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Rational Gauss-Radau and rational Szegő-Lobatto quadrature on the interval and the unit circle respectively

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Abstract

We present a relation between rational Gauss-Radau quadrature formulas with one fixed node in the open interval $(-1,1)$ that approximate integrals of the form $J_\mu(f) = \int_{-1}^{1} f(x)d\mu(x)$, and rational Szegő-Lobatto quadrature formulas with two fixed nodes on the complex unit circle that approximate integrals of the form $I_{\mathring{\mu}}(f) = \int_{-\pi}^{\pi} f(e^{\mathrm{i}\theta})d\mathring{\mu}(\theta)$. The measures μ and $\mathring{\mu}$ are assumed to be positive bounded Borel measures on the interval $[-1,1]$ and the complex unit circle respectively, and are related by $\mathring{\mu}'(\theta) = \mu'(\cos\theta)\,|\sin\theta|$. Further, we include some illustrative numerical examples.

Keywords: Rational Gauss-Radau quadrature, rational Szegő-Lobatto quadrature, quasi-orthogonal rational functions, para-orthogonal rational functions.

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§1. Introduction

In this paper we shall explore the interplay between quadrature formulas for integrals of the form $J_\mu(f) = \int_{-1}^{1} f(x)d\mu(x)$, where μ is a positive Borel measure on $I = [-1,1]$, and integrals of the form $I_{\mathring{\mu}}(f) = \int_{-\pi}^{\pi} f(e^{\mathrm{i}\theta})d\mathring{\mu}(\theta)$, where $\mathring{\mu}$ is supported on $[-\pi,\pi]$. The integral $I_{\mathring{\mu}}$ can also be considered as an integral over the unit circle $\mathbb{T}$ of the complex plane, which we shall currently do to refer to the two different cases. The Joukowski transform is a well known two-to-one mapping relating the interval I and the complex unit circle $\mathbb{T}$, which can be most easily characterized by the relations $z = e^{\mathrm{i}\theta} \in \mathbb{T} \Leftrightarrow x = \cos\theta \in I$. It is used for example in Szegő's book [20] to relate orthogonal polynomials on I with orthogonal polynomials on $\mathbb{T}$ whenever the measures are related by $\mathring{\mu}'(\theta) = \mu'(\cos\theta)|\sin\theta|$. The corresponding quadrature formulas can be related as well. For example in [5], Gaussian quadrature formulas on I correspond to Szegő quadrature formulas on $\mathbb{T}$. Both of them have distinct nodes in the support of the measure, the weights are positive and they have a maximal domain of validity in the space of polynomials on I and the space of Laurent polynomials on $\mathbb{T}$ respectively.

The classical Gauss-Radau and Gauss-Lobatto quadrature formulas correspond to prefixing one and two nodes, respectively, in I, while Szegő-Radau and Szegő-Lobatto quadrature formulas correspond to prefixing one and two nodes, respectively, on $\mathbb{T}$. The remaining nodes are freely chosen inside I, respectively $\mathbb{T}$, to achieve the maximal domain of validity while maintaining positive weights. This was elaborated e.g. in [4] for I and [2, 17] for $\mathbb{T}$.

For integrals with an integrand having singularities close to I, respectively $\mathbb{T}$, it may be advantageous to consider also poles for the space where the quadrature is exact. To obtain these spaces of rational functions, one has to extend the theory of orthogonal polynomials to the theory of orthogonal rational functions with prescribed poles; see [8]. The poles are assumed to be outside the support I, and hence, their Joukowski image is outside $\mathbb{T}$. The relation between the orthogonal functions on I and $\mathbb{T}$ has been described for the case of real poles in [25] and more generally in [14]. Based on this relation, precise one-to-one relations between rational Gaussian quadrature formulas and rational Szegő quadrature formulas were given in [12]. Also the cases of Radau and Lobatto, with one and two prescribed nodes in the endpoints of the interval, respectively, were considered for the rational case as they were in [4, 9, 7, 17].

The aim of this paper is to investigate the connection between rational Gauss-Radau quadrature formulas, this time with a prescribed node inside the open interval $(-1,1)$, and rational Szegő-Lobatto

quadrature formulas. Both rational Gauss-Radau and rational Szegő-Lobatto quadrature rules have already been studied separately in the papers [13] and [7] respectively. Conditions were given to ensure the existence of the quadrature rules, and it has been shown that the nodes and weights in the quadrature rules can be found by solving an eigenvalue problem involving respectively tridiagonal and five-diagonal matrices. Although the rational Gauss-Radau and rational Szegő-Lobatto quadrature rules were studied separately, it is still worthwhile to relate both kinds of quadrature rules because it can be advantageous to exploit this relation. For example: Properties as rate of convergence and error bounds are easily transformed from one to the other (see e.g. [11]); analytic computations for the case of the interval are often more easily performed by considering its transformation onto the complex unit circle (see e.g. [16]) so that more efficient numerical procedures to compute the nodes and weights can be derived (see e.g. [15, 24]); the computational effort to compute the nodes and weights for the case of the complex unit circle can be reduced by considering its transformation onto the interval (see e.g. [3] for the polynomial case).

The outline of the paper is as follows. In Section 2 we give the necessary theoretical background on orthogonal rational functions and on rational quadrature rules. Then, in Section 3 we present the main results regarding the relation between rational Gauss-Radau and rational Szegő-Lobatto quadrature rules. We conclude in Section 4 with some illustrative numerical examples for the case of the Chebyshev weight function of the first kind on the interval, which corresponds to the Lebesgue measure on the complex unit circle.

§2. Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$ and for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Further, the positive half line will be represented by $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $a \in \mathbb{C}$, then $\Re\{a\}$ refers to the real part of a, while $\Im\{a\}$ refers to the imaginary part, and the imaginary unit will be denoted by $\mathbf{i}$. The unit circle and the open unit disc are denoted respectively by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Whenever the value zero is omitted in a set $X \subseteq \overline{\mathbb{C}}$, this will be represented by X_0. Similarly, the complement of a set $Y \subset \overline{\mathbb{C}}$ with respect to a set $X \subseteq \overline{\mathbb{C}}$ will be given by X_Y; i.e., $X_Y = \{t \in X : t \notin Y\}$.

In this paper, we will consider quadrature formulas on the unit circle $\mathbb{T}$ and on the interval $I = [-1, 1]$. Although z and x are both complex variables, we reserve the notation z for the case of the unit circle and x for the case of the interval.

For any complex function $f(t)$, with $t = z$ or $t = x$, we define the involution operation or substar conjugate by $f_*(t) = \overline{f(1/\overline{t})}$. Next, we define the super-c conjugate by $f^c(t) = \overline{f(\overline{t})}$, and consequently f_*^c by $f_*^c(t) = f(1/t)$. Note that, if $f(t)$ has a pole at $t = p$, then $f_*(t)$ (respectively $f^c(t)$ and $f_*^c(t)$) has a pole at $t = 1/\overline{p}$ (respectively $t = \overline{p}$ and $t = 1/p$). Further, with f^{inv} we denote the inverse of the function f, to avoid confusion with the notation $f^{-1} = 1/f$.

2.1. Quadrature rules on I

Let a sequence of complex poles $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$ be fixed. Unless stated otherwise, the poles in $\mathcal{A}$ are arbitrary complex or infinite; hence, they do not have to appear in pairs of complex conjugates. Define the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = b_{k-1}(x) Z_k(x), \quad \text{with} \quad Z_k(x) = \frac{x}{1 - x/\alpha_k}, \quad k \geqslant 1. \tag{2.1}$$

These basis functions generate the nested spaces of rational functions with poles in $\mathcal{A}$ defined by

$$\mathcal{L}_{-1} = \{0\} \quad \text{and} \quad \mathcal{L}_k := \mathcal{L}\{\alpha_1, \ldots, \alpha_k\} = \operatorname{span}\{b_0, \ldots, b_k\}, \; k \geqslant 0.$$

With the definition of the super-c conjugate we define $\mathcal{L}_k^c = \{f : f^c \in \mathcal{L}_k\}$. Note that $\mathcal{L}_k$ and $\mathcal{L}_k^c$ are rational generalizations of the space $\mathcal{P}_k$ of polynomials of degree less than or equal to k. Indeed, if all $\alpha_j = \infty$, the expressions in (2.1) become $Z_k(x) = Z_k^c(x) = x$ and $b_k(x) = b_k^c(x) = x^k$.

Consider the integral

$$J_\mu(f) = \int_{-1}^{1} f(x) d\mu(x),$$

where μ is a positive bounded Borel measure (in short, a measure) on I. To approximate $J_\mu(f)$, where f is a function with singularities (possibly close to, but) outside the interval, positive rational interpolatory quadrature rules (PRIQRs) are often preferred. An nth PRIQR is obtained by integrating an interpolating rational function in $\mathcal{L}_{n-1}$, and is of the form

$$J_n(f) = \sum_{k=1}^{n} \lambda_k f(x_k), \quad \{x_k\}_{k=1}^n \subset I,\ x_j \neq x_k \text{ if } j \neq k, \quad \{\lambda_k\}_{k=1}^n \subset \mathbb{R}_0^+,$$

so that $J_\mu(f) = J_n(f)$ for every $f \in \mathcal{R}_{p,q} = \mathcal{L}_p \cdot \mathcal{L}_q^c = \{f \cdot g^c : f \in \mathcal{L}_p, g \in \mathcal{L}_q\}$, with $0 \leqslant q \leqslant n-1 \leqslant p \leqslant n$. The space $\mathcal{R}_{p,q}$ is then called the domain of validity. The existence of such a PRIQR depends, however, on the sequence of nodes $\{x_k\}_{k=1}^n$, and it is easily verified that whenever the PRIQR exists, as a consequence it should hold that $\mathcal{R}_{p,q} = \mathcal{R}_{p,q}^c$.

Let $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ denote an nth orthonormal rational function (ORF) with respect to the inner product

$$\langle f\,, g\rangle_\mu = \int_{-1}^{1} f(x) g^c(x) d\mu(x);$$

i.e.,

$$\langle \varphi_n\,, \varphi_k\rangle_\mu = \begin{cases} 0, & k < n \\ 1, & k = n. \end{cases}$$

Whenever $\alpha_n \in \overline{\mathbb{R}}_I$, the zeros x_k of $\varphi_n(x)$ are all distinct and in the open interval $(-1, 1)$, and hence, can be chosen as nodes for the quadrature formula $J_n(f)$. In this way, we obtain the n-point rational Gaussian quadrature formula, which is a PRIQR on I that has maximal domain of validity; i.e., the approximation is exact for every function $f \in \mathcal{R}_{n,n-1}$. See also [8, Chapt. 11.6], [22], [15, Thm. 2.3], and [10, Thm. 2.3.5].

In this paper, however, we will be concerned with so-called rational Gauss-type quadrature rules. These are PRIQRs with j nodes fixed in advance, where $0 < j < n$, and with the remaining $n - j$ nodes chosen to achieve the maximal possible domain of validity. Also now, the existence of these PRIQRs depends on the nodes fixed in advance. Whenever it exists, the dimension of the domain of validity will generally decrease by one for each node that is fixed in advance. A special kind of rational Gauss-type quadrature rules are the rational Gauss-Radau quadrature rule ($j = 1$) and the rational Gauss-Lobatto quadrature rule ($j = 2$). In [13] it is proved that the nodes in an n-point rational Gauss-Radau quadrature rule with fixed node in $x_\alpha \in I$ and domain of validity $\mathcal{R}_{n-1,n-1}$ are zeros of the so-called quasi-orthogonal rational function (qORF)

$$Q_{n,\tau_n}(x) = \varphi_n(x) + \tau_n \frac{Z_n(x)}{Z_{n-1}^c(x)} \varphi_{n-1}(x), \quad \tau_n = -\frac{Z_{n-1}^c(x_\alpha)\varphi_n(x_\alpha)}{Z_n(x_\alpha)\varphi_{n-1}(x_\alpha)} \in \overline{\mathbb{C}}. \tag{2.2}$$

Moreover, it has been proved in the same reference that this qORF is orthogonal on $\mathcal{L}_{n-1}$ with respect to measures $\tilde{\mu}$ supported on a subset of $\overline{\mathbb{R}}$.

Note that, due to the domain of validity $\mathcal{R}_{n-1,n-1}$, the remaining nodes in a rational Gauss-Radau quadrature rule with fixed node in $x_\alpha \in I$ do not depend of α_n, and hence, in the remainder we may as well assume that $\alpha_n \in \overline{\mathbb{R}}_I$.[1]

2.2. Quadrature rules on $\mathbb{T}$

Another sequence of basis functions will be used for the unit circle. Given a sequence of complex numbers $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, we define the Blaschke products [2] for $\mathcal{B}$ as

$$B_0(z) \equiv 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad \text{with} \quad \zeta_k(z) = \frac{z - \beta_k}{1 - \overline{\beta}_k z}, \quad k \geqslant 1. \tag{2.3}$$

These Blaschke products generate the nested spaces of rational functions $\mathring{\mathcal{L}}_{-1} = \{0\}$ and $\mathring{\mathcal{L}}_k = \operatorname{span}\{B_0, \ldots, B_k\}$, $k \geqslant 0$.

With the definition of the substar conjugate and the super-c conjugate we can define $\mathring{\mathcal{L}}_{k*} = \{f : f_* \in \mathring{\mathcal{L}}_k\}$, $\mathring{\mathcal{L}}_k^c = \{f : f^c \in \mathring{\mathcal{L}}_k\}$ and $\mathring{\mathcal{L}}_{k*}^c = \{f : f_*^c \in \mathring{\mathcal{L}}_k\}$. Note that $\mathring{\mathcal{L}}_k$ and $\mathring{\mathcal{L}}_k^c$ are rational

[1] The parameter τ_n in (2.2), however, does depend on α_n by means of the ORF φ_n.

[2] The products are named after Wilhelm Blaschke, who introduced these for the first time in [1].

generalizations of $\mathcal{P}_k$ too. Indeed, if all $\beta_j = 0$ (or equivalently, $1/\overline{\beta}_j = \infty$ for every j), the expressions in (2.3) become $\zeta_k(z) = \zeta_k^c(z) = z$ and $B_k(z) = B_k^c(z) = z^k$.

