay, controlled with state-feedback. As the main theoretical results, the governing multiple root configurations are proven to correspond not only to rightmost roots, but also to global minimizers of the spectrum abscissa function. The proofs rely on perturbation theory of nonlinear eigenvalue problems and exploit the quasi-convexity of the spectral abscissa function. In the second part, a computational characterization of minima of the spectral abscissa is made for output feedback, yielding a more complex picture, which includes

configurations with both multiple and simple rightmost roots. In the analysis, the pivotal role of the invariant zeros is highlighted, which translate into restrictions on the tunable parameters in the closed-loop quasi-polynomial.

Keywords: time-delay system, stability, nonlinear eigenvalue problem

1 Introduction

For classes of LTI time-delay systems where the characteristic equation is affine in the controller parameters, recent works have characterized situations where the *multiplicity-induced-dominancy (MID)* property holds, meaning that a sufficiently high multiplicity[1] of a characteristic root implies that it must be a rightmost root, see, e.g., [2–9]. The first systematic study is presented [10], even though some hints are provided in [11]. In [10], an integral representation of the characteristic function has been introduced, which constitutes the methodological basis. More recently, it has been shown in [5] that quasi-polynomials with dominant multiple roots share their remaining roots with some linear combinations of *Kummer hypergeometric functions* [12]. In addition, design approaches have been proposed based on directly assigning multiple roots, which are referred to as *partial pole placement* [3], guaranteeing the exponential stability of closed-loop solutions with the advantage of a prescribed decay. It should be mentioned that to show the dominancy, as discussed in [2, 13] for second-order systems, the argument principle can be used, but then rather as a posterior check. Finally, for further insights on delay-systems stability, see for instance [14–18].

The MID research direction, specific to the delay systems case, is inspired by observed characteristic roots with multiplicity larger than one in minima of the spectral abscissa (i.e., the maximum real part of the characteristic roots). Such minima are revealed by recent algorithms for designing stabilizing controllers with a fixed structure or order that rely on a direct minimization of the spectral abscissa as a function of controller or design parameters [19–21]. Note also that the study of integral proportional-delayed controllers applied to a servo-system in [8] also makes the connection with a multiple characteristic root.

In contrast to the large number of results regarding the relation between multiple roots and rightmost roots, to the best of our knowledge, barely theoretical results are available that address the *connection between multiple roots and minimizers of the spectral abscissa function*, in spite of the computational evidence that motivates the study of MID. A major contributing factor is the complexity of the problem. First, in addition to the fact that the spectral abscissa function is obtained as a maximum over infinitely many characteristic root functions, the interaction between them makes the function non-smooth. Furthermore, for parameter values where a rightmost characteristic root is multiple and defective, the spectral abscissa function may even fail to

[1] In fact, high multiplicity corresponds to a multiplicity larger than the order of the ordinary differential equation corresponding to the delay-free case; sometimes, such a multiplicity is called *over-order* or *intermediate*. Such a multiplicity is finite and is bounded by the Pólya–Szegő bound, see for instance [1].

be locally Lipschitz continuous. Such a situation is often observed in local minimizers [22]. As a consequence, the characteristic equation cannot be supplemented with simple optimality conditions in terms of zero gradient with respect to the controller parameters. Furthermore, the spectral abscissa is in general non-convex and may have multiple local minima, as already observed in the delay-free case [23]. Instead, one may encode the characteristic root configuration in optima. However, the precise configuration(s) of rightmost roots corresponding to minima is difficult to determine a priori (i.e., without executing an optimization algorithm) since it strongly depends on dimensions, model structure and system parameters. Second, the technical arguments used in the proofs of the MID property do not allow making assertions about optimality of the spectral abscissa function. Moreover, in the case of second-order linear systems of retarded type, we may already have several configurations of dominancy by multiple characteristic roots: (i) generic multiplicity[2] 4 (see, e.g., [13]), lowest over-ordered multiplicity 3 (see also [13]), and (iii) the generic double complex-conjugate roots (see some discussions in [24]).

In view of these observations, the theoretical results of the paper are the first where, for a whole class of systems, explicit relations between multiple characteristic roots and global minima of the spectral abscissa as a function of controller parameters are proven. Such relations are relevant, because the ultimate goal of partial pole placement as in [3] is not just to have dominance guaranteed, but to realize a minimum of the spectral abscissa. Note also that an LTI system is exponentially stabilizable if and only the global minimum of the spectral abscissa of the closed-loop system is strictly negative, and that the spectral abscissa characterizes the asymptotic growth or decay rate of solutions. The class of systems considered in this paper consists of *all* SISO controllable second-order systems with input delay, described by

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \quad y(t) = Cx(t), \tag{1}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$, and the control actions considered are both static state feedback and static output feedback. We assume that the plant model is already in controller canonical form,

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_2 & c_1 \end{bmatrix}, \tag{2}$$

with five plant parameters $(a_1, a_2, c_1, c_2, \tau) \in \mathbb{R}^4 \times \mathbb{R}_{>0}$. The corresponding transfer function is given by

$$T(\lambda) = \frac{c_1\lambda + c_2}{\lambda^2 + a_1\lambda + a_2} e^{-\lambda\tau}. \tag{3}$$

Note that the characteristic roots of the open-loop system are equal to

$$\lambda_{1,2} = \begin{cases} -\dfrac{a_1}{2} \pm \sqrt{\dfrac{a_1^2}{4} - a_2}, & \frac{a_1^2}{4} - a_2 \geq 0, \\ -\dfrac{a_1}{2} \pm \sqrt{a_2 - \dfrac{a_1^2}{4}}\;\imath, & \frac{a_1^2}{4} - a_2 < 0, \end{cases}$$

[2]The maximal allowable multiplicity a real characteristic root may have.

where for λ_1 the plus-sign is selected. Hence, the open-loop spectrum can be equivalently described in terms of the pair $(\lambda_{\rm av}, \lambda_{\rm diff})$, defined as

$$\lambda_{\rm av} = \frac{\lambda_1 + \lambda_2}{2}, \quad \lambda_{\rm diff} = \frac{\lambda_1 - \lambda_2}{2}.$$

In the first part of the paper, we consider static state feedback for (1)-(2). It should be noted that the corresponding closed-loop system is represented by a delay-differential equation, see, for instance, [14, 17], for a detailed discussion concerning the fundamental properties and the qualitative behavior of the solutions. In a preliminary step, we revisit the computational characterization of global minima of the spectral abscissa and related rightmost characteristic root configurations from [25]. As a novel element that facilitates the analysis considerably, we show that the plant parameters can be reduced to one parameter, since the delay can be unified by time-scaling, which corresponds to a scaling of the eigenvalue parameter, and only the *relative position* of the two open loop characteristic roots of the scaled system turns out to be determining for the optimized spectrum configuration. Subsequently, we prove that the resulting configurations correspond not only to rightmost roots but also to global minimizers of the spectral abscissa function. Instrumental to the derivation of the latter result is the use of perturbation theory of nonlinear eigenvalue problems [26–28], since the splitting behavior of rightmost multiple roots is key in making assertions about local optimality along given directions. The leap towards global optimality is then based on quasi-convexity of the objective function [25], which on its turns follows from the identification of convex directions of the closed-loop quasi-polynomials in [29]. In this context, we note that it was observed in [13], based on the argument principle (see also [30, 31]), that the sign of the discriminant of the open-loop system plays an important role in the characterization of the MID property in the cases where the multiplicity is 4 (generic case) or 3 (lowest over-ordered multiplicity case). The analysis in the paper allows a better understanding of the underlying properties in terms of parameter variation and optimality.

