Pattern Analysis in Networks of Delayed Coupled Nonlinear Systems

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Abstract— **In this paper, a method for pattern analysis in networks of nonlinear systems of Lur'e type interconnected via time-delayed coupling functions is presented. We consider a class of nonlinear systems which are globally asymptotically stable in isolation. Interconnecting such systems into a network via time-delayed coupling can result in persistent oscillatory behavior, which may lead to pattern formation in the delay-coupled systems. We focus on networks of Lur'e systems in which a Hopf bifurcation causes the instability of the network equilibrium. We develop a numerically efficient method in order to analyze the oscillatory behavior occurring in such networks. Our analysis is based on the harmonic balance method and tested on the network of delay coupled FitzHugh-Nagumo (FHN) model neurons.**

I. INTRODUCTION

In recent years, the investigation of behavior of complex networks, formed by coupled nonlinear dynamical systems, has been a significant subject in mathematical biology, control theory, applied physics, and interdisciplinary fields. Substantial number of natural phenomena can be described by complex networks (e.g. neuronal systems, heart cells synchronization, social networks, flocks of birds and other kinds of collective behavior) [1],[2]. In particular, emergent behavior and pattern formation in systems that interact via time-delay coupling functions, are subjects of intensive study. The problem of emergence of oscillations in coupled inert systems (i.e. systems which are globally asymptotically stable when the coupling is absent) started with Turing's work on morphogenesis [3]. About two decades later, Smale performed a nonlinear analysis of Turing instability or diffusion-driven instability [4]. The author posed the question of determining conditions under which the diffusive coupling leads to oscillatory behavior in the network of initially globally asymptotically stable systems. This question was answered in [5], and [6], and references therein.

Results for different synchronous regimes and pattern formation, such as synchronization [7], partial synchronization [8], and pattern formation [9], are presented in many studies in different fields.

The above-mentioned studies all considered diffusive coupling, which is (usually) symmetric and delay-free. We consider networks of Lur'e systems interconnected via time-delayed coupling functions. Such time-delay coupling functions appear, among others, in network of neurons [10], electrical circuits [11], and networked control systems [12]. In this work, we extend results of previous work [13] and develop a method for determining an oscillatory profile (e.g. DC components, amplitudes, phases and a frequency) of oscillatory patterns emerged in the networks of nonlinear systems of Lur'e form interconnected via time-delayed coupling functions. Lur'e models are widely used in many fields (e.g. computational biology) and simplify the analysis by means of the Harmonic Balance method due to a scalar nonlinearity. We consider the case where a Hopf bifurcation causes the instability of the network dynamics. The coupling strength and time-delay term take the role of bifurcation parameters. It is shown, that the Multivariable Harmonic Balance method for delay coupled systems (DMHB) can be used for analysis of oscillations even if they are of a nonharmonic (relaxation) type. Knowing and understanding the oscillatory profile can be a key to a network design (synthesis) problem.

This paper is organized as follows. In Section 2, the class of systems under consideration and the type of the coupling are introduced. Section 3 presents conditions for oscillatory behavior in networks described by Delayed Differentiation Equations (DDEs) and shows how a bifurcation point can be detected. The harmonic balance method is described in Section 4. The DMHB method is introduced in Section 5. Numerical results are given in Section 6. In Section 7 the conclusions are given.

Throughout the paper, the following notations are used. Let $\mathbb{R}$ and $\mathbb{C}$, denote the real and complex numbers, respectively. j stands for the imaginary unit, $j^2 = -1$. Notation $\mathbb{C}_+$ corresponds to the open right half plane, $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. I_N denotes the identity matrix of the size $N \times N$, $\otimes$ the Kronecker product and $\oplus$ the Kronecker sum (i.e., $A \oplus B = A \otimes I_B + I_A \otimes B$). Symbols $^\top$ and * stand for the transposition and conjugate transposition, respectively. C^1 stands for a set of continuously differentiable functions.

