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STABILISATION OF TIME-DELAY SYSTEMS USING NONSMOOTH OPTIMISATION TECHNIQUES

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Abstract: We present an optimisation based approach for the stabilisation of linear time-delay systems with a linear static state feedback controller. As time-delay systems have infinitely many eigenvalues, a full assignment of the spectrum is impossible. However, if the system is stabilisable with the given controller structure, stability can always be achieved by minimising the real part of the rightmost eigenvalue, or spectral abscissa, as a function of the controller parameters. The spectral abscissa is in general a nonsmooth and nonconvex function, eliminating the use of standard optimisation methods. The proposed stabilisation method consists of a combination of a recently developed bundle gradient algorithm for nonsmooth optimisation on the one hand, and tools to compute the rightmost, stability determining eigenvalues of time-delay systems on the other hand. The resulting algorithm has been applied to several numerical examples, both theoretic and practical. As a by-product, the use of bundle gradient methods to solve eigenvalue optimisation problems for infinite-dimensional systems is illustrated. c

Keywords: stabilisation methods, closed-loop control, time delay, optimisation

1. INTRODUCTION

Time delays are common phenomena in a wide range of scientific domains, e.g. industrial processes, engineering systems, economical systems, biological and chemical processes. They often appear very naturally whenever there is some sort of dependency on the past. For example, they are introduced by transportation or communication lags, originate from feedback delays in control loops, or emerge when distributed parameter systems are approximated by low-order models with delays.

Since these time delays may have a large impact on the stability and dynamic behaviour of a system, it is important to take them into account explicitly in the mathematical models, which leads to the use of delay differential equations (Hale and Verduyn Lunel, 1993; Niculescu, 2001).

In this paper we address the stabilisation of the general linear time-delay system

$$\dot{x}(t) = \sum_{i=0}^{r} A_i x(t-\tau_i) + \sum_{j=0}^{s} B_j u(t-\tau_j), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, A_i and B_j are matrices of appropriate dimensions and $\tau_i, \tau_j \geq 0$ are the fixed delays.

This system is controlled by a linear static state feedback (extension to other types of fixed-order controllers is straightforward) of the form

$$u(t) = Kx(t), \quad K \in \mathbb{R}^{p \times n} \quad (2)$$

with controller parameters contained in K.

The stability of the closed-loop system (1)-(2) is determined by the real part of the rightmost eigenvalue, also called the spectral abscissa $\alpha(K)$:

$$\alpha(K) = \sup\{\Re(\lambda) : \lambda \text{ is an eigenvalue}\}. \quad (3)$$

The proposed stabilisation method consists of solving directly the following nonsmooth optimisation problem

$$\min_K \ \alpha(K). \tag{4}$$

From a pure stabilisation point of view, it suffices to obtain gain values where $\alpha(K) < 0$, since the asymptotic behaviour of the trajectories of the associated dynamical system is characterised by a decay rate of $\mathrm{e}^{\alpha(K)}$. The gain values in which the minimum of (4) is reached therefore yield some sort of 'optimal' stability.

As will be explained in depth, recently developed optimisation algorithms are used to tackle (4), combined with numerical methods to compute the rightmost eigenvalues of the closed-loop time-delay system, needed to be able to evaluate the objective function.

The main difference with a related method, namely the continuous pole placement method (Michiels *et al.*, 2002), lies in the fact that no eigenvalue continuation strategy, tailored to the specific time-delay case, is followed to reduce the objective function $\alpha(K)$. Instead, a *general purpose* optimisation method for dealing with nonsmooth, nonconvex objective functions is used, namely the gradient sampling algorithm developed by Burke *et al.* (2005*a*). This method, as a member of the so-called bundle gradient methods, requires only that the objective function be differentiable almost everywhere.

Methods for nonsmooth, nonconvex optimisation have already been successfully applied to finite-dimensional eigenvalue optimisation problems arising in controller design where the number of controller parameters is much smaller than the problem dimension. See for instance (Burke *et al.*, 2005*a*; Burke *et al.*, 2005*b*; Noll and Apkarian, 2005; Mammadov and Orsi, 2005), and the references therein. Examples include the design of suboptimal $\mathcal{H}_\infty$-controllers with a fixed, low-order controller structure, and static output feedback stabilisation problems. The application of such methods to stabilise time-delay systems can be regarded as an extension to an infinite-dimensional setting, as the number of controller parameters is finite but the number of eigenvalues is infinite.

