A bibliography on semiseparable matrices

Raf Vandebril, Marc Van Barel, Gene Golub, and Nicola Mastronardi

Raf Vandebril
Department of Computer Science
KU Leuven, Belgium
Raf.Vandebril@cs.kuleuven.be

Marc Van Barel
Department of Computer Science
KU Leuven, Belgium
Marc.VanBarel@cs.kuleuven.be

Gene Golub
SCCM Program, Stanford University
Stanford, California, USA

Nicola Mastronardi
Istituto per le Applicazioni del Calcolo
M. Picone, sez. Bari, Italy
N.Mastronardi@ba.iac.cnr.it

Abstract

Currently there is a growing interest in semiseparable matrices and generalized semiseparable matrices. To gain an appreciation of the historical evolution of this concept, we present in this paper an extensive list of publications related to the field of semiseparable matrices. It is interesting to see that semiseparable matrices were investigated in different fields, e.g., integral equations, statistics, vibrational analysis, independently of-each other. Also notable is the fact that leading statisticians at that time Used semiseparable matrices without knowing their inverses to be tridiagonal. During this historical evolution the definition of semiseparable matrices has always been a difficult point leading to misunderstandings; sometimes they were defined as the inverses of irreducible tridiagonal matrices leading to generator representable matrices, while in other cases they were defined as matrices having low rank blocks below the diagonal.

Article information

- Vandebril, Raf; Van Barel, Marc; Golub, G.; Mastronardi, Nicola. *A bibliography on semiseparable matrices*, Calcolo, volume 42, issue 3-4, pages 249-270, 2005.
- The content of this article is identical to the content of the published paper, but without the final typesetting by the publisher.
- Journal's homepage: http://link.springer.com/journal/10092
- Published version: http://dx.doi.org/10.1007/s10092-005-0107-z
- KU Leuven's repository url: https://lirias.kuleuven.be/handle/123456789/124532

KU LEUVEN

www.kuleuven.be

Calcolo manuscript No.
(will be inserted by the editor)

A bibliography on semiseparable matrices

Raf Vandebril[1], **Marc Van Barel**[1], **Gene Golub**[2], **Nicola Mastronardi**[3]

[1] K.U.Leuven, Dept. Computer Science, Email: {raf.vandebril,marc.vanbarel}@cs.kuleuven.ac.be
[2] SCCM Program, Gates 2B, Stanford University, CA, USA, Email: golub@stanford.edu
[3] Istituto per le Applicazioni del Calcolo M. Picone, sez. Bari, Email:irmanm21@area.ba.cnr.it

Received: December 2004 / Revised version: date

Abstract Currently there is a growing interest in semiseparable matrices and generalized semiseparable matrices. As it is interesting to know the historical evolution of this concept, we present in this paper an extensive list of publications related to the field of semiseparable matrices. It is interesting to see that semiseparable matrices were investigated in different fields, e.g. integral equations, statistics, vibrational analysis, independently of each other. Also interesting to know is that the leading statisticians at that time used semiseparable matrices, without knowing their inverses to be tridiagonal. During this historical evolution the definition of semiseparable matrices has always been a difficult point leading to misunderstandings, as they were sometimes defined as the inverses of irreducible tridiagonal matrices leading to generator representable matrices, while in other cases they were defined as matrices having low rank blocks below the diagonal.

In this overview we present a list of interesting results which contributed to the evolution of the concept of semiseparable matrices as we know them nowadays.

Keywords: semiseparable matrices, tridiagonal matrices, historical overview.

1. Semiseparable matrices

As semiseparable matrices were defined in different ways, we will clearly distinguish the different definitions.

Definition 1 *A matrix S is called a generator representable semiseparable matrix of semiseparability rank r if there exist two matrices R_1 and R_2, both of rank r, such that*

$$\begin{aligned} \text{triu}(S) &= \text{triu}(R_1) \\ \text{tril}(S) &= \text{triu}(R_2) \end{aligned}$$

$triu(R_1)$ and $tril(R_2)$ denote respectively the upper triangular part of the matrix R_1 and the strictly lower triangular part of the matrix R_2. Suppose the semiseparability rank to be equal to 1. This means that R_1 and R_2 are two rank one matrices. Therefore they can both be written as the outer

The research of the first and second author was partially supported by the Research Council K.U.Leuven, project OT/00/16 (SLAP: Structured Linear Algebra Package), by the Fund for Scientific Research–Flanders (Belgium), projects G.0078.01 (SMA: Structured Matrices and their Applications), G.0176.02 (ANCILA: Asymptotic aNalysis of the Convergence behavior of Iterative methods in numerical Linear Algebra), G.0184.02 (CORFU: Constructive study of Orthogonal Functions) and G.0455.0 (RHPH: Riemann-Hilbert problems, random matrices and Padé-Hermite approximation), and by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture, project IUAP V-22 (Dynamical Systems and Control: Computation, Identification & Modelling). The scientific responsibility rests with the authors. The research of the third author was partially supported by the NSF. The research of the fourth author was partially supported by MIUR, grant number 2004015437.

product of two vectors, respectively u and v for R_1 and s and t for R_2. These vectors are also called the generators of the semiseparable matrix S.

Generator representable semiseparable matrices can be seen as the inverses of irreducible tridiagonal matrices.

The following definition is more general than the previous one. To define the class of semiseparable matrices we use Matlab style notation. The submatrix of the matrix Z consisting of the rows $i, i+1, \ldots, n$ and the columns $1, 2, \ldots i$ is denoted by $S(i:n; 1:i)$.

Definition 2 *A matrix S is called a semiseparable matrix of semiseparability rank r if the following two conditions are satisfied:*

$$\begin{aligned} \operatorname{rank}(S(i:n, 1:\min(i+r-1, n))) &\leq r \qquad \forall i \in \{1, \ldots, n\}. \\ \operatorname{rank}(S(1:\min(i+r-1, n), i:n)) &\leq r \qquad \forall i \in \{1, \ldots, n\}. \end{aligned}$$

This means that all the subblocks taken out of the part of the matrix below the rth superdiagonal have rank at most r. The similar statement is true for the parts taken out above the rth subdiagonal. Semiseparable matrices of semiseparability rank 1 *are just called semiseparable matrices.*

This class of matrices is a generalization of the class of generator representable semiseparable matrices as is shown in the next theorem:

Theorem 1. *The inverse of an invertible*

- *generator representable semiseparable matrix is an irreducible tridiagonal matrix;*
- *semiseparable matrix is a tridiagonal matrix.*

Proofs can be found for example in [Strang and Nguyen, 2004; Asplund, 1959a,b; Fiedler and Markham, 1986]. For example also a diagonal matrix belongs to the class of semiseparable matrices, whereas it does not belong to the class of generator representable semiseparable matrices. In fact the class of semiseparable matrices as presented above, cannot be represented in a easy way as one represents the class of generator representable matrices. Moreover one can also prove that the inverse of band matrices (with a symmetric bandwidth) is the class of semiseparable matrices as defined above. This is in fact the motivation for the definition used in this paper. (The same definition can be found for example in [Strang and Nguyen, 2004]).

In fact there are different classes of matrices closely related to the class of semiseparable matrices and one can easily change the definitions above to come to slightly changed versions. Below some examples are given:

- A straightforward generalization is to give both the upper and lower part a different rank, in this way we obtain the class of (p,q)-semiseparable matrices.
- In the definition of (p,q)-semiseparable matrices, the semiseparable structure expands above (and below) the diagonal. We can omit for example this condition, and come to a class of matrices for which the semiseparable structure extends just up to the diagonal.
- We can even exclude the diagonal. In this way we obtain the class of (p,q)-quasiseparable matrices. Matrices belonging to this class have p as an upper bound for the rank of all submatrices taken out of the strictly lower triangular part, and q as an upper bound for all matrices taken out of the strictly upper triangular part. This class includes for example the semiseparable matrices, and also tridiagonal matrices belong to this class. Quasiseparable matrices are sometimes also called weakly semiseparable matrices [Tyrtyshnikov, 2000].
- Also extensions where only the lower (or upper) part has a semiseparable structure are possible. These matrices are sometimes called Hessenberg-like or lower(upper)-semiseparable.
- One can generalize all the above representations to come to block forms. Such as recursively semiseparable, sequentially semiseparable matrices or low Hankel rank matrices.