II. PROBLEM SETTING

Consider N single-input-single-output (SISO) nonlinear systems of Lur'e type

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \\ u_i &= u_{ci} + \psi(y_i), \end{aligned} \tag{1}$$

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where $i \in 1,\ldots,N$ stands for the system index, $x_i(t) \in \mathbb{R}^n$ is the state of i-th system, $u_i(t) \in \mathbb{R}^1$ is the combined input of the i-th system which consists of two components: $\psi(\cdot)$ which is a C^1 scalar nonlinear time-invariant memoryless function and $u_{ci} \in \mathbb{R}$ which is the input receiving data from the coupling of multiple systems, $y_i(t) \in \mathbb{R}$ is the output of the i-th system used for the coupling and A, B, C are constant matrices of appropriate dimension with CB a positive constant. We assume that

Assumption 1: The isolated system (1), described by $\dot{x}_i := f(x_i) := Ax_i + B\psi(Cx_i)$, has unique equilibrium x_{i0}, which is globally asymptotically stable and locally exponentially stable.

Note that local exponential stability of the equilibrium is equivalent to the Jacobian matrix $J_0 = J(x_{i0}) = \frac{\partial f}{\partial x_i}(x_{i0})$ at x_{i0} being Hurwitz, i.e., all eigenvalues of J_0 have negative real part.

The N systems (1) are interconnected via linear time-delay coupling functions of the form

$$u_{ci}(t) = \gamma \sum_l \gamma_{il}[y_l(t-\tau) - y_i(t)], \tag{2}$$

where positive constant γ is the coupling strength, positive constant τ is the time delay, and nonnegative constants γ_{il} are the interconnection weights. Note that γ_{il} is positive if and only if there is a connection from system l to system i. Define the $N \times N$ matrix $\Gamma = (\gamma_{il})$. Matrix Γ is the (weighted) adjacency matrix of the graph that specifies the interaction structure. We allow the graph to be directed. The graph is simple in the sense that self-connections are forbidden. Moreover, every pair of systems can be joined by a sequence of directed edges, which is equivalent to strong connectivity of the graph. In addition, we make the following assumption.

Assumption 2: The sum of each row of matrix Γ equals 1.

This assumption was used for the delay-coupled nonlinaer system analysis in [14], [15] and ensures that a synchronous (oscillatory) state exists. Moreover Assumption 2 implies, by the Gershgorin Disc Theorem, that all eigenvalues of Γ are located in the closed unit disc in $\mathbb{C}$.

III. PRELIMINARIES

The network (1)-(2) is described by DDEs of retarded type. It is important to mention that for the considered class of delay systems, even though they are infinite-dimensional, a Hopf bifurcation theorem holds [16], and the center manifold corresponding to a bifurcation is finite-dimensional. When a linearized equilibrium becomes unstable, the number of unstable modes is always finite. This allows a direct extension of tools for finite-dimensional systems to detect/compute bifurcations and periodic solutions nearby [16].

Since our goal is to analyse oscillatory patterns, we establish conditions for oscillatory behavior of the network. Suppose Assumption 1 and Assumption 2 hold true and

- the solutions of the network (1)-(2) are uniformly bounded and uniformly ultimately bounded [17];
- the network (1)-(2) has a unique but unstable equilibrium at $x_0 = 1_N \otimes x_{10}$.

Clearly, if both additional assumptions are satisfied, the coupled systems will exhibit oscillatory behavior. Uniqueness of the equilibrium is not necessary for the existence of oscillations. However, the stability analysis of additional equilibria could be cumbersome as the locations of these equilibria depend on coupling strength γ.

The first condition is often satisfied for semi-passive systems. For instance, Lemma 7.1 of [18] states that if an isolated system of the form (1) is strictly C^1-semipassive, then for each fixed γ and τ the solution of the coupled systems (1)-(2) are uniformly bounded and uniformly ultimately bounded. Note that, unlike in the ODE (finite dimensional) case, uniform ultimately boundedness does not imply uniform boundedness in the DDE case [19].