Consider now the integral

$$I_{\mathring{\mu}}(\mathring{f}) = \int_{-\pi}^{\pi} \mathring{f}(z) d\mathring{\mu}(\theta),\ z = e^{\mathbf{i}\theta},$$

where $\mathring{\mu}$ is a positive bounded Borel measure (in short, a measure) on $\mathbb{T}$.[3] The PRIQRs to approximate $I_{\mathring{\mu}}(\mathring{f})$ are then of the form

$$I_n(\mathring{f}) = \sum_{k=1}^{n} \mathring{\lambda}_k \mathring{f}(z_k), \quad \{z_k = e^{\mathbf{i}\theta_k}\}_{k=1}^n \subset \mathbb{T},\ z_j \neq z_k \text{ if } j \neq k, \quad \{\mathring{\lambda}_k\}_{k=1}^n \subset \mathbb{R}_0^+,$$

so that $I_{\mathring{\mu}}(\mathring{f}) = I_n(\mathring{f})$ for every $\mathring{f} \in \mathring{\mathcal{R}}_p = \mathring{\mathcal{L}}_p \cdot \mathring{\mathcal{L}}_{p*}$, with $(n-1)/2 \leq p \leq n-1$.

Let $\phi_n \in \mathring{\mathcal{L}}_n \setminus \mathring{\mathcal{L}}_{n-1}$ denote an nth ORF with respect to the inner product

$$\left\langle \mathring{f}, \mathring{g} \right\rangle_{\mathring{\mu}} = \int_{-\pi}^{\pi} \mathring{f}(z) \mathring{g}_*(z) d\mathring{\mu}(\theta).$$

The leading coefficient $\mathring{\kappa}_n$ (i.e., the coefficient of $B_n(z)$ in the expansion of $\phi_n(z)$ in the basis $\{B_0, \ldots, B_n\}$) is then given by $\mathring{\kappa}_n = \overline{\phi_n^*(\beta_n)}$, where $\phi_n^*(z) = B_n(z)\phi_{n*}(z)$. In the remainder we will assume $\mathring{\kappa}_n \in \mathbb{R}_0^+$. Further, let $\delta_k \in \mathbb{D}$, $e_k \in \mathbb{R}_0^+$ and $\mathring{\rho}_k \in \mathbb{T}$ be given by

$$\begin{aligned} \delta_k &= -\frac{\left\langle \left(\frac{z-\beta_{k-1}}{1-\overline{\beta}_k z}\right) \phi_{k-1}, \phi_j \right\rangle_{\mathring{\mu}}}{\left\langle \left(\frac{1-\overline{\beta}_{k-1} z}{1-\overline{\beta}_k}\right) \phi_k^*, \phi_j \right\rangle_{\mathring{\mu}}},\ 0 \leqslant j < k, \qquad e_k = \sqrt{\left(\frac{1-|\beta_k|^2}{1-|\beta_{k-1}|^2}\right) \frac{1}{(1-|\delta_k|^2)}}, \\ \text{and} \quad \mathring{\rho}_k &= \frac{\overline{\delta}_k(\beta_k - \beta_{k-1})\phi_{k-1}(\beta_k) + (1-\overline{\beta}_{k-1}\beta_k)\phi_{k-1}^*(\beta_k)}{\left|\overline{\delta}_k(\beta_k - \beta_{k-1})\phi_{k-1}(\beta_k) + (1-\overline{\beta}_{k-1}\beta_k)\phi_{k-1}^*(\beta_k)\right|}. \end{aligned}$$

In [8, Chapt. 4.1] it is proved then that the ORFs ϕ_k, $k > 0$, with respect to the measure $\mathring{\mu}$ satisfy a recurrence relation of the form:

$$\begin{bmatrix} \phi_k(z) \\ \phi_k^*(z) \end{bmatrix} = e_k \left(\frac{1-\overline{\beta}_{k-1} z}{1-\overline{\beta}_k z}\right) \begin{bmatrix} \mathring{\rho}_k & 0 \\ 0 & \overline{\mathring{\rho}}_k \end{bmatrix} \begin{bmatrix} 1 & \delta_k \\ \overline{\delta}_k & 1 \end{bmatrix} \begin{bmatrix} \zeta_{k-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(z) \\ \phi_{k-1}^*(z) \end{bmatrix}. \tag{2.4}$$

Furthermore, it holds that

$$\mathring{\rho}_k^2 \delta_k = \frac{\phi_k(\beta_{k-1})}{\phi_k^*(\beta_{k-1})}. \tag{2.5}$$

When replacing the nth coefficients δ_n, $\mathring{\rho}_n$ and e_n in the recurrence relation with respectively

$$\begin{aligned} \xi_n &= \frac{\mathring{\rho}_n \delta_n + \mathring{\tau}_n \overline{\mathring{\rho}}_n}{\mathring{\rho}_n + \overline{\mathring{\rho}}_n \mathring{\tau}_n \overline{\delta}_n} \in \mathbb{T} \qquad (\mathring{\tau}_n \in \mathbb{T}), \\ \varrho_n &= \frac{\mathring{\rho}_n + \mathring{\tau}_n \overline{\mathring{\rho}}_n \overline{\delta}_n}{\left|\mathring{\rho}_n + \mathring{\tau}_n \overline{\mathring{\rho}}_n \overline{\delta}_n\right|} \in \mathbb{T} \quad \text{and} \quad \varepsilon_n = e_n \left|\mathring{\rho}_n + \mathring{\tau}_n \overline{\mathring{\rho}}_n \overline{\delta}_n\right| \in \mathbb{R}_0^+, \end{aligned}$$

we obtain the so-called para-orthogonal rational function (pORF) for $\mathring{\mu}$:

$$\mathring{Q}_{n,\mathring{\tau}_n}(z) = \varepsilon_n \varrho_n \frac{1-\overline{\beta}_{n-1} z}{1-\overline{\beta}_n z} \left[\zeta_{n-1}(z)\phi_{n-1}(z) + \xi_n \phi_{n-1}^*(z)\right] = \phi_n(z) + \mathring{\tau}_n \phi_n^*(z) = \overline{\mathring{\tau}}_n \mathring{Q}_{n,\mathring{\tau}_n}^*(z). \tag{2.6}$$

The zeros z_k of $\mathring{Q}_{n,\mathring{\tau}_n}(z)$ are all distinct and on the unit circle $\mathbb{T}$, and hence, can be chosen as nodes for the quadrature formula $I_n(\mathring{f})$. In this way, we obtain an n-point rational Szegő quadrature formula, which is a PRIQR on $\mathbb{T}$ that has maximal domain of validity ($p = n-1$). See also [8, Chapt. 5].

Due to the presence of the parameter $\mathring{\tau}_n$ in (2.6), the nodes and weights in an n-point rational Szegő quadrature formula are (unlike in the case of the interval) not unique. For the same reason,

[3] The measure $\mathring{\mu}$ on $\mathbb{T}$ induces a measure on $[-\pi, \pi]$ for which we shall use the same notation $\mathring{\mu}$.

an n-point rational Szegő-Radau quadrature formula (one node fixed in advance) always exists and has maximal domain of validity too, while an n-point rational Szegő-Lobatto quadrature formula (two nodes fixed in advance) has domain of validity $\mathring{\mathcal{R}}_{n-2}$ whenever it exists. In [7] it is proved that the nodes in an n-point rational Szegő-Lobatto quadrature rule with two distinct fixed nodes z_α and z_β on $\mathbb{T}$ are zeros of a pORF $\mathring{\tilde{Q}}_{n,\mathring{\tau}_n}(z) = \tilde{\phi}_n(z) + \mathring{\tau}_n \tilde{\phi}_n^*(z) \in \hat{\mathring{\mathcal{L}}}_n$, $\mathring{\tau}_n \in \mathbb{T}$, of degree n for which the sequence of ORFs $\{\tilde{\phi}_n\}_{k=0}^n$ is generated by means of the recurrence relation (2.4) with the sequence of parameters $\{\delta_1, \delta_2, \ldots, \delta_{n-2}, \tilde{\delta}_{n-1}, \tilde{\delta}_n\} \subset \mathbb{D}$,[4] where $\tilde{\delta}_{n-1}$ and

$$\tilde{\xi}_n = \frac{\mathring{\tilde{\rho}}_n \tilde{\delta}_n + \mathring{\tau}_n \overline{\mathring{\tilde{\rho}}}_n}{\mathring{\tilde{\rho}}_n + \overline{\mathring{\tilde{\rho}}}_n \mathring{\tau}_n \overline{\tilde{\delta}}_n} \in \mathbb{T} \tag{2.7}$$

are uniquely determined by the nodes z_α and z_β. Also now, it is easy to see that the sequence of ORFs $\{\tilde{\phi}_n\}_{k=0}^n$ form an orthonormal system in $\hat{\mathring{\mathcal{L}}}_n$ with respect to 'modified' measures $\mathring{\tilde{\mu}}$ on $\mathbb{T}$; e.g., with respect to

$$d\mathring{\tilde{\mu}}(z) = \left\{1 + \mathring{c}_{\alpha,\beta} B_{(n-1)*}(z) \mathring{Q}_{2(n-1),\mathring{\tau}_{2(n-1)}}(z)\right\} d\mathring{\mu}(z), \quad \mathring{\tau}_{2(n-1)} \in \mathbb{T}, \tag{2.8}$$

where $\mathring{Q}_{2(n-1),\mathring{\tau}_{2(n-1)}}$ is a pORF in $\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{n-1}$ with respect to the measure $\mathring{\mu}$, and $\mathring{c}_{\alpha,\beta}$ is a constant depending on z_α and z_β so that $\mathring{c}_{\alpha,\beta} B_{(n-1)*}(z) \mathring{Q}_{2(n-1),\mathring{\tau}_{2(n-1)}}(z) \geqslant -1$ for every $z \in \mathbb{T}$.

2.3. Relation between I and $\mathbb{T}$

We denote the Joukowski Transformation $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}_I$ and the unit circle $\mathbb{T}$ onto the interval I. When $z = e^{\mathrm{i}\theta}$, then $x = J(z) = \cos\theta$. In this paper we will assume that x and z are related by this transformation. The inverse mapping is denoted by $z = J^{inv}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}_I$. With the sequence $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$ we associate a sequence $\mathcal{B} = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, so that $\beta_k = J^{inv}(\alpha_k)$, and $\hat{\mathcal{B}} = \{\hat{\beta}_1, \hat{\beta}_2, \ldots\} \subset \mathbb{D}$ with

$$\hat{\beta}_{2k} = \overline{\hat{\beta}}_{2k-1} = \beta_k, \quad k \geqslant 1. \tag{2.9}$$

In the remainder we will consistently use the hat in our notation to refer to the sequence of numbers (2.9). Furthermore, due to our earlier assumption that $\alpha_n \in \overline{\mathbb{R}}_I$, we will assume in the remainder that $\beta_n \in (-1, 1)$.

A connection between quadrature formulas on the unit circle and the interval I is given in e.g. [5] and [6]. If μ is a measure on I, we obtain a measure on $\mathbb{T}$ by setting

$$\mathring{\mu}(E) = \mu\left(\{\cos\theta, \theta \in E \cap [0, \pi)\}\right) + \mu\left(\{\cos\theta, \theta \in E \cap [-\pi, 0)\}\right), \tag{2.10}$$

which can also be written as $\mathring{\mu}(E) = \int_E |d\mu(\cos\theta)|$. Using the Lebesgue decomposition of μ and the change-of-variables theorem (see e.g. [19, p. 153]) it is not difficult to see that $\mathring{\mu}'(\theta) = \mu'(\cos\theta)\,|\sin\theta|$ (see also [20]). Clearly, this measure $\mathring{\mu}$ is then symmetric (i.e.; $d\mathring{\mu}(-\theta) = -d\mathring{\mu}(\theta)$), so that $I_{\mathring{\mu}}(\mathring{f}_*^c) = I_{\mathring{\mu}}(\mathring{f})$ for every function $\mathring{f}$ on $\mathbb{T}$. In [12, Lem. 3.1] the following is proved for symmetric measures on $\mathbb{T}$.

Theorem 2.1. *Suppose the numbers $\{\beta_1, \ldots, \beta_{n-1}\}$ are real and/or appear in complex conjugate pairs. Assume $\mathring{\mu}$ is a symmetric measure on $\mathbb{T}$ and let $\mathring{Q}_{n,\mathring{\tau}_n}(z) = \phi_n(z) + \mathring{\tau}_n \phi_n^*(z)$, $\mathring{\tau}_n \in \mathbb{T}$, be an nth pORF with respect to $\mathring{\mu}$. Then the following holds:*

1. *The zeros of $\mathring{Q}_{n,\mathring{\tau}_n}(z)$ appear in complex conjugate pairs iff $\mathring{\tau}_n = \pm 1$.*
2. *$\mathring{Q}_{n,\mathring{\tau}_n}(z)$ has a zero in*
   - *(a) $z = 1$ iff $\mathring{\tau}_n = -1$,*
   - *(b) $z = -1$ iff $\mathring{\tau}_n = (-1)^{n+1}$.*

[4] Note that once the δ_k's are fixed in the recurrence relation, so are the $\mathring{\rho}_k$'s and e_k's.

3. *Let $I_n(\mathring{f}) = \sum_{k=1}^{n} \mathring{\lambda}_k \mathring{f}(z_k)$ be an n-point rational Szegő quadrature formula for $I_{\mathring{\mu}}(\mathring{f})$, based on the zeros of the pORF $\mathring{Q}_{n,\pm 1}(z)$. Then for $k = 1, \ldots, n$, the weight $\mathring{\lambda}_k$ corresponding to the node z_k is equal to the weight $\mathring{\lambda}_j$ corresponding to the node $z_j = \overline{z}_k$.*

Finally, note that by the Joukowski Transform, a function $f(x)$ transforms into a function $\mathring{f}(z) = (f \circ J)(z)$, so that $\mathring{f}(z) = \mathring{f}(z^{-1})$ and $J_\mu(f) = \frac{1}{2} I_{\mathring{\mu}}(\mathring{f})$. Moreover, from [16, Lem. 3.1] it follows that every function $f \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ transforms into a function $\mathring{f} \in (\mathring{\mathcal{L}}_k^c \cdot \mathring{\mathcal{L}}_{k*}) \setminus (\mathring{\mathcal{L}}_{k-1}^c \cdot \mathring{\mathcal{L}}_{(k-1)*})$; see also [10, Chapt. 3.2]. Further, the following relation between the ORFs on I and on $\mathbb{T}$ has been proved in [14, Thm. 4.2].