Matlab is a registred trademark of the Mathworks inc.

2. The overview

In this section an historical overview of papers and books closely related to the topic of semiseparable matrices is given. For all the references a small summary of the results is included. Also different types of applications and extensions of the class of semiseparable matrices are mentioned. It can be seen, that interesting results are often rediscovered in different fields. Unfortunately we were not able to retrieve the following papers: [Ukita, 1955] cited in [Greenberg and Sarhan, 1959; Lewis, 1982]; the papers cited in [Szynal and Szynal, 1972]; [Faddeev, 1981]; some citations of [Tyrtyshnikov, 2000]; the papers [Bevilacqua, 1991; Bevilacqua and Capovani, 1972].

The name semiseparable matrix finds its origin in the discretization of kernels. A separable kernel $k(x,y)$ means that the kernel is of the following form $k(x,y) = g(x)f(y)$. Associating a matrix with this kernel gives a rank 1 matrix. This matrix can be written as the outer product of two vectors. A semiseparable kernel satisfies the following properties (sometimes also called a Green's kernel [Karlin, 1968, p. 110]):

$$k(x,y) = \begin{cases} g(x)f(y) \text{ if } x \leq y \\ g(y)f(x) \text{ if } y \leq x \end{cases}.$$

Associating a matrix with this semiseparable kernel, gives us a generator representable semiseparable matrix.

1937 [Gantmacher and Kreĭn, 1937] In this paper of Gantmacher and Krein, they prove that the inverse of a symmetric Jacobi matrix (this corresponds to a symmetric irreducible tridiagonal matrix) is a one-pair matrix (this corresponds to a symmetric generator representable semiseparable matrix) via explicit calculations.

1950 [Gantmacher and Kreĭn, 2002] In the book of Gantmacher and Krein, the theory of the previously published paper is included. This book is often referred to as the first book in which the inverse of an irreducible tridiagonal matrix is calculated.

1953 [Berger and Saibel, 1953] Berger and Saibel provide in this paper an explicit formula for calculating the inverse of a continuant matrix (this is a tridiagonal matrix, not necessarily symmetric), based on the LU-decomposition of the original continuant matrix. Moreover, no conditions of nonzeroness are placed on the elements of the continuant matrix.

1956 [Roy and Sarhan, 1956] In this paper the authors Roy and Sarhan invert very specific matrices arising in statistical applications, e.g., a lower triangular semiseparable matrix, 2 types of specific semiseparable matrices and also a semiseparable plus diagonal matrix are inverted.

1957 [Asplund, 1959b] In this paper, S. O. Asplund, the father of E. Asplund, proves the same as Gantmacher and Krein, by calculating the inverse via techniques for solving finite boundary value problems. A brief remark states that higher order band matrices have as inverses higher order Green's matrices (The Green's matrix can be considered as a generator representable semiseparable matrix).

1957 [Asplund, 1959a] E. Asplund formulates for the first time a theorem stating that the inverse of an invertible lower $\{p\}$-Hessenberg matrix (with no restrictions on the elements and called a $\{p\}$-band matrix in the paper) is an invertible lower $\{p\}$-Hessenberg-like matrix (with the definition of the ranks of subblocks) and vice versa. A $\{p\}$-Hessenberg-like matrix is referred to as a Green's matrix of order p in the paper. No explicit formulas for the inverse are presented, only theoretical results about the structure. More precisely, he also proves, then if the elements on the pth superdiagonal of the lower $\{p\}$-Hessenberg matrix are different from zero, that the inverse is a generator representable lower $\{p\}$-Hessenberg-like matrix.

1959 [Greenberg and Sarhan, 1959] In this paper by Greenberg and Sarhan, the papers [Ukita, 1955; Roy and Sarhan, 1956] are generalized and applied to several types of matrices arising in statistical applications. A relation is introduced which needs to be satisfied, such that the inverse of the matrix is a diagonal matrix of type r (these diagonal matrices of type r correspond to band matrices of width r). Different semiseparable matrices are given and the relation is investigated for r equal to $1,2$ and 3, thereby proving that these matrices have a banded inverse.

1960 [Roy et al., 1960] As the statistical research at that time was interested in fast calculations for so-called patterned matrices, the authors Roy, Greenberg and Sarhan designed an order

n algorithm for calculating the determinant. The patterned matrices are semiseparable and semiseparable plus diagonal matrices.

1961 [Schechter, 1960] Schechter provides a method for inverting nonsymmetric block tridiagonal matrices for blocks which have the same size and are invertible, based on the LDU-decomposition of these matrices. The applications of this method can be found in the field of partial differential equations.

1963 [Golub, 1963] Golub investigates the optimal scaling of some variance matrices, which are of semiseparable form, based on the scaling of their inverse.

1964 [Ting, 1964] Ting describes, based on the LUdecomposition of a tridiagonal matrix T (not necessarily symmetric) a special decomposition of T^{-1} as a sum of inverses of bidiagonal matrices, multiplied with a diagonal matrix. This can be used for solving various systems of equations with the same tridiagonal matrix T.

1965 [Torii, 1966a] Published in a Japanese Journal, and translated afterwards to [Torii, 1966b].

1966 [Torii, 1966b] Torii, provides explicit formulas for calculating the inverse of a nonsymmetric tridiagonal matrix. The stability of solving a tridiagonal system of equations via inversion is investigated.

1967 [Chandan, 1967] The author Chandan inverts a symmetric semiseparable plus identity matrix, and concludes that it is again a symmetric semiseparable matrix plus the identity. This matrix results from a statistical problem.

1968 [Kounias, 1968] The author Kounias provides explicit formulas for inverting some patterned matrices. For example, a tridiagonal Toeplitz matrix and a matrix completely filled with one element plus a diagonal are inverted.

1969 [Uppuluri and Carpenter, 1969] Uppuluri and Carpenter present an exact formula to compute the inverse of a specific covariance matrix, which is a symmetric tridiagonal Toeplitz matrix.

1970 [Allgower, 1973] Allgower provides a method for calculating the inverse of banded Toeplitz matrices.

1970 [Kershaw, 1970] Kershaw provides bounds between which the elements of the inverse of a tridiagonal matrix with positive off-diagonal elements will lie. The elements of the super and the subdiagonal of the tridiagonal are related in the following way: denote with T the tridiagonal matrix, then we have: $T_{i,i+1} = 1 - T_{i-1,i}$ for $1 < i < n$.

1970 [Uppuluri and Carpenter, 1970] Uppuluri and Carpenter calculate the inverse of a tridiagonal Toeplitz matrix (not necessarily symmetric) via the associated difference equation. The results are based on [Kounias, 1968].

1970 [Capovani, 1970] Explicit formulas for the inverse of a nonsymmetric tridiagonal matrix are given by Capovani (no demands are placed on the elements of the tridiagonal). The formulas are adapted for the tridiagonal Toeplitz case and the formulas are applicable to block tridiagonal matrices (for which all the blocks have the same size) as well.

1971 [Capovani, 1971] Capovani proves that the inverse of a nonsymmetric tridiagonal matrix with the super and subdiagonal elements different from zero, also has as inverse a generator representable matrix, where the upper and lower triangular part have different generators. Using these results the author proves that a pentadiagonal matrix can be written as the product of two tridiagonal matrices, for which one of them is symmetric.

1971 [Baranger and Duc-Jacquet, 1971] Baranger and Duc-Jacquet prove that the inverse of a generator representable semiseparable matrix (called "une matrice factorisable" in the paper) is a tridiagonal matrix. They explicitely calculate the inverse of a generator representable semiseparable matrix.