In order to verify the second condition, we need to show that the equilibrium point x_0 is unique and unstable. Using Assumption 2, we can write the network dynamics as

$$\dot{x}(t) = F(x(t)) + \gamma[(\Gamma \otimes BC)x(t-\tau) - (I_N \otimes BC)x(t)]. \tag{3}$$

The linearization of (3) around the network equilibrium x_0 results in

$$\dot{\hat{x}}(t) = [I_N \otimes (J_0 - \gamma BC)]\hat{x}(t) + (\gamma\Gamma \otimes BC)\hat{x}(t-\tau). \tag{4}$$

It is well known that the zero solution of the linear system (4) is unstable for some $\gamma > 0$ and $\tau > 0$ if its associated characteristic equation

$$\Delta(\lambda; \gamma, \tau) = 0 \tag{5}$$

with

$$\Delta(\lambda; \gamma, \tau) := \det(\lambda I_{Nn} - I_N \otimes (J_0 - \gamma BC) - (\gamma\Gamma \otimes BC)e^{-\lambda\tau})$$

has a root in $\mathbb{C}_+$ for that γ and τ [20]. However, computing the roots of the characteristic equation above in the (γ, τ)-parameter space is cumbersome for large N. To address this issue, we use sufficient conditions for instability of the network equilibrium presented in [18]. In order to do so, we denote

$$\mathscr{H}(s) = C(sI_n - J_0)^{-1}B = \frac{p(s)}{q(s)}$$

the transfer function from u_i to y_i of system (1), linearized around its equilibrium. Here p and q are polynomials of degree $n-1$ and n, respectively. Note that system (1) has relative degree one ($CB > 0$). Following [18], we formulate two Lemmas.

Lemma 1: (Instability) Suppose that Assumption 2 holds true. Let

$$\nu := \inf_{\omega > 0} \mathscr{R}(\mathscr{H}(i\omega)).$$

If $\nu < 0$, then for each $\gamma \geq \frac{-1}{2\nu}$ there exists a $\tau > 0$ such that the characteristic equation (5) has a root in $\mathbb{C}_+$.

Note that using Lemma 1 a value of $\gamma_{min} := \frac{-1}{2\nu}$ can be computed. We continue with conditions for uniqueness of the network equilibrium.

Lemma 2: (Uniqueness of the network equilibrium) Let Assumption 1 hold true and denote the eigenvalues of Γ by $\bar{\lambda}_i$, $i = 1,...,N$. Let λ^* be the smallest real-valued eigenvalue of Γ. Choose $\bar{\gamma} \in (0,\infty]$ as the largest number for which the matrix

$$J(\chi) - \gamma(1-\lambda^*)BC$$

is nonsingular for all $\chi \in \mathbb{R}^n$ and all $\gamma \in [0,\bar{\gamma})$. Then the network equilibrium solution $x_0 = 1_N \otimes x_{10}$ is the unique equilibrium solution of (3) for $\gamma \in [0,\bar{\gamma})$.

The proofs of Lemma 1 and Lemma 2 are provided in the Appendix of [18]. Using conditions from Lemmas 1 and 2, one can easily determine a range of coupling strengths $\gamma \in [\gamma_{min},\bar{\gamma})$ for which there exists τ such that oscillations are present.

In order to determine a bifurcation point, we need to compute critical values of τ. The critical delays of a delay-differential equation can be computed by solving a nonlinear two-parameter eigenvalue problem [21]. The solution of this two-parameter problem can be translated to solving a quadratic eigenvalue problem. Recall the characteristic equation (5). Since, we seek for a bifurcation point, λ and γ are taken as follows $\lambda = j\omega$ and $\gamma = \gamma_{min}$.

Defining

$$\mathscr{P} := \exp(-j\omega\tau) \tag{6}$$

a quadratic eigenvalue problem is constructed as follows

$$\det[(\mathscr{A} + \mathscr{B}\mathscr{P}) \oplus (\mathscr{A}^\top + \mathscr{B}^\top(\frac{1}{\mathscr{P}}))] = 0, \tag{7}$$

where $\mathscr{A} = I_n \otimes (J_0 - \gamma_{min}BC)$ and $\mathscr{B} = (\gamma_{min}\Gamma) \otimes (BC)$.