In the second part of the paper, we use static output feedback to control (1)-(2). A major difference with state feedback is the role played by (invariant) zeros, which are known to severely restrict how characteristic roots can be moved by feedback control, see, e.g., the book [32] and the references therein. This is illustrated with a computational analysis of (1)-(2), where the zero $-c_2/c_1$ is added to the set of plant parameters. As we shall see, the presence of the zero may counteract the tendency of clustering of rightmost characteristic roots towards (a) root(s) of high multiplicity. This results in a range of optimized characteristic root configurations involving both multiple and single roots, depending on the plant parameters. Such a diverse picture has not been observed in previous studies of the MID, because so far restrictions on the tunable parameters in the closed-loop quasi-polynomial, induced by open-loop zeros, have not been taken into account.

Finally, we wish to stress the synergies of analytical and computational methods on which the paper relies. The main focus in Section 2 is to prove properties of global minima of the spectral abscissa function. As a prerequisite, (candidate) global minimizers must be identified among the gain values. The numerical exploration and

observation in [25] suggested us to consider as candidates these gains obtained by directly assigning multiple characteristic roots.

The rest of the paper is organized as follows. In Sections 2 and 3, we address the study of state and output feedback, respectively. In the concluding Section 4, we return to the complexity of the problem and justification of the restriction to second-order system, and we describe perspectives of the work. Throughout the article, the synergies of analytic and computational techniques are highlighted.

2 State feedback

In this section, we consider static state feedback for (1)-(2),

$$u(t) = -K_2 x_1(t) - K_1 x_2(t), \tag{4}$$

such that the characteristic equation of the closed-loop systems is given by

$$\lambda^2 + \left(a_1 + K_1 e^{-\lambda\tau}\right)\lambda + \left(a_2 + K_2 e^{-\lambda\tau}\right) = 0. \tag{5}$$

The roots of the equation (5) are called characteristic roots (see, e.g., [14]) and we refer to [33] and [30] for a deeper discussion of their location as a function of the system's parameters.

2.1 Reduction to one plant parameter

The transformation

$$\lambda = \lambda_{\text{av}} + \frac{\mu}{\tau} \quad \left(= -\frac{a_1}{2} + \frac{\mu}{\tau}\right), \tag{6}$$

applied to (5), leads us to the equation $H(\mu) = 0$, where the characteristic function H is described by

$$H(\mu) = \mu^2 - p + (k_1\mu + k_2)e^{-\mu}, \tag{7}$$

with

$$p = \tau^2\lambda_{\text{diff}}^2 = \frac{(a_1\tau)^2}{4} - a_2\tau^2 \tag{8}$$

and

$$k_1 = K_1\tau e^{-\lambda_{\text{av}}\tau}, \quad k_2 = \left(K_2\tau^2 + K_1\tau^2\lambda_{\text{av}}\right)e^{-\lambda_{\text{av}}\tau}. \tag{9}$$

Note that

$$\mu = \tau(\lambda - \lambda_{\text{av}}). \tag{10}$$

Since there is an invertible linear transformation between the original controller parameters, (K_1, K_2), and the transformed ones, (k_1, k_2), the latter can be considered as free parameters. The spectral abscissa associated with H is the function

$$(k_1, k_2) \mapsto \alpha(k_1, k_2) := \max_{\mu\in\mathbb{C}} \{\Re(\mu) : \ H(\mu;\ k_1, k_2) = 0\}. \tag{11}$$

Recall that, since (1)-(2) and (4) describes a time-delay system of retarded type, the maximum in (11) is well-defined, since the supremum of the real parts of the roots of $H(\cdot;\ k_1, k_2)$ is attained by some root $\mu \in \mathbb{C}$ (see, e.g., [33, Section 1.1.5]).

When characterizing minima of the spectral abscissa function, the plant parameters (a_1, a_2, τ) can thus be reduced to only one real parameter, namely parameter p in (7), whose sign determines whether the plant has two real characteristic roots or a pair of complex conjugate roots. For example, if c is the minimal spectral abscissa of H, as a function of (k_1, k_2), then, by (6), $\lambda_{\text{av}} + \frac{c}{\tau}$ is the minimal spectral abscissa of (1)-(2) and (4), as a function of (K_1, K_2).

Remark 1. The transformation (9) of the gain parameters is dependent on the plant parameters. Hence, if two sets of plants $\left(a_1^{(i)}, a_2^{(i)}, \tau^{(i)}\right)$, $i = 1, 2$, give rise to the same normalized parameter p, then the gains (k_1^*, k_2^*) inducing a globally minimal spectral abscissa c^* in (11), give rise to two set of feedback gains,

$$\begin{bmatrix} K_1^{(i)} \\ K_2^{(i)} \end{bmatrix} = \begin{bmatrix} (\tau^{(i)})^{-1} e^{\lambda_{\text{av}}^{(i)} \tau^{(i)}} & 0 \\ -(\tau^{(i)})^{-1} \lambda_{\text{av}}^{(i)} e^{\lambda_{\text{av}}^{(i)} \tau} & (\tau^{(i)})^{-2} e^{\lambda_{\text{av}}^{(i)} \tau^{(i)}} \end{bmatrix} \begin{bmatrix} k_1^* \\ k_2^* \end{bmatrix},$$

with $\lambda_{\text{av}}^{(i)} = -\frac{a_1^{(i)}}{2}$, $i = 1, 2$. Common is that the feedback with gains $\left(K_1^{(i)}, K_2^{(i)}\right)$ applied to the i-th plant, leads to the minimal closed-loop spectral abscissa $\lambda_{\text{av}}^{(i)} + \frac{c^*}{\tau}$, for $i = 1, 2$.

If $\tau = 1$, it is easy to observe that the transformation from λ to μ is a pure translation. Then, said otherwise, the qualitative change of the spectrum before and after spectral abscissa optimization does not depend on the *absolute* location of the open-loop spectrum (while the relative position of the open-loop characteristic roots can be described with one parameter). However, the required feedback gains depend on the absolute location.

2.2 Description of rightmost root configurations in the optima

As numerically observed in [25] using the software package TDS-CONTROL [21], two generic configurations of rightmost characteristic roots are encountered in the global optimum of (11), depending on the plant parameters. Note that, in [25], the number of plant parameters was reduced to two by a scaling of time, whereas considering one parameter as in this work is sufficient.

The first configuration corresponds to a rightmost real root with multiplicity three, referred to as TR in what follows ('triple real'). Solving the equations

$$H(c;\ k_1, k_2) = H'(c;\ k_1, k_2) = H''(c;\ k_1, k_2) = 0$$

leads us to two branches described by

$$c^{\pm} = \pm\sqrt{2 + p} - 2 \tag{12}$$

and

$$\begin{cases} k_1^{\pm} = \left(\pm 2\sqrt{p+2} - 2\right) e^{\pm c}, \\ k_2^{\pm} = \left(\pm 10\sqrt{p+2} - 2p - 14\right) e^{\pm c}, \end{cases} \tag{13}$$

defined for parameter $p \in [-2, \infty)$. These branches are visualized in the right part of Figure 1. Only the upper branch corresponds to a rightmost root (c^+). In addition, the parameters (k_1^+, k_2^+) globally minimize the spectral abscissa function. Proofs will be provided in Section 2.3.