Theorem 2.2. *Suppose μ is a measure on I and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (2.10). Let φ_k be an ORF with respect to μ, and let $\hat{\phi}_k$ be the ORF with respect to $\mathring{\mu}$ so that $\hat{\kappa}_k \in \mathbb{R}_0^+$. Then there exist non-zero constants c_k and d_k so that*

$$\varphi_k(x) = c_k B_{k*}(z)\{\hat{\phi}_{2k}^c(z) + \hat{\phi}_{2k}^*(z)\} = d_k B_{k*}(z)\{\zeta_k(z)\hat{\phi}_{2k-1}(z) + \hat{\phi}_{2k-1}^{c*}(z)\}. \tag{2.11}$$

The constant c_k is explicitly given by

$$c_k = \rho_k \left\{ 1 + \frac{\Re\{\hat{\phi}_{2k}^c(\beta_k)\}}{\hat{\kappa}_{2k}} \right\}^{-1/2}, \quad \rho_k \in \mathbb{T}.$$

§3. Relating rational Gauss-Radau with rational Szegő-Lobatto rules

In [12, Sect. 3] the n-point rational Gauss-Radau quadrature rules with fixed node in $x_\alpha \in \{\pm 1\}$ were related with $(2n-1)$-point rational Szegő-Radau quadrature rules with fixed node in $z_\alpha \in \{\pm 1\}$. In this section we will consider n-point rational Gauss-Radau quadrature rules with all nodes (including the fixed node x_α) in $(-1, 1)$. Our aim is then to relate them with $2n$-point rational Szegő-Lobatto quadrature rules with fixed nodes in $z_\alpha = \overline{z}_\beta = J^{inv}(x_\alpha) \in \mathbb{T}$, and to find a relation between the parameter τ_n in the qORF Q_{n,τ_n} and the parameter $\tilde{\delta}_{2n-1}$ in the pORF $\tilde{\mathring{Q}}_{2n,\mathring{\tau}_n} \in \hat{\mathring{\mathcal{L}}}_{2n}$. Note that $\tilde{\mathring{\rho}}_{2n} \in \{\pm 1\}$ and $\tilde{\delta}_{2n} \in (-1, 1)$ due to $\tilde{\phi}_k^c(z) \equiv \tilde{\phi}_k(z)$ for $k \in \{2n-1, 2n\}$, while it immediately follows from Theorem 2.1 that $\mathring{\tau}_n$ should be equal to one, so that $\tilde{\xi}_{2n}$, defined as before in (2.7), should be equal to one too. We now have the following.

Theorem 3.1. *Suppose μ is a measure on I and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (2.10). Consider the sequence of ORFs $\{\tilde{\phi}_k\}_{k=0}^{2n}$, based on the sequence of numbers $\hat{\mathcal{B}}_{2n}$, that are generated by means of the recurrence relation (2.4) with the sequence of parameters $\{\delta_1, \delta_2, \ldots, \delta_{n-2}, \tilde{\delta}_{n-1}, \tilde{\delta}_n\} \subset \mathbb{D}$, and let $\hat{I}_{2n}(\mathring{f}) = \mathring{\lambda}_\alpha \mathring{f}(z_\alpha) + \mathring{\lambda}_\beta \mathring{f}(\overline{z}_\alpha) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k \mathring{f}(z_k)$ be a $2n$-point rational Szegő-Lobatto quadrature formula with fixed nodes in $\{z_\alpha, \overline{z}_\alpha\} \subset \mathbb{T} \setminus \{-1, 1\}$ for $I_{\mathring{\mu}}(\mathring{f}) = \int_{-\pi}^{\pi} \mathring{f}(z) d\mathring{\mu}(\theta)$, whose nodes are the zeros z_k of the pORF $\tilde{\mathring{Q}}_{2n,1}(z) = \tilde{\phi}_{2n}(z) + \tilde{\phi}_{2n}^*(z)$. Further, suppose the zeros are ordered in such a way that $z_{n-1+k} = \overline{z}_k$ for $k = 1, \ldots, n-1$, with $z_k \neq \overline{z}_j$ for every $1 \leq k < j \leq n-1$. Then, when taking $x_\alpha = J(z_\alpha)$, $\lambda_\alpha = \mathring{\lambda}_\alpha$, and $x_k = J(z_k)$ and $\lambda_k = \mathring{\lambda}_k$ for $k = 1, \ldots, n-1$, the formula $J_n(f) = \lambda_\alpha f(x_\alpha) + \sum_{k=1}^{n-1} \lambda_k f(x_k)$ coincides with the n-point rational Gauss-Radau quadrature formula with fixed node in $x_\alpha \in (-1, 1)$ for $J_\mu(f) = \int_{-1}^{1} f(x) d\mu(x)$, based on the zeros of the qORF $Q_{n,\tau_n} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, given by*

$$Q_{n,\tau_n}(x) = \tilde{K}_n B_{n*}(z) \tilde{\mathring{Q}}_{2n,1}(z), \quad \tilde{K}_n \in \mathbb{C}_0, \quad \textit{with} \quad \tau_n = -\frac{Z_{n-1}^c(x_\alpha)\varphi_n(x_\alpha)}{Z_n(x_\alpha)\varphi_{n-1}(x_\alpha)}.$$

Proof. (The proof is similar to the proof of [12, Thm. 3.2].) From the first and second parts of Lemma 2.1 it follows that $\tilde{\mathring{Q}}_{2n,1}(z)$ has $2n$ zeros, all different from 1 and -1, appearing in complex conjugate pairs. Next, from the third part of Theorem 2.1 it then follows that $\mathring{\lambda}_\alpha = \mathring{\lambda}_\beta$ and that $\mathring{\lambda}_{n-1+k} = \mathring{\lambda}_k$ for $k = 1, \ldots, n-1$. Consider now an arbitrary function $f \in \mathcal{R}_{n-1,n-1} = \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$.

Clearly, its corresponding function $\mathring{f}(z) = (f \circ J)(z)$ is then in $(\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}^c_{n-1})^c \cdot (\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}^c_{n-1})_* = \mathring{\mathcal{R}}_{2n-2}$. Consequently,

$$J_n(f) = \lambda_\alpha f(x_\alpha) + \sum_{k=1}^{n-1} \lambda_k f(x_k) = \mathring{\lambda}_\alpha \mathring{f}(z_\alpha) + \sum_{k=1}^{n-1} \mathring{\lambda}_k \mathring{f}(z_k) =$$
$$\frac{1}{2}\left\{ \mathring{\lambda}_\alpha(z_\alpha) + \mathring{\lambda}_\beta \mathring{f}(\overline{z}_\alpha) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k \mathring{f}(z_k) \right\} = \frac{1}{2}\hat{I}_{2n}(\mathring{f}) = \frac{1}{2} I_{\mathring{\mu}}(\mathring{f}) = J_\mu(f).$$

Finally, because the equality holds for every $f \in \mathcal{R}_{n-1,n-1}$, the n-point formula is the n-point rational Gauss-Radau quadrature formula with fixed node in $x_\alpha \in (-1,1)$. This concludes the proof. ■

Recall from the Subsection 2.1 that the qORF Q_{n,τ_n}, given by (2.2), is orthogonal with respect to measures $\tilde{\mu}$ supported on a set $S \subsetneq \overline{\mathbb{R}}$. However, it has not been proved so far that (at any time) one or more of these measures $\tilde{\mu}$ are supported on I whenever the zeros of Q_{n,τ_n} are all in $(-1,1)$. So, supposing the qORF Q_{n,τ_n} has a fixed zero in $x_\alpha \in (-1,1)$, we first need to verify whether it is possible to construct a pORF $\mathring{\tilde{Q}}_{2n,1}$ with fixed zeros in $z_\alpha = \overline{z}_\beta = J^{inv}(x_\alpha)$ before we can make any statement in the opposite direction as in Theorem 3.1. In other words, we are interested now in the relation between the parameters τ_n in the qORF and $\tilde{\delta}_{2n-1}$ in the pORF, to find out whether $\tilde{\delta}_{2n-1} \in \mathbb{D}$ for a given τ_n. For this, let us assume that the qORF Q_{n,τ_n} is orthogonal with respect to modified measures $\tilde{\mu}$ supported on I; e.g., with respect to

$$d\tilde{\mu}(x) = \{1 + c_\alpha \varphi_{2n-1}(x)\}\, d\mu(x), \tag{3.1}$$

where $\varphi_{2n-1} \in \mathcal{R}_{n,n-1}$ is orthonormal on $\mathcal{R}_{n-1,n-1}$ with respect to the measure μ, and c_α is a constant depending on x_α so that $c_\alpha \varphi_{2n-1}(x) \geqslant -1$ on I. Further, let $\tilde{\mathring{\mu}}$ be the corresponding measure on $\mathbb{T}$ for $\tilde{\mu}$, given by (2.10), and let $\tilde{\phi}_{2n} \in \tilde{\mathring{\mathcal{L}}}_n \cdot \mathring{\mathcal{L}}_{n*}$ denote the ORF with respect to $\tilde{\mathring{\mu}}$ so that $\tilde{\kappa}_{2n} \in \mathbb{R}_0^+$. Then it follows from Theorem 2.2 that there exist constants $k_n \in \mathbb{C}_0$ and $\tilde{c}_n = \tilde{\rho}_n \left\{1 + \frac{\tilde{\phi}_{2n}(\beta_n)}{\tilde{\kappa}_{2n}}\right\}^{-1/2} \in \mathbb{C}_0$ so that

$$k_n Q_{n,\tau_n}(x) = \tilde{c}_n B_{n*}(z) \mathring{\tilde{Q}}_{2n,1}(z) = \tilde{d}_n B_{n*}(z)\{\zeta_n(z)\tilde{\phi}_{2n-1}(z) + \tilde{\phi}^*_{2n-1}(z)\}. \tag{3.2}$$

In Subsection 2.2 the parameter $\tilde{\delta}_{n-1}$ in the pORF $\mathring{\tilde{Q}}_{n,1}(z)$ is just assumed to be in $\mathbb{D}$. However, in [7] it has been proved that the rational Szegő-Lobatto quadrature rule with fixed nodes in z_α and z_β on $\mathbb{T}$ can only exist whenever $\tilde{\delta}_{n-1}$ lies on a circle segment in $\mathbb{D}$ with a certain midpoint and radius. In Lemma 3.3 we will give simple expressions for this midpoint and radius for the special case of symmetric measures and the sequences of complex numbers $\{\beta_j\}_{j=1}^n$ that are real and/or appear in complex conjugate pairs, with $\beta_n \in (-1,1)$. But first, we need the following lemma. The proof of all the lemmas that follow can be found in Section 5.

Lemma 3.2. *Suppose $\mathring{\mu}$ is a symmetric measure on $\mathbb{T}$, and let ϕ_j denoted the jth ORFs with respect to $\mathring{\mu}$ so that their leading coefficients $\mathring{\kappa}_j \in \mathbb{R}_0^+$. Assume that the numbers $\{\beta_j\}_{j=1}^{k-1}$ are real and/or appear in complex conjugate pairs, and that $\beta_k \in (-1,1)$. Then there exist constants $b_k \in \mathbb{C}_0$ and a_k, with $\Re\{a_k\} = 0$, so that*

$$\phi^c_{k-1}(z) = \left(\frac{1 - \overline{\beta}_{k-1} z}{z - \overline{\beta}_{k-1}}\right) \left\{ b_k \zeta_{k-1}(z) \phi_{k-1}(z) + a_k \phi^*_{k-1}(z) \right\}. \tag{3.3}$$

The constants satisfy the following equalities:

$$a_k = \left(\frac{\mathring{\rho}_k^2 \delta_k - \overline{\mathring{\rho}}_k^2 \overline{\delta}_k}{1 - |\delta_k|^2} \right) \quad \textit{and} \quad b_k = \mathring{\rho}_k^2 + \overline{\delta}_k a_k.$$

Clearly, $\phi^c_{k-1}(z)$ in (3.3) does not depend on the parameter δ_k. Therefore, also the constants a_k and b_k in Lemma 3.2 have to be independent of δ_k; i.e., for every parameter $\tilde{\delta}_k \in \mathbb{D}$ that corresponds with a modified symmetric measure $\tilde{\mathring{\mu}}$ on $\mathbb{T}$, it should hold that

$$a_k = \left(\frac{\tilde{\mathring{\rho}}_k^2 \tilde{\delta}_k - \overline{\tilde{\mathring{\rho}}}_k^2 \overline{\tilde{\delta}}_k}{1 - |\tilde{\delta}_k|^2} \right) \quad \text{and} \quad b_k = \tilde{\mathring{\rho}}_k^2 + \overline{\tilde{\delta}}_k a_k;$$

or, equivalenty,

$$a_k = \frac{b_k\tilde{\delta}_k - \overline{b}_k\overline{\tilde{\delta}}_k}{1+|\tilde{\delta}_k|^2} \quad \text{and} \quad \mathring{\tilde{\rho}}_k^2 = b_k - \overline{\tilde{\delta}}_k a_k. \tag{3.4}$$

Meanwhile, from the recurrence relation (2.4) it follows that

$$\mathring{\tilde{\rho}}_k = \frac{\overline{\tilde{\delta}}_k(\beta_k-\beta_{k-1})\phi_{k-1}(\beta_k) + (1-\overline{\beta}_{k-1}\beta_k)\phi^*_{k-1}(\beta_k)}{\left|\overline{\tilde{\delta}}_k(\beta_k-\beta_{k-1})\phi_{k-1}(\beta_k) + (1-\overline{\beta}_{k-1}\beta_k)\phi^*_{k-1}(\beta_k)\right|}, \tag{3.5}$$

which leaves us with three conditions for just two parameters $\tilde{\delta}_k$ and $\mathring{\tilde{\rho}}_k$. The following lemma, however, shows that these three condition are not linear independent.

