ONDERZOEKSRAPPORT NR 9215

The Optimal Sequencing and Frequency of Tests in the Inspection of Multicharacteristic Components

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D/1992/2376/18

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In this paper we deal with a quality control problem in the inspection of multicharacteristic components. Suppose that a product component has several characteristics, any of which may go defective and hence result in the failure of the component. Assume also that each characteristic can be separately inspected (tested) and a 100% inspection policy will be followed. By 100% inspection we mean that if a component is tested on a characteristic, then all components will be tested on that characteristic. Now, questions of theoretical and practical interests are: What characteristics should be selected for inspection ? How many times a characteristic of a component should be tested ? And in what sequence selected inspections should be carried out ? The number (frequency) of inspections of each characteristic and the sequence of inspections are issues of concern because of the following underlying assumptions. The inspection is not perfect with known probability of classifying a nondefective as defective (Type I error) and of classifying a defective as nondefective (Type II error). (Reliability is often the word used in practice to mean the same thing.) Once a component is rejected by an inspection, it will not enter inspections arranged afterwards. Rejected components are assumed worthless. (This assumption will be relaxed later.) The costs of such an inspection program include the cost due to false rejection of non-defective components, the cost due to false acceptance of defective components, and finally the cost of inspections. The objective is to find an inspection plan that minimizes the total costs involved.

Raouf, Jain and Sathe [1] developed a model which describes the cost implication of a **multi-cycle** inspection plan. In their model, the inspection process is conducted in cycles. In an **inspection**

cycle. every characteristic will be inspected (tested) once. A component is rejected as soon as it does not pass a characteristic test and hence is classified as a defective. Accepted components may, if necessary, go to the next cycle of inspection, which involves the inspection of **all** characteristics again. The whole process is repeated for n cycles. The components that pass all the inspection stages are the accepted ones finally. Under the objective function of minimizing the average cost per **accepted** component, Raouf et al. developed an algorithm for determining the sequence in which characteristics are to be inspected in a cycle.

The computational aspect of the Raouf et al. procedure is very involved. Hau L. Lee [2] therefore simplified the model, and provided a formal proof of the optimality of the sequence generated by the Raouf-Jain-Sathe algorithm. He also gave an upper boundary of the optimal number of cycles.

However, the Raouf-Jain-Sathe model is actually an "all-or-nothing" offer: either all of the characteristics must be tested in a cycle, or none of them will be inspected. Why should all characteristics be tested or even repeatedly tested if only some of the tests are needed ? This is exactly the question that motivated this paper. The reasoning goes along the following lines. Since the inspection costs, the defective rates, and the testing errors are different across characteristics, it is perfectly possible that for certain tests it is not cost effective at all to be conducted, whereas for some others it is worthwhile to be carried out or even repeated. Thus, if we drop the "all-or-nothing" constraint, and develop **an inspection plan** by only selecting those tests that are cost effective, we would then improve the cost of the inspection program.

To make it clear the difference between a test and a test type, the following terminology will be used in this paper. **The test of type i**, or simply **test i**, is meant the inspection of characteristic i. Whereas **a test** is meant the process of carrying an inspection. The meaning will be clear in the context.

For easier understanding of the notation and the expressions thereafter, an example of inspection plan is illustrated in Figure 1.

Inspection Plan	Explanation
P = { 2, 1, 1, 4, 2 }	Test 2 is conducted first, followed by two tests of type 1, then test 4 and test 2. Total number of tests in the plan = 5 Number of different test types in the plan = 3

Figure 1. An Inspection Plan Example

In this example, a component is supposed to have 4 characteristics, i.e., there are 4 different types of test. However, the inspection of characteristic 3 (or test 3) is excluded from the inspection plan because of, say, the cost reason. While test 1 and test 2 are conducted twice.

In general, assume a component has N characteristics, and let P be an inspection plan consisting of n tests. Denote i_j as the test type of the jth test in sequence of P. Thus, for instance, $i_1 = 2$, $i_2 = 1$, in the above example. Note that since each of the N test types may be selected as each of the n tests in P, hence there are N^n different inspection plans even if n, yet not known, is fixed. Complete enumeration for finding the optimal plan is in other words completely out of question. After a through analysis of the problem, we will develop an efficient algorithm for finding the inspection plan that minimizes the total costs involved. Clearly, the optimal plan will also minimize the average cost per component **inspected**. The objective function is another important issue. In the Raouf-Jain-Sathe and Hau model the the objective function to be minimized is the average cost per component **accepted**. We will provide a justification for chosing total costs as the objective function.

Finally, the results obtained will be extended to two cases of more practical values. In the first case, a salvage value is assumed for a rejected component. While in the second, we assume that rejected components can be repaired in the price of repair costs.

NOTATION

In this paper we shall, as far as possible, use the same notation as the Hau L. Lee model.

N	number of characteristics in each component, or the number of tests of different type
i, k	index for characteristics or test types, i,k = 1, ..., N
n	total number of tests in an inspection plan
j, τ	j^{th} (τ^{th}) position in sequence of an inspection plan
i_j	j^{th} test in sequence of an inspection plan being the test of type i_j.
n_{ij}	number of times that the test of type i has been selected **before** the jth test of an inspection plan
M_j	number of components entering the jth test
p_i	probability of characteristic i being defective entering the inspection
P_{1i}	Type I error of test i, i.e., the probability of classifying nondefective characteristic i as defective
P_{2i}	Type II error of test i, i.e., the probability of classifying defective characteristic i as nondefective
^jp_i	probability of characteristic i being defective entering the jth test of an inspection plan
PG_j	number of components being nondefective entering the jth test of an inspection plan
TA(j)	total number of accepted components at the end of jth test of an inspection plan
TCFR(j)	total cost of false rejection in the first j tests
TCFA(j)	total cost of false acceptance just after the first j tests

TCI(j)	total cost of inspection in the first j tests
C_r	unit cost due to false rejection of a nondefective component
C_a	unit cost due to false acceptance of a defective component
C_i	unit cost of test i

OBJECTIVE FUNCTION

As denoted, C_r is the unit cost due to false rejection of a nondefective component, it should be equal to the value or price of a nondefective component if it is sold. By the same token, C_a is the penalty cost per defective sold. Assume that we have M_1 components entering the inspection, and there are PG_1 nondefectives among them. Suppose further that after the execution of an inspection plan which consists of n tests, TA(n) components are finally accepted, and among them are D_{n+1} defective components and PG_{n+1} nondefective components. Let TCI(n) be the cost of the inspection (cost of n tests in the plan), then the "profit" after following the inspection plan is

$$\begin{aligned} F(n) &= C_r\, PG_{n+1} - C_a\, D_{n+1} - TCI(n) \\ &= C_r\, PG_1 - \{\, C_r\, [\, PG_1 - PG_{n+1}\,] + C_a\, D_{n+1} + TCI(n)\,\} \end{aligned} \qquad (1)$$

Since F(n) is a function of the inspection plan followed, we would like to find an inspection plan such that F(n) is maximized. Two points should be noted here. First, the profit calculation is not restricted to the case in which the components, as products, are sold in the market. When components are made or bought for assembly, C_a would be interpreted as the penalty cost (trouble-shooting cost, called in practice) if a defective component is assembled in a final product (e.g., damages to the final product, repair cost), and C_r would be the opportunity cost of obtaining a nondefective component. Second, PG_1, TA(n), D_{n+1}, PG_{n+1}, TCI(n), F(n) should be taken as **expected** values, since the tests are not perfect, and the number of nondefective components in a lot of size M_1 is a random variable (changing from lot to lot). For this reason,

the numbers and variables henceforth should be interpreted in the same fashion, though, the word "expected" will be dropped for simplicity. Now, observe that PG_1 is independent of the inspection plan followed, hence Maximizing F(n) is equivalent to minimizing TC(n), where TC(n) is the total cost of the inspection program defined as

$$TC(n) = C_r [PG_1 - PG_{n+1}] + C_a D_{n+1} + TCI(n) \qquad (2)$$

Note that $[PG_1 - PG_{n+1}]$ is the number of false rejection of non-defective components, and hence the first term in TC(n) is the total cost of false rejection, TCFR(n). The second term in TC(n) is the total cost of false acceptance TCFA(n). Thus,

$$TC(n) = TCFR(n) + TCFA(n) + TCI(n). \qquad (3)$$

An inspection plan is called optimal if it minimizes TC(n).