The second configuration is associated with a pair of complex conjugate rightmost characteristic roots of multiplicity two, referred to as DP in what follows ('double pair'). Solving the equations

$$\begin{cases} \Re H(c + \imath\omega;\ k_1, k_2) = 0 \\ \Im H(c + \imath\omega;\ k_1, k_2) = 0 \\ \Re H'(c + \imath\omega;\ k_1, k_2) = 0 \\ \Im H'(c + \imath\omega;\ k_1, k_2) = 0 \end{cases}$$

in the unknowns (k_1, k_2, p, c, ω), leads us to the parametric curve described by

$$p = -\frac{2\,\omega^2\,\left(\cos(2\,\omega) + 2\,\omega\,\sin(2\,\omega) - 2\,\omega^2 + 2\,\omega^4 - 1\right)}{4(\omega^2 - \sin(\omega))^2}, \tag{14}$$

$$c = -\frac{\omega\,\left(\omega - \frac{\sin(2\,\omega)}{2}\right)}{\omega^2 - \sin(\omega)^2} \tag{15}$$

and

$$\begin{cases} k_1 = -\dfrac{2\,\omega\,(\sin(\omega) - \omega\,\cos(\omega))}{\omega^2 - \sin(\omega)^2}\,e^c, \\ k_2 = \dfrac{2\,\omega^3\,\sin(\omega)}{\sin(\omega)^2 - \omega^2}\,e^c - \dfrac{2\,\omega^2\,(\omega - \cos(\omega)\,\sin(\omega))\,(\sin(\omega) - \omega\,\cos(\omega))}{\left(\omega^2 - \sin(\omega)^2\right)^2}\,e^c, \end{cases} \tag{16}$$

with $\omega \in [0,\ \infty)$. Note here that, for $\omega = 0$, one has to consider the limit $(p, c, k_1, k_2) = \left(-2, -2, -2\,e^{-2}, -10\,e^{-2}\right)$. Expressions (14)-(15) give rise to the curved segment in the left part of Figure 1. This segment emanates from $(p, c) = (-2, -2)$ for $\omega = 0$ and stretches out in the direction of the negative real axis. In $(p, c) = (-2, -2)$, where all branches interact, there is also continuity of the gain values. The figure also clarifies that all values of $p \in \mathbb{R}$ are covered by the two configurations, which is easily shown from (12) and (14).

We note that in [25], a third configuration was identified for specific parameter values, characterized by a rightmost characteristic real root of multiplicity four. This situation occurs for $p = -2$. This spectrum configuration is merely a consequence of the interaction/transition between branches DP and TR.

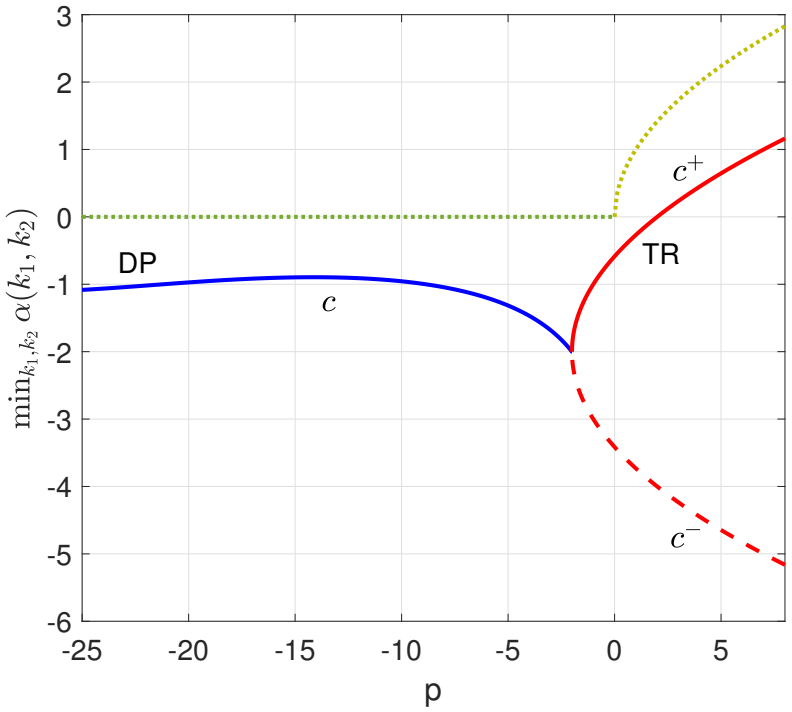


Fig. 1 Global minimum of the spectral abscissa (11) as a function of (normalized) plant parameter p. The segment to the left of $p = -2$ relates to a complex conjugate pair of rightmost characteristic roots with multiplicity two, the two branches to the right to the presence of a real root of multiplicity three. The top branch corresponds to a rightmost root and global minimizer of the spectral abscissa function. The dotted segments depict the spectral abscissa of the open loop system.

2.3 Proofs of dominance of rightmost roots and global optimality

Before stating and proving the main theoretical results, we highlight a property on the behavior of the zeros of (7), relevant to the continuation argument that will be employed in the proofs of Theorems 1-2. Given $\xi \in \mathbb{R}$, any zero μ of (7) with $\Re(\mu) \geq \xi$ necessarily satisfies $|\mu|^2 \leq |p| + e^{-\xi}|k_1||\mu| + e^{-\xi}|k_2|$, showing that

$$|\mu| \leq \frac{e^{-\xi}|k_1|}{2} + \sqrt{|p| + e^{-\xi}|k_2| + \frac{e^{-2\xi}|k_1|^2}{4}}.$$

This means that, as long as the parameters (p, k_1, k_2) remain bounded, zeros of (7) in the right half-plane $\{\mu \in \mathbb{C} : \ \Re(\mu) > \xi\}$ remain in a bounded set, and, hence, cannot go to complex infinity.

In particular, if (p^*, k_1^*, k_2^*) are such that (7) admits no zero μ with $\Re(\mu) > \xi$, then the only possibility for (7) to have a zero μ with $\Re(\mu) > \xi$ for some other values of parameters $(\widetilde{p}, \widetilde{k}_1, \widetilde{k}_2)$ is that, for every continuous path $[0, 1] \ni s \mapsto (p(s), k_1(s), k_2(s))$

in the parameter space with $(p(0), k_1(0), k_2(0)) = (p^*, k_1^*, k_2^*)$ and $(p(1), k_1(1), k_2(1)) = (\widetilde{p}, \widetilde{k}_1, \widetilde{k}_2)$, there exists $s^* \in [0, 1]$ such that, with the parameters $(p(s^*), k_1(s^*), k_2(s^*))$, (7) admits a zero μ with $\Re(\mu) = \xi$, i.e., the zero arrives at the half-plane $\{\mu \in \mathbb{C} : \Re(\mu) > \xi\}$ by crossing the vertical axis $\{\mu \in \mathbb{C} : \Re(\mu) = \xi\}$.

We are now in position to prove the dominance of the multiple rightmost roots corresponding to TR and DP. We start with configuration TR.

Theorem 1. *Let the plant parameter $p \geq -2$ be given. Then the triple real characteristic root c^+, induced by controller parameters (k_1^+, k_2^+) in (13), is a rightmost root.*

Proof. The proof relies on a continuation argument along the branch defined by plant parameter p and controller parameters $(k_1^+(p), k_2^+(p))$, with $p \geq -2$. For $p = -2$, the root $c^+(p)$ has multiplicity four, otherwise it has multiplicity three. Considering $d \in \mathbb{R}$ as a variable, the equation

$$0 = H(c^+(p) + \imath d;\ k_1^+(p), k_2^+(p))$$

implies

$$0 = \left|(c^+(p) + \imath d)^2 - p\right|^2 - \left|k_1^+(p)(c^+(p) + \imath d) + k_2^+(p)\right|^2 e^{-2c^+(p)} = d^4,$$

from which we conclude $d = 0$. Hence, along the branch TR, a complex conjugate pair of characteristic roots cannot cross the line $\Re(\mu) = c^+(p)$. At the same time, for $p > -2$ a crossing of a real eigenvalue at $\mu = c^+(p)$ is not possible, as the multiplicity of $\mu = c^+(p)$ is invariant (always three). Hence, the assertion of the theorem holds if for an arbitrary $p > -2$, the characteristic root $c^+(p)$ is rightmost. The latter has been shown for $p = 0$ (the double integrator case) in [13]. □

Theorem 2. *Let $\omega > 0$ be given. Let the plant parameter p be according to (14), and let c be given by (15). Then the double characteristic roots $c \pm \imath\omega$, which are induced by the controller parameters (16), are the rightmost roots.*

Proof. We follow, once again, an argument based on continuation in ω, inspired by [24].