Lemma 3.3. *The constants a_k and b_k, defined as before in Lemma 3.2, satisfy the following equality: $|b_k|^2 = 1+|a_k|^2$. Further, whenever $a_k = 0$, the equalities in (3.4)–(3.5) are satisfied with $\tilde{\delta}_k = \tilde{\delta}_k^a\overline{b}_k$, for every $\tilde{\delta}_k^a \in (-1,1)$. It then holds that $\mathring{\tilde{\rho}}_k^2 = \mathring{\rho}_k^2$. If, on the other hand, $a_k \neq 0$, then the equalities in (3.4)–(3.5) are satisfied for every $\tilde{\delta}_k \in C \cap \mathbb{D}$, where C represents the circle with center $\left(\frac{\Im\{b_k\}}{\Im\{a_k\}}, \frac{\Re\{b_k\}}{\Im\{a_k\}}\right)$ and radius $|a_k|^{-1}$.*

The following lemma now provides an expression for Q_{n,τ_n} in terms of $\hat{\phi}_{2n-2}$ and $\hat{\phi}^*_{2n-2}$.

Lemma 3.4. *Suppose μ is a measure on I and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (2.10). Let Q_{n,τ_n} be a qORF with respect to μ, and let $\hat{\phi}_k$ be the ORF with respect to $\mathring{\mu}$ so that $\hat{\kappa}_k \in \mathbb{R}_0^+$. Further, assume Q_{n,τ_n} is orthogonal with respect to modified measures $\tilde{\mu}$ supported on I, and let $\tilde{C}_n \in \mathbb{C}_0$ and $\tilde{w}_n(z)$ be defined by*

$$\begin{aligned} \tilde{C}_n &= \tilde{\rho}_n\mathring{\tilde{\rho}}_{2n}\sqrt{\left(\frac{1-\beta_n^2}{1-|\beta_{n-1}|^2}\right)\frac{1}{(1-|\tilde{\delta}_{2n-1}|^2)(1-\tilde{\delta}_{2n})}} \quad \textit{and} \\ \tilde{w}_n(z) &= \mathring{\tilde{\rho}}_{2n-1}(z-\beta_n) + \overline{\mathring{\tilde{\rho}}}_{2n-1}\overline{\tilde{\delta}}_{2n-1}(1-\beta_n z), \end{aligned}$$

with $\tilde{\rho}_n \in \mathbb{T}$, $\mathring{\tilde{\rho}}_{2n} \in \{\pm 1\}$, $\tilde{\delta}_{2n} \in (-1,1)$, and the parameters $\tilde{\delta}_{2n-1}$ and $\mathring{\tilde{\rho}}_{2n-1}$ satisfying

$$\hat{a}_{2n-1} = \frac{\hat{b}_{2n-1}\tilde{\delta}_{2n-1} - \overline{\hat{b}}_{2n-1}\overline{\tilde{\delta}}_{2n-1}}{1+|\tilde{\delta}_{2n-1}|^2} \quad \textit{and} \quad \mathring{\tilde{\rho}}_{2n-1}^2 = \hat{b}_{2n-1} - \overline{\tilde{\delta}}_{2n-1}\hat{a}_{2n-1},$$

where $\hat{a}_{2n-1}$ and $\hat{b}_{2n-1}$ are defined as before in Lemma 3.2. Then there exists a non-zero constant k_n so that

$$k_n Q_{n,\tau_n}(x) = \tilde{C}_n B_{n*}(z)\frac{1-\overline{\beta}_{n-1}z}{(1-\beta_n z)^2}\left\{\tilde{w}_n(z)\zeta_{n-1}(z)\hat{\phi}_{2n-2}(z) + \tilde{w}_n^*(z)\hat{\phi}^*_{2n-2}(z)\right\}. \tag{3.6}$$

On the other hand, based on (2.2) we can also prove the following expression for Q_{n,τ_n} in terms of $\hat{\phi}_{2n-2}$ and $\hat{\phi}^*_{2n-2}$.

Lemma 3.5. *Suppose μ is a measure on I and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (2.10). Let Q_{n,τ_n} be a qORF with respect to μ, and let $\hat{\phi}_k$ be the ORF with respect to $\mathring{\mu}$ so that $\hat{\kappa}_k \in \mathbb{R}_0^+$. Further, let $\hat{C}_n \in \mathbb{C}_0$ and $\hat{w}_n(z)$ be defined by*

$$\begin{aligned} \hat{C}_n &= \rho_n\mathring{\hat{\rho}}_{2n}\sqrt{\left(\frac{1-\beta_n^2}{1-|\beta_{n-1}|^2}\right)\frac{1}{(1-|\hat{\delta}_{2n-1}|^2)(1-\hat{\delta}_{2n})}} \quad \textit{and} \\ \hat{w}_n(z) &= \mathring{\hat{\rho}}_{2n-1}(z-\beta_n) + \overline{\mathring{\hat{\rho}}}_{2n-1}\overline{\hat{\delta}}_{2n-1}(1-\beta_n z), \end{aligned}$$

with $\rho_n \in \mathbb{T}$, $\mathring{\hat{\rho}}_{2n} \in \{\pm 1\}$, $\hat{\delta}_{2n} \in (-1,1)$, and $\hat{\delta}_{2n-1}$ and $\mathring{\hat{\rho}}_{2n-1}$ the coefficients in the recurrence relation (2.4). Then it holds that

$$\begin{aligned} Q_{n,\tau_n}(x) = \hat{C}_n B_{n*}(z)\tfrac{1-\overline{\beta}_{n-1}z}{(1-\beta_n z)^2}\Bigg\{ &\left[\hat{w}_n(z) + \breve{\tau}_n\hat{b}_{2n-1}(1-\overline{\beta}_{n-1}z)\right]\zeta_{n-1}(z)\hat{\phi}_{2n-2}(z) + \\ &\left[\hat{w}_n^*(z) + \breve{\tau}_n(1-\overline{\beta}_{n-1}\hat{a}_{2n-1})\left(z - \tfrac{\overline{\beta}_{n-1}-\hat{a}_{2n-1}}{1-\overline{\beta}_{n-1}\hat{a}_{2n-1}}\right)\right]\hat{\phi}^*_{2n-2}(z)\Bigg\}. \end{aligned} \tag{3.7}$$

where

$$\breve{\tau}_n = \tau_n \frac{\rho_{n-1}}{\rho_n \hat{\overline{\rho}}_{2n}} \left(\frac{1+\beta_n^2}{1+\overline{\beta}_{n-1}^2} \right) \sqrt{\left(\frac{1-|\beta_{n-1}|^2}{1-\beta_n^2} \right) \frac{(1-\left|\hat{\delta}_{2n-1}\right|^2)(1-\hat{\delta}_{2n}^2)}{(1+\hat{\gamma}_{2n-2})}},$$

$$\rho_{n-1} \in \mathbb{T}, \quad \hat{\gamma}_{2n-2} = \begin{cases} \hat{\delta}_{2n-2}, & \beta_{n-1} \in (-1,1) \\ \frac{\Im\{\hat{\overline{\rho}}_{2n-1}\hat{\delta}_{2n-1}\}}{\Im\{\beta_{n-1}\}} \left(\frac{1-|\beta_{n-1}|^2}{1-\left|\hat{\delta}_{2n-1}\right|^2} \right), & \beta_{n-1} \notin (-1,1) \end{cases}, \tag{3.8}$$

and $\hat{a}_{2n-1}$ and $\hat{b}_{2n-1}$ are defined as before in Lemma 3.2.

In the previous two lemmas we obtained two expressions for Q_{n,τ_n} in terms of $\hat{\phi}_{2n-2}$ and $\hat{\phi}^*_{2n-2}$. The second equality (3.7) always holds, while the first equality (3.6) only holds true under the assumption that Q_{n,τ_n} is orthogonal with respect to modified measures supported on I. Comparing the expressions for Q_{n,τ_n} in (3.6) and (3.7), we find that the parameters τ_n and $\tilde{\delta}_{2n-1}$ should satisfy the following equalities:

$$\begin{cases} K_n \tilde{w}_n(z) - \hat{w}_n(z) &= \breve{\tau}_n \hat{b}_{2n-1}(1-\overline{\beta}_{n-1}z) \\ K_n \tilde{w}^*_n(z) - \hat{w}^*_n(z) &= \breve{\tau}_n(1-\overline{\beta}_{n-1}\hat{a}_{2n-1}) \left(z - \frac{\overline{\beta}_{n-1}-\hat{a}_{2n-1}}{1-\overline{\beta}_{n-1}\hat{a}_{2n-1}} \right), \end{cases} \tag{3.9}$$

where $\tilde{w}_n(z)$, $\hat{w}_n(z)$ and $\breve{\tau}_n$ are defined as above, and

$$K_n = \frac{\tilde{C}_n}{k_n \hat{C}_n} = \frac{\tilde{\rho}_n \hat{\tilde{\rho}}_{2n}}{k_n \rho_n \hat{\overline{\rho}}_{2n}} \sqrt{\frac{(1-|\hat{\delta}_{2n-1}|^2)(1-\hat{\delta}_{2n})}{(1-|\tilde{\delta}_{2n-1}|^2)(1-\tilde{\delta}_{2n})}} \in \mathbb{C}_0.$$

We now have the following lemma.

Lemma 3.6. *Suppose μ is a measure on I and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (2.10). Let Q_{n,τ_n} be a qORF with respect to μ and with a fixed zero in $x_\alpha \in (-1,1)$, and assume Q_{n,τ_n} is orthogonal with respect to modified measures $\tilde{\mu}$ supported on I. Further, let $\hat{\phi}_k$ be the ORF with respect to $\mathring{\mu}$ so that $\mathring{\kappa}_k \in \mathbb{R}_0^+$, generated by means of the recurrence relation (2.4) with coefficients $\hat{\delta}_k \in \mathbb{D}$ and $\hat{\overline{\rho}}_k \in \mathbb{T}$. Suppose $\rho_k \in \mathbb{T}$ and $\hat{a}_k$ and $\hat{b}_k$ are defined as above in Theorem 2.2 and Lemma 3.2 respectively, and let $\tilde{\delta}_{2n-1} \in \mathbb{D}$ and $\hat{\tilde{\rho}}_{2n-1} \in \mathbb{T}$ denote the modified coefficients in the construction of the pORF $\mathring{Q}_{2n,1}$ with fixed zeros in $z_\alpha = \overline{z}_\beta = J^{inv}(x_\alpha)$, satisfying the equalities in (3.4). Then the parameters τ_n in the qORF and $\tilde{\delta}_{2n-1}$ in the pORF are related by*

$$\overline{\tilde{\delta}}_{2n-1} = \frac{\breve{\tau}_n \hat{\overline{\rho}}_{2n-1} \hat{b}_{2n-1}(1-\overline{\beta}_{n-1}\beta_n) + \overline{\hat{\delta}}_{2n-1}(1-\beta_n^2)}{\breve{\tau}_n \hat{\overline{\rho}}_{2n-1}[(\beta_n - \overline{\beta}_{n-1}) + \hat{a}_{2n-1}(1-\overline{\beta}_{n-1}\beta_n)] + (1-\beta_n^2)}, \tag{3.10}$$

where $\breve{\tau}_n$ is given by (3.8). Or, equivalently,

$$\breve{\tau}_n = \frac{\overline{\hat{\overline{\rho}}}_{2n-1}(\overline{\tilde{\delta}}_{2n-1} - \overline{\hat{\delta}}_{2n-1})}{\tilde{\overline{\rho}}^2_{2n-1}(1-\overline{\beta}_{n-1}\beta_n) + \overline{\tilde{\delta}}_{2n-1}(\overline{\beta}_{n-1} - \beta_n)}.$$

The following lemma now shows that (3.10) provides a fast and easy way to check whether for a given τ_n the zeros of the qORF Q_{n,τ_n} are all in $(-1,1)$.

Lemma 3.7. *Consider the qORF Q_{n,τ_n}, with $\tau_n \in \mathbb{C}$, and let $\tilde{\delta}_{2n-1}$ be given by (3.10). Then the zeros of the qORF Q_{n,τ_n} are all in $(-1,1)$ iff $\tilde{\delta}_{2n-1} \in \mathbb{D}$.*

Based on the previous two lemmas, we can now prove the following.

Theorem 3.8. *Suppose μ is a measure on I and assume $\mathring{\mu}$ is the corresponding measure on $\mathbb{T}$, given by (2.10). Let $J_n(f) = \lambda_\alpha f(x_\alpha) + \sum_{k=1}^{n-1} \lambda_k f(x_k)$ be an n-point rational Gauss-Radau quadrature formula with fixed node in $x_\alpha \in (-1,1)$ for $J_\mu(f) = \int_{-1}^{1} f(x)d\mu(x)$, whose nodes are the zeros of the qORF $Q_{n,\tau_n} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$. Set $x_k = \cos\theta_k$, and define $\{z_k\}_{k=1}^{2n-2}$ and $\{\mathring{\lambda}_k\}_{k=1}^{2n-2}$ by means of*

$$\left.\begin{array}{ll} z_k = e^{\mathbf{i}\theta_k}, & \mathring{\lambda}_k = \lambda_k \\ z_{n-1+k} = e^{-\mathbf{i}\theta_k}, & \mathring{\lambda}_{n-1+k} = \lambda_k \end{array}\right\}, \; k = 1, \ldots, n-1.$$

Further, let $z_\alpha = \overline{z}_\alpha = J^{inv}(x_\alpha)$ *and* $\mathring{\lambda}_\alpha = \lambda_\alpha$. *Then* $\hat{I}_{2n}(\mathring{f}) = \mathring{\lambda}_\alpha \mathring{f}(z_\alpha) + \mathring{\lambda}_\alpha \mathring{f}(\overline{z}_\alpha) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k \mathring{f}(z_k)$ *coincides with the* $2n$*-point rational Szegő-Lobatto quadrature formula with fixed nodes in* $\{z_\alpha, \overline{z}_\alpha\} \subset \mathbb{T} \setminus \{-1, 1\}$ *for* $I_{\mathring{\mu}}(\mathring{f}) = \int_{-\pi}^{\pi} \mathring{f}(z) d\mathring{\mu}(\theta)$, *based on the zeros of the pORF* $\mathring{\tilde{Q}}_{2n,1}$ *of the form*

$$\mathring{\tilde{Q}}_{2n,1} = \tilde{K}_{2n} \left[\zeta_{n-1}(z) \tilde{\phi}_{2n-1}(z) + \tilde{\phi}^*_{n-1}(z) \right], \quad \tilde{K}_{2n} \in \mathbb{C}_0,$$

with

$$\tilde{\phi}_{2n-1}(z) = \tilde{\mathring{\rho}}_{2n-1} \tilde{e}_{2n-1} \frac{1 - \overline{\beta}_{n-1} z}{1 - \beta_n z} \left[\zeta_{n-1}(z) \hat{\phi}_{2n-2}(z) + \tilde{\delta}_{2n-1} \hat{\phi}^*_{2n-1}(z) \right],$$

where $\tilde{\delta}_{2n-1} \in \mathbb{D}$ *is given by* (3.10), *and* $\tilde{e}_{2n-1} \in \mathbb{R}_0^+$ *and* $\tilde{\mathring{\rho}}_{2n-1} \in \mathbb{T}$ *are the corresponding recurrence coefficients defined as above in* (2.4).