Other stabilisation approaches for time-delay systems include, amongst others, Lyapunov based methods (where stabilisability conditions and the resulting controller are typically expressed by the solvability of algebraic Riccati equations or by the feasibility of linear matrix inequalities), algebraic methods (delay systems viewed as systems over rings etc.), methods based on prediction (where an infinite-dimensional controller yields a finite number of closed-loop eigenvalues) and methods based on a spectral decomposition. See (Gu *et al.*, 2003; Niculescu, 2001; Richard, 2003) for an overview.

The main advantage of the eigenvalue optimisation approach is that it does not introduce any conservatism, which means that stabilising gain values can in principle be found if they exist. Furthermore, it uses a simple controller structure and is generally applicable to systems involving multiple, independent delays in state, inputs and outputs.

Another advantage is that a broad class of model transformations to bring a system into the form (1)-(2) can be dealt with, despite the fact that these may introduce additional dynamics (Gu and Niculescu, 2000). Indeed, the non-physical eigenvalues, introduced for example by the transformation taking care of distributed delays, are known a priori and can be removed explicitly after an eigenvalue computation. (Luzyanina and Roose, 2004)

The structure of this paper is as follows. We first discuss some general properties of the spectral abscissa function (3). These will ratify the choice of the optimisation algorithm. The stabilisation method is then outlined, and the used tools and algorithms are further highlighted. Finally some numerical examples illustrate the applicability of the proposed method.

2. THE STABILISATION METHOD

2.1 *Properties of the objective function* $\alpha(K)$

Due to the presence of the supremum operator in (3), $\alpha(K)$, the function to be minimised, is not smooth. To recognise this, let us first define an *active* eigenvalue λ as an eigenvalue for which $\Re\{\lambda\} = \alpha(K)$. At gain values where more than one eigenvalue is active, it is easy to see that the objective function is not differentiable. Moreover, if an active eigenvalue has an algebraic multiplicity larger than its geometric multiplicity, then $\alpha(K)$ is not even Lipschitz continuous. The system $\ddot{x} = kx$ illustrates this, since it yields

$$\alpha(k) = \begin{cases} 0 & k \leq 0, \\ \sqrt{k} & k \geq 0, \end{cases} \tag{5}$$

which indeed exhibits a non-Lipschitz point at zero, corresponding to two coinciding active eigenvalues. As in this example, it is often the case that the minimum of the optimisation problem will occur at a point where the eigenvalues have a multiplicity larger than 1. This is also well known in the context of eigenvalue optimisation problems for finite-dimensional systems. Typical for the delay case is that the number of active eigenvalues and even their multiplicity may be larger than the system's dimension (Michiels and Roose, 2002).

Since eigenvalues with higher multiplicities are numerically ill conditioned, computing them with high accuracy is hard. However, because these eigenvalues are very sensitive w.r.t. changes of the controller parameters or the delay(s), a high precision of the minimum is not desirable, since large sensitivities have a negative impact on the stability in terms of robustness. Therefore, a rough approximation to the minimum is mostly satisfactory, which can then for example be used as a starting point for a robust stabilisation process. See for example (Michiels and Roose, 2003; Burke *et al.*, 2003) for a more elaborate explanation of this two-stage approach.

Although $\alpha(K)$ is nonsmooth, as described above, it is important to note that the nonsmoothness only occurs on a set with measure zero. Indeed, when K is chosen randomly in the controller parameter space, the derivative of $\alpha(K)$ will exist with probability 1, meaning that the function is differentiable a.e. A sufficient constraint regarding differentiability is that there is only one active eigenvalue (pair), or, in other words, that the rightmost eigenvalue (pair) is isolated. If this is the case, the derivative can even be computed analytically. Indeed, let u and v be the respective left and right eigenvector belonging to a single active eigenvalue λ at a given point K. Then differentiation of the characteristic equation results for the single input case in the following direct formula for the gradient of $\alpha(K)$ at K:

$$\nabla\alpha(K) = \Re\left(\frac{\left\langle u, \sum_{j=0}^{s} B_j^d \right\rangle v}{\left\langle u, \left[I + \sum_{i=0}^{r}\tau_i A_i^d + \sum_{j=0}^{s}\tau_j B_j^d K\right] v\right\rangle}\right),$$

where $\langle\cdot,\cdot\rangle$ is the dot product and the notation A_i^d (B_j^d) is used to denote $A_i e^{-\lambda\tau_i}$ ($B_j e^{-\lambda\tau_j}$).