THE COST IMPLICATION OF AN INSPECTION PLAN

In this section, we derive some relationships that will be used for further analysis. Since our model is different from the one in [2], we shall provide all necessary inductions to facilitate the understanding of the results.
Let P denote an inspection plan, and assume that

$$P = \{ i_1 , i_2 , i_3 , \ldots , i_n \} . \qquad (4)$$

Associated with P is the information of n_{ij}, i = 1, ..., N; j = 1, ..., n+1. (See NOTATION for the definition.)

We assume that the p_i's are independent. The same assumption is also made in [1] and [2]. For simplicity, we will use the notion "defect in i" to mean the ith characteristic of a component is defective. The term "nondefect in i " is similarly defined.

Suppose that the inspection program is conducted according to P. Then M_j, the number of components entering the jth test of P, *under the assumption that p_i's are independent*, can be calculated as follows:

$M_j = M_1$ Prob (passing the first (j-1) tests of P)
$= M_1$ Prob (passing n_{ij} tests of type i, i = 1, 2, ..., N).

Since for a given i,

Prob (passing n_{ij} tests of type i)
= Prob (defect in i **and** passing n_{ij} tests of type i, **or**
nondefect in i **and** passing n_{ij} tests of type i)
$= p_i P_{2i}^{n_{ij}} + (1 - p_i)(1-P_{1i})^{n_{ij}}$

Note that if test i has not been selected before the jth test of P, then $n_{ij} = 0$, and hence above probability is equal to 1. Thus, the probability calculation above can be applied to any i. Consequently,

$$M_j = M_1 \prod_{i=1}^{N} \{ p_i P_{2i}^{n_{ij}} + (1 - p_i)(1-P_{1i})^{n_{ij}} \} \qquad (5)$$

Observe that TA(n), the number of components finally accepted under P is simply M_{n+1}, i. e.,

$$TA(n) = M_1 \prod_{i=1}^{N} [p_i P_{2i}^{n_{i\,n+1}} + (1-p_i)(1-P_{1i})^{n_{i\,n+1}}] \qquad (6)$$

To simplify the expression, define Q_{ij} as

$$\begin{aligned} Q_{ij} &= p_i P_{2i}^{j} + (1-p_i)(1-P_{1i})^{j}, && j \geq 1,\ i = 1, 2, ..., N; \\ Q_{i0} &= 1 && i = 1, 2, ..., N; \\ Q_{0j} &= 1 && j \geq 1. \end{aligned} \qquad (7)$$

Then, we have

$$M_j = M_1 \prod_{i=1}^{N} Q_{i\,n_{ij}} \qquad (8)$$

$$TA(n) = M_1 \prod_{i=1}^{N} Q_{i\,n_{i\,n+1}} \qquad (9)$$

The number of nondefective components entering the jth test of P, PG_j, can be similarly calculated:

$PG_j = M_1$ Prob (a component is nondefective and
passes the first (j-1) tests of P)

$$= M_1 [\prod_{i=1}^{N} (1-p_i)] [\prod_{k=1}^{j-1} (1-p_{1i_k})] \qquad (10)$$

$$= M_1 \prod_{i=1}^{N} (1 - p_i)(1-P_{1i})^{n_{ij}} \qquad (11)$$

Note that the number of nondefective components that are finally accepted under P is just PG_{n+1}, i.e.,

$$PG_{n+1} = M_1 \prod_{i=1}^{N}(1-p_i)(1-P_{1i})^{n_{i\,n+1}} \qquad (12)$$

Thus, the number of nondefective components that are falsely rejected in the inspection process is:

$$M_1 \prod_{i=1}^{N}(1-p_i)[1-\prod_{i=1}^{N}(1-P_{1i})^{n_{i\,n+1}}] .$$

The total cost of false rejection under P, TCFR(n), can now be calculated directly by

$$TCFR(n) = C_r M_1 \prod_{i=1}^{N}(1-p_i)[1-\prod_{i=1}^{N}(1-P_{1i})^{n_{i\,n+1}}] . \qquad (13)$$

We next consider TCFA(n). Note that the difference between TA(n) and PG_{n+1} should be the number of false acceptance finally, and hence we have

$$TCFA(n) = C_a M_1 \{\prod_{i=1}^{N} Q_{in_{i\,n+1}} - \prod_{i=1}^{N}(1-p_i)(1-P_{1i})^{n_{i\,n+1}}\} . \qquad (14)$$

Finally, the total cost of inspection under P can be calculated directly by definition:

$$TCI(n) = \sum_{j=1}^{n} C_{1j} M_j = M_1 \sum_{j=1}^{n} C_{1j} \prod_{i=1}^{N} Q_{in_{ij}} . \qquad (15)$$

Before closing this section, we would like to point out that if P is a multi-cycle inspection plan with, say, n cycles, then $n_{i\,n+1} = n$ for all i, and the cost calculations are reduced to those given in [2].

MARGINAL ANALYSIS

By examining (9), (13), and (14), it can be easily seen that TA(n), TCFR(n), TCFA(n) are sequence independent. That is, they can be calculated immediately once the frequencies of the tests of different types in P, $n_{i\,n+1}$, i = 1, 2, ..., N, are given. However, TCI(n) ((15)) is sequence dependent. It is not difficult to understand the dependency, though, since those components that are rejected by earlier tests in P will not require the tests sequenced later in P, and hence the corresponding inspection costs are saved, but how

many components will be rejected earlier and how much costs will be saved later depend on what tests are arranged earlier and what tests are arranged later in sequence of P. Thus, in searching for the optimal inspection plan, we need to consider both the frequency and the sequencing of the tests. We will tackle this problem by first analyzing what tests could be selected and then deciding what sequence they should be.

Suppose that the first (j-1) tests of P, i_1 , i_2 , ..., i_{j-1} , have already been fixed, and we are considering to choose the test of type k, i.e., test k, as the jth test of P. Before going on, we first define R_{ij} as the rejection rate in the jth test if test i is selected as the jth test. A_{ij} is similarly defined as the acceptance rate. Hence, if k is selected as the jth test, then

$$A_{kj} = M_{j+1}/M_j , \quad \text{and} \quad R_{kj} = 1 - A_{kj} . \qquad (16)$$

To see that M_{j+1}/M_j is really dependent on k, we only need to remind the fact that if k is selected as the jth test, then

$$n_{ij+1} = n_{ij} , \qquad \text{for} \quad i \neq k ;$$

and

$$n_{kj+1} = n_{kj} + 1 .$$

Using (5) and (7), we have

$$A_{kj} = \frac{M_{j+1}}{M_j} = \frac{p_k P_{2k}^{n_{kj}+1} + (1-p_k)(1-P_{1k})^{n_{kj}+1}}{p_k P_{2k}^{n_{kj}} + (1-p_k)(1-P_{1k})^{n_{kj}}}$$

$$= \frac{p_k P_{2k}^{n_{kj}+1} + (1-p_k)(1-P_{1k})^{n_{kj}+1}}{Q_{kn_{kj}}}$$

$$= \frac{p_k P_{2k}^{n_{kj}}}{Q_{kn_{kj}}} P_{2k} + \left(1 - \frac{p_k P_{2k}^{n_{kj}}}{Q_{kn_{kj}}}\right)(1-P_{1k}) . \qquad (17)$$