For $\omega \to 0+$, the pair of roots collapses in a real root with multiplicity four, which satisfies the generic MID property and is therefore a rightmost root [4]. If we now increase ω from zero, adapt the plant parameter in a continuous way, according to $p(\omega)$, and adapt the gain parameters according to $(k_1(\omega), k_2(\omega))$, then the number of characteristic roots with real part equal to $c(\omega)$ is always equal to four (counting multiplicity). To see this, considering $d \in \mathbb{R}$ as a variable, the equation

$$0 = H(c(\omega) + \imath d;\ k_1(\omega), k_2(\omega))$$

implies

$$0 = \left|(c(\omega) + \imath d)^2 - p(\omega)\right|^2 - \left|k_1(\omega)(c(\omega) + \imath d) + k_2(\omega)\right|^2 e^{-2c(\omega)} = (d^2 - \omega^2)^2, \quad (17)$$

where the second equality follows from analytic computations. Therefore, the double roots $c(\omega) \pm \imath d(\omega)$ must be rightmost roots for all $\omega \geq 0$. □

We are now ready to state the two main theoretical results of the paper.

Theorem 3. *For any $p \geq -2$, the controller parameters $\left(k_1^+, k_2^+\right)$ in (13) are strict global minimizers of the spectral abscissa function (11).*

Proof. We first consider the case where p satisfies the strict inequality $p > -2$. The proof is divided into two parts.

Part 1. We show that in *any* direction in the controller parameter space, taken from (k_1^+, k_2^+), the spectral abscissa exhibits a strict increase locally.

We denote by $(d_1, d_2) \in \mathbb{R}^2,\ (d_1, d_2) \neq (0, 0)$, such a direction and by $\epsilon \in \mathbb{R}_{>0}$ a perturbation parameter that determines the step taken in the considered direction. For fixed (d_1, d_2), we then analyze the zeros of the perturbed quasi-polynomial

$$F(\mu;\ \epsilon) = \mu^2 - p + \left((k_1^+ + \epsilon d_1)\mu + (k_2^+ + \epsilon d_2)\right) e^{-\mu}$$

as a function of ϵ, around $\epsilon = 0$. More precisely, we study the root chains that emerge from the splitting of the triple root c^+, which is rightmost by Theorem 1, when ϵ is increased from zero. If along one of these root chains the real part strictly increases, then the spectral abscissa necessarily exhibits a strict increase. We make distinction between three cases.

- In the generic case where the direction satisfies

  $$\frac{\partial F}{\partial \epsilon}(c^+;\ 0) = \left(d_1 c^+ + d_2\right) e^{-c^+} \neq 0,$$

  the triple root exhibits a so-called completely regular splitting [27, 34], which implies that the root functions emerging from c^+ can be expanded as

  $$\mu_i(\epsilon) = c^+ + \left(-3! \frac{\frac{\partial F}{\partial \epsilon}(c^+;\ 0)}{\frac{\partial^3 F}{\partial \mu^3}(c^+;\ 0)}\right)^{\frac{1}{3}} e^{\imath \frac{2\pi}{3}}\ \epsilon^{\frac{1}{3}} + \mathcal{O}\left(\epsilon^{\frac{2}{3}}\right), \quad i = 1, \ldots, 3. \tag{18}$$

  Consequently, there is a real number $\bar{\epsilon} > 0$ such that

  $$\left(\max_{i \in \{1,\ldots,3\}} \Re(\mu_i(\epsilon))\right) > c^+,\ \forall \epsilon \in (0,\ \bar{\epsilon}),$$

  hence, the spectral abscissa is strictly increasing at $\epsilon = 0$. Thus, we can restrict ourselves to the case where $d_1 c^+ + d_2 = 0$ in what follows.
- In case $(d_1, d_2) = (-1, c^+)$, it can be observed that a root at c^+ is *invariant* with respect to ϵ, and we can factorize

  $$F(\mu;\ \epsilon) = (\mu - c^+)\left(\hat{F}(\mu) - \epsilon\, e^{-\mu}\right),$$

where the entire function $\hat{F}$ is the analytic extension of the function $\mu \mapsto \frac{F(\mu;\ 0)}{\mu - c^+}$, which has a removable singularity at c^+. As a result of the deflation, $\hat{F}$ has a double zero at c^+. Clearly, the roots of

$$\hat{F}(\mu) - \epsilon e^{-\mu} = 0 \tag{19}$$

satisfy the completely regular splitting property at $\mu = c^+$ and $\epsilon = 0$. Consequently, they can be expanded as Puiseux series in powers of $\sqrt{\epsilon}$. Substituting a (truncated) series in the equation $F(\mu;\ \epsilon) = 0$ and specifying the values for k_1^+, k_2^+ and c^+ in terms of p lead to

$$\begin{aligned} \mu_{1,2}(\epsilon) &= \sqrt{p+2} - 2 \pm \sqrt{\frac{3}{\sqrt{p+2}} e^{2-\sqrt{p+2}} \sqrt{\epsilon}} + \mathcal{O}(\epsilon), \\ \mu_3(\epsilon) &= \sqrt{p+2} - 2. \end{aligned} \tag{20}$$

Thus, for small $\epsilon > 0$, one of the characteristic roots is shifted to the right, which causes again a strict increase of the spectral abscissa.

- In case $(d_1, d_2) = (1, -c^+)$, the factorization of F leads us to

$$F(\mu;\ \epsilon) = (\mu - c^+) \left(\hat{F}(\mu) + \epsilon e^{-\mu} \right)$$

and we can repeat the same procedure as for the previous case. This brings us to the expansions

$$\begin{aligned} \mu_{1,2}(\epsilon) &= \sqrt{p+2} - 2 \pm \sqrt{\frac{3}{\sqrt{p+2}} e^{2-\sqrt{p+2}} \sqrt{\epsilon}\, \imath + \frac{3 + 6\sqrt{p+2}}{e^{\sqrt{p+2}-2}(16+8p)} \epsilon} + \mathcal{O}(\epsilon^{\frac{3}{2}}), \\ \mu_3(\epsilon) &= \sqrt{p+2} - 2. \end{aligned}$$

Now three terms are needed in the expansion of $\mu_{1,2}$ to be conclusive about a strict increase of the spectral abscissa function. This is the case as the coefficient of ϵ is positive for $p > -2$.
Qualitatively, one characteristic root is invariant if ϵ is increased from zero, while the two other ones split along the imaginary axis but bend towards the open right half-plane.

Part 2. We are now ready to prove strict global optimality at (k_1^+, k_2^+). The argument is by contradiction.

Assume there are coefficients $(\hat{k}_1, \hat{k}_2) \neq (k_1^+, k_2^+)$ such that $\alpha(\hat{k}_1, \hat{k}_2) \leq \alpha(k_1^+, k_2^+) = c^+$. Relying on the notion of convex directions for quasi-polynomials, it has been shown in [25, Proposition 1] that for the considered 2nd order system, the spectral abscissa function is quasi-convex, meaning that every sublevel set is convex. Clearly, the sublevel set $\mathcal{S}_{c^+} = \{(k_1, k_2) :\ \alpha(k_1, k_2) \leq c^+\}$ contains both $(\hat{k}_1, \hat{k}_2)$ and (k_1^+, k_2^+).

As this set is convex, if follows that

$$(k_1^+, k_2^+) + \epsilon(d_1, d_2) \in \mathcal{S}_{c^+},\ \forall \epsilon \in [0,\ 1],$$

with $(d_1, d_2) = (\hat{k}_1 - k_1^+, \hat{k}_2 - k_2^+)$, which implies on its turn that

$$\alpha\left((k_1^+, k_2^+) + \epsilon(d_1, d_2)\right) \leq c^+,\ \forall \epsilon \in [0,\ 1].$$

This property is in contradiction with the strict local increase of the spectral abscissa in direction (d_1, d_2).