1972 [Hoskins and Ponzo, 1972] Hoskins and Ponzo provide formulas for calculating the inverse of a specific band matrix of dimension $2r+1$, with the binomial coefficients in the expansion of $(x-1)^{2r}$ in each row and column. Such a matrix arises for example in the solution of partial difference equations.

1972 [Rehnqvist, 1972] Rehnqvist presents here a method for inverting a very specific Toeplitz band matrix. Via multiplication with another invertible Toeplitz matrix and reordering of the elements one can calculate the inverse of this matrix.

1972 [Szynal and Szynal, 1972] D. Szynal and J. Szynal present two theorems for the existence of the inverse of a Jacobi matrix and two methods for calculating the inverse. The results can be rewritten in block form such that they can be applied to specific band matrices. No specific demands of symmetry or nonzeroness of the elements have to be fulfilled.

1973 [Hoskins and Thurgur, 1973] Hoskins and Thurgur invert a specific type of band matrix (see [Hoskins and Ponzo, 1972]). The inversion formulas are obtained by calculating the LU-decomposition of the band matrix and formulas are given to compute the determinant and the infinity norm of this inverse.

1973 [Kounadis, 1973] Kounadis deduces in this paper a recursive formula for inverting symmetric block tridiagonal matrices, for which all the blocks are of the same dimension. This technique is also applied to the class of tridiagonal matrices.

1973 [Bukhberger and Emel'yanenko, 1973] Bukhberger and Emel'yanenko present in this paper a computational method for inverting symmetric tridiagonal matrices, for which all the elements are different from zero. The theoretical results are applicable in a physical application which studies the motion of charged particles in a particular environment. The results are identical to the ones in the book [Gantmacher and Kreĭn, 2002].

1974 [Trench, 1974] Trench presents a method for inverting $\{p,q\}$-banded Toeplitz matrices by exploiting the banded structure.

1976 [Bevilacqua and Capovani, 1976] Bevilacqua and Capovani extend the results of the papers [Greenberg and Sarhan, 1959] and [Allgower, 1970, 1973] to band matrices and to block band matrices (not necessarily symmetric). Formulas are presented for inverting band matrices whose elements on the extreme diagonals are different from zero. The results are extended to block band matrices.

1976 [Hoskins and McMaster, 1976] The summary of the paper, as provided by the authors Hoskins and McMaster: for a symmetric positive definite Toeplitz matrix of band width five and order n, those regions where the elements of the inverse alternate in sign are determined.

1977 [Hoskins and McMaster, 1977] Hoskins and McMaster investigate properties of the inverse of banded Toeplitz matrices as in [Hoskins and McMaster, 1976]. More precisely, they investigate the properties of a band matrix of width 4 and a band matrix of width 5 coming from a boundary value problem.

1977 [Berg, 1977] Berg provides in this paper for the first time an explicit technique for inverting a band matrix that is not symmetric, and has for the upper and lower part different bandwidths. We could already assume these results based on the paper by Asplund, but here explicit formulas are given. A disadvantage is again the strong assumption that the elements on the extreme super and subdiagonals have to be different from zero.

1977 [Demko, 1977] Demko proves theorems bounding the size of the elements of the inverse of band matrices in terms of the norm of the original matrix, the distance towards the diagonal and the bandwidth. In particular it is shown that the size of the elements decays exponentially to zero if one goes further and further away from the diagonal.

1977 [Neuman, 1977] Neuman provides algorithms for inverting tri and pentadiagonal matrices (not necessarily symmetric) via the UL decomposition. Under some additional conditions the numerical stability of the algorithm is proved.

1977 [Valvi, 1977] Valvi determines several explicit formulas for inverting specific patterned matrices arising in statistical applications. These matrices are specific types of semiseparable and semiseparable plus diagonal matrices.

1978 [Hoskins and McMaster, 1978] Bounds on the infinity norm of the inverse of Toeplitz band matrices with band width 5 are derived by Hoskins and McMaster.

1978 [Greville, 1978] Greville provides conditions on band matrices such that the inverse of the band matrix becomes a Toeplitz matrix. This results in Toeplitz matrices which are also semiseparable.

1978 [Oohashi, 1978] Oohashi proves that the elements of the inverse of a band matrix can be expressed in terms of the solution of a homogeneous difference equation, related to the original band matrix. In this way explicit formulas for calculating the inverse are obtained. The band matrix does not need to have the same lower and upper band size, but the elements on the

extreme diagonals need to be different from zero. The results are an extension of the results proved in [Torii, 1966b] for tridiagonal matrices.

1978 [Barrett and Feinsilver, 1978] Barrett and Feinsilver provide a probabilistic proof, stating that the inverse of a symmetric generator representable semiseparable matrix, is a symmetric tridiagonal matrix. Note that the generator representable matrices are not characterized by two generators, but via the rank 1 assumptions in the lower and upper triangular part. The results are restricted to positive definite and symmetric matrices.

1979 [Ikebe, 1979] Ikebe provides in this paper an algorithm for inverting an upper Hessenberg matrix under the assumption that the subdiagonal elements are different from zero. It is proved that the inverse matrix has the lower triangular part representable with two generators. An extension towards block Hessenberg matrices is included.

1979 [Yamamoto and Ikebe, 1979] Yamammoto and Ikebe propose formulas for inverting band matrices with different bandwidths under the assumption that the elements on the extreme diagonals are different from zero.

1979 [Barrett, 1979] Barrett formulates another type of theorem connected to the inverses of tridiagonal matrices. In most of the preceding papers one assumed the sub and superdiagonal elements of the corresponding tridiagonal matrix to be different from zero. In this paper only one condition is left, it assumes that the diagonal elements of the symmetric semiseparable matrix are different from zero. Moreover, the proof is also suitable for nonsymmetric matrices. The theorems presented in this paper are very close to the final version result, stating that the inverse of a tridiagonal matrix is a semiseparable matrix, satisfying the rank definition.

1979 [Singh, 1979] Singh gives explicit formulas for inverting a lower block bidiagonal matrix. The blocks on the diagonal have to be invertible.

1980 [Haley, 1980] The author Haley splits the band system $Bx = u$ into two blocks, which can via recurrence divided once more. This leads to a method for calculating explicit inverses of banded matrices. As example tridiagonal and Toeplitz tridiagonal matrices are considered.

1981 [Barrett and Feinsilver, 1981] This paper should be considered as one of the most important papers concerning the inverses of band matrices. Barrett and Feinsilver provide a general framework. General theorems and proofs considering the vanishing of minors when looking at the matrices and their inverses are given, thereby characterizing the complete class of band and semiseparable matrices, without excluding cases in which there appear zeros. The results are a straightforward consequence of paper [Barrett, 1979].

1982 [Lewis, 1982] Lewis provides an explicit formula which can be used to compute the inverse of tridiagonal matrices. The matrix does not necessarily need to be symmetric, nor all elements have to be different from zero. Interesting is also a new kind of representation for nonsymmetric semiseparable matrices, which does not use 4 but three vectors x,y and z, where the elements of S are of the following form:

$$s_{ij} = \begin{cases} x_i y_j z_j, & i \geq j, \\ y_i x_j z_j, & i \leq j. \end{cases}$$

The explicit formula is a generalization of the theorem for symmetric matrices by Yoshimasa [Ukita, 1955].

1983 [Graybill, 1983] The author Graybill presents in his book several specific matrices, which are semiseparable, and inversion formulas for these matrices are given.

1984 [Demko et al., 1984] Decay rates for the inverse of band matrices are obtained by Demko, Moss and Smith. It is shown that the decay rate depends on the spectrum of the matrix AA^T, for a band matrix A.

1984 [Rizvi, 1984] Rizvi derives, based on the formulas proposed by Barrett and Feinsilver and on the LU-decomposition, formulas for inverting a quasi-tridiagonal matrix, which is in fact a block tridiagonal matrix (for which all the blocks have the same size). Necessary and sufficient conditions, for which a special block matrix has as inverse a quasi-tridiagonal matrix are derived.