Solving equation (7) for $\mathscr{P}$ and computing the corresponding values of ω_p using (5), the critical values of τ_c can be derived from (6).

IV. HARMONIC BALANCE METHOD

In this section the Harmonic Balance method is introduced. The method gives an approximation to a periodic solution of ODEs. In order to show how the Harmonic balance equation for a SISO nonlinear system is derived, we consider a system of the Lur'e type:

$$\begin{aligned} \dot{\xi} &= A_L\xi + B_L w, \\ w &= \zeta(\eta), \\ \eta &= C_L\xi, \end{aligned} \tag{8}$$

where ξ is the state of the system, w is the input of the system, η is the output of the system, $\zeta(\cdot)$ is C^1 scalar nonlinear time-invariant memoryless function and A_L, B_L, C_L are constant matrices of appropriate dimension with $C_LB_L > 0$.

Suppose $\eta(\omega t)$ is periodic with frequency ω and let

$$\eta(\omega t) = \sum_{k=-\infty}^{+\infty} a_k e^{kj\omega t}, \quad a_k = \bar{a}_{-k} \tag{9}$$

and look for a set of Fourier coefficients and a frequency ω which satisfy the system's equations. We now use Fourier series to rewrite the system's input as follows

$$w = \zeta(\eta) = \sum_{k=-\infty}^{+\infty} c_k e^{kj\omega t}, \quad c_k = \bar{c}_{-k}. \tag{10}$$

Using the orthogonality of the functions $\{\exp(kj\omega t)\}_k$, we find that the Fourier coefficients must satisfy

$$a_k = W(kj\omega)c_k, \tag{11}$$

for all integers k and $W(s) = C_L(sI - A_L)^{-1}B_L$ is the transfer function representing the linear time-invariant part of the dynamics. Since $W(kj\omega)$ is strictly proper, the most information of the system behavior is contained in the first harmonic ($k = 1$). A DC component appears when, for instance, a nonlinearity is not odd. According to this, we consider equations for $k = 0,1$ [22]. Equation (11) is reduced to

$$\begin{aligned} [1 - W(0)K_0(a_0,a_1)]\,a_0 &= 0, \\ [1 - W(j\omega)K_1(a_0,a_1)]\,a_1 &= 0, \end{aligned} \tag{12}$$

where $K_0(a_0,a_1)$ and $K_1(a_0,a_1)$ are describing functions which are computed as follows

$$\begin{aligned} K_0(a_0,a_1) &= \frac{c_0}{a_0} = \frac{1}{2a_0\pi}\int_0^{2\pi} \zeta(\eta(\omega t))(\omega t), \\ K_1(a_0,a_1) &= \frac{c_1}{a_1} = \frac{1}{a_1\pi}\int_0^{2\pi} \zeta(\eta(\omega t))\sin(\omega t)(\omega t). \end{aligned} \tag{13}$$

Note that describing functions K_0 and K_1 are real due to the fact that the nonlinear function $\zeta(\cdot)$ is time-invariant and memoryless [22].

V. MULTIVARIABLE HARMONIC BALANCE FOR DELAYED SYSTEMS

In this section, we show how the Harmonic Balance method can be applied to a network of delayed differential equations (DDEs). The Multivariable Harmonic Balance method for DDE is a direct extension of the Harmonic Balance method for networks of Lur'e systems described in [23]. The idea of the method is to turn the problem of determining an oscillatory profile consisting of DC components q_{0i}, amplitudes α_i, phases ϕ_i and frequency ω, into an eigenvalue problem.

Using the Laplace transformation, the time-invariant linear part of the network (1)-(2) is rewritten into the transfer function representation, leading to the following description of the closed loop system (note that s stands for the differential operator)

$$\begin{aligned} y &= W(s)I_N u, \\ u &= M(\Delta_\tau)y + \nu, \\ \nu &= \Psi(y), \end{aligned} \tag{14}$$

where $I_N \in \mathbb{R}^{N\times N}$ is an identity matrix, $W(s) = C(sI - A)^{-1}B$, $M(\Delta_\tau) = \gamma\Delta_\tau\Gamma - \gamma I_N$ represents the coupling structure, Δ_τ is the delay operator ($\Delta_\tau f(t) = f(t-\tau)$) and $\Psi(y) \in \mathbb{R}^N$.