Proof. The statement directly follows from Lemmas 3.6–3.7 and (3.2). ■

We conclude this section with the following remark.

Remark 3.9. In (3.1) we have given an example of a modified measure in the case of the interval. Here, the modified measure $\tilde{\mu}$ was assumed to be absolutely continuous with respect to the measure μ. The results in this section do not depend on this assumption made in example (3.1), but it is clear that, if the qORF Q_{n,τ_n} is orthogonal with respect to modified measures $\tilde{\mu}$ that are absolutely continuous with respect to μ, then one of these modified measures will be of the form given by (3.1). Lemma 3.7 proves the existence of modified measures that are supported on I whenever the zeros of the qORF Q_{n,τ_n} are all in $(-1, 1)$, but it doesn't prove that one or more of them are absolutely continuous with respect to μ. Hence, it may be interesting to further investigate whether the latter holds too whenever the zeros of the qORF Q_{n,τ_n} are all in $(-1, 1)$. Of course, the same remark can be given for example (2.8) in the case of the complex unit circle.

§4. Illustrative examples

In this section we will illustrate the results from the previous section by means of some numerical experiments. For this, we consider the Chebyshev weight function of the first kind $d\mu(x) = dx/\sqrt{1 - x^2}$ on I, for which the corresponding measure $\mathring{\mu}$ on $\mathbb{T}$ is the Lebesgue measure $d\mathring{\mu}(z) = dz/\mathbf{i}z$. Our purpose then is to obtain a characterization of rational Gauss-Radau rules associated with μ by giving explicit expressions for the corresponding nodal rational functions and weights in the quadrature formulas.

Given the sequence of complex numbers $\mathcal{B}_n = \{\beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{D}$, the ORFs with respect to the Lebesgue measure $\mathring{\mu}$ are obtained by applying the recurrence relation (2.4) with $\delta_k = 0$ and $\mathring{\rho}_k = 1$ for every $k > 0$, and with initial conditions $\phi_0(z) \equiv \phi_0^*(z) \equiv \frac{1}{\sqrt{2\pi}}$ and $\zeta_0(z) = z$ (or, equivalently, $\beta_0 = 0$), so that

$$\phi_k(z) = \sqrt{\frac{1 - |\beta_k|^2}{2\pi}} \frac{z B_{k-1}(z)}{1 - \overline{\beta}_k z}, \qquad \text{and hence} \qquad \phi_k^*(z) = \sqrt{\frac{1 - |\beta_k|^2}{2\pi}} \frac{1}{1 - \overline{\beta}_k z}, \qquad k > 0. \tag{4.1}$$

Recall that for the rational Gauss-Radau quadrature rule we work with the sequence $\hat{\mathcal{B}}_{2n} = \{\hat{\beta}_1, \ldots, \hat{\beta}_{2n}\} \subset \mathbb{D}$ given by (2.9), and with $\beta_n \in (-1, 1)$, so that

$$\hat{B}_{2k-1}(z) = B_k^c(z) B_{k-1}(z) \qquad \text{and} \qquad \hat{B}_{2k}(z) = B_k(z) B_k^c(z). \tag{4.2}$$

Further, the kth ORF with respect to the measure μ on I is given by (see also [16, 18] and [10, Chapt. 3]):

$$\varphi_k(x) = \sqrt{\frac{1 - |\beta_k|^2}{2\pi}} \left\{ \frac{z B_{k-1}^c(z)}{1 - \beta_k z} + \frac{B_{(k-1)*}(z)}{z - \beta_k} \right\} = B_{k*}(z) \{ \hat{\phi}_{2k}^c(z) + \hat{\phi}_{2k}^*(z) \}, \quad k > 0;$$

i.e., $c_k = \rho_k = 1$ for every $k > 0$ in Theorem 2.2. Hence, (3.6) along with (4.1)–(4.2) gives us the following expression for an nth qORF with respect to the measure μ on I:

$$Q_{n,\tau_n}(x) = C_n \frac{\tilde{w}_n(z) B_{n-1}^c(z) + \tilde{w}_{n*}(z) B_{(n-1)*}(z)}{(1 - \beta_n/z)(1 - \beta_n z)}, \quad C_n \in \mathbb{C}_0,$$

where it follows from Lemma 3.3, (3.8) and (3.10) that

$$\tilde{w}_n(z) = \pm \left[z(1 - \tilde{\delta}_{2n-1}\beta_n) - (\beta_n - \tilde{\delta}_{2n-1}) \right],$$

with

$$\tilde{\delta}_{2n-1} = \frac{\breve{\tau}_n(1 - \overline{\beta}_{n-1}\beta_n)}{\breve{\tau}_n(\beta_n - \overline{\beta}_{n-1}) + (1 - \beta_n^2)} \in (-1,1), \quad \text{and} \quad \breve{\tau}_n = \tau_n \left(\frac{1+\beta_n^2}{1+\overline{\beta}_{n-1}^2} \right) \sqrt{\left(\frac{1 - |\beta_{n-1}|^2}{1-\beta_n^2} \right)}.$$

As a result, the nodes $x_k = J(z_k)$ of the n-point rational Gauss-Radau quadrature formula $J_n(f)$ satisfy

$$z_k \left(\frac{z_k - \tilde{\beta}_n}{1 - \tilde{\beta}_n z_k} \right) B_{n-1}(z_k) B_{n-1}^c(z_k) = -1, \qquad \tilde{\beta}_n := \frac{\beta_n - \tilde{\delta}_{2n-1}}{1 - \tilde{\delta}_{2n-1}\beta_n}, \tag{4.3}$$

where it should hold that

$$\tilde{\beta}_n = \frac{1 + z_\alpha^2 B_{n-1}(z_\alpha) B_{n-1}^c(z_\alpha)}{z_\alpha \left[1 + B_{n-1}(z_\alpha) B_{n-1}^c(z_\alpha) \right]}, \tag{4.4}$$

in order to have a fixed node in $x_\alpha = J(z_\alpha) \in (-1,1)$. Note that for $\tau_n = 0$ (and hence, $\tilde{\delta}_{2n-1} = 0$), it holds that $\tilde{\beta}_n = \beta_n$. Furthermore, from (4.3) it is easily verified that $\tilde{\beta}_n \in (-1,1)$ iff $\tilde{\delta}_{2n-1} \in (-1,1)$. Consequently, the nodes of an n-point rational Gauss-Radau quadrature formula with respect to the Chebyshev weight function of the first kind are in fact the nodes of the n-point rational Gaussian quadrature formula with respect to the same weight function, but with maximal domain of validity $\mathcal{L}\{\alpha_1, \ldots, \alpha_{n-1}, \tilde{\alpha}_n\} \cdot \mathcal{L}_{n-1}^c$ (see also [13, Sect. 6]).[5] Since $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$ is a subspace of $\mathcal{L}\{\alpha_1, \ldots, \alpha_{n-1}, \tilde{\alpha}_n\} \cdot \mathcal{L}_{n-1}^c$, it follows that the weights λ_k of the n-point rational Gauss-Radau quadrature formula $J_n(f)$ with fixed node in $x_\alpha \in (-1,1)$ are the weights of this n-point rational Gaussian quadrature formula too; hence, they are given by (see e.g. [16]):

$$\lambda_k = 2\pi \left\{ 1 + \sum_{j=1}^{n-1} \left[P(z_k, \beta_j) + P(z_k, \overline{\beta}_j) \right] + P(z_k, \tilde{\beta}_n) \right\}^{-1},$$

where $P(z,\beta)$ denotes the Poisson kernel: $P(z,\beta) = \frac{1-|\beta|^2}{|z-\beta|^2}$, with $z \in \mathbb{T}$ and $\beta \in \mathbb{D}$.

In [24] a numerical procedure is described to compute the nodes and weights in the rational Gauss-Chebyshev quadrature rule up to machine precision in $\mathcal{O}(m \cdot n)$ flops, where n denotes the number of interpolation points and m represents the number of different poles (two poles α_j and α_k were assumed to be different if $\alpha_j \neq \alpha_k$ and $\alpha_j \neq \overline{\alpha}_k$). This numerical procedure was implemented in MATLAB®[6]. In the numerical examples that follow, we used this implementation together with the above expressions to compute the nodes and weights in the rational Gauss-Radau quadrature rules on I. Unless stated otherwise, all the computations were done in double precision using MATLAB® 7.

Example 4.1. Define the function $f^{[\alpha]}(x) = \frac{\alpha x - 1}{x - \alpha}$, with $\alpha = J(\beta) \in \overline{\mathbb{C}}_I$. In [21, Thm. 3.2(2)] it is proved that

$$J_\mu(f^{[\alpha]}) = \int_{-1}^{1} f^{[\alpha]}(x) \frac{dx}{\sqrt{1-x^2}} = \pi\beta.$$

Consider now the sequence of poles $\{\alpha_1, \ldots, \alpha_{13}\}$, with

$$\alpha_k = \begin{cases} J\left(1.15 - 0.2k + \mathbf{i} \sin\left[\frac{(k-1)\pi}{9}\right]\right), & k \leqslant 10 \\ \infty, & k > 10. \end{cases}$$

We then approximate $J_\mu(f_1)$ and $J_\mu(f_2)$, with

$$f_1(x) = \frac{1}{\pi} \Re \left\{ \sum_{k=1}^{10} f^{[\alpha_k]}(x) \right\} \qquad \text{and} \qquad f_2(x) = f_1(x) + \frac{x}{\pi}, \tag{4.5}$$

[5] This observation does not hold for every measure; e.g. it does not hold for the Chebyshev weight function of the second kind $d\mu(x) = \sqrt{1-x^2}dx$ (see [13, Sect. 6]).

[6] MATLAB is a registered trademark of The MathWorks, Inc.

by means of an n-point rational interpolatory quadrature rule. Note that the exact solution is $J_\mu(f_1) = J_\mu(f_2) = 0.5$. Since the n-point rational Gaussian quadrature rule, based on the sequence of poles $\{\alpha_1, \ldots, \alpha_n\}$, does not exist for $1 < n < 10$, we consider an n-point rational Gauss-Radau quadrature rule instead, based on the sequence of poles

$$\{\alpha_1, \ldots, \alpha_{n-1}, \tilde{\alpha}_n\}, \quad \tilde{\alpha}_n = J(\tilde{\beta}_n) \in \overline{\mathbb{R}}_I, \quad 2 \leqslant n \leqslant 13.$$

First, we consider the case in which the last pole is fixed by $\tilde{\alpha}_n = \infty$, and then we consider the case in which $\tilde{\beta}_n$, given by (4.4), is chosen in such a way that the quadrature rule has a fixed node in $x_\alpha \in \{0, 0.1, -0.5\}$. Tables 1–2 then give the relative error on the approximation:

$$\text{Error i} \ = \left| \frac{J_n(f_i) - J_\mu(f_i)}{J_\mu(f_i)} \right|, \qquad i = 1, 2.$$

A '/' means that the quadrature rule does not exists (i.e., one of the nodes of the qORF is outside I because $\tilde{\delta}_{2n-1} \notin \mathbb{D}$).

These tables clearly show that the approximation by means the n-point rational Gauss-Radau quadrature rule (whenever it exists) is exact for $n \geqslant 11$ if $i = 1$ and for $n \geqslant 12$ if $i = 2$, respectively. Further, in the special case of $\tilde{\alpha}_n = \infty$, the n-point rational Gauss-Radau quadrature rule with $n > 10$ is in fact the n-point rational Gaussian quadrature rule, which explains the exactness of this quadrature rule for $n = 11$ if $i = 2$.

	$\tilde{\alpha}_n = \infty$	$x_\alpha = 0$		$x_\alpha = 0.1$		$x_\alpha = -0.5$	
n	Error 1	Error 1	$\tilde{\beta}_n$	Error 1	$\tilde{\beta}_n$	Error 1	$\tilde{\beta}_n$
2	$1.29e+01$	$2.53e+01$	$5.13e-02$	$2.44e+01$	$1.56e-01$	$1.47e+01$	$-4.74e-01$
3	$5.93e+00$	$2.43e+01$	$2.68e-01$	$1.87e+01$	$3.99e-01$	$9.83e+00$	$-3.71e-01$
4	$9.17e+00$	$2.37e+01$	$5.66e-01$	$1.24e+01$	$7.63e-01$	$6.26e+00$	$-2.58e-01$
5	$6.63e+00$	$2.34e+01$	$8.34e-01$	/	$1.19e+00$	$4.62e+00$	$-1.93e-01$
6	$5.36e+00$	$2.80e+01$	$8.78e-01$	/	$1.37e+00$	$3.88e+00$	$-1.88e-01$
7	$1.79e+00$	$8.65e+00$	$4.84e-01$	/	$1.15e+00$	$1.55e+00$	$-1.62e-01$
8	$1.41e+00$	$1.83e+00$	$9.21e-02$	$2.75e+00$	$7.04e-01$	$2.45e+00$	$3.11e-01$
9	$6.99e-01$	$4.37e-01$	$-3.22e-01$	$8.07e-01$	$3.02e-01$	/	$-1.03e+00$
10	$1.47e-01$	$5.08e-02$	$-7.64e-01$	$1.47e-01$	$-5.10e-03$	/	$-2.68e+00$
11	$4.22e-15$	/	$-1.06e+00$	$1.64e-14$	$-1.56e-01$	/	$-9.24e+00$
12	$3.80e-14$	$2.29e-14$	$9.44e-01$	/	$6.62e+00$	$2.44e-14$	$-8.92e-01$
13	$7.59e-14$	/	$-1.06e+00$	$7.11e-15$	$4.89e-02$	$9.04e-14$	$1.21e-01$

Table 1: Relative errors in the rational Gauss-Radau quadrature formulas with fixed last pole $\tilde{\alpha}_n$ or with fixed node in x_α for the estimation of $J_\mu(f_1)$, where f_1 is given by (4.5).