1984 [Gohberg and Kaashoek, 1984] An explicit inversion formula for semiseparable plus diagonal matrices was presented by Gohberg and Kaashoek.

1985 [Gohberg et al., 1985, 1987] Gohberg, Kailath and Koltracht provide a method for solving higher order semiseparable plus diagonal systems of equations (and inverting them). The class of matrices needs to be strongly regular.

1986 [Mattheij and Smooke, 1986] Mattheij and Smooke provide in this paper a method for deriving the explicit inverse of tridiagonal matrices, as coming from nonlinear boundary value problems. The formulas are also suitable for the block tridiagonal case.

1986 [Romani, 1986] Romani investigates the demands on a symmetric band matrix, such that its inverse can be written as the sum of inverses of irreducible tridiagonal matrices, because this is not true in general.

1986 [Rózsa, 1986] Rózsa investigates sufficient conditions to be placed on a semiseparable matrix, in order to have a nonsingular $\{p,q\}$-semiseparable matrix. This paper is to our knowledge the first paper in which there appeared the now well-known name of semiseparable matrices. A theorem is included stating that the inverse of a strict $\{p,q\}$-band matrix is a generator representable $\{p,q\}$-semiseparable matrix.

1986 [Fiedler and Markham, 1986] In this paper, Fiedler translates the abstract formulation in [Gustafson, 1984] towards the matrix case, this theorem is often referred to as the nullity theorem. This theorem is very useful in the field of semiseparable and tridiagonal matrices as all possible relations connected to the inverse fit in this nullity theorem.

1986 [Cao and Stewart, 1986] A generalization of the papers [Ikebe, 1979; Yamamoto and Ikebe, 1979] is presented by Cao and Stewart, for Hessenberg matrices with a larger bandwidth and block Hessenberg matrices, for which the blocks do not necessary need to have the same dimension. The implementation described is suitable for parallel computations.

1987 [Rózsa, 1987] Rózsa proves, based on linear difference equations, the same results as in [Rózsa, 1986]. A small section is dedictated to band Toeplitz matrices.

1987 [Fiedler and Markham, 1987] In this paper Fiedler proves that the off-diagonal rank of a matrix is maintained under inversion.

1988 [Bevilacqua et al., 1988] Bevilacqua, Codenotti and Romani present a method to solve a block tridiagonal system in a parallel way, by exploiting the structure of the inverse of the block tridiagonal matrix.

1988 [Eijkhout and Polman, 1988] Eijkhout and Polman provide decay rates for the inverse of band matrices which are close to Toeplitz matrices.

1989 [Rózsa et al., 1989] The authors Rózsa, Bevilacqua, Favati and Romani, present generalizations of their previous papers towards methods for computing the inverse of block tridiagonal and block band matrices. The results are very general and the blocks do not all need to be square or of the same dimension. The paper contains a lot of interesting references connected to the theory of semiseparable and tridiagonal matrices.

1990 [Bevilacqua et al., 1990a] The authors Bevilacqua, Lotti and Romani, present two types of algorithms for reducing the total amount of storage locations needed by the inverse of a band matrix.

1990 [Bevilacqua et al., 1990b] The authors Bevilacqua, Romani and Lotti provide a method for parallel inversion of band matrices. No other assumption is required than the nonsingularity of some of the principal submatrices.

1991 [Favati et al., 1991] Favati, Lotti, Romani and Rózsa provide theorems and formulas for the inverse of generalized block band matrices. The paper covers many different cases in a very general framework.

1991 [Rózsa et al., 1991] A new proof is included by Rózsa, Bevilacqua, Romani and Favati stating that the inverse of a $\{p,q\}$-semiseparable matrix is a strict $\{p,q\}$-band matrix. This new proof leads to a recursive scheme for calculating the inverse.

1992 [Meurant, 1992] In this paper by Meurant many references connected to the inverse of semiseparable and band matrices are included. Some results concerning the inverse of symmetric tridiagonal and block tridiagonal matrices are reviewed, based on the Cholesky decomposition. Also results concerning the decay of the elements of the inverse are obtained.

1993 [Fiedler, 1993] Fiedler formulates general theorems connected to the structured ranks of matrices and their inverses.

1993 [Brualdi and Massey, 1993] The authors Brualdi and Massey generalize some of the results of [Fiedler, 1993], for structures in which the diagonal is not included.

1994 [Rózsa et al., 1994] Rózsa, Romani and Bevilacqua provide results similar to the ones in [Favati et al., 1991]. But the results are now proved via the nullity theorem, which is also proved in this paper.

1996 [Eidelman and Gohberg, 1996] Eidelman and Gohberg present an order $O(n)$ algorithm for calculating the inverse of a generator representable plus diagonal semiseparable matrix.

1997 [Eidelman and Gohberg, 1997b] Eidelman and Gohberg present a look ahead recursive algorithm to compute the triangular factorization of generator representable semiseparable matrices plus a diagonal.

1997 [Eidelman and Gohberg, 1997a] Eidelman and Gohberg present a fast and numerically stable algorithm for inverting semiseparable plus diagonal matrices. No conditions, except nonsingularity, are placed on the matrices.

1998 [McDonald et al., 1998] McDonald, Nabben, Neumann, Schneider and Tsatsomeros pose properties on the class of generator semiseparable matrices such that their tridiagonal inverses belong to the class of Z-matrices.

1998 [Dewilde and van der Veen, 1998] The authors Dewilde and van der Veen present in their book a new type of representation for sequentially semiseparable matrices or low Hankel rank matrices (quasiseparable matrices also belong to this class), and also a QR-factorization and inversion formulas for this class of matrices are presented.

1999 [Elsner, 1997] Elsner investigates in more detail the inverses of band and Hessenberg matrices. The different cases, concerning generator representable semiseparable matrices and semiseparable matrices for which the special subblocks have low rank are included.

1997 [Koltracht, 1997] Koltracht provides in this paper a new method for solving higher order semiseparable systems of equations. The semiseparable matrix is thereby transformed into a narrow band matrix.

1999 [Nabben, 1999b] Nabben provides upper and lower bounds for the entries of the inverse of diagonally dominant tridiagonal matrices.

1999 [Nabben, 1999a] Decay rates for the inverse of special tridiagonal and band matrices are given by Nabben.

1999 [Eidelman and Gohberg, 1999] The authors Eidelman and Gohberg investigate here a generalization of the class of semiseparable matrices, namely the class of quasiseparable matrices. They show that the class of quasiseparable matrices is closed under inversion, and they present a linear complexity inversion method.

2000 [Chandrasekaran and Gu, 2000] Chandrasekaran and Gu present an algorithm to transform a generator representable semiseparable matrix plus band matrix into a similar tridiagonal matrix.

2000 [Tyrtyshnikov, 2000] Tyrtyshnikov expands the class of generator representable matrices towards a class called weakly semiseparable matrices. He proves that the inverse of a (p,q)-band matrix is a (p,q)-weakly semiseparable matrix, but the converse does not necessarily holds. This means that the class of (p,q)-weakly semiseparable matrices is a little more general than our class of (p,q)-semiseparable matrices. The class of weakly semiseparable matrices as defined by Tyrtishnikov corresponds to the class of quasiseparable matrices.

2000 [Hackbusch and Khoromskij, 2000a,b] In these papers Hackbusch and Khoromskij, define a new class of matrices representing integral operators, the so called class of $\mathcal{H}$ matrices. These matrices are special types of rank structured matrices, this means that specific parts in the matrix satisfy rank properties. Also the arithmetic with these matrices is investigated.

2000 [Dewilde and van der Veen, 2000] The authors Dewilde and van der Veen present in this paper a method for inverting a class of infinite systems of equations. The technique described here can also be used to invert the so called 'low Hankel rank' and 'sequentially semiseparable matrices'.

2000 [Eidelman, 2000] In this paper the author Eidelman provides a recursive algorithm for solving systems of quasiseparable form.