This result is interesting. By notation, ^jp_i is the probability of a component that is defect in i entering the jth test. Since test k is selected as the jth test, hence the probability that a component will be accepted in the jth test (i.e., acceptance rate A_{kj}) is:

A_{kj} = Prob (defect in k and accepted or
nondefect in k and accepted)

$$= {}^{j}p_k\, P_{2k} + (1 - {}^{j}p_k)(1 - P_{1k}) \qquad (18)$$

By comparing (18) with (17), we obtain

$${}^{j}p_k = \frac{p_k P_{2k}^{n_{kj}}}{Q_{kn_{kj}}} = \frac{p_k P_{2k}^{n_{kj}}}{p_k P_{2k}^{n_{kj}} + (1 - p_k)(1 - P_{1k})^{n_{kj}}} \qquad (19)$$

This result is expected. Indeed, since $Q_{kn_{kj}}$ is the probability that a component passes n_{kj} tests of test k, and $p_k P_{2k}^{n_{kj}}$ is the probability that a component is defect in k and passes n_{kj} tests of test k, and hence the ratio of the two must be the defective rate of characteristic k just before the jth test. Such an argument is not rigorous, though, since it ignores the possibility that other tests might affect the quality level of characteristic k as well. We obtained ${}^{j}p_k$ independently, however, and it turns out to be as expected. In return, it proves that the quality level of characteristic k is only dependent on the number of times that test k has been carried out, it does not depend on how many times the tests of other types having been conducted. This is an important result, and will be referred to as the ***independence property*** of ${}^{j}p_k$ for later uses. Clearly, this property of ${}^{j}p_k$ comes from the fact that p_i's are independent. Note that in our discussion k is an arbitrary test type, and hence above results hold for any characteristic i or test i.

If for test k, $P_{1k} = 1.0$, then all components that are nondefect in k would be rejected if test k is selected, and hence all nondefective components would be rejected by test k. Consequently, test k should never be selected in the optimal plan. (Interestingly, we could redefine the test results of such a test so that its P_{1k} equals zero. This can be done by only accepting those components that do not pass the test.) Hence in the sequel of the paper, we will assume that for every test k, $P_{1k} < 1.0$.

The second case that will not be considered further is the one for which $p_k = 1.0$ for a characteristic k. Since in this case, all components are defect in k, and we need not to carry test k at all. Thus, we may assume $p_k < 1.0$ for every k.

The following Lemmas are useful in developing the optimal plan.

Lemma 1

a. If $p_k = 0$, then ${}^j p_k \equiv 0$ for all n_{kj};

b. If $P_{1k} + P_{2k} < 1$ and $p_k > 0$, then ${}^j p_k$ is a **decreasing** function of n_{kj}, and ${}^j p_k$ approaches 0 when $n_{kj} \to \infty$.
Moreover, ${}^j p_k = O[(\alpha)^{n_{kj}}]$,
where, $\alpha = P_{2k}/(1 - P_{1k}) < 1$;

c. If $P_{1k} + P_{2k} = 1$, then ${}^j p_k \equiv p_k$ for all n_{kj};

d. If $P_{1k} + P_{2k} > 1$ and $p_k > 0$, then ${}^j p_k$ is a **increasing** function of n_{kj}, and ${}^j p_k$ approaches 1.0 if $n_{kj} \to \infty$.
Moreover, ${}^j p_k = O(1-\beta^{n_{kj}})$,
where $\beta = (1 - P_{1k})/P_{2k} < 1$.

Proof: By (19), ${}^j p_k$ can be rewritten as

$$ {}^j p_k = \frac{p_k \alpha^{n_{kj}}}{p_k \alpha^{n_{kj}} + 1 - p_k}, \quad \text{where, } \alpha = P_{2k}/(1 - P_{1k}). $$

Thus, a and c are obvious. For b and d, note first, if $P_{1k} + P_{2k} < 1$, then $\alpha < 1$, and hence $\alpha^{n_{kj}}$ decreases as n_{kj} increases; second, if $P_{1k} + P_{2k} > 1$, then $\alpha > 1$, and hence $\alpha^{n_{kj}}$ increases as n_{kj} increases. Let function f(x) be defined as

$$ f(x) = \frac{p_k x}{p_k x + 1 - p_k}, $$

then, since $f'(x) = p_k(1 - p_k)/(p_k x + 1 - p_k)^2 > 0$, so f(x) is an increasing function of x. Now, b and d are obvious conclusions if we set $x = \alpha^{n_{kj}}$.

Note that if there is no component that is defect in k (i.e. $p_k = 0$), then the only reason for invoking test k is to reduce the number of components to go to later tests (only if $P_{1k} > 0$), but this could be better done by throwing away an equal number of components that would be rejected by the test. Hence we may assume that corresponding to each test k, $p_k > 0$. Lemma 1 tells us then that if for a test k, $P_{1k} + P_{2k} < 1$, then it will continuously improve the quality level if such tests are conducted. Interestingly, only a few tests will reap most of the benefit, since ${}^j p_k = O[(\alpha)^{n_{kj}}]$. However, if $P_{1k} + P_{2k} \geq 1$ for test k, then it will never improve the quality level if test k is carried out. Thus, roughly as the same line of reasoning as in the case of $p_k = 0$, such a test should not be selected in the

optimal plan. Of course, we will provide a formal proof for this claim.

Since $R_{kj} = 1 - A_{kj}$, and by (18), we have

$$\begin{aligned} R_{kj} &= {}^{j}p_k\,(1 - P_{2k}) + (1 - {}^{j}p_k)\,P_{1k} \\ &= {}^{j}p_k\,(1 - P_{1k} - P_{2k}) + P_{1k}. \end{aligned} \qquad (20)$$

Combining (20) with Lemma 1, we obtain the following important lemma:

Lemma 2

e. $R_{kj} \equiv P_{1k}$, if $P_{1k} + P_{2k} = 1$;

f. R_{kj} is a **decreasing** function of n_{kj} if $P_{1k} + P_{2k} \neq 1$, and

$$\lim_{n_{kj}\to\infty} R_{jk} = \begin{cases} P_{1k} & \text{if } P_{1k} + P_{2k} < 1; \\ 1 - P_{2k} & \text{if } P_{1k} + P_{2k} > 1. \end{cases}$$

By the **independence property** of ${}^{j}p_k$ noted earlier, ${}^{j}p_k$ is only dependent on n_{kj}, and hence by (20), R_{kj} is also only dependent on n_{kj}. This is the **independence property** of R_{kj}. Moreover, since ${}^{j}p_k$ decreases (if $P_{1k} + P_{2k} < 1$) or increases (if $P_{1k} + P_{2k} > 1$) almost exponentially as n_{kj} grows, hence R_{kj} decreases almost exponentially.

Using the definition of R_{ij}, we may express M_j in following useful form:

$$M_j = M_1 \prod_{\tau=1}^{j-1} (1 - R_{i_\tau \tau}). \qquad (21)$$

Now, we should come to the discussion of the cost implications of selecting test k as the jth test. Of course, the purpose of the test is to remove the defectives among the incoming lot of components. But, it should be justified to do so, since it is not cost free. The reduction in the number of defectives and hence the reduction in the penalty costs to be paid is the encouraging side of having one more test, while the increasing in inspection costs and in the lost "revenues" due to false rejection of non-defectives is the discouraging side of having one more test. To be more explicit, let us put them in mathematical terms.