Finally, we address the special case where $p = -2$, for which c^+ is a rightmost root with multiplicity four. Similar arguments as in the proof for $p > -2$ can be used. The analysis even simplifies considerably since (18) and (20) have to be replaced by expansions in $\epsilon^{\frac{1}{4}}$ and $\epsilon^{\frac{1}{3}}$, respectively. As a consequence, two terms in the expansions are already sufficient to conclude about a strict increase of the spectral abscissa in the considered direction. The proof is finished. □

Theorem 4. *For any $\omega > 0$, the controller parameters (16) are strict global minimizers of the spectral abscissa function (11) when the plant parameter p is specified as (14).*

Proof. It is sufficient to show that in any direction in the parameter space, taken from $(k_1(\omega), k_2(\omega))$, the spectral abscissa exhibits a strict increase locally, in view of the argument used in Part 2. of the proof of Theorem 3.

For direction $(d_1, d_2) \in \mathbb{R}^2,\ (d_1, d_2) \neq (0, 0)$, we consider the perturbed quasi-polynomial

$$F(\mu;\ \epsilon) = \mu^2 - p(\omega) + ((k_1(\omega) + \epsilon d_1)\mu + (k_2(\omega) + \epsilon d_2))\, e^{-\mu} \tag{21}$$

and study the root chains that emerge from the splitting of the root $c(\omega) + \imath\omega$, with $c(\omega)$ given by (15). Note that this root is rightmost, following from Theorem 2. The multiplicity of this root is exactly two and cannot be larger, following from (17). The conditions for a completely regular splitting are always satisfied, since $d_1(c(\omega)+\imath\omega)+d_2$ cannot be zero. It follows that the two emerging characteristic root chains can be expanded as

$$\mu^{\pm}(\epsilon) = c(\omega) + \imath\omega \pm \left(-2!\frac{\frac{\partial F}{\partial \epsilon}(c(\omega) + \imath\omega;\ 0)}{\frac{\partial^2 F}{\partial \mu^2}(c(\omega) + \imath\omega;\ 0)}\right)^{\frac{1}{2}} \sqrt{\epsilon} + \mathcal{O}(\epsilon)\,.$$

A strict increase of the spectral abscissa follows from this first-order expansion, unless the direction is such that

$$\left(-2!\frac{\frac{\partial F}{\partial \epsilon}(c(\omega) + \imath\omega;\ 0)}{\frac{\partial^2 F}{\partial \mu^2}(c(\omega) + \imath\omega;\ 0)}\right) \in \mathbb{R}_{<0}.$$

That (degenerate) case requires

$$d_1(\omega) = \frac{2\,\omega^3\,\cos(\omega) - 2\,\sin(\omega) + 2\cos(\omega)^2\,\sin(\omega)}{\omega^2 + \cos(\omega)^2 - 1},$$
$$d_2(\omega) = \frac{2\,\omega\left(2\,\cos(\omega) + 3\,\omega^3\,\sin(\omega) - \omega^5\,\sin(\omega) - 4\cos(\omega)^3 + 2\cos(\omega)^5 - 3\,\omega\,\sin(\omega) - 2\,\omega^3\cos(\omega)^2\,\sin(\omega) + 3\,\omega\cos(\omega)^2\,\sin(\omega)\right)}{\left(\omega^2 + \cos(\omega)^2 - 1\right)^2},$$

up to multiplication with a positive scalar. Substituting these expressions in (21) and expanding one of the two emanating root chains (denoted by μ_1) in a Puiseux series yields, after tedious computations,

$$\mu_1(\epsilon) = c(\omega) + \imath\sqrt{2\omega}\sqrt{\epsilon} + \kappa(\omega)\epsilon + \mathcal{O}\left(\epsilon^{\frac{3}{2}}\right),$$

with

$$\Re(\kappa(\omega)) = \frac{27\,\omega - \frac{15\,\sin(2\,\omega)}{4} + 3\,\sin(4\,\omega) - \frac{3\,\sin(6\,\omega)}{4} - 36\,\omega\,\cos(2\,\omega) + 9\,\omega\,\cos(4\,\omega) - 8\,\omega^3\,\cos(2\,\omega) - 10\,\omega^3\,\cos(4\,\omega)}{6N(\omega)}$$
$$+ \frac{40\,\omega^5\,\cos(2\,\omega) - 24\,\omega^2\,\sin(2\,\omega) + 12\,\omega^2\,\sin(4\,\omega) - 36\,\omega^4\,\sin(2\,\omega) + 24\,\omega^6\,\sin(2\,\omega) + 18\,\omega^3 - 16\,\omega^5 + 16\,\omega^7}{6N(\omega)}$$

and

$$N(\omega) = 2\,\omega\,\sin(4\,\omega) - 4\,\omega\,\sin(2\,\omega) - 2\,\cos(2\,\omega) - 8\,\omega^2\,\cos(2\,\omega) + 10\,\omega^4\,\cos(2\,\omega)$$
$$-8\,\omega^3\,\sin(2\,\omega) + 4\,\omega^5\,\sin(2\,\omega) + \cos(2\,\omega)^2 - 2\,\omega^2\cos(2\,\omega)^2 + 10\,\omega^2 - 6\,\omega^4 + 4\,\omega^6 + 1. \tag{22}$$

The function $\omega \mapsto \Re(\kappa(\omega))$, which satisfies the expansion

$$\Re(\kappa(\omega)) = \frac{76\,\omega^3}{525} - \frac{452\,\omega^5}{39375} + \frac{125528\,\omega^7}{227390625} + O\left(\omega^9\right)$$

and $\lim_{\omega\to\infty} \Re(\kappa(\omega))/\omega = 4$, is positive for $\omega > 0$, so the root chains bend towards the open right half-plane. Hence, also in the direction corresponding to the degenerate case, the spectral abscissa necessarily increases. □

2.4 Back to the original plant parametrization

The split between the two characteristic root configurations in the minimum of the spectral abscissa is at $p = -2$, or $a_2\tau^2 - \frac{(a_1\tau)^2}{4} = 2$. The following result is a direct corollary from Theorems 3-4.

Corollary 5. *Consider system (1)-(2), controlled with static state feedback (4). Then the spectral abscissa function is quasi-convex. Moreover, the following results hold:*

- *if $a_2\tau^2 - \frac{(a_1\tau)^2}{4} > 2$, then the global minimum of the spectral abscissa function is characterized by a complex conjugate pair of rightmost characteristic roots of multiplicity two;*

- *if* $a_2\tau^2 - \frac{(a_1\tau)^2}{4} < 2$, *then the global minimum of the spectral abscissa function is characterized by a rightmost real root of multiplicity three;*
- *If* $a_2\tau^2 - \frac{(a_1\tau)^2}{4} = 2$, *then the global minimum of the spectral abscissa function is characterized by a rightmost real root of multiplicity four.*

Remark 2. Corollary 5 emphasizes the role of the (scaled) discriminant of the open-loop system in the classification of minima of the spectral abscissa. Note that in the study of dominancy of multiple characteristic roots (over-ordered real and/or generic complex-conjugates) the discriminant plays a similar role [13].

In terms of the original plant parameters a_1, a_2 and τ, the minimal spectral abscissa of (1)-(2) and (4), called C in what follows, can be determined from the results in this section, combined with (6) and (8). It suffices to compute p from the plant parameters (a_1, a_2, τ), evaluate the minimal spectral abscissa of H, visualized in Figure 1, and transform the resulting value c back via the relation $C = \lambda_{\text{av}} + \frac{c}{\tau}$. A translation of Figure 1 to the original plant parameters gives rise to Figure 2, obtained in [25] from numerical optimization experiments with the software TDS-CONTROL. We also represent in Figure 2, in red, the curve $a_2\tau^2 - \frac{(a_1\tau)^2}{4} = 2$, corresponding to $p = -2$ and characterizing the changes of regimes described in Corollary 5. The line segments in the level sets originate from the case $p \geq -2$. To conclude the session, we derive an explicit expression for these segments.