2001 [Fasino and Gemignani, 2001] This paper by Fasino and Gemignani provides for semiseparable matrices, also singular ones, a sparse structured representation. More precisely, it is shown that a semiseparable matrix A, always can be written as the inverse of a block tridiagonal matrix plus a sparse, low rank matrix Z.

2001 [Fasino and Gemignani, 2003] Fasino and Gemignani, study in this paper the direct and the inverse eigenvalue problem of diagonal plus semiseparable matrices.

2001 [Mastronardi et al., 2001] Mastronardi, Chandrasekaran and Van Huffel present an order $O(n)$ algorithm to solve a system of equations where the coefficient matrix is a generator representable semiseparable plus diagonal matrix. The algorithm is suitable for an implementation on two processors.

2001 [Chandrasekaran and Gu, 2001] Chandrasekaran and Gu present a fast and stable solver for recursively semiseparable systems of equations.

2002 [Van Camp et al., 2004] Van Camp, Van Barel and Mastronardi provide two fast algorithms for solving diagonal plus generator representable semiseparable systems of equations. The solution method consists of an effective calculation of the QR-factorization of this type of matrices.

2002 [Van Barel et al., 2002] In this paper, Van Barel, Fasino, Gemignani and Mastronardi investigate the relation between orthogonal rational functions and generator representable semiseparable plus diagonal matrices.

2002 [Mullhaupt and Riedel, 2002] Mullhaupt and Riedel derive properties for matrices having weakly lower triangular rank equal to d. Decompositions of these matrices as a product of a unitary matrix and an upper $\{p\}$-Hessenberg matrix are provided.

2002 [Gemignani and Fasino, 2002] Fasino and Gemignani describe an order $O(n)$ solver for banded plus semiseparable systems of equations. The algorithm exploits the structure of the inverse of the semiseparable matrix.

2002 [Chandrasekaran et al., 2002] The authors Chandrasekaran, Dewilde, Gu, Pals and van der Veen present in this paper a backward stable method for solving sequentially semiseparable systems of equations.

2002 [Eidelman and Gohberg, 2002b] The authors Eidelman and Gohberg present a method for solving a system of equations with a quasiseparable matrix based on a modification of the algorithms presented in [Dewilde and van der Veen, 1998, 2000]. Also inversion and factorization formulas are presented.

2003 [Chandrasekaran and Gu, 2003] Chandrasekaran and Gu present a method for solving systems whose coefficient matrix is a semiseparable plus band matrix.

2003 [Chandrasekaran and Gu, 2004] Chandrasekaran and Gu present a divide and conquer method to calculate the eigendecomposition of a symmetric generator representable semiseparable plus a block diagonal matrix.

2003 [Van Barel et al., 2003] The authors, Van Barel, Vandebril and Mastronardi construct in this paper an orthogonal similarity transformation to transform symmetric matrices in semiseparable form.

2003 [Vandebril et al., 2003a] The same authors as above construct an implicit QR-algorithm for semiseparable matrices, in order to compute their eigendecomposition.

2003 [Mastronardi et al., 2003a] Mastronardi, Chandrasekaran and Van Huffel, provide an algorithm to transform a symmetric generator representable semiseparable plus diagonal matrix into a similar tridiagonal one. Also a second algorithm to reduce an unsymmetric generator representable semiseparable plus diagonal matrix to a bidiagonal one by means of orthogonal transformations is included.

2003 [Mastronardi et al., 2003b] Mastronardi, Van Camp and Van Barel present a divide and conquer algorithm to compute the eigendecomposition of diagonal plus generator representable semiseparable matrices.

2003 [Fasino et al., 2003] Fasino, Mastronardi and Van Barel propose two new algorithms for transforming diagonal plus generator representable semiseparable matrices to tridiagonal or bidiagonal form. See also [Mastronardi et al., 2003a].

2003 [Fiedler, 2003] Fiedler investigates the properties of so-called basic matrices, these matrices have weakly lower and weakly upper triangular rank 1, in fact this class is closely related to the matrices.
2003 [Vandebril et al., 2003b] Vandebril, Van Barel and Mastronardi translate in this paper the traditional algorithm for computing the singular values via bidiagonal matrices towards their analogue in the semiseparable case.
2003 [Bini et al., 2004] Bini, Gemignani and Pan derive an algorithm for performing a step of QR on a generalized semiseparable matrix. The lower part of this generalized matrix is in fact lower triangular semiseparable. The authors present an alternative representation, consisting of 3 vectors, to represent this part.
2003 [Bevilacqua and Del Corso, 2003] Bevilacqua and Del Corso investigate in this report the existence and the uniqueness of a unitary similarity transformation of a symmetric matrix into semiseparable form. Also the implementation of a QR-step on a semiseparable matrix without shift is investigated.
2004 [Strang and Nguyen, 2004] Strang and Nguyen investigate in this paper the nullity theorem as provided by Fiedler and Markham, and also the theoretical results from Barrett and Feinsilver.
2004 [Van Barel et al., 2004] Van Barel, Van Camp and Mastronardi provide an orthogonal similarity transformation to reduce symmetric matrices into a similar semiseparable one of rank k.
2004 [Vandebril et al., 2005] Vandebril, Van Barel and Mastronardi give a clear overview of the differences between generator representable semiseparable matrices and semiseparable matrices. Also a new type of representation for semiseparable matrices is presented.
2004 [Fiedler and Vavřín, 2004] Fiedler investigates the properties of a class called Generalized Hessenberg matrices, these matrices have weakly lower triangular rank 1.
2004 [Fasino, 2004] Fasino proves that any Hermitian matrix, with pairwise distinct eigenvalues can be transformed into a similar diagonal plus semiseparable matrix, with a prescribed diagonal. Moreover, it is proved that the unitary transformation in this similarity reduction is the unitary matrix in the QR-factorization of a suitably defined Krylov subspace.

In the remaining part of this section we will briefly mention some new classes of matrices closely related to semiseparable matrices. In fact they are generalizations in the sense that specific parts in the matrix satisfy now specific rank conditions. In this way one can develop more general classes of rank-structured matrices. As these classes of matrices are more general than the simple class of semiseparable matrices, we do not include all the references related to these subjects. Some important references, related to these classes are however included in our list, this should make it possible for the reader to retrieve other related references.

Currently the class of sequentially semiseparable matrices is investigated thoroughly [Chandrasekaran et al., 2002; Dewilde and van der Veen, 1998, 2000; Eidelman, 2000; Eidelman and Gohberg, 1999, 2002b]. This class is very close to the class of $\mathcal{H}$-matrices (see [Börm et al., 2003; Hackbusch and Khoromskij, 2000a,b] and the references therein). The class of recursively semiseparable matrices [Chandrasekaran and Gu, 2001] is more general than the class of sequentially semiseparable matrices, as the different rank-structured blocks do not interact with each other which is the case for the sequentially semiseparable.

Semiseparable matrices also appear in various types of applications, e.g., the field of integral equations [Gonzales et al., 1997; Petersen and Ran, 2004; Kang et al., 1999], operator theory [Eidelman and Gohberg, 2002a, 2003; Groenewald et al., 2005], boundary value problems [Gonzales et al., 1997; Greengard and Rokhlin, 1991; Lee and Greengard, 1997; Starr, 1992], in the theory of Gauss-Markov processes [Kavčić and Moura, 2000], time varying linear systems [Dewilde and van der Veen, 1998; Gohberg and Kaashoek, 1984], in statistics [Graybill, 1983], acoustic and electromagnetic scattering theory [Colton and Kress, 1998], signal processing [Mullhaupt and Riedel, 2001, 2002], numerical integration and differentiation [Starr, 1992] and rational interpolation [Van Barel et al., 2002]. Also in the biological field applications exist resulting directly in semiseparable matrices [Elsner and Giersch, 1998].

3. Conclusions

We hope the list we presented here gives a clear overview of publications related to semiseparable matrices. We are well aware that it is not possible to trace all related publications. Therefore we appreciate missing references and feedback. In this way we can maintain and update our reference list at our website.