Since $M_j \prod_{i=1}^{N}(1 - {}^{j}p_i)$ is the number of nondefective components before the jth test of test k, and P_{1k} is the proportion of the nondefective components that will be rejected in the jth test. Hence, the additional cost due to false rejection if test k is selected as the jth test (ACFR(k,j)) is

$$ACFR(k,j) = C_r M_j P_{1k} \prod_{i=1}^{N}(1 - {}^{j}p_i) \qquad (22)$$

Note that since the number of components that will be rejected in the jth test is $M_j R_{kj}$, hence the number of defective components that will be removed in the jth test is

$$M_j R_{kj} - P_{1k} M_j \prod_{i=1}^{N}(1 - {}^{j}p_i) ,$$

and the **additional reduction** in the cost due to false acceptance (ARCFA (k,j)) thus is

$$ARCFA(k,j) = C_a M_j\{ R_{kj} - P_{1k} \prod_{i=1}^{N}(1 - {}^{j}p_i) \}. \qquad (23)$$

Finally, the **additional cost** of inspection (ACI(k,j)) is simply

$$ACI(k,j) = C_k M_j . \qquad (24)$$

Now, let ARTC(k,j) be the **additional (or marginal) reduction** in total cost in the jth test if test k is selected, i.e.,

$$ARTC(k,j) = ARCFA(k,j) - \{ ACFR(k,j) + ACI(k,j) \}, \qquad (25)$$

then it will improve the cost of the inspection program if we carry one more test of type k in addition to those already done, if ARTC(k,j) > 0. To highlight this point, we will say that test k is **eligible at j,** if and only if ARTC(k,j) > 0.

Using the equations from (22) to (25), we obtain

$$ARTC(k,j) = M_j\{C_a R_{kj} - C_k - (C_a + C_r) P_{1k} \prod_{i=1}^{N}(1 - {}^{j}p_i) \}. \qquad (26)$$

Thus, test k is eligible at j if and only if

$$C_a R_{kj} - C_k > (C_a + C_r) P_{1k} \prod_{i=1}^{N}(1 - {}^{j}p_i). \qquad (27)$$

Inequality (27) has an important implication. By Lemma 1 and Lemma 2, we know that both R_{kj} and ${}^{j}p_i$ are decreasing exponentially if $P_{1k} + P_{2k} < 1$. Hence the marginal reduction in total cost decreases and quickly diminishes as n_{kj} grows. Later, we will show

that it is never optimal to have a test for which $P_{1i} + P_{2i} \geq 1$. Hence we may restrict our choices among those tests that satisfy $P_{1k} + P_{2k} < 1$. Consequently, if test k is not eligible at j, then it will never become eligible afterwards. Since this result will be used later in developing the optimal plan, we shall state it formally as a lemma. Let E_j be the set defined as

$$E_j = \{ k \mid \text{test } k \text{ is eligible at } j \}. \qquad (28)$$

Lemma 3

Assume for test k, $P_{1k} + P_{2k} < 1$. Then, if $k \notin E_j$, then $k \notin E_\tau$, $\tau > j$.

Given the inspection plan P in (4), it is natural to define the **total reduction in total cost** (TRTC(n)) of P as

$$TRTC(n) = \sum_{j=1}^{n} ARTC(i_j, j) . \qquad (29)$$

(See (25) for the definition of $ARTC(i_j,j)$.)

Lemma 4

Maximizing F(n) (profit) is equivalent to maximizing TRTC(n).

Proof: We only need to write F(n) in terms of TRTC(n). Repeatedly using the following relationships:

$$M_j - M_{j+1} = M_j R_{i_j j}, \quad PG_j - PG_{j+1} = P_{1i_j} PG_j = P_{1i_j} M_j \prod_{i=1}^{N} (1 - {}_j p_i),$$

we have, $F(n) = C_r PG_{n+1} - C_a (M_{n+1} - PG_{n+1}) - \sum_{j=1}^{n} M_j C_{i_j}$

$$= (C_a + C_r) PG_{n+1} - C_a M_{n+1} - \sum_{j=1}^{n} M_j C_{i_j}$$

$$= (C_a + C_r) PG_n - C_a M_n - \sum_{j=1}^{n-1} M_j C_{i_j}$$
$$+ C_a R_{i_n n} M_n - M_n C_{i_n} - (C_a + C_r) P_{1i_n} PG_n$$

$$= F(n-1) + ARTC(i_n, n)$$
$$= \ldots$$
$$= (C_a + C_r) PG_1 - C_a M_1 + TRTC(n). \qquad (30)$$

Observe that $((C_a + C_r) PG_1 - C_a M_1)$ is the "profit" if not a single test is conducted, which is independent of P. Hence Lemma 4 is true.

Thus, P is optimal if it maximizes TRTC(n), or vice versa.

Lemma 5

If for test k, $P_{1k} + P_{2k} \geq 1$, then test k should not be selected in the optimal plan.

Proof: Suppose that test k, for which $P_{1k} + P_{2k} \geq 1$, is selected in the optimal plan P. Assume that P contains n tests, and the latest position at which test k appears is j. That is, test k is not repeated after the jth test. Thus, P can be written as

$$P = \{ i_1 , i_2 , ..., i_{j-1}, k, i_{j+1}, ..., i_n \}.$$

Let $$P' = \{ i_1 , i_2 , ..., i_{j-1}, i_{j+1}, ..., i_n \}.$$

That is, P' is made of P simply by dropping the jth test, test k, from P. Basically, since test k does not improve the quality level (see c and d of Lemma 1), it should not be selected in the optimal plan. But this is not a rigorous argument, because the removing of test k from P would increase the inspection costs for carrying those tests after the jth of P. However, the increased inspection costs should be covered by increased "revenues", since more components would be finally accepted after dropping the jth test. To verify it mathematically, let TCI(i, τ) be the total inspection costs occured during the tests from the ith to the τth of P. Thus, for instance, TCI(j+1, n) can be calculated as

$$TCI(j+1, n) = \sum_{\gamma=j+1}^{n} M_\gamma C_{i_\gamma} = (1 - R_{kj}) M_1 \sum_{\gamma=j+1}^{n} [\prod_{\substack{\tau=1 \\ \tau \neq j}}^{\gamma-1} (1 - R_{i_\tau \tau})] C_{i_\gamma} , \qquad (31)$$

and $$TCI(n) = TCI(1,j-1) + TCI(j,j) + TCI(j+1,n).$$

Equation (21) is used to replace M_γ in the above calculation. Since P is an optimal plan, we may assume that the "profit" under P is a nonnegative value, otherwise all components should be thrown away without any inspection. The "profit" under P can be easily derived by using (10), (21), and (31) as follows:

$$0 \leq F(n) = (1 - P_{1k}) A - (1 - R_{kj}) B - TCI(1,j-1) - TCI(j,j) - (1 - R_{kj}) C$$

where $$A = M_1 [\prod_{i=1}^{N} (1 - p_i) \prod_{\substack{\tau=1 \\ \tau \neq j}}^{n} (1 - P_{li_\tau})] (C_a + C_r) ;$$

$$B = M_1 [\prod_{\substack{\tau=1 \\ \tau \neq j}}^{n} (1 - R_{i_\tau \tau})] C_a ; \quad \text{and} \quad C = M_1 \sum_{\gamma=j+1}^{n} [\prod_{\substack{\tau=1 \\ \tau \neq j}}^{\gamma-1} (1 - R_{i_\tau \tau})] C_{i_\tau} .$$

Note that since $i_\tau \neq k$, $\tau = j+1, \ldots, n$, hence dropping test k from the jth position of P would not affect the value of $R_{i_\tau \tau}$, $\tau = j+1, \ldots, n$, by the independence property. Thus, C is equal to the total inspection costs for carrying the last (n - j) tests of P', test i_τ, $\tau = j+1, \ldots, n$. Notice also the fact that $R_{kj} \leq P_{1k}$ by (20), since $P_{1k} + P_{2k} \geq 1$. Thus, we have

$$0 \leq F(n) \leq (1- R_{kj}) (A - B - C) - TCI(1,j-1) - TCI(j,j)$$
$$< A - B - TCI(1,j-1) - C . \qquad (32)$$

Observe that (A - B - TCI(1,j-1) - C) is nothing else than the "profit" under P', and which is higher than the "profit" under the optimal plan P by inequality (32). This contradiction implies that Lemma 5 must be true.