Using (8), the condition $p \geq -2$ can be rephrased as

$$a_2\tau^2 - \frac{(a_1\tau)^2}{4} \leq 2.$$

Note that the parabola visualized in Figure 2 is described by $a_2\tau^2 - \frac{(a_1\tau)^2}{4} = 2$. Hence, we are considering plant parameters corresponding to points below this parabola. From (6) and (12) the relation between C and c is described by

$$C = \lambda_{\text{av}} + \frac{\sqrt{p+2} - 2}{\tau},$$

leading to

$$C\tau = -\frac{a_1\tau}{2} + \sqrt{\frac{(a_1\tau)^2}{4} - a_2\tau^2 + 2} - 2$$

and

$$a_2\tau^2 = -(2 + C\tau)(a_1\tau) - \left((C\tau)^2 + 4C\tau + 2\right),$$

which obviously represents a line segment in the $(a_1\tau, a_2\tau^2)$-parameter space.

3 Output feedback

In this section we computationally analyze the minima of the spectral abscissa function of the feedback interconnection of (1)-(2) and the output feedback controller

$$u(t) = -Ky(t),$$

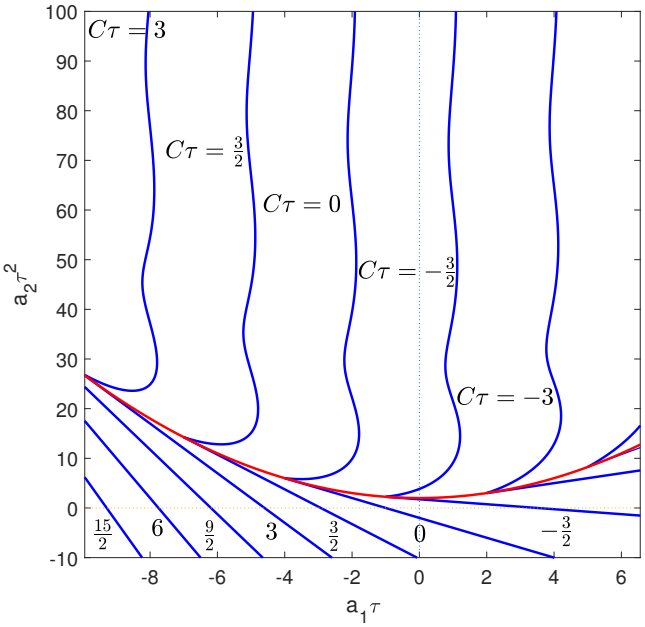


Fig. 2 Level sets of the global minimum of the spectral abscissa of (1)-(2) and (4) in the plant-parameter space (a_1, a_2, τ). The level is indicated with the value of $C\tau$.

where we reveal the strong influence of the zero of the transfer function. The control law induces the characteristic equation

$$\lambda^2 + \left(a_1 + Kc_1e^{-\lambda\tau}\right)\lambda + \left(a_2 + Kc_2e^{-\lambda\tau}\right) = 0.$$

We assume that the zero of (3) is finite or, equivalently, $c_1 \neq 0$ (some comments on the case $c = 0$ will be formulated in Remark 3).

The transformation (6) now leads to the equation $H(\mu) = 0$, with

$$H(\mu) = \mu^2 - p + k(\mu - z)e^{-\mu},$$

with normalized plant parameter p given by (8) and

$$k = Kc_1\tau e^{-\lambda_{\mathrm{av}}\tau}, \quad z = -\frac{c_2\tau}{c_1} + \frac{a_1\tau}{2}.$$

Clearly k can be considered as a free parameter. Note that $z = \tau(-c_2/c_1 - \lambda_{\mathrm{av}})$, which is consistent with (10). By the transformation, the plant parameters are condensed from $(a_1, a_2, c_1, c_2, \tau)$ into the pair (p, z).

The movement of characteristic roots is restricted by the presence of the zero, which is indicated by the following proposition.

Proposition 6. *If the transfer function of the plant has no pole-zero cancellation, then for any k, there cannot be a characteristic root equal to the zero.*

Proof. The argument is by contradiction. Suppose that $H(\mu;\ k) = 0$ for $\mu = z$ and some $k = \hat{k}$. Then

$$0 = H(z;\ \hat{k}) = z^2 - p = H(z;\ 0).$$

Hence, z is also a characteristic root of the uncontrolled system, whose transfer function must have a pole-zero cancellation. This contradicts the assumption. □

By Proposition 6, the loci of characteristic roots as a function of k cannot pass through the zero. As we shall see, this may affect the type of minima of the spectral abscissa function.

If we minimize the spectral abscissa as a function of k for different given couples (p, z), the following three generic rightmost characteristic root configurations occur, which will be illustrated later on:

(NS1) A non-smooth minimum characterized by (the splitting of) a double real root, governed by the equations

$$H(c;\ k) = H'(c;\ k) = 0$$

in unknowns (k, c).

(NS2) A non-smooth minimum characterized by a simple real root and a pair of complex conjugate simple roots overtaking each other as rightmost roots. This situation necessitates the satisfaction of

$$H(c;\ k) = 0,\ \Re H(c + \imath\omega;\ k) = 0,\ \ \Im H(c + \imath\omega;\ k) = 0$$

in unknowns (k, c, ω).

(SO1) A smooth minimum where the real part of an isolated pair of complex conjugate characteristic roots exhibits a minimum as a function of k, described by

$$\Re H(c + \imath\omega;\ k) = 0,\ \Im H(c + \imath\omega;\ k) = 0,\ \Re\left(\frac{\frac{\partial H}{\partial k}(c + \imath\omega;\ k)}{\frac{\partial H}{\partial \lambda}(c + \imath\omega;\ k)}\right) = 0.$$

Note that a smooth minimum where an isolated real root exhibits a minimum as a function of k is *not* a generic situation, as the necessary conditions

$$H(c;\ k) = 0,\ \frac{\partial H}{\partial k}(c;\ k) = 0$$

lead to the special condition $z^2 = p$ on the plant parameters and a pole-zero cancellation. The configurations DP and TR for state-feedback are also not generic situations. For example, requiring a triple real roots leads us to three equations in two unknowns (c, k). However, these configurations may appear when *freeing* a plant parameter. In

the (p, z)-parameter space, configuration DP is described by

$$p = -\frac{\omega^2 \left(\cos(2\omega) + 2\omega \sin(2\omega) - 2\omega^2 + 2\omega^4 - 1\right)}{2\left(\omega^2 - \sin(\omega)^2\right)^2},$$

$$z = -\frac{\omega^2 \sin(\omega)}{\sin(\omega) - \omega \cos(\omega)} - \frac{\omega \left(\omega - \cos(\omega) \sin(\omega)\right)}{\omega^2 - \sin(\omega)^2},$$

parametrized by $\omega \in [0, \infty)$ and shown in the left part of Figure 3 (again the limit to be considered for $\omega = 0$). For given $p \leq -2$, there is one value of the zero, which we call z^*, for which DP occurs. Stated otherwise, only when $z = z^*$, the optimal state-feedback can be realized by output feedback. The configuration TR on its turn is described by

$$z = \frac{k^2 - 6k + 4}{2k}, \; p = \frac{k^2}{4} + k - 1, \; c = \frac{k}{2} - 1$$

and visualized in the right part of Figure 3. The full curves correspond to $c = c^+$ $(k \geq -2)$, the dashed one to $c = c^-$ $(k < -2)$. Vertical asymptotes of the functions corresponding DP and TR occur if the optimal state feedback corresponds to pure proportional feedback, which can be interpreted as output feedback with the transfer function zero located at infinity. The curves corresponding DP and TR in Figure 3 form relevant separatrices of configurations of rightmost roots in the global minimum of the spectral abscissa. A similar role is played by the curve $z = \pm\sqrt{p}$, corresponding to a pole-zero cancellation, and also displayed in the figure.