Alternatively, the system of equations

$$\begin{bmatrix} \lambda I - \sum_{i=0}^{m} A_i^d - \sum_{j=0}^{n} B_j^d K & \left(I + \sum_{i=0}^{m}\tau_i A_i^d + \sum_{j=0}^{n}\tau_j B_j^d K\right) v \\ v^* & 0 \end{bmatrix} \cdot \begin{bmatrix} \nabla v(K) \\ \nabla\alpha(K)^T \end{bmatrix} = \begin{bmatrix} \left(\sum_{j=0}^{m} B_j^d\right) v^T \\ 0 \end{bmatrix}$$

can be solved to attain the same value. This requires some more work, because also the derivative of the eigenvector is calculated, but the numerical stability is in general better.

Remark that $\alpha(K)$ is also nonconvex, and therefore it may have many local minima. The optimisation algorithm that we will look at in the following section is able to find such local minima, but it gives no guarantee about whether or not this is also the global minimum.

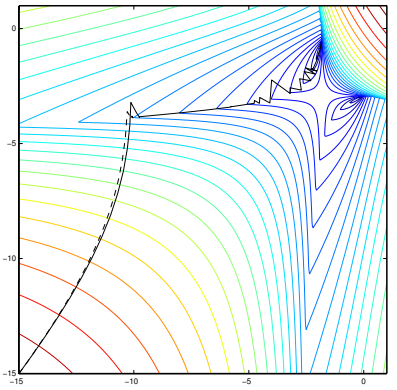


Fig. 1. Behaviour of the gradient sampling algorithm vs. the steepest descent algorithm.

2.2 *Optimisation technique*

We will now take a closer look at the sample gradient algorithm which we have used to perform the nonsmooth, nonconvex optimisation. It essentially generalises the steepest descent method to deal with nonsmoothness, as it generates at every iteration point x_k an approximation to the *nonsmooth* steepest descent direction, given by

$$\arg\min_{||d||\leq 1}\ \max_{z\in\bar{\partial}f(x_k)} \langle z, d\rangle,$$

where $\bar{\partial}f(x_k)$ denotes the generalised gradient or Clarke subdifferential of f at x_k. It does so by, as its name indicates, first sampling a number of additional gradients in the near neighbourhood of the iteration point, which can be calculated efficiently using the previously mentioned formulas. A sort of 'collective downward direction' is then constructed by choosing the vector with smallest norm that lies in the convex hull of these sampled gradients. Notice that this vector will be nonzero only if a common descent direction exists, which can be used as a criterion to evaluate whether a point is optimal or not. A standard line search along this direction finally yields the new point. For detailed information, see (Burke *et al.*, 2005*a*).

Figure 1 shows a typical behaviour of the sample gradient method (full line) as opposed to the steepest descent method (dashed line). It is observed that the two algorithms highly resemble each other in areas without nonsmoothness. Near points of nonsmoothness however, the steepest descent method gets stuck, whereas the sample gradient method, following the computed direction of common descent, is able to proceed. It is also seen that the latter tends to follow the line of nonsmoothness, which is a positive property in the light of eigenvalue optimisation, since minimisers are likely to be points of nondifferentiability.

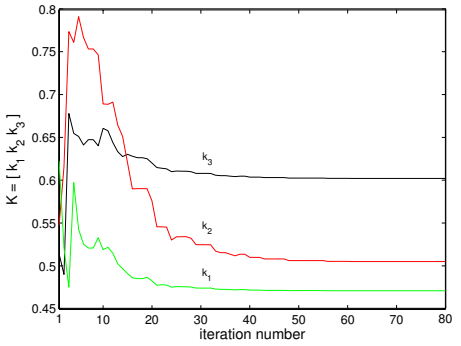


Fig. 2. Evolution of the controller parameters of (7) during the optimisation process.

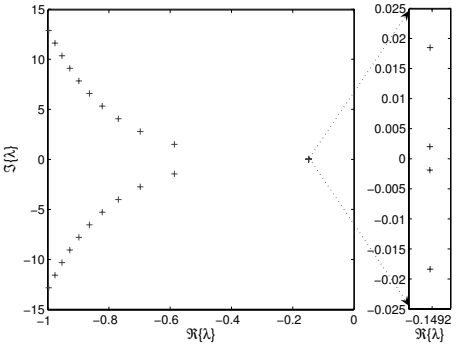


Fig. 3. Spectrum of the stabilised system (7)-(8)

To evaluate the objective function $\alpha(K)$ itself, we must determine the supremum of all solutions of the nonlinear characteristic time-delay equation

$$\det\Big(\lambda I - \sum\nolimits_{i=0}^{m} A_i^d - \sum\nolimits_{j=0}^{n} B_j^d K\Big) = 0. \quad (6)$$

For this we have DDE-BIFTOOL at our disposal, a MATLAB package developed by Engelborghs *et al.* (2001) for the bifurcation analysis of nonlinear equations. It includes a routine that, using a linear multistep approach (Engelborghs and Roose, 2002), computes an approximation to the eigenvalues of a given time-delay system with real part larger than a user specified value. Using Newton iterations on the original equation (6), these approximations are then corrected in order to obtain accurate values for the stability determining eigenvalues, from which the value of $\alpha(K)$ is of course easily extracted.