Indeed, it can be very easily shown that even the "throwing-away" strategy is superior to the plan P. Under the "throwing-away" strategy, the jth test of P is replaced by an action of throwing away $R_{kj} M_j$ components, selected at random. The number $R_{kj} M_j$ is chosen because $R_{kj} M_j$ components would be rejected by the jth test of P. As a result, the inspection cost of the last (n-j) tests and the number of components finally accepted will remain unchanged by the action taken. Yet, the quality level of finally accepted components under the "throwing-away" strategy is higher than that under P by Lemma 1.

The following corollaries follow immediately by the proof (with a slight change) of Lemma 5.

Corollary 1 If $p_k = 0$, then test k should not be selected in the optimal plan P.

Corollary 2 If $P_{2k} = 0$, then test k should not be selected in the optimal plan **more** than once.

Only a note is needed here for the proofs of Corollary 1 and Corollary 2. If $p_k = 0$, then $R_{kj} = P_{1k}$, and hence (32) is still true. If $P_{2k} = 0$, then by (19) the defective rate of characteristic k is zero after a test of type k (as it should be).

With Lemma 5 on hand, we may assume in the following that for each test k, $P_{1k} + P_{2k} < 1$. Hence by Lemma 1, the more the tests,

the higher the quality level will be. The next lemma, Lemma 6, is trying to give a stopping point for the number of tests to be carried out, if the cost of the inspection program is to be considered. But before showing it, we have to rewrite ARTC(k,j) (26).

Note first that $\prod_{i=1}^{N}(1 - {}^{j}p_i)$ is the quality level of the lot entering the jth test. Thus, by (10)

$$\prod_{i=1}^{N}(1 - {}^{j}p_i) = PG_j / M_j = M_1 \{ [\prod_{i=1}^{N}(1-p_i)] [\prod_{\tau=1}^{j-1}(1-P_{1i_\tau})] \} / M_j.$$

(Indeed, it can be derived directly.) While by the definition of R_{ij}, $M_j = M_1 \prod_{\tau=1}^{j-1}(1-R_{i_\tau\tau})$, hence ARTC(k,j) can be rewritten as

$$ARTC(k,j) = M_1 \{ \prod_{\tau=1}^{j-1}(1-R_{i_\tau\tau}) [C_a R_{kj} - C_k] -$$

$$P_{1k} [\prod_{i=1}^{N}(1-p_i)] [\prod_{\tau=1}^{j-1}(1-P_{1i_\tau})] [(C_a + C_r)] \}. \qquad (33)$$

Lemma 6

Suppose that the inspection plan P given in (4) is an optimal one, then test i_j must be eligible at j, i.e. $i_j \in E_j$, j = 1, ..., n.

Proof: First of all, by Lemma 4, the optimal plan P should maximize TRTC(n), the total reduction in total cost, which is the sum of the marginal reductions of total cost in the n tests of P. Now, suppose that, on the contrary to Lemma 6, there exists a position, say, k, $1 \le k \le n$, such that $i_k \notin E_k$, i.e., $ARTC(i_k,k) \le 0$. Without a loss of generality, we may assume that k is the latest position at which Lemma 6 is violated. That is, $i_j \in E_j$ for all $j > k$ if $k < n$. If $k = n$, i.e., $ARTC(i_n,n) \le 0$, then, the total reduction in total cost would be higher (at least equal) if we drop the last test of P (test i_n). (It can be seen in the sequel of the proof that the last position (n) is the only position at which ARTC could be zero in the optimal plan. However since in such a case, the last test gives no improvement in total cost, we may thus drop it and leave Lemma 6 unchanged.) If $k < n$, then we again consider the alternative of dropping the kth test from P, and see what happens to TRTC. First of all, it is clear that the reductions of total cost in the first (k-1) tests of P will not be affected by the dropping of the kth test.

Secondly, consider ARTC(i_j,j) for **any** j > k. Since $i_j \in E_j$ for all j > k, hence by Lemma 3, $i_k \neq i_j$ for **all** j > k. Consequently, by the the **independence property** of R_{ij}, $R_{i_\tau \tau}$ remains unchanged for any τ > k after dropping the kth test of P. For **any** j > k, by (33), and using the fact that $i_j \in E_j$ and the kth test of P is i_k, we have

$$ARTC(i_j,j) = M_1 \{(1 - R_{i_k k}) \prod_{\substack{\tau=1 \\ \tau \neq k}}^{j-1} (1 - R_{i_\tau \tau}) [Ca\, R_{i_j j} - C_{i_j}] -$$

$$(1 - P_{1i_k}) P_{1i_j} [\prod_{i=1}^{N} (1 - p_i)] [\prod_{\substack{\tau=1 \\ \tau \neq k}}^{j-1} (1 - P_{1i_\tau})] [(Ca + Cr)]\} \geq 0$$

For simplicity, let

$$A = M_1 \prod_{\substack{\tau=1 \\ \tau \neq k}}^{j-1} (1 - R_{i_\tau \tau}) [Ca\, R_{i_j j} - C_{i_j}], \quad \text{and}$$

$$B = M_1\, P_{1i_j} [\prod_{i=1}^{N} (1 - p_i)] [\prod_{\substack{\tau=1 \\ \tau \neq k}}^{j-1} (1 - P_{1i_\tau})] [(Ca + Cr)]$$

Then, $ARTC(i_j,j) = (1 - R_{i_k k})\, A - (1 - P_{1i_k})\, B \geq 0$

Note that by (20), $R_{i_k k} > P_{1i_k}$, hence,

$$0 \leq ARTC(i_j,j) < (1 - P_{1i_k})(A - B) < A - B .$$

Observe that (A - B) is nothing but the marginal reduction in total cost resulted from carrying test i_j if the kth test of P (test i_k) is dropped. Thus, **any** of the marginal reductions achieved from those tests after the kth of P will be **rigidly** improved if the kth test of P (test i_k) is dropped from P. Finally, dropping the kth test of P will itself improve the TRTC by a value equal to $|ARTC(i_k,k)|$, since $ARTC(i_k,k) \leq 0$. Thus, even $ARTC(i_k,k) = 0$, the TRTC will be higher if the kth test of P is dropped. This contradicts to the optimality of P and hence Lemma 6 is true.

Lemma 6 tells us that we may restrict our choice among E_j in the selection of the test type for the jth test. It is very important to note here that, by (33), ARTC(k,j) declines exponentially (even faster), and hence the number of candidates in E_j becomes smaller and smaller. Moreover, as noted earlier, if test k is not eligible at j, then it will never become eligible afterwards, i.e., $E_{j+1} \subseteq E_j$, j = 1, 2, ... Once a candidate is gone, it will never enter the competition again. These important properties provide us powerful insights in the composition of the optimal inspection plan.