In Figure 4 we depict the minimal value of the spectral abscissa as a function of the transfer function's normalized zero z, for $p = 5$, inducing open loop poles $\pm\sqrt{5}$. For this value of p and different values of z, indicated in Figure 4, we show in Figure 5 the real parts of the rightmost characteristic roots and their root locus as a function of controller parameter k, around the minimizer. For $z = 2.8$ (panel (a)) the two poles of the transfer function are to the left of the zero. Shifting the rightmost pole to the left leads to an interaction with the other pole of type NS1. For $z = 1$ (panel (b)) the rightmost pole lies slightly to the right of the zero. When shifting this pole to the left by decreasing k from zero, it can neither interact with another real characteristic root because of the (blocking) zero, nor jump over this zero (as a consequence of Proposition 6). Hence, a minimum of the spectral abscissa is naturally reached when a complex pair takes over the role of rightmost characteristic roots, inducing a minimum of type NS2. For $z = z^* \approx 0.747$ (panel (c)), four interacting roots create configuration TR. For $z = -1$ (panel (d)), the starting point is similar to the one for $z = 1$, with the difference that the rightmost pole is at a larger distance from the zero. This allows the complex pair coming from the left to bifurcate on the real axis, before one of the emanating real roots interacts with the rightmost real root (minimum of type NS1). Finally, for $z = -3$ (panel (e)), the two poles are to the right of the zero, which does not prevent them interacting, and we get a minimum of type NS1, as in the first case. Clearly, the occurrence of NS2 in some parameter range is caused by the presence of the blocking zero.

Finally, we consider situations where p is fixed to -10, inducing the open loop poles $\pm\sqrt{10}\imath$. In Figure 6 we show the minimal value of the spectral abscissa as a function of

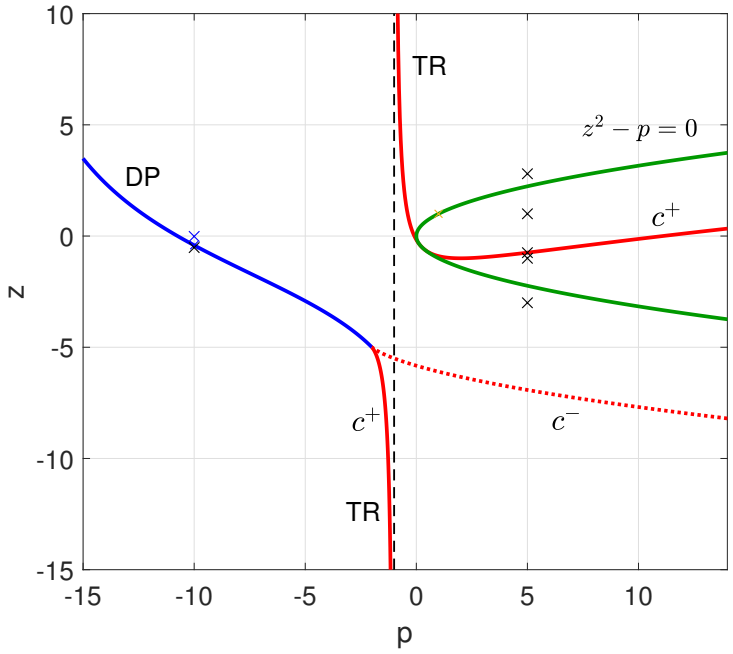


Fig. 3 Curves corresponding to configurations DP, TR and a pole-zero cancellation in the (normalized) plant parameter space (p, z). The plant parameters indicated with crosses refer to the experiments reported in Figures 5 and 7 and the accompanying text.

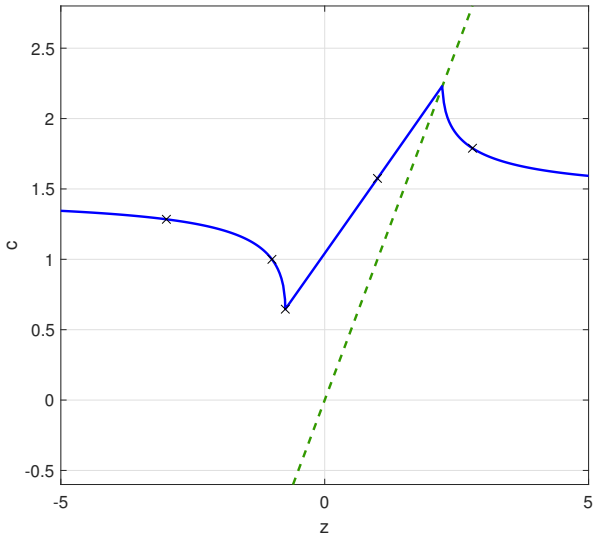


Fig. 4 Minimum spectral abscisa c as a function of plant parameter z, for $p = 5$. The dashed line is described by $c = z$. The values of the zeros, indicated with crosses, refer to the experiments reported in Figure 5 and the accompanying text.

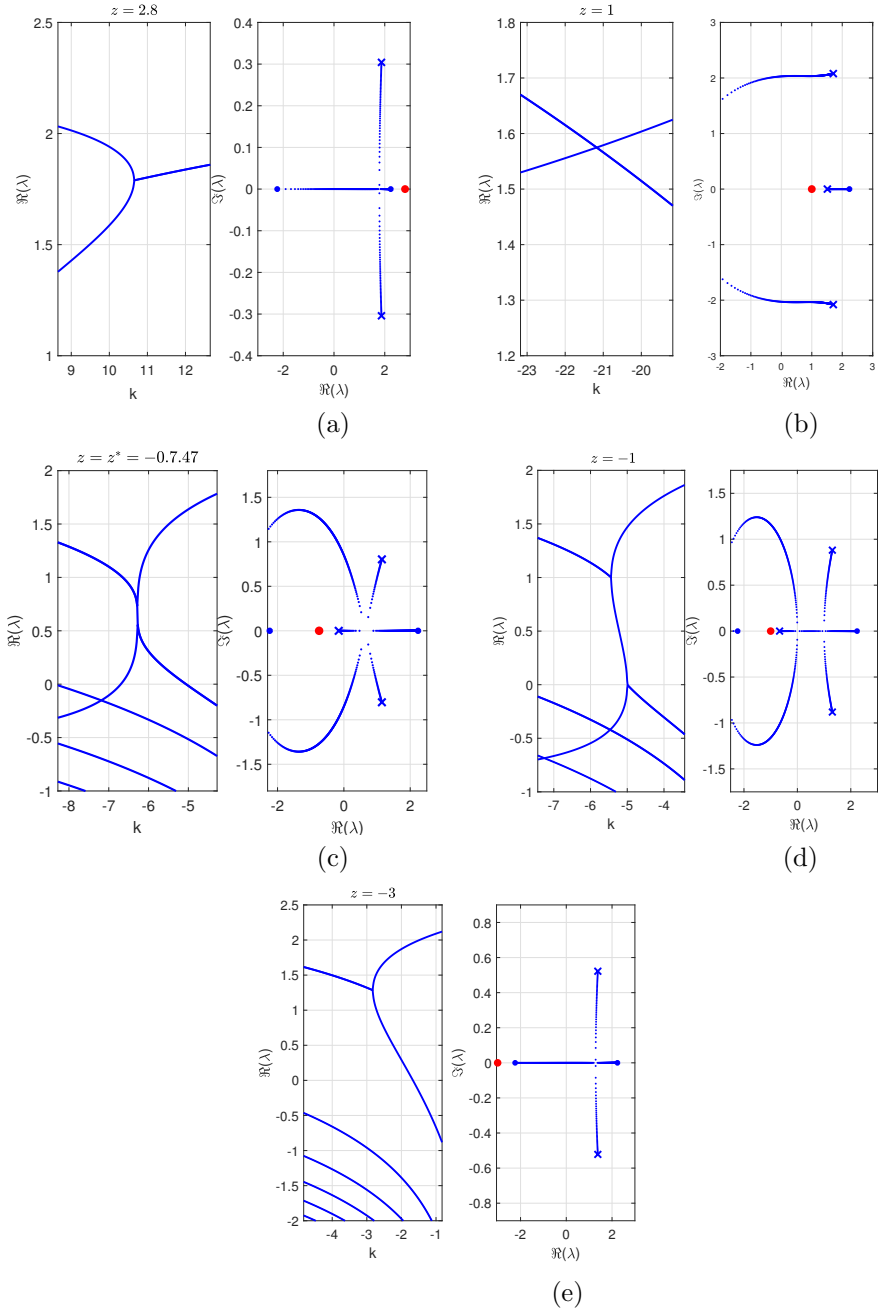


Fig. 5 Real parts of the rightmost characteristic roots as a function of controller parameter k and associated root locus around the global minimum, for $p = 5$ and different values of z, indicated in Figure 4. In the root locus, the branches go from the blue dots to the blue crosses for increasing $|k|$. The red dot indicates the plant's zero location.