Example 4.2. Finally, consider the function

$$f_3(x) = \frac{1}{\pi} \sin\left(\frac{1}{(x-\upsilon)^2 + \omega^2} \right), \quad \upsilon \in I, \quad \omega \in \mathbb{R}_0^+, \tag{4.6}$$

which is similar to the function considered in [23, Ex. 5.7]. This function has essential singularities in $x = \upsilon \pm \omega\mathbf{i}$. For ω very close to 0, this function is extremely oscillating near these singularities. Since an essential singularity can be viewed as a pole of infinity multiplicity, this suggests taking

$$\alpha_k = \upsilon + (-1)^k \omega\mathbf{i}, \ k = 1, \ldots, n. \tag{4.7}$$

For $\upsilon = \Re\{J(0.20 + 0.75\mathbf{i})\} = \frac{641}{2410}$ and $\omega = |\Im\{J(0.20 + 0.75\mathbf{i})\}| = \frac{477}{1928}$ we obtain with the aid of MAPLE®[7] 9.5 that $J_\mu(f_3) \approx 0.4259620458467829$.

[7] MAPLE is a registered trademark of Waterloo Maple, Inc.

	$\tilde{\alpha}_n = \infty$	$x_\alpha = 0$		$x_\alpha = 0.1$		$x_\alpha = -0.5$	
n	Error 2	Error 2	$\tilde{\beta}_n$	Error 2	$\tilde{\beta}_n$	Error 2	$\tilde{\beta}_n$
2	$1.29e+01$	$2.52e+01$	$5.13e-02$	$2.47e+01$	$1.56e-01$	$1.39e+01$	$-4.74e-01$
3	$5.93e+00$	$2.40e+01$	$2.68e-01$	$1.92e+01$	$3.99e-01$	$9.38e+00$	$-3.71e-01$
4	$9.17e+00$	$2.32e+01$	$5.66e-01$	$1.31e+01$	$7.63e-01$	$6.04e+00$	$-2.58e-01$
5	$6.63e+00$	$2.28e+01$	$8.34e-01$	/	$1.19e+00$	$4.47e+00$	$-1.93e-01$
6	$5.36e+00$	$2.73e+01$	$8.78e-01$	/	$1.37e+00$	$3.74e+00$	$-1.88e-01$
7	$1.79e+00$	$8.29e+00$	$4.84e-01$	/	$1.15e+00$	$1.43e+00$	$-1.62e-01$
8	$1.41e+00$	$1.77e+00$	$9.21e-02$	$2.32e+00$	$7.04e-01$	$2.26e+00$	$3.11e-01$
9	$6.99e-01$	$5.56e-01$	$-3.22e-01$	$6.95e-01$	$3.02e-01$	/	$-1.03e+00$
10	$1.47e-01$	$2.03e-01$	$-7.64e-01$	$1.48e-01$	$-5.10e-03$	/	$-2.68e+00$
11	$4.44e-15$	/	$-1.06e+00$	$2.25e-02$	$-1.56e-01$	/	$-9.24e+00$
12	$3.97e-14$	$2.25e-14$	$9.44e-01$	/	$6.62e+00$	$2.25e-14$	$-8.92e-01$
13	$7.62e-14$	/	$-1.06e+00$	$6.22e-15$	$4.89e-02$	$8.90e-14$	$1.21e-01$

Table 2: Relative errors in the rational Gauss-Radau quadrature formulas with fixed last pole $\tilde{\alpha}_n$ or with fixed node in x_α for the estimation of $J_\mu(f_2)$, where f_2 is given by (4.5).

Because $\alpha_k \notin \overline{\mathbb{R}}$ for every $k > 0$, the n-point rational Gaussian quadrature rule, based on the sequence of poles $\{\alpha_1, \ldots, \alpha_n\}$, does not exist for every $n > 0$. For this reason we consider an n-point rational Gauss-Radau quadrature rule instead, based on the sequence of poles

$$\{\alpha_1, \ldots, \alpha_{n-1}, \tilde{\alpha}_n\}, \quad \tilde{\alpha}_n = J(\tilde{\beta}_n) \in \overline{\mathbb{R}}_I, \quad 2 \leqslant n \leqslant 33.$$

First, we consider the case in which the last pole is fixed by $\tilde{\alpha}_n = \infty$, and then we consider the case in which $\tilde{\beta}_n$, given by (4.4), is chosen in such a way that the quadrature rule has a fixed node in $x_\alpha \in \{0, 0.1, 0.2\}$. Figure 1 then graphically shows the relative error

$$\text{Error 3} = \left| \frac{J_n(f_3) - J_\mu(f_3)}{J_\mu(f_3)} \right|$$

as a function of the number of nodes in the quadrature rules, while the numerical results, together with the parameter $\tilde{\beta}_n$, are given in Table 3 for some values of n. The graphs clearly show that the quadrature rules globally perform equally well, but that locally one rule may perform better over the other or some of the rules may not exist. Note that the rational Gauss-Radau quadrature does not exist for

$$n \in \begin{cases} \{3,4,7,8,11,12,15,16,19,20,23,24,27,28,31,32\} & \text{if} \quad x_\alpha = 0 \\ \{2,3,5,6,8,9,11,12,14,15,18,21,24,27,30,33\} & \text{if} \quad x_\alpha = 0.1 \\ \{2,4,7,9,11,13,14,16,18,20,23,25,27,30,32\} & \text{if} \quad x_\alpha = 0.2. \end{cases}$$

Thus, to ensure the existence of the quadrature rule for every n, it may be better to fix the last pole $\tilde{\alpha}_n$ rather than to fix the node x_α.

§5. Proofs

Proof of Lemma 3.2

Note that under the conditions given on the measure $\mathring{\mu}$ and the numbers $\{\beta_j\}_{j=1}^k$, it holds that $\phi_k^c(z) \equiv \phi_k(z)$. Thus, it follows from the recurrence relation (2.4) that

$$\phi_k(z) = \mathring{\rho}_k e_k \tfrac{1-\overline{\beta}_{k-1} z}{1-\beta_k z} \left[\zeta_{k-1}(z)\phi_{k-1}(z) + \delta_k \phi_{k-1}^*(z)\right] = \overline{\mathring{\rho}}_k e_k \tfrac{1-\beta_{k-1} z}{1-\beta_k z} \left[\zeta_{k-1}^c(z)\phi_{k-1}^c(z) + \overline{\delta}_k \phi_{k-1}^{c*}(z)\right] = \phi_k^c(z)$$

$$\phi_k^*(z) = \overline{\mathring{\rho}}_k e_k \tfrac{1-\overline{\beta}_{k-1} z}{1-\beta_k z} \left[\overline{\delta}_k \zeta_{k-1}(z)\phi_{k-1}(z) + \phi_{k-1}^*(z)\right] = \mathring{\rho}_k e_k \tfrac{1-\beta_{k-1} z}{1-\beta_k z} \left[\delta_k \zeta_{k-1}^c(z)\phi_{k-1}^c(z) + \phi_{k-1}^{c*}(z)\right] = \phi_k^{c*}(z),$$

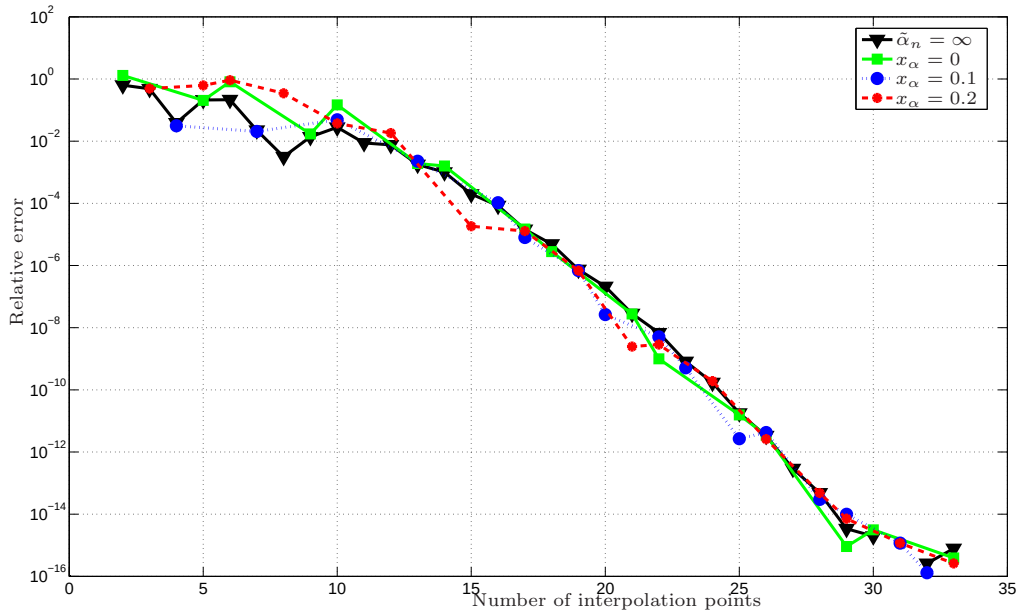

Figure 1: Relative errors in the rational Gauss-Radau quadrature formulas with fixed last pole in $\tilde{\alpha}_n$ or with fixed node in x_α for the estimation of $J_\mu(f_3)$, where f_3 is given by (4.6). (For $n = 31$, we obtained with fixed last pole in $\tilde{\alpha}_n = \infty$ that Error 3 $= 0$.)

	$\tilde{\alpha}_n = \infty$	$x_\alpha = 0$		$x_\alpha = 0.1$		$x_\alpha = 0.2$	
n	Error 3	Error 3	$\tilde{\beta}_n$	Error 3	$\tilde{\beta}_n$	Error 3	$\tilde{\beta}_n$
2	$6.32e-01$	$1.30e+00$	$9.94e-01$	/	$1.74e+00$	/	$5.00e+00$
3	$4.85e-01$	/	$1.60e+02$	/	$-1.81e+00$	$4.92e-01$	$-2.17e-01$
4	$3.87e-02$	/	$-1.02e+00$	$3.13e-02$	$3.62e-02$	/	$1.62e+00$
5	$2.12e-01$	$2.04e-01$	$-1.25e-02$	/	$1.53e+00$	$6.22e-01$	$-8.20e-01$
8	$3.15e-03$	/	$-1.04e+00$	/	$1.35e+00$	$3.46e-01$	$3.59e-01$
9	$1.37e-02$	$1.71e-02$	$-2.51e-02$	/	$-2.60e+00$	/	$2.46e+01$
16	$8.41e-05$	/	$-1.10e+00$	$1.05e-04$	$-2.30e-01$	/	$-7.84e+00$
17	$1.44e-05$	$1.49e-05$	$-5.02e-02$	$7.99e-06$	$9.48e-01$	$1.28e-05$	$1.21e-01$
32	$2.61e-16$	/	$-1.22e+00$	$1.30e-16$	$5.03e-01$	/	$-2.36e+01$
33	$7.82e-16$	$3.91e-16$	$-1.01e-01$	/	$6.28e+00$	$2.61e-16$	$4.16e-02$

Table 3: Relative errors in the rational Gauss-Radau quadrature formulas with fixed last pole in $\tilde{\alpha}_n$ or with fixed node in x_α for the estimation of $J_\mu(f_3)$, where f_3 is given by (4.6).

and hence, that

$$e_k \frac{1-\beta_{k-1}z}{1-\beta_k z}(1-|\delta_k|^2)\zeta^c_{k-1}(z)\phi^c_{k-1}(z) = \mathring{\rho}_k\phi_k(z) - \overline{\mathring{\rho}}_k\overline{\delta}_k\phi^*_k(z) = e_k \frac{1-\overline{\beta}_{k-1}z}{1-\beta_k z}\left[(\mathring{\rho}_k^2 - \overline{\mathring{\rho}}_k^2\overline{\delta}_k^2)\zeta_{k-1}(z)\phi_{k-1}(z) + (\mathring{\rho}_k^2\delta_k - \overline{\mathring{\rho}}_k^2\overline{\delta}_k)\phi^*_{k-1}(z)\right].$$

The equality in (3.3) now easily follows.