3. NUMERICAL EXAMPLES

3.1 *Third-order system*

Consider the system

$$\dot{x} = Ax(t) + Bu(t-\tau), \quad u = Kx, \quad (7)$$

with an input delay $\tau = 5$ and system matrices

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & -0.2 & -0.07 \end{bmatrix}, \; B = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}. \quad (8)$$

For $K = 0$, the spectral abscissa is positive, rendering the open-loop system unstable. Its stabilisation by means of the continuous pole placement method served as an illustrative example in the paper of Michiels *et al.* (2002), where it was found that the minimum of $\alpha(K)$ is characterised by four coinciding active eigenvalues $\lambda_{1..4} = -0.15$.

Applying the direct optimisation approach should obviously result in the same minimum. Figure 2 shows the evolution of the control parameters as a function of the iterations of the gradient sampling algorithm. Figure 3 depicts the system's spectrum at the gain values that were found as a minimiser for $\alpha(K)$, namely $K = \begin{bmatrix} 0.472 & 0.505 & 0.603 \end{bmatrix}$. We indeed retrieve a quadruple eigenvalue at -0.149. The narrow plot on the right however, zooming in on this eigenvalue, reveals that actually four distinct active eigenvalues are found, due to the ill condition of computing coinciding eigenvalues.

3.2 *Wind tunnel model*

A model for a closed-air high-speed wind tunnel is presented in (Manitius, 1984). It consists of the following delay differential equations:

$$\dot{x}_1(t) = -x_1(t) - akx_2(t-\tau), \quad (9)$$

$$\ddot{x}_2(t) = -2\xi w\dot{x}_2(t) - w^2(x_2(t) - u(t)), \quad (10)$$

with $a = 1/1.964$, $k = -0.117$, $\xi = 0.8$, $w = 6$ and a delay $\tau = 0.33$. The variable x_1 denotes the Mach number and x_2 denotes the position-guiding vane angle of a driving fan. Rewriting in the standard form yields a linear time-delay system of dimension 3, with open-loop eigenvalues

$$\lambda_1 = -a = -0.59165,$$

$$\lambda_{2,3} = -\xi w \pm w\sqrt{\xi^2 - 1} = -4.8 \pm 3.6\mathrm{i},$$

implying the open system is stable. In the referenced paper, a special feedback controller is used that maintains the finiteness of the spectrum, so that a full assignment method can be applied.

Here, we try to minimise the spectral abscissa with a linear static state feedback controller of the form (2). Starting from random gain values, the gradient sampling algorithm repeatedly converged to $K = \begin{bmatrix} 13.353 & -1.213 & 0.880 \end{bmatrix}$, for which $\alpha(K) = -5.3435$, a tenfold improvement with respect to the original spectral abscissa. The decrease of the spectral abscissa as a function of the minimisation method's iteration number is shown in Figure 4. Its convergence rate is faster than linear, to be seen in contrast with for example the continuous pole placement method.

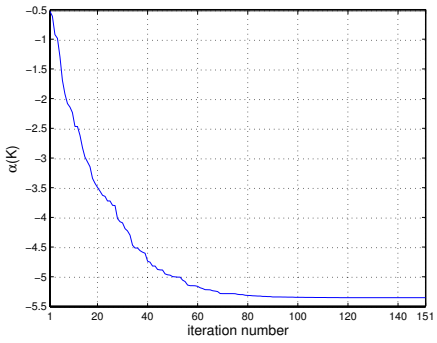


Fig. 4. Evolution of the spectral abscissa of (9)-(10) during its minimisation process.