Recall (14) and rewrite input u as

$$u = M(\Delta_\tau)W(s)I_N u + \nu, \tag{15}$$

$$W(s)\left[\frac{1}{W(s)}I_N - M(\Delta_\tau)\right]u = \nu. \tag{16}$$

Defining $R(s,\Delta_\tau) = \left[\frac{1}{W(s)}I_N - M(\Delta_\tau)\right]^{-1}$, the input u takes the form

$$u = \frac{1}{W(s)}R(s,\Delta_\tau)\nu. \tag{17}$$

Substituting (17) in (14), we obtain

$$y - R(s,\Delta_\tau)\Psi(y) = 0. \tag{18}$$

Premultiplying by $R(s,\Delta_\tau)^{-1}$, we derive

$$\left[\frac{1}{W(s)} I_N - M(\Delta_\tau)\right] y - \Psi(y) = 0. \tag{19}$$

The nonlinearity $\Psi(y)$ is approximated by its describing function, whose components can be computed using (13). The network output signal y is approximated by a sum of the offsets and signal's first harmonic

$$y \cong q_0 + q_1,\ q_0 = \text{const},\ q_1 = \alpha \sin(\omega t + \phi_0 + \phi). \tag{20}$$

Recalling the Fourier expansion and substituting s by $kj\omega$, and Δ_τ by $\exp(-kj\omega\tau)$, we obtain

$$\left[\frac{1}{W(kj\omega)} I_N - M(kj\omega) - K_k(q_{0i},\alpha_i) I_N\right] q_k = 0. \tag{21}$$

We now have two Multivariable Harmonic Balance equations for the networks with delay coupling (DMHB) from (21) for $k = 0, 1$, respectively. These can be expressed as

$$J_0(q_0,\alpha) q_0 := \left[\left(\frac{1}{W(0)} I_N - M(0)\right) - K_0(q_{0i},\alpha_i) I_N\right] q_0 = 0, \tag{22}$$

and

$$J_1(q_0,\alpha,\omega) q_1 := \\ = \left[\left(\frac{1}{W(j\omega)} I_N - M(j\omega)\right) - K_1(q_{0i},\alpha_i) I_N\right] q_1 = 0. \tag{23}$$

Our goal is to determine an oscillatory profile which consists of the following parameters: q_0 is a constant offset vector, α is a vector of amplitudes, ϕ is a vector of phase shifts and ω stands for the frequency of oscillations. All parameters above are encoded in the relations (22) and (23). These equations are nonlinear due to the describing functions (13). In order to determine these parameters, we formulate an optimization problem. Since the trivial solution $q_1 = 0$ corresponds to an unstable equilibrium, we look for a set of phases ϕ_i, amplitudes α_i, DC components q_{0i} and frequency ω, which solve nonlinear equations (22) and (23). We define the objective function to minimize as

$$F(g) := Q_0^* Q_0 + Q_1^* Q_1 + p^\top p, \tag{24}$$

where

$$g := [\phi; \alpha; q_0; \omega] \in \mathbb{R}^{3N+1}, \qquad p := \alpha - |q_1| \in \mathbb{R}^N,$$
$$Q_0 := J_0(q_0,\alpha) q_0 \in \mathbb{C}^N, \qquad Q_1 := J_1(q_0,\alpha,\omega) q_1 \in \mathbb{C}^N,$$

where $|q_1| = [|q_{11}|, |q_{12}|, ..., |q_{1N}|]^\top$.