References

E. L. Allgower. Criteria for positive definiteness of some band matrices. *Numerische Mathematik*, 16:157–162, 1970.

E. L. Allgower. Exact inverse of certain band matrices. *Numerische Mathematik*, 21:279–284, 1973.

E. Asplund. Inverses of matrices a_{ij} which satisfy $a_{ij} = 0$ for $j > i + p$. *Mathematica Scandinavica*, 7:57–60, 1959a.

S. O. Asplund. Finite boundary value problems solved by Green's matrix. *Mathematica Scandinavica*, 7:49–56, 1959b.

J. Baranger and M. Duc-Jacquet. Matrices tridiagonales symmetriques et matrices factorisables. *Revue Française d'Informatique et de Recherche Operationelle*, 5(R-3):61–66, 1971.

W. W. Barrett. A theorem on inverse of tridiagonal matrices. *Linear Algebra and Its Applications*, 27:211–217, 1979.

W. W. Barrett and P. J. Feinsilver. Gaussian families and a theorem on patterned matrices. *Journal of Applied Probability*, 15:514–522, 1978.

W. W. Barrett and P. J. Feinsilver. Inverses of banded matrices. *Linear Algebra and Its Applications*, 41:111–130, 1981.

L. Berg. Auflösung von gleichungssystemen mit einer bandmatrix. *Zeitschrift für Angewandte Mathematik und Mechanik*, 57:373–380, 1977.

W. J. Berger and E. Saibel. On the inversion of continuant matrices. *Journal of the Franklin Institute*, 6:249–253, 1953.

R. Bevilacqua. Structural and computational properties of band matrices. *appeared in R. Bevilacqua et al., Complexity of structured computational problems, Applied mathematics monographs, CNR, giardini editori e stanpatori, Pisa*, pages 131–188, 1991.

R. Bevilacqua and M. Capovani. Proprietà delle matrici tridiagonali ad elementi ed a blocchi. Monografie dell'I.E.I., 1972.

R. Bevilacqua and M. Capovani. Proprietà delle matrici a banda ad elementi ed a blocchi. *Bolletino Unione Matematica Italiana*, 5(13-B):844–861, 1976.

R. Bevilacqua, B. Codenotti, and F. Romani. Parallel solution of block tridiagonal linear systems. *Linear Algebra and Its Applications*, 104:39–57, 1988.

R. Bevilacqua and G. M. Del Corso. Existence, uniqueness and algorithms for matrix unitary reduction to semiseparable form. Tech. Report TR-03-12, University of Pisa, July 2003.

R. Bevilacqua, G. Lotti, and F. Romani. Storage compression of inverses of band matrices. *Computers & Mathematics with Applications*, 20(8):1–11, 1990a.

R. Bevilacqua, F. Romani, and G. Lotti. Parallel inversion of band matrices. *Computers and Artificial Intelligence*, 9(5):493–501, 1990b.

D. A. Bini, L. Gemignani, and V. Y. Pan. *QR*-like algorithms for generalized semiseparable matrices. Technical Report 1470, Department of Mathematics, University of Pisa, 2004.

S. Börm, L. Grasedyck, and W. Hackbusch. Introduction to hierarchical matrices with applications. *Engineering Analysis with Boundary Elements*, 27:405–422, 2003.

R. A. Brualdi and J. J. Q. Massey. More on structure-ranks of matrices. *Linear Algebra and Its Applications*, 183:193–199, 1993.

B. Bukhberger and G. A. Emel'yanenko. Methods of inverting tridiagonal matrices. *Computational Mathematics and Mathematical Physics (translated from Zhurnal Vychislitel'noĭ Matematiki i Matematicheskoĭ Fiziki)*, 13:10–20, 1973.

W. L. Cao and W. J. Stewart. A note on inverses of Hessenberg-like matrices. *Linear Algebra and Its Applications*, 76:233–240, 1986.

M. Capovani. Sulla determinazione della inversa delle matrici tridiagonali e tridiagonali a blocchi. *Calcolo*, 7:295–303, 1970.

M. Capovani. Su alcune proprietà delle matrici tridiagonali e pentadiagonali. *Calcolo*, 8:149–159, 1971.

K. M. Chandan. The inverse of a certain matrix with an application. *Annals of Mathematical Statistics*, 38:1289–1292, 1967.

S. Chandrasekaran, P. Dewilde, M. Gu, T. Pals, and A.-J. van der Veen. Fast stable solver for sequentially semi-separable linear systems of equations. *Lecture Notes in Computer Science*, 2552:545–554, 2002.

S. Chandrasekaran and M. Gu. Fast and stable eigendecomposition of symmetric banded plus semi-separable matrices. *Linear Algebra and Its Applications*, 313:107–114, 2000.

S. Chandrasekaran and M. Gu. A fast and stable solver for recursively semi-separable systems of linear equations. *Contemporary Mathematics*, 281:39–53, 2001.

S. Chandrasekaran and M. Gu. Fast and stable algorithms for banded plus semiseparable systems of linear equations. *SIAM Journal on Matrix Analysis and its Applications*, 25(2):373–384, 2003.

S. Chandrasekaran and M. Gu. A divide and conquer algorithm for the eigendecomposition of symmetric block-diagonal plus semi-separable matrices. *Numerische Mathematik*, 96(4):723–731, Feb. 2004.

D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*. Springer-Verlag, second edition, 1998.

S. Demko. Inverses of band matrices and local convergence of spline projections. *SIAM Journal on Numerical Analysis*, 14(4):616–619, 1977.

S. Demko, W. F. Moss, and P. W. Smith. Decay rates for inverses of band matrices. *Mathematics of Computation*, 43:491–499, 1984.

P. Dewilde and A.-J. van der Veen. *Time-varying systems and computations*. Kluwer academic publishers, Boston, June 1998.

P. Dewilde and A.-J. van der Veen. Inner-outer factorization and the inversion of locally finite systems of equations. *Linear Algebra and Its Applications*, 313:53–100, Feb. 2000.

Y. Eidelman. Fast recursive algorithm for a class of structured matrices. *Applied Mathematics Letters*, 13:57–62, 2000.

Y. Eidelman and I. C. Gohberg. Inversion formulas and linear complexity algorithm for diagonal plus semiseparable matrices. *Computers & Mathematics with Applications*, 33(4):69–79, Aug. 1996.

Y. Eidelman and I. C. Gohberg. Fast inversion algorithms for diagonal plus semiseparable matrices. *Integral Equations and Operator Theory*, 27(2):165–183, 1997a.

Y. Eidelman and I. C. Gohberg. A look-ahead block Schur algorithm for diagonal plus semiseparable matrices. *Computers & Mathematics with Applications*, 35(10):25–34, 1997b.

Y. Eidelman and I. C. Gohberg. On a new class of structured matrices. *Integral Equations and Operator Theory*, 34:293–324, 1999.

Y. Eidelman and I. C. Gohberg. Algorithms for inversion of diagonal plus semiseparable operator matrices. *Integral Equations and Operator Theory*, 44(2):172–211, 2002a.

Y. Eidelman and I. C. Gohberg. A modification of the Dewilde-van der Veen method for inversion of finite structured matrices. *Linear Algebra and Its Applications*, 343-344:419–450, Apr. 2002b.

Y. Eidelman and I. C. Gohberg. Fast inversion algorithms for a class of structured operator matrices. *Linear Algebra and Its Applications*, 371:153–190, 2003.

V. Eijkhout and B. Polman. Decay rates of inverses of banded M-matrices that are near to Toeplitz matrices. *Linear Algebra and Its Applications*, 109:247–277, 1988.

L. Elsner. Some observations on inverses of band matrices and low rank perturbations of triangular matrices. *Acta Technica Academiae Scientiarum Hungaricae*, 108(1-2):41–48, 1997.