Now, we should come to the issue of the optimal sequencing of the tests in an inspection plan. The sequencing problem in our model can be stated as follows: Given the types and the corresponding frequencies of the tests in a plan P of n tests, how to arrange these tests in the n positions of P so that the total cost implied by P is minimized ? The answer to this problem can be obtained by similar derivation to that in [2].

Lemma 7

The optimal sequence of tests of P is achieved by arranging them in descending order of the ratio C_{i_j}/R_{i_jj}. That is, the tests of P should be sequenced such that

$$C_{i_1}/R_{i_11} \leq C_{i_2}/R_{i_22} \leq \ldots \leq C_{i_n}/R_{i_nn} .$$

Proof: Since only TCI(n) is sequencing dependent, and thus the optimal sequence should be the one that minimizes TCI(n).
Suppose that the optimal test sequence of P is not arranged according to the descending order of C_{i_j}/R_{i_jj}, then there must exists two consecutive positions, say, the jth and $(j+1)^{th}$, such that

$$C_{i_j}/R_{i_jj} > C_{i_{j+1}}/R_{i_{j+1}j+1}. \qquad (34)$$

For simplicity, let us assume that $i_j = i$, $i_{j+1} = k$. First of all, it can be assured that $i \neq k$, since otherwise $n_{i\,j+1} = n_{ij} + 1$, and the fact that R_{ij} is a decreasing function of n_{ij} would mean that

$$C_{i_j}/R_{i_jj} = C_i/R_{ij} < C_i/R_{i\,j+1} = C_{i_{j+1}}/R_{i_{j+1}j+1}, \qquad (35)$$

which contradicts to (34). Thus $i \neq k$, and hence $n_{k\,j+1} = n_{kj}$, and $R_{k\,j+1} = R_{kj}$. Consider now the interchange between j^{th} and $(j+1)^{th}$ test of P. That is, suppose test k is assigned as the jth test, and test i as the $(j+1)^{th}$. Note first that such an interchange would not affect the inspection costs of the remaining tests (i.e., all tests other than j^{th} and $(j+1)^{th}$), because M_τ remains the same for all $\tau \neq j+1$ (see (21)). Let TCI'(n) denote the total cost of inspection after the interchange, then we have by (34):

$$\begin{aligned} TCI(n) - TCI'(n) &= C_i M_j + C_k M_j(1 - R_{ij}) - [C_k M_j + C_i M_j (1 - R_{kj})] \\ &= M_j (C_i R_{kj} - C_k R_{ij}) = M_j (C_i R_{k\,j+1} - C_k R_{ij}) > 0 \end{aligned}$$

This contradicts to the optimality of the sequence before the interchange, and thus Lemma 7 cannot be violated.

Corollary 3 If the tests from the jth to the $(j+m)^{th}$ of an optimal plan P happen to be as such that

$$C_{i_j}/R_{i_j j} = C_{i_{j+1}}/R_{i_{j+1} j+1} = \ldots = C_{i_{j+m}}/R_{i_{j+m} j+m},$$

then,

g. $i_j \neq i_{j+1} \neq \ldots \neq i_{j+m}$;

h. changing the sequence of the tests from the jth to the $(j+m)^{th}$ will affect neither the eligibility of each test in P, nor the optimality of P.

Proof: g can be obtained by the contradiction of (35). As to h, we only need to note that (TCI(n) - TCI'(n)) will be zero by a pairwise interchange (see the proof of Lemma 7). Hence, the interchange will not affect the total cost, thus the optimality of P remains. With pairwise interchanges, we may get any sequence of the tests from the jth to the $(j+m)^{th}$. Finally, the proof for h can be completed by applying Lemma 6.

The rationale underlying Lemma 7 is very simple: those tests with low unit inspection costs and high rejection rates should be arranged earlier, so that "defectives" are removed by earlier cheap tests and less components go to later expensive tests.

A short summary of the main results obtained so far will be helpful for continuation:

1. ${}^{j}p_k$, R_{kj} are only dependent on n_{kj}, the **independence property**;
2. ${}^{j}p_k$ is decreasing almost exponentially if $P_{1k} + P_{2k} < 1$, and increasing almost exponentially if $P_{1k} + P_{2k} > 1$ (Lemma 1);
3. R_{kj} is decreasing almost exponentially (Lemma 2);
4. If $P_{1k} + P_{2k} \geq 1$, then test k should not be selected in the optimal plan (Lemma 5);
5. If $P_{2k} = 0$, then test k should not be selected in the optimal plan **more** than once (Corollary 2);
6. If P is optimal, then i_j must be eligible at j, i.e. $i_j \in E_j$, $j = 1, \ldots, n$ (Lemma 6);
7. $E_{j+1} \subseteq E_j$, $j = 1, 2, \ldots, n$ (Lemma 3);
8. The optimal sequence of tests of P must be as such that their $C_{..}/R_{..}$ ratios satisfy (Lemma 7):

$$C_{i_1}/R_{i_1 1} \le C_{i_2}/R_{i_2 2} \le \ldots \le C_{i_n}/R_{i_n n} ;$$

9. Tests with equal C.. /R.. ratios can be arranged in any sequence (Corollary 3).

THE "NON-ALGORITHM" FOR SEARCHING THE OPTIMAL PLAN

As pointed earlier, there are N^n possible inspection plans even if n, yet not known, is fixed. Because the tests selected earlier affect the eligibility and the marginal reductions of later selected ones, the decision to what test should be selected next is really very hard to make. The results obtained so far, however, make it possible to have an efficient algorithm to find the optimal inspection plan. The "**now**-or-**never**" (**NON**) algorithm to be proposed in this paper is an algorithm that selects the best one in the class of cost effective inspection plans. An inspection plan P is called a cost effective inspection plan (or simply, an effective plan) if it satisfies: (Suppose P is given in the form of (4).)

1. All tests are eligible at corresponding positions of P, i.e., $i_j \in E_j$, j = 1, ..., n;
2. The tests in P are optimally sequenced.

Hence, by Lemma 6, the best one in the class of effective plans is also the optimal one among all possible inspection plans.

The NON-algorithm can be briefly described by the "**now**-or-**nver**" (**NON**) rule. To understand the notion of "**now**-or-**never**", let us assume that we are constructing an effective plan, and suppose that we have already built up the first (j-1) tests and are considering the candidates for the jth test. By Lemma 6, the test for the jth test is in E_j. Let test i_j be the test that

$$C_{i_j}/R_{i_j j} = \min_{k \in E_j} C_k/R_{kj} ,$$

and suppose that there is no tie in the C. /R. $_j$ ratios among the tests in E_j. If test i_j is not to be selected **now** (for the jth test), then it should **never** be selected afterwards, because otherwise a test with a C. /R. $_j$ ratio higher than $C_{i_j}/R_{i_j j}$ would be put in j^{th} position, while test i_j with the C. /R. . ratio **equal to** $C_{i_j}/R_{i_j j}$ is put in a position later than j^{th}, and which does not conform to the second requirement for a plan to be effective by Lemma 7. Thus for test i_j, we have two choices: either select it **now**, or **never** consider it again. Now, if we