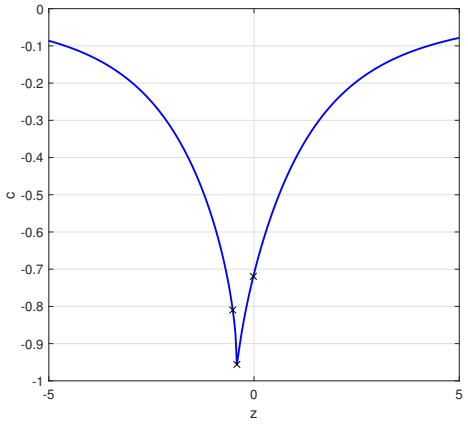


Fig. 6 Minimum spectral abscissa c as a function of plant parameter z, for $p = -10$. The values of the zeros, indicated with crosses, refer to the experiments reported in Figure 7 and the accompanying text.

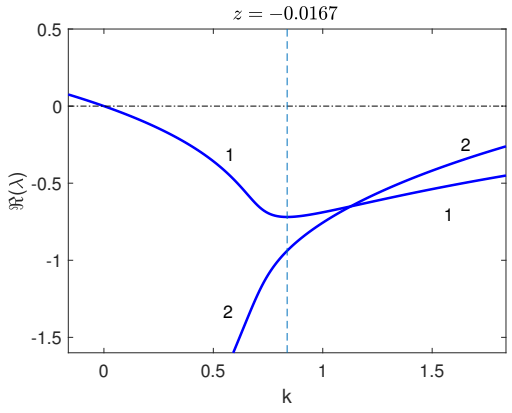
z, around $z = z^*$, where situation DP occurs. In Figure 7 we then visualize the spectral abscissa as a function of k, around the global minimizere. For both $z = -0.0167 > z^*$ (panel (a)) and $z = -0.5184 < z^*$ (panel (c)), a smooth minimum of type SO1 occurs, i.e., a pair of characteristic roots that moves to the left in the complex plane, bends and moves back to the right, without interaction with other roots. Interestingly, at $z = z^* \approx -0.4156$ (corresponding to DP and panel (b)), two branches of complex pairs are broken into segments that are reconnected differently, as indicated by the numbers in panel (a) and (c).

Remark 3. The case $c_1 = 0$ can be rephrased in terms of the study in [8] on proportional-retarded controllers, provided a_2 is related to a fixed proportional gain and τ is considered as a tunable controller parameter. For $a_1 > 0$, $a_2 > 0$, the minimum of the spectral abscissa as a function of k and τ was related to a real characteristic root of multiplicity three in this reference.

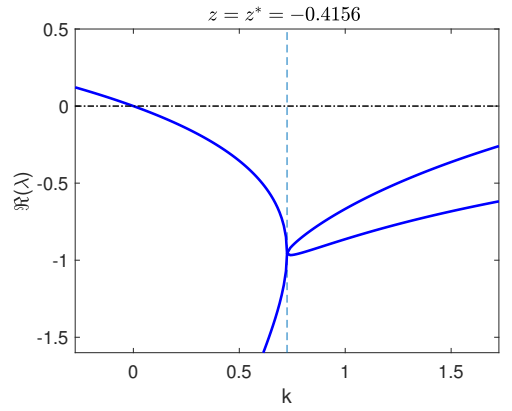
4 Concluding remarks

In the paper, we established connections between global optima of the spectral abscissa function and configurations of rightmost characteristic roots, supported by theory and/or numerical experiments.

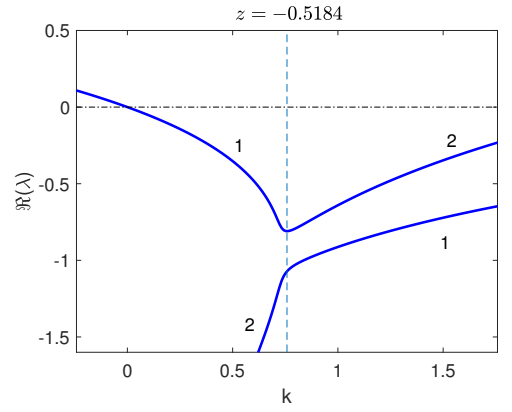
The analysis was restricted to the class of all second-order LTI plants with input delay. This is justified given the high complexity of the problem, which is barely explored, even in the context of stability optimization of delay-free systems. In particular, as a major theoretical contribution in Section 2, the observations on global optima of the spectral abscissa and associated spectrum configurations involving multiple characteristic roots were all confirmed by mathematical proofs. The combination of observation from numerical results, formalizing observations and proving them demonstrates synergies of analytical and computational approaches in the study



(a)



(b)



(c)

Fig. 7 Real parts of the rightmost characteristic roots as a function of controller parameter k around the global minimum, for $p = -10$ and different values of z, indicated in Figure 6. The branch numbers indicate how segments are reconnected at $z = z^*$.

of time-delay systems. Furthermore, many high-dimensional SISO systems can be accurately approximated by a second-order system plus an input-output delay

For $n > 2$, the characterization of global optima is even more challenging. The starting point in our analysis was the determination of possible spectrum configurations in optima. The analysis in Section 3 already showed that this is not trivial even for one tunable controller parameter. It is expected that the number of configurations quickly grows in the dimension n and the number of controller parameters, while also multiple local optima may co-exist [25]. Even when restricting the analysis to one particular class, namely the multiple integrator with input delay, the adopted techniques are challenged for $n > 2$. For this class, numerical experiments up to $n = 10$ and for the state feedback case indicate that there is still a unique global minimizer of the spectral abscissa, characterized by a real rightmost root of multiplicity $n + 1$. The argument for local optimality along fixed directions in the proof of Theorem 3, which is based on the splitting of a multiple root, can most likely be generalized to any n, by use of recursion. However, the argument to turn such local optimality into global optimality, relying on quasi-convexity, already fails for $n = 3$, where the characteristic equation of the closed-loop system is of the form

$$\lambda^3 + (K_1\lambda^2 + K_2\lambda + K_3)e^{-\lambda\tau} = 0.$$

For $\tau = 0$ the stability region in the controller parameter space can be determined analytically using Routh's criterion, which results in the conditions $K_1 > 0, K_2 > 0, K_3 > 0$ and $K_3 < K_1K_2$. As the stability region is obviously non-convex, the spectral abscissa function cannot be quasi-convex.

On the other hand, the determination of gains by imposing a specific characteristic roots configuration, was only used instrumental in the methodology of proving connections between multiple roots and minimizers of the spectral abscissa of (1) under control law (4). For the control design of generic LTI time-delay systems, recently spectrum based approaches have been integrated in the package TDS-CONTROL [21], which can handle retarded type, neutral type and descriptor systems with discrete delays in state, inputs and outputs, and a broad class of controller structures. This package is conceptually similar to the package HIFOO [23] for designing fixed-order controllers for LTI delay-free systems, focusing on the case where the order of the controller is smaller than the order of the system. The stabilization functionality of these packages is based on numerically minimizing the spectral abscissa function using dedicated algorithms for non-smooth optimization. Even though only convergence to local minimizers can be guaranteed (a set of initial controllers is randomly chosen and/or supplied by the user), these tools turn out to be very performant in applications. They do not require any prior knowledge about spectrum configurations in optima.

Acknowledgments

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