L. Elsner and C. Giersch. Metabolic control analysis: Separable matrices and interdependence of control coefficients. *Journal of theoretical Biology*, 193:649–661, 1998.

D. K. Faddeev. Properties of the inverse of a Hessenberg matrix. *Numerical methods and questions in the organizations of calculations 5, Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta imeni V. A. Steklova*, 111:177–179, 1981. (in Russian).

D. Fasino. Rational Krylov matrices and QR-steps on Hermitian diagonal-plus-semiseparable matrices. To appear in Numerical Linear Algebra with Applications (DOI 10.1002/nla.448), 2004.

D. Fasino and L. Gemignani. Structural and computational properties of possibly singular semiseparable matrices. *Linear Algebra and Its Applications*, 340:183–198, 2001.

D. Fasino and L. Gemignani. Direct and inverse eigenvalue problems, for diagonal-plus-semiseparable matrices. *Numerical Algorithms*, 34:313–324, 2003.

D. Fasino, N. Mastronardi, and M. Van Barel. Fast and stable algorithms for reducing diagonal plus semi separable matrices to tridiagonal and bidiagonal form. *Contemporary Mathematics*, 323:105–118, 2003.

P. Favati, G. Lotti, F. Romani, and P. Rózsa. Generalized band matrices and their inverses. *Calcolo*, 28:45–92, 1991.

M. Fiedler. Structure ranks of matrices. *Linear Algebra and Its Applications*, 179:119–127, 1993.

M. Fiedler. Basic matrices. *Linear Algebra and Its Applications*, 373:143–151, 2003.

M. Fiedler and T. L. Markham. Completing a matrix when certain entries of its inverse are specified. *Linear Algebra and Its Applications*, 74:225–237, 1986.

M. Fiedler and T. L. Markham. Rank-preserving diagonal completions of a matrix. *Linear Algebra and Its Applications*, 85:49–56, 1987.

M. Fiedler and Z. Vavřín. Generalized Hessenberg matrices. *Linear Algebra and Its Applications*, 380:95–105, 2004.

F. R. Gantmacher and M. G. Kreĭn. Sur les matrices oscillatoires et complètement non négatives. *Compositio Mathematica*, 4:445–476, 1937.

F. R. Gantmacher and M. G. Kreĭn. Oscillation matrices and kernels and small vibrations of mechanical systems. AMS Chelsea publishing, 2002.

L. Gemignani and D. Fasino. Fast and stable solution of banded-plus-semiseparable linear systems. *Calcolo*, 39(4):201–217, 2002.

I. C. Gohberg and M. A. Kaashoek. Time varying linear systems with boundary conditions and integral operators, I. the transfer operator and its properties. *Integral Equations and Operator Theory*, 7:325–391, 1984.

I. C. Gohberg, T. Kailath, and I. Koltracht. Linear complexity algorithms for semiseparable matrices. *Integral Equations and Operator Theory*, 8(6):780–804, 1985.

I. C. Gohberg, T. Kailath, and I. Koltracht. A note on diagonal innovation matrices. *Acoustics Speech and Signal Processing*, 7:1068–1069, 1987.

G. H. Golub. Comparison of the variance of minimum variance and weighted least squares regression coefficients. *Annals of Mathematical Statistics*, 34(3):984–991, Sept. 1963.

R. A. Gonzales, J. Eisert, I. Koltracht, M. Neumann, and G. Rawitscher. Integral equation method for the continuous spectrum radial Schrödinger equation. *Journal of Computational Physics*, 134:134–149, 1997.

F. A. Graybill. *Matrices with applications in statistics*. Wadsworth international group, Belmont, California, 1983.

B. G. Greenberg and A. E. Sarhan. Matrix inversion, its interest and application in analysis of data. *Journal of the American Statistical Association*, 54:755–766, 1959.

L. Greengard and V. Rokhlin. On the numerical solution of two-point boundary value problems. *Communications on Pure and Applied Mathematics*, 44:419–452, 1991.

T. N. E. Greville. On a problem concerning band matrices with Toeplitz inverses. *Proceedings of the eigth Manitoba Converence on Numerical Mathematics and Computing*, pages 275–283, 1978.

G. J. Groenewald, M. A. Petersen, and A. C. M. Ran. Characterization of integral operators with semi–separable kernel with symmetries. *Journal of Functional Analysis*, 219:255–284, 2005.

W. H. Gustafson. A note on matrix inversion. *Linear Algebra and Its Applications*, 57:71–73, 1984.

W. Hackbusch and B. N. Khoromskij. A sparse $\mathcal{H}$-matrix arithmetic, part I: general complexity estimates. *Journal of Computational and Applied Mathematics*, 125(1-2):479–501, 2000a.

W. Hackbusch and B. N. Khoromskij. A sparse $\mathcal{H}$-matrix arithmetic, part II: Application to multi-dimensional problems. *Computing*, 64(1):21–47, 2000b.

S. B. Haley. Solution of band matrix equations by projection-recurrence. *Linear Algebra and Its Applications*, 32:33–48, 1980.

W. D. Hoskins and G. E. McMaster. On the inverses of a class of Toeplitz matrices of band width five. *Linear and Multilinear Algebra*, 4(2):103–106, 1976.

W. D. Hoskins and G. E. McMaster. Properties of the inverses of a set of band matrices. *Linear and Multilinear Algebra*, 5(3):183–196, 1977.

W. D. Hoskins and G. E. McMaster. On the infinity norm of the inverse of a class of Toeplitz matrices of band width five. *Linear and Multilinear Algebra*, 6(2):153–156, 1978.

W. D. Hoskins and P. J. Ponzo. Some properties of a class of band matrices. *Mathematics of Computation*, 26:393–400, 1972.

W. D. Hoskins and M. C. Thurgur. Determinants and norms for the inverses of a set of band matrices. *Utilitas Mathematica*, 3:33–47, 1973.

Y. Ikebe. On inverses of Hessenberg matrices. *Linear Algebra and Its Applications*, 24:93–97, 1979.

S.-Y. Kang, I. Koltracht, and G. Rawitscher. High accuracy method for integral equations with discontinuous kernels, 1999. http://arxiv.org/abs/math.NA/9909006, Submitted for publication.

S. Karlin. *Total positivity*, volume 1. Stanford University Press, Stanford, California, 1968.

A. Kavčić and M. F. Moura. Matrices with banded inverses: inversion algorithms and factorization of Gauss-Markov processes. *IEEE Transactions on Information Theory*, 46(4):1495–1509, July 2000.

D. Kershaw. Inequalities on the elements of the inverse of a certain tridiagonal matrix. *Mathematics of computation*, 24:155–158, 1970.

I. Koltracht. Linear complexity algorithm for semiseparable matrices. *Integral Equations and Operator Theory*, 29(3):313–319, 1997.

Kounadis. Inversion de matrices tridiagonales symétriques. *RAIRO Recherche Opérationelle*, 7: 91–100, 1973.

E. G. Kounias. An inversion technique for certain patterned matrices. *Journal of Mathematical Analysis and its Applications*, 21:695–698, 1968.

J.-Y. Lee and L. Greengard. A fast adaptive numerical method for stiff two-point boundary value problems. *SIAM Journal on Scientific Computing*, 18(2):403–429, Mar. 1997.

J. W. Lewis. Inversion of tridiagonal matrices. *Numerische Mathematik*, 38:333–345, 1982.

N. Mastronardi, S. Chandrasekaran, and S. Van Huffel. Fast and stable two-way algorithm for diagonal plus semi-separable systems of linear equations. *Numerical Linear Algebra with Applications*, 8(1):7–12, 2001.

N. Mastronardi, S. Chandrasekaran, and S. Van Huffel. Fast and stable algorithms for reducing diagonal plus semiseparable matrices to tridiagonal and bidiagonal form. *BIT*, 41(1):149–157, 2003a.