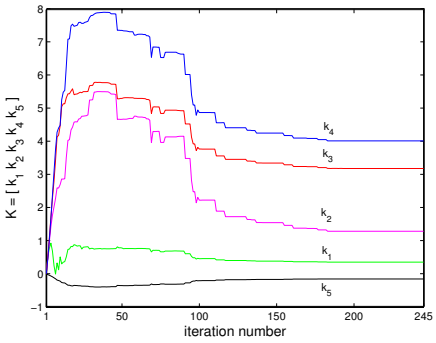


Fig. 6. Evolution of the controller parameters of the heat transfer model

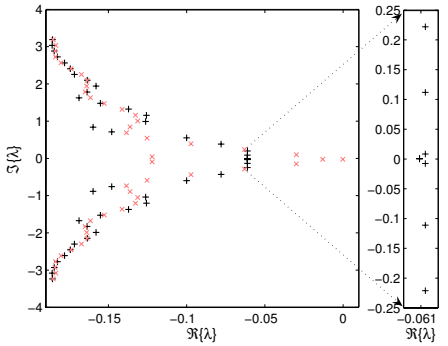


Fig. 5. Spectrum of the heat transfer model before (×) and after (+) stabilisation.

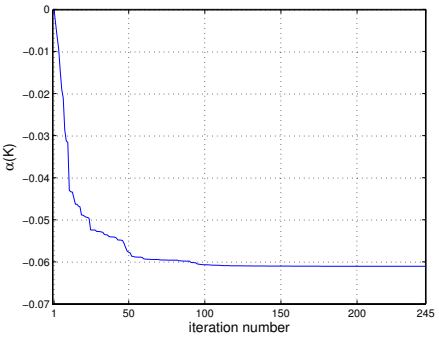


Fig. 7. Spectral abscissa during minimisation.

3.3 *Heat transfer model*

In (Vyhlídal, 2003), a water-based heating system is introduced. The equations that model this heat transfer process are:

$$T_h\dot{x}_h = -x_h(t-\eta_h) + K_b x_a(t-\tau_b) + K_u x_{h,set}(t-\tau_u), \quad (11)$$

$$T_a\dot{x}_a = -x_a(t) + x_c(t-\tau_e) + K_a\Big(x_h(t) - \frac{1+q}{2}x_a(t) - \frac{1-q}{2}x_c(t-\tau_e)\Big), \quad (12)$$

$$T_d\dot{x}_d = -x_d(t) + K_d x_a(t-\tau_d), \quad (13)$$

$$T_c\dot{x}_c = -x_c(t-\eta_c) + K_c x_d(t-\tau_c), \quad (14)$$

$$\dot{x}_e = -x_c(t) + x_{c,set}(t), \quad (15)$$

where the x-variables denote the temperature measured at different places in the circuit, and the parameter values are listed in the table below.

$T_h = 14$	$K_b = 0.24$	$\tau_b = 40$	$\eta_h = 6.5$
$T_a = 3$	$K_a = 1$	$\tau_e = 13$	$q = 1$
$T_d = 3$	$K_d = 0.94$	$\tau_d = 18$	
$T_c = 25$	$K_c = 0.81$	$\tau_c = 2.8$	$\eta_c = 9.2$
	$K_u = 0.39$	$\tau_u = 13.2$	

The set-point value $x_{h,set}(t)$ is determined by the static state feedback controller

$$x_{h,set}(t) = K\begin{bmatrix} x_h & x_a & x_d & x_c & x_e \end{bmatrix}^T.$$

The (uncontrolled) open loop system has an eigenvalue at zero and is consequently not stable. Using the continuous pole placement method, stabilisation was established by Vyhlídal (2003) up to a spectral abscissa of $\alpha(K) = -0.0413$.

After applying the stabilisation method proposed in this paper, a spectrum was reached with a rightmost eigenvalue with real part equal to -0.06977, at gain values $K = \begin{bmatrix} 0.364 & 1.30 & 3.20 & 4.03 & -1.46 \end{bmatrix}$. The system's spectrum for these optimal gain values is plotted in Figure 5, along with the spectrum of the initial, uncontrolled system. The number of active eigenvalues is now larger, since the dimension of the system is, too. In Figure 6, the variation of the controller parameters during the nonsmooth optimisation process is shown, while Figure 7 plots the evolution of the spectral abscissa. It is seen that the decrease of $\alpha(K)$ towards its minimum again happens quite fast. When only an approximate value is required, for example as a starting point for robust stabilisation, a hundred iterations already give a satisfactory result.

4. CONCLUSIONS

The study and development of nonsmooth, nonconvex optimisation methods has opened a wide range of possibilities, since objective functions with these properties arise commonly in practice. In this paper, they were successfully combined with tools for the bifurcation analysis of nonlinear equations in order to address the problem of stabilising linear time-delay systems with a linear static state feedback. In doing so, the application range of these methods was expanded to solving control problems, still using a finite number of controller parameters, yet dealing with infinite-dimensional systems.

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