The vector g consist of phases, amplitudes, DC components and frequency, the component p represents the second equation of (20). To simplify the problem, the phase of the first node is fixed at $\phi_1 = 0$ without loss of generality, and boundaries are chosen as: $\phi_i \in [0, 2\pi] \quad \forall i \neq 1,\ \alpha_i \in [0, \beta] \quad \forall i,\ q_{0i} \in [-\beta, \beta] \quad \forall i,\ \omega \in [0, 2\omega_{bif}]$, where β is a heuristically chosen number. A smart choice of β allows a substantial reduction in convergence time.

Solutions to nonlinear equations (22) and (23) are found by minimizing the squared residual norm of the objective function (24). The local solver `fmincon` included in `MATLAB` environment is used. The solver uses the *interior-point* algorithm [24] in order to find a minimum of a constrained nonlinear multivariable function satisfying the imposed constraints. Since equations (22) and (23) are nonlinear and multiple solutions exist, these solutions heavily depend on the initial conditions. Multiple values of the initial conditions are generated by the `MultiStart` algorithm within the feasible region [25].

Having a set of parameters g, oscillations can be reconstructed using (20).

VI. NUMERICAL EXAMPLE

In this section, our results are illustrated in a network of inert FitzHugh-Nagumo (FHN) model neurons [26]. The dynamics of this model neuron are given by the following equations:

$$\begin{aligned} \dot{x}_{i1} &= 0.08(x_{i2} - 0.8x_{i1}), \\ \dot{x}_{i2} &= x_{i2} - (x_{i2})^3/3 - x_{i1} - 0.559 + u_i, \\ y_i &= x_{i2}. \end{aligned} \tag{25}$$

One can easily verify that the FHN model has a locally exponentially stable equilibrium at $\bar{x}_i = (-1.225 - 0.980)^\top$. Moreover, the FHN model neuron is strictly C^∞-semipassive with a quadratic storage function, see [27]. Hence, the solutions of any network consisting of FHN neurons are uniformly bounded for any nonnegative γ and τ.

In order to apply our method to the network of FHN neurons, we need to perform a change of coordinates. For that, we define new variables $z_{i1} = x_{i1} - \bar{x}_{i1}$ and $z_{i2} = x_{i2} - \bar{x}_{i2}$. The system (25) is equivalent to the system:

$$\begin{aligned} \dot{z}_{i1} &= -0.064 z_{i1} + 0.08 z_{i2}, \\ \dot{z}_{i2} &= -z_{i1} + (1 - \bar{x}_{i2}^2) z_{i2} - (z_{i2}^3 + 3\bar{x}_{i2} z_{i2}^2)/3 + u_i, \\ y_i &= z_{i2}. \end{aligned} \tag{26}$$

It is easy to see that this system has an equilibrium at the origin and is of form (1) with

$$A = \begin{bmatrix} -0.064 & 0.08 \\ -1 & 0.0391 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$\psi(y_i) = -y_i^3/3 + 0.98 y_i^2,$$

and $i = 1, 2, 3$. The nonlinear function $\psi(y_i)$ is not odd, which leads to a nonzero offset component of signal y_i. This implies that $y_0 \neq 0$ and both equations (22) and (23) have to be solved.

The graph topology is described by the following adjacency matrix,

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{27}$$

The describing functions for nonlinearity $\psi(y_i)$ are given as

$$K_0(q_{0i},\alpha_i) = 0.98 q_{0i} - 0.5\alpha_i^2 - 0.33 q_{0i}^2 + 49\alpha_i^2/(100 q_{0i}),$$
$$K_1(q_{0i},\alpha_i) = -0.25\alpha_i^2 - q_{0i}^2 + 1.96 q_{0i}.$$

In order to determine the bifurcation point, we recall the condition from Lemma 1. It is found that $\nu = \inf_{\omega>0}\mathbb{R}(\mathscr{H}(i\omega)) = -0.197$, which is attained at $\omega = \omega^* = 0.4202$. Thus the minimal coupling strength is $\gamma_{min} = 2.5376$. The critical value of time delay $\tau_c = 4.7643$ is computed by solving quadratic eigenvalue problem (7).