N. Mastronardi, M. Van Barel, and E. Van Camp. Divide and conquer type algorithms for computing the eigendecomposition of diagonal plus semiseparable matrices. Technical Report 7 (5/2003), Istituto per le Applicazioni del Calcolo "M. Picone", Consiglio Nazionale delle Ricerche, Rome, Italy, 2003b. To appear in Numerical Algorithms.

R. Mattheij and M. Smooke. Estimates for the inverse of tridiagonal matrices arising in boundary-value problems. *Linear Algebra and Its Applications*, 73:33–57, 1986.

J. J. McDonald, R. Nabben, M. Neumann, H. Schneider, and M. J. Tsatsomeros. Inverse tridiagonal Z-matrices. *Linear and Multilinear Algebra*, 45(1):75–97, 1998.

G. Meurant. A review of the inverse of symmetric tridiagonal and block tridiagonal matrices. *SIAM Journal on Matrix Analysis and its Applications*, 13:707–728, 1992.

A. P. Mullhaupt and K. S. Riedel. Banded matrix representation of triangular input normal pairs. *IEEE Transactions on Automatic Control*, 46(12):2018–2022, 2001.

A. P. Mullhaupt and K. S. Riedel. Low grade matrices and matrix fraction representations. *Numerical Linear Algebra with Applications*, 342:187–201, 2002.

R. Nabben. Decay rates of the inverse of nonsymmetric tridiagonal and band matrices. *SIAM Journal on Matrix Analysis and its Applications*, 20(3):820–837, 1999a.

R. Nabben. Two sided bounds on the inverse of diagonally dominant tridiagonal matrices. *Linear Algebra and Its Applications*, 287:289–305, July 1999b.

E. Neuman. The inversion of certain band matrices. *Roczniki Polskiego Towarzystwa Matematycznego*, 3(9):15–24, 1977.

T. Oohashi. Some representation for inverses of band matrices. *TRU Mathematics*, 14(2):39–47, 1978.

J. Petersen and A. C. M. Ran. LU- versus UL-factorization of integral operators with semiseparable kernel. *Integral Equations and Operator Theory*, 50:549–558, 2004.

L. Rehnqvist. Inversion of certain symmetric band matrices. *BIT*, 12:90–98, 1972.

S. A. H. Rizvi. Inverses of quasi-tridiagonal matrices. *Linear Algebra and Its Applications*, 56: 177–184, 1984.

F. Romani. On the additive structure of the inverses of banded matrices. *Linear Algebra and Its Applications*, 80:131–140, 1986.

S. N. Roy, B. G. Greenberg, and A. E. Sarhan. Evaluation of determinants, characteristic equations and their roots for a class of patterned matrices. *Journal of the Royal Statistical Society. Series B. Methodological*, 22:348–359, 1960.

S. N. Roy and A. E. Sarhan. On inverting a class of patterned matrices. *Biometrika*, 43:227–231, 1956.

P. Rózsa. Band matrices and semi-separable matrices. *Colloquia Mathematica Societatis Janos Bolyai*, 50:229–237, 1986.

P. Rózsa. On the inverse of band matrices. *Integral Equations and Operator Theory*, 10:82–95, 1987.

P. Rózsa, R. Bevilacqua, P. Favati, and F. Romani. On the inverse of block tridiagonal matrices with applications to the inverses of band matrices and block band matrices. *Operator Theory: Advances and Applications*, 40:447–469, 1989.

P. Rózsa, R. Bevilacqua, F. Romani, and P. Favati. On band matrices and their inverses. *Linear Algebra and Its Applications*, 150:287–295, 1991.

P. Rózsa, F. Romani, and R. Bevilacqua. On generalized band matrices and their inverses. In J. D. Brown, editor, *Proceedings of the Lanczos international centary conference*, pages 109–121, Philadelphia PA, 1994. SIAM Press.

S. Schechter. Quasi tridiagonal matrices and type insensitive difference equations. *Quarterly of Applied Mathematics*, 18:285–295, 1960.

V. N. Singh. The inverse of a certain block matrix. *Bulletin of the Australian Mathematical Society*, 20:161–163, 1979.

H. P. J. Starr. *On the numerical solution of one-dimensional integral and differential equations*. PhD thesis, Yale University, 1992. Research Report YALEU/DCS/RR-888.

G. Strang and T. Nguyen. The interplay of ranks of submatrices. *SIAM Review*, 46(4):637–646, 2004.

D. Szynal and J. Szynal. À propos de l'inversion des matrices généralisées de Jacobi. *Aplikace Matematiky*, 17:28–32, 1972.

T. Ting. A method of solving a system of linear equations whose coefficients form a tridiagonal matrix. *Quarterly of Applied Mathematics*, 22(2):105–106, 1964.

T. Torii. Inversion of tridiagonal matrices and the stability of tridiagonal systems of linear equations. *Information processing in Japan (Joho Shori)*, 6:41–46, 1966a. (In Japanese).

T. Torii. Inversion of tridiagonal matrices and the stability of tridiagonal systems of linear equations. *Technology Reports of the Osaka University*, 16:403–414, 1966b.

W. F. Trench. Inversion of Toeplitz band matrices. *Mathematics of Computation*, 28(128):1089–1095, 1974.

E. E. Tyrtyshnikov. Mosaic ranks for weakly semiseparable matrices. In M. Griebel, S. Margenov, and P. Yalamov, editors, *Large-Scale Scientific Computations of Engineering and Environmental Problems II*, volume 73 of *Notes on numerical fluid mechanics*, pages 36–41. Vieweg, 2000.

Y. Ukita. Characterization of 2-type diagonal matrices with an application to order statistics. *Journal of the Hokkaido College of Art and Literature*, 6:66–75, 1955.

V. R. R. Uppuluri and J. A. Carpenter. the inverse of a matrix occurring in first-order moving-average models. *Sankhyā The Indian Journal of Statistics Series A*, 31:79–82, 1969.

V. R. R. Uppuluri and J. A. Carpenter. An inversion method for band matrices. *Journal of Mathematical Analysis and its Applications*, 31:554–558, 1970.

F. Valvi. Explicit presentation of the inverses of some types of matrices. *Journal of the Institute of Mathematics and its Applications*, 19(1):107–117, 1977.

M. Van Barel, D. Fasino, L. Gemignani, and N. Mastronardi. Orthogonal rational functions and diagonal plus semiseparable matrices. In F. T. Luk, editor, *Advanced Signal Processing Algorithms, Architectures, and Implementations XII*, volume 4791 of *Proceedings of SPIE*, pages 167–170, 2002.

M. Van Barel, E. Van Camp, and N. Mastronardi. Orthogonal similarity transformation into semiseparable matrices of semiseparability rank k. Report TW 380, Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Leuven (Heverlee), Jan. 2004. To appear in Numerical Linear Algebra with Applications.

M. Van Barel, R. Vandebril, and N. Mastronardi. An orthogonal similarity reduction of a matrix into semiseparable form. To appear in SIAM Journal on Matrix Analysis and its Applications. (Report TW360), May 2003.

E. Van Camp, N. Mastronardi, and M. Van Barel. Two fast algorithms for solving diagonal-plus-semiseparable linear systems. *Journal of Computational and Applied Mathematics*, 164-165: 731–747, 2004.

R. Vandebril, M. Van Barel, and N. Mastronardi. An implicit QR algorithm for semiseparable matrices to compute the eigendecomposition of symmetric matrices. Report TW 367, Department of Computer Science, K.U.Leuven, Leuven, Belgium, Aug. 2003a. To appear in Numerical Linear Algebra with Applications (DOI 10.1002/nla.425).

R. Vandebril, M. Van Barel, and N. Mastronardi. A *QR*-method for computing the singular values via semiseparable matrices. *Numerische Mathematik*, 99:163–195, Oct. 2003b.

R. Vandebril, M. Van Barel, and N. Mastronardi. A note on the representation and definition of semiseparable matrices. *Numerical Linear Algebra with Applications*, Mar. 2005.

T. Yamamoto and Y. Ikebe. Inversion of band matrices. *Linear Algebra and Its Applications*, 24: 105–111, 1979.