make the second choice, that is, do not select test i_j as the jth test, then we may consider the test in E_j with the second lowest $C./R._j$ ratio as the candidate for the jth test. But again, we face two choices: either select it now, or remove it from further consideration. This process repeats until all tests in E_j are considered. Each test of E_j may be put in the jth position, or not. If not, then this test, together with all other tests in E_j that have lower $C./R._j$ ratios, will be eliminated from further consideration. Thanks to the "now-or-never" property, it allows us to save a great deal of work in searching for the optimal plan. For instance, if the test, say, test k, with the maximum $C./R._j$ ratio is to be selected as the jth test, then the only choice for the $(j+1)^{th}$ test (and tests after $(j+1)^{th}$) is test k, if $k \in E_{j+1}$ ($\subseteq E_j$). If $k \notin E_{j+1}$, then we do not need any additional test, even if E_{j+1} is not empty, and an effective plan has emerged. Think of the case otherwise without the NON rule. We would have to list all possibilities that each test in E_{j+1} (which is already much "less then" the set of N possible tests) could be put in the $(j+1)^{th}$ position, and for each test in the $(j+1)^{th}$ position we have to enumerate all eligible tests in the $(j+2)^{th}$ position, and so on until $E_{j+\tau}$ is empty for some τ. All of these possibilities will be eliminated or greatly reduced by following the NON rule. For this reason, we shall call a test **effective eligible** at j if it is in E_j **and** has not been eliminated so far (before stage j) by the NON rule. Thus, the selection for the jth test can be restricted among those tests that are effective eligible at j. Roughly speaking, the number of possible plans is reduced exponentially by using Lemma 6, and it is further reduced by 50% **at each stage** (position) of the selection process by sticking to the NON rule. One problem that still remains unsolved now is how to handle those tests in E_j that have equal $C./R._j$ ratios. For instance, suppose that both test i and test k reaches the minimum $C./R._j$ ratio in E_j, and assume further that test i is considered first. The first branch then is to put test i in the jth position; while the second branch is to assign test k as the jth test; and the third branch would be to put a test in E_j other than test i and test k in the jth position, and so on. If the second branch is chosen, that is, test i is not selected now for the jth position, which is occupied by test k, then we cannot forget test i, since test i may be put in the $(j+1)^{th}$ position without violating the optimal

sequencing rule. Thus, it seems that the NON rule no longer applies here. Fortunately, if it is indeed optimal to put test k in the jth and test i in the $(j+1)^{th}$ position, then their order can be reversed, namely, test i in the jth and test k in the $(j+1)^{th}$, without destroying the optimality of the plan by Corollary 3. But the last alternative--test i in the jth and test k in the $(j+1)^{th}$-- is included in the first branch. Since in the first branch, test k will be considered to be put in the $(j+1)^{th}$ position after test i is selected in the jth position because first, $k \in E_{j+1}$ by corollary 3; second, test k has the minimum $C_{\cdot}/R_{\cdot j+1}$ ratio in E_{j+1}. Consequently, we do not have to consider the alternative in the second branch. That is, test i may be removed from further consideration if test i is not selected in j^{th}. Which means that the NON rule can still be used. Indeed, it can be elaborated in detail that the NON rule is valid in all circumstances, e.g., no matter how many tests that have the same $C_{\cdot}/R_{\cdot}$ ratio, and how many ties at a time. Note that if there are m tests that tie at the same $C_{\cdot}/R_{\cdot}$ ratio at a time, we only need to consider one (arbitrary) sequence of them, instead of m! possible sequences ! How significantly the searching work may be reduced by applying Corollary 3 can be clearly seen here.

Now, developing an efficient algorithm by using the NON rule described above, combined with readily available **branch and bound** techniques, is an easy matter. Indeed, how to branch at a stage is already specified by the NON rule. Some points have to be stressed here, though. First, by Corollary 2, once a test with $P_2 \cdot = 0$ is selected, it should be removed from further consideration. Second, some stop-rules should be incorporated in the bound calculations to guarantee ultimate stoppage of the searching process in any branch. Normally a branch will be terminated and hence an effective plan is generated when there are no effective eligible tests left at a stage, which is particularly so when E_j is empty for some j.

Lemma 8

If the quality level of the incoming components is so high that, even without any inspection, the "profit" is positive if it is "sold", then the searching process will stop in finite steps.

Proof: We only need to show that E_j will be empty for some j in any branch. By the assumption in the lemma, we have

$$0 < C_r\, PG_1 - C_a\,(M_1 - PG_1) = (C_a + C_r)\, PG_1 - C_a\, M_1, \quad \text{or}$$
$$(C_a + C_r) \prod_{i=1}^{N} (1 - p_i) > C_a\ .$$

Thus, there exists $\varepsilon > 0$, such that

$$(C_a + C_r) \prod_{i=1}^{N} (1 - p_i) > C_a(1+\varepsilon) > C_a\ . \qquad (36)$$

Since by Lemma 2, $R_{kj} \to P_{1k}$, when $n_{kj} \to \infty$, hence there exists an integer N_k, such that when $n_{kj} \geq N_k$, the following is true:

$$R_{kj} < P_{1k}\,(1 + e). \qquad (37)$$

Combining (36) and (37) with the fact that

$$\prod_{i=1}^{N} (1 - {}^{j}p_i) \geq \prod_{i=1}^{N} (1 - p_i),$$

we obtain, when $n_{kj} \geq N_k$,

$$(C_a + C_r)\, P_{1k} \prod_{i=1}^{N} (1 - {}^{j}p_i) > C_a\, R_{kj} \geq C_a\, R_{kj} - C_k\ .$$

Which means, by (27), $k \notin E_j$ if $n_{kj} \geq N_k$. That is, test k can not be selected more than N_k times in an effective plan. Consequently, E_j will be empty for $j \geq M = \sum_{k=1}^{N} N_k$. That is, any of the branches, which is corresponding to an effective plan, will certainly be terminated in M steps.

The implicit assumption made in the proof of Lemma 8 is that $P_{1k} \neq 0$. If $P_{1k} = 0$, then we shall assume that $C_k \neq 0$, otherwise it would mean that test k is totally cost-free to be conducted any number of times, and hence it would be optimal to carry test k enough times so that all components that are defect in k are removed out before any other inspection starts, and thus such a test may not be considered in the inspection plan we are constructing. Now, suppose that $P_{1k} = 0$, and $C_k \neq 0$, then, since $R_{kj} \to P_{1k} = 0$, there exists N_k, such that when $n_{kj} \geq N_k$,

$$C_a\, R_{kj} < C_k,$$

which implies, by (27), $k \notin E_j$, and Lemma 8 still holds.

Note that if the quality of the incoming components is so bad that it is not "profitable" to "sell" before any inspection, but it becomes "profitable" after some tests of a plan (or a branch), then it can be

easily seen by the proof of Lemma 8 that the searching process in that branch will stop in finite steps. Now, if it never becomes "profitable" in a branch, then certainly the plan corresponding to that branch will not be optimal. To stop the search in such a branch, if it is not otherwise kicked out by the eligibility requirement or other bound calculations, the following rule is suggested: if, after a test is added in a position, say, the jth, of an inspection plan being constructed, the following inequality holds:

$$C_r M_1 \prod_{i=1}^{N} (1 - p_i)(1 - P_{1i})^{n_{i\,j+1}} \leq \sum_{\tau=1}^{j} M_\tau C_{i_\tau},$$

then, the corresponding plan should be eliminated, and the search can be switched to one of the other branches. The implication of above inequality should be clear now. It can be easily shown that the above inequality will be eventually held in finite steps, if again for each k, $C_k + P_{1k} \neq 0$.