Proof of Lemma 3.3

Suppose $\tilde{\delta}_k \in \mathbb{D}$ satisfies the first equality in (3.4). First, we prove that if $\tilde{\mathring{\rho}}_k \in \mathbb{T}$ is given by (3.5), then $\tilde{\delta}_k$ and $\tilde{\mathring{\rho}}_k$ satisfy the second equality in (3.4) too. For this, let us assume that $\sigma_k^2 = b_k - \overline{\tilde{\delta}}_k a_k$, and define

$$\tilde{\phi}_k(z) = \sigma_k \tilde{e}_k \frac{1-\overline{\beta}_{k-1}z}{1-\beta_k z}\left[\zeta_{k-1}(z)\phi_{k-1}(z) + \tilde{\delta}_k \phi_{k-1}^*(z)\right], \quad \beta_k \in (-1,1),\ \tilde{e}_k \in \mathbb{R}_0^+.$$

Then it follows from Lemma 3.2 that

$$\begin{aligned}\sigma_k\tilde{\phi}_k^c(z) - \overline{\sigma}_k\overline{\tilde{\delta}}_k\tilde{\phi}_k^{c*}(z) &= \tilde{e}_k\frac{1-\beta_{k-1}z}{1-\beta_k z}(1-|\tilde{\delta}_k|^2)\zeta_{k-1}^c(z)\phi_{k-1}^c(z)\\ &= \tilde{e}_k\frac{1-\overline{\beta}_{k-1}z}{1-\beta_k z}\left[(\sigma_k^2 - \overline{\sigma}_k^2\overline{\tilde{\delta}}_k^2)\zeta_{k-1}(z)\phi_{k-1}(z) + (\sigma_k^2\tilde{\delta}_k - \overline{\sigma}_k^2\overline{\tilde{\delta}}_k)\phi_{k-1}^*(z)\right] = \sigma_k\tilde{\phi}_k(z) - \overline{\sigma}_k\overline{\tilde{\delta}}_k\tilde{\phi}_k^*(z).\end{aligned}$$

Thus, setting $f_k(z) = \tilde{\phi}_k(z) - \tilde{\phi}_k^c(z)$ and $C_k = \frac{\overline{\sigma}_k}{\sigma_k}\overline{\tilde{\delta}}_k \in \mathbb{D}$, we obtain that

$$f_k(z) = C_k f_k^*(z) \Leftrightarrow f_k^*(z) = \overline{C}_k[f_k^*(z)]^* = \overline{C}_k f_k(z),$$

and hence,

$$f_k(z) = |C_k|^2 f_k(z).$$

Since $C_k \in \mathbb{D}$, it follows that $f_k(z) \equiv 0$. Consequently, $\tilde{\phi}_k(z) \equiv \tilde{\phi}_k^c(z)$ and $\tilde{\phi}_k^*(\beta_k) \in \mathbb{R}$, which means that either $\sigma_k = \tilde{\mathring{\rho}}_k$ or $\sigma_k = -\tilde{\mathring{\rho}}_k$, where $\tilde{\mathring{\rho}}_k$ is given by (3.5).

Next, assuming that $a_k = 0$, it follows that $\Im\{b_k\tilde{\delta}_k\} = 0$ which holds true for every $\tilde{\delta}_k = \tilde{\delta}_k^a\overline{b}_k$ with $\tilde{\delta}_k^a \in \mathbb{R}$. Since $\tilde{\delta}_k \in \mathbb{D}$, we find that $\tilde{\delta}_k^a \in (-|b_k|^{-1}, |b_k|^{-1})$, where $\tilde{\mathring{\rho}}_k^2 = b_k = \mathring{\rho}_k^2 \in \mathbb{T}$.

Finally, for $a_k \neq 0$ we have that

$$a_k = \frac{b_k\tilde{\delta}_k - \overline{b}_k\overline{\tilde{\delta}}_k}{1+|\tilde{\delta}_k|^2} \Leftrightarrow \left|\tilde{\delta}_k + \frac{\overline{b}_k}{a_k}\right|^2 = \frac{|b_k|^2 - |a_k|^2}{|a_k|^2},$$

where we used the fact that $\overline{a}_k = -a_k$. On the other hand we have that

$$\tilde{\mathring{\rho}}_k^2 = b_k - \overline{\tilde{\delta}}_k a_k \Rightarrow \left|\tilde{\delta}_k + \frac{\overline{b}_k}{a_k}\right|^2 = \frac{1}{|a_k|^2}.$$

Thus, we obtain two circles with the same center $\left(\frac{\Im\{b_k\}}{\Im\{a_k\}}, \frac{\Re\{b_k\}}{\Im\{a_k\}}\right)$ and at least one point δ_k in common. For this reason, their radii have to be the same, which concludes the proof.

Proof of Lemma 3.4

Before we can prove Lemma 3.4, we first need the following.

Lemma 5.1. *In the special case in which $\beta_k \in (-1,1)$, the constants c_k and d_k in Theorem 2.2 are given by*

$$c_k = \rho_k\left\{1+\hat{\delta}_{2k}\right\}^{-1/2} \quad \text{and} \quad d_k = \rho_k\hat{\mathring{\rho}}_{2k}\left\{1-\hat{\delta}_{2k}\right\}^{-1/2}, \quad \hat{\mathring{\rho}}_{2k} \in \{\pm 1\}.$$

Proof. Since the measure $\mathring{\mu}$ in Theorem 2.2 is symmetric, and $\hat{\beta}_{2k} = \beta_k = \hat{\beta}_{2k-1} \in (-1,1)$, it follows from (2.5) that $\hat{\delta}_{2k} = \hat{\mathring{\rho}}_{2k}^2\hat{\delta}_{2k} = \frac{\hat{\phi}_{2k}(\beta_k)}{\hat{\kappa}_{2k}} = \frac{\hat{\phi}_{2k}^c(\beta_k)}{\hat{\kappa}_{2k}} \in (-1,1)$. As a result

$$c_k = \rho_k\left\{1+\hat{\delta}_{2k}\right\}^{-1/2}.$$

Next, by means of the recurrence relation (2.4) we obtain, with $\hat{\phi}_{2k}^c(z) \equiv \hat{\phi}_{2k}(z)$ that

$$c_k B_{k*}(z)\{\hat{\phi}_{2k}(z) + \hat{\phi}_{2k}^*(z)\} = c_k\hat{e}_{2k}\hat{\mathring{\rho}}_{2k}(1+\hat{\delta}_{2k})B_{k*}(z)\{\zeta_k(z)\hat{\phi}_{2k-1}(z) + \hat{\phi}_{2k-1}^{c*}(z)\},$$

and hence, that

$$d_k = c_k \hat{e}_{2k} \hat{\mathring{\rho}}_{2k}(1+\hat{\delta}_{2k}) = \rho_k \hat{\mathring{\rho}}_{2k} \sqrt{\frac{1+\hat{\delta}_{2k}}{1-\left|\hat{\delta}_{2k}\right|^2}} = \rho_k \hat{\mathring{\rho}}_{2k} \left\{1-\hat{\delta}_{2k}\right\}^{-1/2}.$$

■

From (3.2) and Lemma 5.1 it follows that

$$k_n Q_{n,\tau_n}(x) = \tilde{\rho}_n \tilde{\mathring{\rho}}_{2n} \left\{1-\tilde{\delta}_{2n}\right\}^{-1/2} B_{n*}(z)\{\zeta_n(z)\tilde{\phi}_{2n-1}(z)+\tilde{\phi}^*_{2n-1}(z)\}, \qquad \tilde{\mathring{\rho}}_{2n} \in \{\pm 1\},\ \tilde{\delta}_{2n} \in (-1,1).$$

Next, applying the recurrence relation (2.4) on $\tilde{\phi}_{2n-1}$ and $\tilde{\phi}^*_{2n-1}$ we obtain that

$$\begin{aligned} k_n Q_{n,\tau_n}(x) = \tilde{\rho}_n \tilde{\mathring{\rho}}_{2n} \left\{1-\tilde{\delta}_{2n}\right\}^{-1/2} \tilde{e}_{2n-1} B_{n*}(z) \frac{(1-\overline{\beta}_{n-1}z)}{(1-\beta_n z)^2} \times \\ \left\{ \left[\tilde{\mathring{\rho}}_{2n-1}(z-\beta_n) + \overline{\tilde{\mathring{\rho}}}_{2n-1}\overline{\tilde{\delta}}_{2n-1}(1-\beta_n z)\right] \zeta_{n-1}(z)\hat{\phi}_{2n-2}(z) + \right. \\ \left. \left[\tilde{\mathring{\rho}}_{2n-1}\tilde{\delta}_{2n-1}(z-\beta_n) + \overline{\tilde{\mathring{\rho}}}_{2n-1}(1-\beta_n z)\right] \hat{\phi}^*_{2n-2}(z) \right\}, \end{aligned}$$

where

$$\tilde{\rho}_n \tilde{\mathring{\rho}}_{2n} \left\{1-\tilde{\delta}_{2n}\right\}^{-1/2} \tilde{e}_{2n-1} = \tilde{C}_n.$$

Finally, with $\tilde{w}_n(z)$ as defined above, we have that

$$\tilde{w}^*_n(z) = \overline{\tilde{\mathring{\rho}}}_{2n-1}(1-\beta_n z) + \tilde{\mathring{\rho}}_{2n-1}\tilde{\delta}_{2n-1}(z-\beta_n),$$

which ends the proof.

Proof of Lemma 3.5

First, note that with $x = J(z)$ and $\alpha_k = J(\beta_k)$ we have that

$$\frac{Z_n(x)}{Z^c_{n-1}(x)} = \left(\frac{1+\beta_n^2}{1+\overline{\beta}^2_{n-1}}\right) \frac{(1-\overline{\beta}_{n-1}z)(z-\overline{\beta}_{n-1})}{(1-\beta_n z)(z-\beta_n)}.$$

Hence, from (2.2) and (2.11) it follows that

$$\begin{aligned} Q_{n,\tau_n}(x) = d_n B_{n*}(z)\{\zeta_n(z)\hat{\phi}_{2n-1}(z) + \hat{\phi}^*_{2n-1}(z)\} + \\ \tau_n \left(\frac{1+\beta_n^2}{1+\overline{\beta}^2_{n-1}}\right) \frac{(1-\overline{\beta}_{n-1}z)(z-\overline{\beta}_{n-1})}{(1-\beta_n z)^2} c_{n-1} B_{n*}(z)\{\hat{\phi}^c_{2n-2}(z) + \hat{\phi}^*_{2n-2}(z)\}. \end{aligned}$$

Next, using the recurrence relation (2.4) for $\hat{\phi}_{2n-1}$ and $\hat{\phi}^*_{2n-1}$, together with Lemma 3.2 for $\hat{\phi}^c_{2n-2}$, and setting $\hat{C}_n = d_n \hat{e}_{2n-1}$ and

$$\breve{\tau}_n = \tau_n \left(\frac{1+\beta_n^2}{1+\overline{\beta}^2_{n-1}}\right) \frac{c_{n-1}}{\hat{e}_{2n-1} d_n}$$

then yields

$$\begin{aligned} Q_{n,\tau_n}(x) = \hat{C}_n B_{n*}(z) \times \\ \left\{ \left[\left(\frac{1-\overline{\beta}_{n-1}z}{1-\beta_n z}\right) (\hat{\mathring{\rho}}_{2n-1}\zeta_n(z) + \overline{\hat{\mathring{\rho}}}_{2n-1}\overline{\hat{\delta}}_{2n-1}) + \breve{\tau}_n \hat{b}_{2n-1} \left(\frac{1-\overline{\beta}_{n-1}z}{1-\beta_n z}\right)^2 \right] \zeta_{n-1}(z)\hat{\phi}_{2n-2}(z) + \right. \\ \left. \left[\left(\frac{1-\overline{\beta}_{n-1}z}{1-\beta_n z}\right) (\hat{\mathring{\rho}}_{2n-1}\hat{\delta}_{2n-1}\zeta_n(z) + \overline{\hat{\mathring{\rho}}}_{2n-1}) + \breve{\tau}_n \frac{1-\overline{\beta}_{n-1}z}{(1-\beta_n z)^2}[(1-\overline{\beta}_{n-1}z)\hat{a}_{2n-1} + (z-\overline{\beta}_{n-1})] \right] \hat{\phi}^*_{2n-2}(z) \right\}. \end{aligned}$$

The equality in (3.7) now easily follows.

Finally, from (3.3) we deduce that

$$\Re\{\hat{\phi}^c_{2n-2}(\beta_{n-1})\} = \begin{cases} \hat{\phi}_{2n-2}(\beta_{n-1}), & \beta_{n-1} \in (-1,1) \\ \hat{a}_{2n-1}\left(\frac{1-|\beta_{n-1}|^2}{\beta_{n-1}-\overline{\beta}_{n-1}}\right)\hat{\kappa}_{2n-2}, & \beta_{n-1} \notin (-1,1). \end{cases}$$

This concludes the proof.

Proof of Lemma 3.6

Note that (3.9) is equivalent with

$$\begin{cases} K_n\overline{\tilde{\rho}}_{2n-1}(\overline{\tilde{\delta}}_{2n-1} - \beta_n\tilde{\rho}^2_{2n-1}) - \overline{\hat{\rho}}_{2n-1}(\overline{\hat{\delta}}_{2n-1} - \beta_n\hat{\rho}^2_{2n-1}) = \breve{\tau}_n\hat{b}_{2n-1} \\ K_n\overline{\tilde{\rho}}_{2n-1}(\overline{\tilde{\delta}}_{2n-1}\beta_n - \tilde{\rho}^2_{2n-1}) - \overline{\hat{\rho}}_{2n-1}(\overline{\hat{\delta}}_{2n-1}\beta_n - \hat{\rho}^2_{2n-1}) = \breve{\tau}_n\hat{b}_{2n-1}\overline{\beta}_{n-1} \\ K_n\overline{\tilde{\rho}}_{2n-1}(\tilde{\rho}^2_{2n-1}\tilde{\delta}_{2n-1} - \beta_n) - \overline{\hat{\rho}}_{2n-1}(\hat{\rho}^2_{2n-1}\hat{\delta}_{2n-1} - \beta_n) = \breve{\tau}_n(1 - \overline{\beta}_{n-1}\hat{a}_{2n-1}) \\ K_n\overline{\tilde{\rho}}_{2n-1}(\tilde{\rho}^2_{2n-1}\tilde{\delta}_{2n-1}\beta_n - 1) - \overline{\hat{\rho}}_{2n-1}(\hat{\rho}^2_{2n-1}\hat{\delta}_{2n-1}\beta_n - 1) = \breve{\tau}_n(\overline{\beta}_{n-1} - \hat{a}_{2n-1}) \, . \end{cases} \tag{5.1}$$

Hence, we obtain in this way four equalities in two unknowns $\tilde{\delta}_{2n-1}$ and K_n. With

$$\tilde{\rho}^2_{2n-1} = \hat{b}_{2n-1} - \overline{\tilde{\delta}}_{2n-1}\hat{a}_{2n-1} = \left[\overline{\hat{b}}_{2n-1} + \tilde{\delta}_{2n-1}\hat{a}_{2n-1}\right]^{-1} \in \mathbb{T}$$

and

$$\hat{\rho}^2_{2n-1} = \hat{b}_{2n-1} - \overline{\hat{\delta}}_{2n-1}\hat{a}_{2n-1} = \left[\overline{\hat{b}}_{2n-1} + \hat{\delta}_{2n-1}\hat{a}_{2n-1}\right]^{-1} \in \mathbb{T},$$

the first two equalities in (5.1) become

$$\begin{cases} K_n\overline{\tilde{\rho}}_{2n-1}[\overline{\tilde{\delta}}_{2n-1}(1+\beta_n\hat{a}_{2n-1}) - \beta_n\hat{b}_{2n-1}] - \overline{\hat{\rho}}_{2n-1}[\overline{\hat{\delta}}_{2n-1}(1+\beta_n\hat{a}_{2n-1}) - \beta_n\hat{b}_{2n-1}] = \breve{\tau}_n\hat{b}_{2n-1} \\ K_n\overline{\tilde{\rho}}_{2n-1}[\overline{\tilde{\delta}}_{2n-1}(\beta_n+\hat{a}_{2n-1}) - \hat{b}_{2n-1}] - \overline{\hat{\rho}}_{2n-1}[\overline{\hat{\delta}}_{2n-1}(\beta_n+\hat{a}_{2n-1}) - \hat{b}_{2n-1}] = \breve{\tau}_n\hat{b}_{2n-1}\overline{\beta}_{n-1} \, , \end{cases} \tag{5.2}$$

and the last two equalities in (5.1) become

$$\begin{cases} K_n\overline{\tilde{\rho}}_{2n-1}\left[\frac{\tilde{\delta}_{2n-1}(1-\beta_n\hat{a}_{2n-1})-\beta_n\overline{\tilde{b}}_{2n-1}}{\overline{\tilde{b}}_{2n-1}+\tilde{\delta}_{2n-1}\hat{a}_{2n-1}}\right] - \overline{\hat{\rho}}_{2n-1}\left[\frac{\hat{\delta}_{2n-1}(1-\beta_n\hat{a}_{2n-1})-\beta_n\overline{\tilde{b}}_{2n-1}}{\overline{\tilde{b}}_{2n-1}+\hat{\delta}_{2n-1}\hat{a}_{2n-1}}\right] = \breve{\tau}_n(1-\overline{\beta}_{n-1}\hat{a}_{2n-1}) \\ K_n\overline{\tilde{\rho}}_{2n-1}\left[\frac{\tilde{\delta}_{2n-1}(\beta_n-\hat{a}_{2n-1})-\overline{\tilde{b}}_{2n-1}}{\overline{\tilde{b}}_{2n-1}+\tilde{\delta}_{2n-1}\hat{a}_{2n-1}}\right] - \overline{\hat{\rho}}_{2n-1}\left[\frac{\hat{\delta}_{2n-1}(\beta_n-\hat{a}_{2n-1})-\overline{\tilde{b}}_{2n-1}}{\overline{\tilde{b}}_{2n-1}+\hat{\delta}_{2n-1}\hat{a}_{2n-1}}\right] = \breve{\tau}_n(\overline{\beta}_{n-1}-\hat{a}_{2n-1}) \, . \end{cases} \tag{5.3}$$

Solving the system of equations (5.2) for $K_n\overline{\tilde{\rho}}_{2n-1}$ and $\overline{\tilde{\delta}}_{2n-1}$, eventually leads to

$$K_n\overline{\tilde{\rho}}_{2n-1} = \overline{\hat{\rho}}_{2n-1}\left\{1 + \breve{\tau}_n\hat{\rho}_{2n-1}\left(\frac{(\beta_n - \overline{\beta}_{n-1}) + \hat{a}_{2n-1}(1-\overline{\beta}_{n-1}\beta_n)}{1-\beta_n^2}\right)\right\}, \tag{5.4}$$

and $\overline{\tilde{\delta}}_{2n-1}$ given by (3.10). On the other hand, solving the system of equations (5.3) for $K_n\overline{\tilde{\rho}}_{2n-1}$ and $\tilde{\delta}_{2n-1}$, we obtain that

$$\begin{aligned} \tilde{\delta}_{2n-1} &= \overline{\tilde{b}}_{2n-1}\frac{\breve{\tau}_n\overline{\tilde{\rho}}_{2n-1}[(\overline{\beta}_{n-1}-\beta_n)\hat{a}_{2n-1} - (1-\overline{\beta}_{n-1}\beta_n)] - \hat{\delta}_{2n-1}(1-\beta_n^2)}{\breve{\tau}_n\overline{\tilde{\rho}}_{2n-1}(\overline{\beta}_{n-1}-\beta_n)(1-\hat{a}^2_{2n-1}) - \overline{\tilde{b}}_{2n-1}(1-\beta_n^2)} \\ &= \frac{\breve{\tau}_n\overline{\tilde{\rho}}_{2n-1}[(\overline{\beta}_{n-1}-\beta_n)\hat{a}_{2n-1} - (1-\overline{\beta}_{n-1}\beta_n)] - \hat{\delta}_{2n-1}(1-\beta_n^2)}{\breve{\tau}_n\overline{\tilde{\rho}}_{2n-1}(\overline{\beta}_{n-1}-\beta_n)\hat{b}_{2n-1} - (1-\beta_n^2)}, \end{aligned} \tag{5.5}$$

while $K_n\overline{\tilde{\rho}}_{2n-1}$ again is given by (5.4). So, it remains to prove that the equalities given by (3.10) and (5.5) are equivalent. Eliminating $\breve{\tau}_n$ in (3.10) and (5.5), we find that it should hold that

$$\frac{\overline{\tilde{\rho}}_{2n-1}(\overline{\tilde{\delta}}_{2n-1} - \overline{\hat{\delta}}_{2n-1})}{(\hat{b}_{2n-1} - \overline{\tilde{\delta}}_{2n-1}\hat{a}_{2n-1})(1-\overline{\beta}_{n-1}\beta_n) + \overline{\tilde{\delta}}_{2n-1}(\overline{\beta}_{n-1}-\beta_n)} =$$

$$\frac{\hat{\rho}_{2n-1}(\hat{\delta}_{2n-1} - \tilde{\delta}_{2n-1})}{(\hat{a}_{2n-1} - \tilde{\delta}_{2n-1}\hat{b}_{2n-1})(\overline{\beta}_{n-1}-\beta_n) - (1-\overline{\beta}_{n-1}\beta_n)},$$

or, equivalently,

$$\left\{\overline{\hat{\rho}}^2_{2n-1}(\overline{\hat{\delta}}_{2n-1}-\overline{\tilde{\delta}}_{2n-1})-\tilde{\hat{\rho}}^2_{2n-1}(\hat{\delta}_{2n-1}-\tilde{\delta}_{2n-1})\right\}\left\{(1-\overline{\beta}_{n-1}\beta_n)+\overline{\hat{\rho}}^2_{2n-1}\overline{\tilde{\delta}}_{2n-1}(\overline{\beta}_{n-1}-\beta_n)\right\}=0.$$

We now have that

$$\begin{aligned}
&\overline{\hat{\rho}}^2_{2n-1}(\overline{\hat{\delta}}_{2n-1}-\overline{\tilde{\delta}}_{2n-1})-\tilde{\hat{\rho}}^2_{2n-1}(\hat{\delta}_{2n-1}-\tilde{\delta}_{2n-1})=\\
&\qquad(\overline{\hat{b}}_{2n-1}+\hat{\delta}_{2n-1}\hat{a}_{2n-1})(\overline{\hat{\delta}}_{2n-1}-\overline{\tilde{\delta}}_{2n-1})-(\hat{b}_{2n-1}-\overline{\tilde{\delta}}_{2n-1}\hat{a}_{2n-1})(\hat{\delta}_{2n-1}-\tilde{\delta}_{2n-1})=\\
&-(\hat{b}_{2n-1}\hat{\delta}_{2n-1}-\overline{\hat{b}}_{2n-1}\overline{\hat{\delta}}_{2n-1})+(\hat{b}_{2n-1}\tilde{\delta}_{2n-1}-\overline{\hat{b}}_{2n-1}\overline{\tilde{\delta}}_{2n-1})+\hat{a}_{2n-1}\left(\left|\hat{\delta}_{2n-1}\right|^2-\left|\tilde{\delta}_{2n-1}\right|^2\right)=0,
\end{aligned}$$

which ends the proof.

Proof of Lemma 3.7

Note that, due to the equality in (3.2), it suffice to prove that one of the zeros of $r_{2n-1}(z):=\frac{1}{\tilde{e}_{2n-1}}\{\zeta_n(z)\tilde{\phi}_{2n-1}(z)+\tilde{\phi}^*_{2n-1}(z)\}$ tends to either 1 or -1 whenever $\left|\tilde{\delta}_{2n-1}\right|$ tends to one. Or, equivalently, either $r_{2n-1}(1)=0$ or $r_{2n-1}(-1)=0$ if $\tilde{\delta}_{2n-1}\in\mathbb{T}$.

So, let $\nu\in\{\pm1\}$. Then we have that $r_{2n-1}(\nu)=0$ iff

$$\begin{aligned}
&\frac{1}{\tilde{e}_{2n-1}}\left\{\zeta_n(\nu)\tilde{\phi}_{2n-1}(\nu)+\hat{B}_{2n-1}(\nu)\tilde{\phi}_{(2n-1)*}(\nu)\right\}=0\\
\Leftrightarrow\quad&\frac{\zeta_n(\nu)}{\tilde{e}_{2n-1}}\left\{\tilde{\phi}_{2n-1}(\nu)+\hat{B}_{2n-2}(\nu)\tilde{\phi}_{(2n-1)*}(\nu)\right\}=0\\
\Leftrightarrow\quad&\frac{1}{\tilde{e}_{2n-1}}\left\{\tilde{\phi}_{2n-1}(\nu)+\left|\hat{B}_{n-1}(\nu)\right|^2\tilde{\phi}^c_{2n-1}(\nu)\right\}=0\\
\Leftrightarrow\quad&\frac{\tilde{\phi}_{2n-1}(\nu)}{\tilde{e}_{2n-1}}=0\\
\Leftrightarrow\quad&\tilde{\hat{\rho}}_{2n-1}\frac{1-\overline{\beta}_{n-1}\nu}{1-\overline{\beta}_n\nu}\left[\zeta_{n-1}(\nu)\hat{\phi}_{2n-2}(\nu)+\tilde{\delta}_{2n-1}\hat{\phi}^*_{2n-2}(\nu)\right]=0,
\end{aligned}\tag{5.6}$$

which implies that $\frac{\tilde{\phi}_{2n-1}(z)}{\tilde{e}_{2n-1}}$ is (up to a nonzero multiplicative factor) a pORF. Consequently, from (2.6) together with (5.6) and Theorem 2.1 we deduce that $r_{2n-1}(\nu)=0$ iff

$$\tilde{\delta}_{2n-1}=\hat{\xi}_{2n-1}:=\frac{\hat{\tilde{\rho}}_{2n-1}\hat{\delta}_{2n-1}-\nu\overline{\hat{\tilde{\rho}}}_{2n-1}}{\hat{\tilde{\rho}}_{2n-1}-\nu\overline{\hat{\tilde{\rho}}}_{2n-1}\overline{\hat{\delta}}_{2n-1}}\in\mathbb{T}.\tag{5.7}$$

Since $\tilde{\delta}_{2n-1}$ and $\tilde{\hat{\rho}}_{2n-1}$ satisfy the equalities in (3.4), it follows from Lemma 3.3 that the equality in (5.7) can only hold if

$$\hat{\xi}_{2n-1}\in\{\pm1\}\ \text{ for }\ \hat{a}_{2n-1}=0\qquad\text{or}\qquad\hat{\xi}_{2n-1}\in C\cap\mathbb{T}\ \text{ for }\ \hat{a}_{2n-1}\neq0,$$

where C is defined as before in Lemma 3.3. If $\hat{a}_{2n-1}=0$ it follows from (5.7) that $\hat{\xi}_{2n-1}=-\nu$, which indeed is in $\{\pm1\}$. This proves the statement for the case of $\hat{a}_{2n-1}=0$.

Finally, consider the case in which $\hat{a}_{2n-1}\neq0$. From the first equality in (3.4) we then deduce that $\hat{\xi}_{2n-1}\in C\cap\mathbb{T}$ iff

$$\hat{b}_{2n-1}\hat{\xi}_{2n-1}-\overline{\hat{b}}_{2n-1}\overline{\hat{\xi}}_{2n-1}=2\hat{a}_{2n-1}.\tag{5.8}$$

Taking into account that

$$\hat{a}_{2n-1}=\left(\frac{\hat{\tilde{\rho}}^2_{2n-1}\hat{\delta}_{2n-1}-\overline{\hat{\tilde{\rho}}}^2_{2n-1}\overline{\hat{\delta}}_{2n-1}}{1-|\hat{\delta}_{2n-1}|^2}\right)\quad\text{and}\quad\hat{b}_{2n-1}=\hat{\tilde{\rho}}^2_{2n-1}+\overline{\hat{\delta}}_{2n-1}\hat{a}_{2n-1},$$

we find that (5.8) is equivalent with

$$\left\{\hat{\mathring{\rho}}_{2n-1}^2\hat{\xi}_{2n-1}-\overline{\hat{\mathring{\rho}}}_{2n-1}^2\overline{\hat{\xi}}_{2n-1}\right\}(1-|\hat{\delta}_{2n-1}|^2)= \\ (2-\overline{\hat{\delta}}_{2n-1}\hat{\xi}_{2n-1}-\hat{\delta}_{2n-1}\overline{\hat{\xi}}_{2n-1})\left\{\hat{\mathring{\rho}}_{2n-1}^2\hat{\delta}_{2n-1}-\overline{\hat{\mathring{\rho}}}_{2n-1}^2\overline{\hat{\delta}}_{2n-1}\right\}. \tag{5.9}$$

After replacing $\hat{\xi}_{2n-1}$ with the expression on the right hand side of (5.7) and simplifying, it becomes clear that the equality in (5.9) indeed holds true for every $\hat{\delta}_{2n-1}\in\mathbb{D}$ and $\hat{\mathring{\rho}}_{2n-1}\in\mathbb{T}$, which ends the proof.

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