EXAMPLE

We have developed an algorithm by using the NON rule and the **depth-first search** technique. We coded it in FORTRAN, and implemented it on the IBM-mainframe VM/370. The following problem is served as an illustration:

Test	p_i	P_{1i}	P_{2i}	C_i
1	0.100	0.050	0.050	5.
2	0.050	0.040	0.010	20.
3	0.050	0.080	0.000	8.
4	0.150	0.000	0.100	6.
5	0.051	0.520	0.300	8.
6	0.060	0.000	0.000	10.
	$C_r = 100.$		$C_a = 400.$	

Figure 2 An Example Problem

In the problem, there are 6 **possible** tests (or characteristics to be tested), and test 6 is error free. After 41 effective plans searched, the program found the following optimal inspection plan for this problem:

$$P^* = \{ 1, 4, 3, 6, 4 \}.$$

The values for some important measures under P* are given in Figure 3, where ACPA means average cost per accepted component.

Optimal Plan		1	4	3	6	4
Accpt. Rate %	100.000*	86.000	74.390	65.017	61.116	60.162
Quality Level	0.61589*	0.6803	0.7865	0.8279	0.8808	0.8947
Total Cost **	15364.4*	11804.2	7676.2	6862.4	5952.2	5937.3
ACPA	153.64*	137.26	103.19	105.55	97.39	98.69

* Corresponding values before inspection

** Assuming 100 components before inspection

Figure 3 Results of the Example Problem

Note first that even if we had known the optimal number of tests n, which is 5 in this example, there would have been $N^n = 6^5 = 7776$ possible plans to enumerate. The NON rule effectively reduced this number to 41 effective plans. The second interesting point is that test 4 is selected twice in P*, while both test 2 and test 5 do not appear in P* at all. By a close look of the data given in Figure 2, it can be seen that the defective rate of characteristic 4 is quite high and test 4 has a high probability of making type II errors, and thus it is reasonable to repeat test 4 so as to produce a high quality level of accepted components. On the other hand, test 2 is not selected because apparently its inspection cost is too high. Test 5 is not selected either because it is very unreliable.

From Figure 3, we see that about 60.16% of incoming components will be finally accepted after the execution of P*. The acceptance rate is "low" because the quality of the incoming components is poor (approximately 32% defective rate without considering characteristics 2 and 5). The defective rate of finally accepted components is about 0.76% (or quality level 99.24%) without taking into account characteristics 2 and 5, and about 10.53% if both characteristic 2 and characteristic 5 are accounted.

It should be noted here that the total cost includes not only the inspection cost, the penalty cost, but also the opportunity cost for the false rejection of good components. The total cost before any inspection is 15364.43, which is wholly the penalty cost if components are sold without any inspection. Thus, it is clear that it is not "profitable" at all to "sell" components without inspection. In fact, it can be easily calculated that for each 100 components sold with no inspection at all there would be -9205 in net "profit". Yet, after the inspection following the optimal plan P*, it will be about 221 in net "profit" after the deduction of the inspection cost incurred and of the penalty cost left over. In deed, the "profit" can be calculated directly by (1):

$$F = C_r PG_1 - TC = 6158.9 - 5937.3 = 221.6 .$$

To compare our model with the one in [1] and [2], we also coded the algorithm in [1] and [2] in FORTRAN, and solved the problem. It is found that the optimal number of cycles to be conducted is 1, if total cost is to be minimized; and is 0 if the average cost per accepted component is to be minimized. The optimal plan if one cycle is to be carried out is P = {5, 1, 4, 3, 6, 2}, and the total cost corresponding is 6699.61. Hence, it is clear that under both objective functions, the best solution to the problem based on the model in [1] and [2] is to "throw" the components away.

The last point interesting to make is that the optimal plan under the TC objective function does not generally coincide with the optimal plan under ACPA objective function, both in our model and in the model in [1] and [2].

EXTENSIONS

The model discussed so far assumes that the rejected components have no value, and will be "thrown away". The same assumption was made in [1] and [2]. While certainly it may find its own practical uses (e.g., in magnetism tape production), it can be extended to some cases of more practical values. In this section, we will consider two such cases: in the first one, a salvage value S is assumed for each rejected component; while in the second, we assume that rejected components can be "repaired" at the price of repair costs.

Case 1. The Salvage Case

Now, we assume that a rejected component, either good or defective, has a salvage value S. This will be the case if rejected components are degraded and sold in the market at a lower price, or if rejected components are considered as raw materials (like, e.g., chips production, glass production).
The profit function in (1) in this case thus is

$$F(n) = C_r\, PG_{n+1} - C_a\, D_{n+1} - TCI(n) + S\,[\, M_1 - PG_{n+1} - D_{n+1}\,]$$

$$= S\, M_1 + (\, C_r - S)\, PG_1 - \{\, (\, C_r - S\,)\,[\, PG_1 - PG_{n+1}\,] + (\, C_a + S\,)\, D_{n+1} + TCI(n)\,\}$$

By comparing this function with that in (1), it is clear that if we set

$$C_r' = C_r - S \qquad \text{and} \qquad C_a' = C_a + S\,,$$

then, we can apply the same model and hence the same algorithm we discussed earlier to solve the problem.

Case 2. The Repair Case

Sometimes in practice once a component is found "defective", it is immediately removed from the test line and moved to a repair station, where repair-workers are responsible to test it thoroughly and repair all of its defective characteristics if any. Suppose that the "repair" cost for a nondefective component is R_g, and the repair cost for a defective component is R_d, where R_d should be accounted as the average cost of repairing a defective. Then, the profit function (1) becomes:

$$F(n) = C_r\, PG_{n+1} + C_r\,[\, M_1 - PG_{n+1} - D_{n+1}\,] - C_a\, D_{n+1} - TCI(n) - [\, PG_1 - PG_{n+1}\,]\, R_g - [\, M_1 - PG_1 - D_{n+1}\,]\, R_d$$

$$= (\, C_r - R_d\,)\, M_1 + R_d\, PG_1 - \{\, R_g\,[\, PG_1 - PG_{n+1}\,] + (\, C_a + C_r - R_d\,)\, D_{n+1} + TCI(n)\,\}.$$

Which reveals that the model and the algorithm presented earlier can be applied to solve this problem too, if we set

$$C_r' = R_g \qquad \text{and} \qquad C_a' = C_a + C_r - R_d\,.$$

Note that $C_a' > 0$ is guarantied in practice, otherwise there would be no incentive to repair a defective.

III. 11. A PRACTICAL APPLICATION

The model has been applied (via a student thesis [3]) in BELL TLCATEL Company to find out what components and what characteristics of the components should be selected for the incoming inspection. The model output results, based on the data collected from the company, are implemented. Preliminary results from the application show substantial cost savings to the company. In addition, through the data collection and model application process, and by exchanging of ideas, the company gained deeper understanding of the production process and cost implications. Further application of the model to the other stages of the production process is under consideration.

III. 12. SUMMARY and CONCLUSIONS

Based on the objective of minimizing the total costs, a model is proposed to find the optimal sequence and frequency of inspections of multicharacteristic components. The inspection plan suggested does not imply cyclical inspection nor the requirement that all characteristics are tested. The "now-or-never" rule, applied in a branch-and-bound based algorithm, drastically reduces the number of plans to be evaluated. Despite the rather complicated statistical analysis, powerful insights are gained that enable the practitioners to implement the algorithm easily. The extension of the model to the case with repair or salvage possibility further enhances the practical use of the algorithm.

REFERENCES

[1] Raouf, A., J. K. Jain, and P. T. Sathe, "A Cost-Minimization Model for Multicharacteristic Component Inspection," IIE Transactions, 15(3), 187-194 (1983).

[2] Hau L. Lee, "On the Optimality of a Simplified Multicharacteristic Component Inspection Model," IIE Transactions, 20(4), 393-398 (1988).

[3] OOMS Nancy, "The Optimal Frequency and Sequencing of Tests in a multicharacteristic Component Inspection Model", Thesis voorgedragen tot het behalen van de graad van handelsingenieur, ETEW, K.U.Leuven