In order to verify our calculations, the method of computing rightmost characteristics roots of DDEs is used [28]. Fig.1 shows that the most right characteristic roots located on the imaginary axes, which corresponds to a Hopf bifurcation (Note that the bifurcation leading to instability of the network equilibria must be a Hopf bifurcation because otherwise the condition of Lemma 2 would be violated). The oscillatory

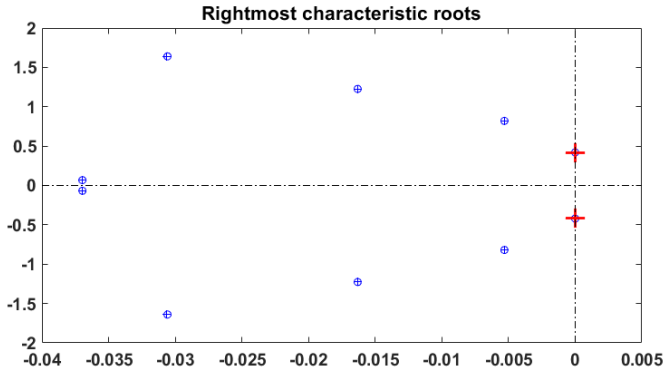

Fig. 1. Rightmost characteristic roots computed for fixed $\gamma_{min} = 2.5376$ and $\tau_c = 4.7643$

profile obtained by simulating DDEs is shown in the Fig. 3. The results are obtained by simulating DDEs using the `dde23` solver in `MATLAB` environment. We use the standard set of options and the initial data is chosen as a constant function defined on $[-\tau, 0]$ whose value is a (uniformly distributed) random number and values of the bifurcation parameters are taken as $\gamma = 3.0451$ and $\tau = 4.4064$. For this pair of parameters, there is one pair of eigenvalues with $\mathrm{Re}(\lambda) > 0$ while all others are confined to the open left half plane $\mathrm{Re}(\lambda) < 0$, which is shown in Fig. 2. It

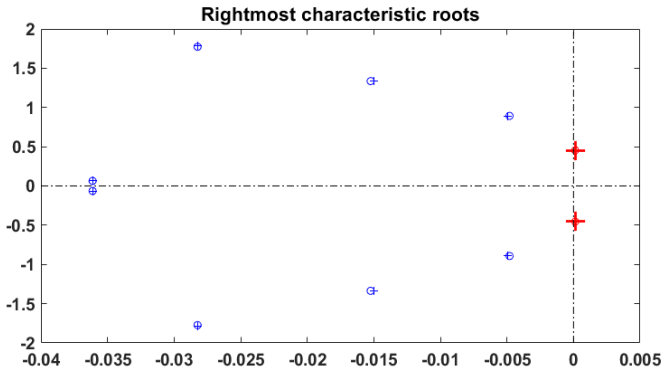

Fig. 2. Rightmost characteristic roots computed for fixed $\gamma = 3.0451$ and $\tau = 4.4064$

is easy to see that the output signals are oscillations of

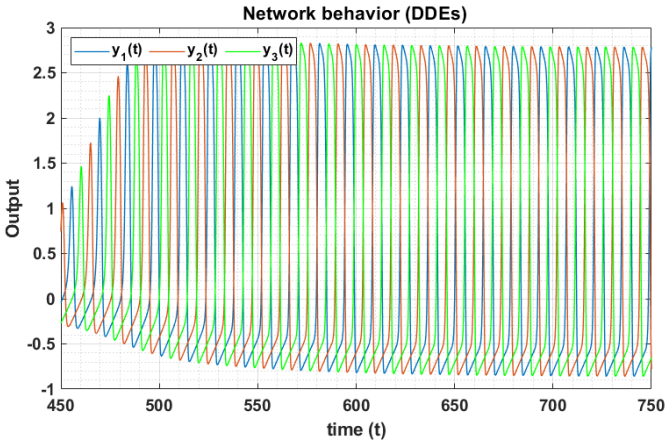

Fig. 3. Network behavior obtained by simulating DDEs

the relaxation type. According to normal form theory [29], oscillations in the neighbourhood of the Hopf bifurcation must be sinusoidal like with zero offsets. Thus we can conclude that the Hopf bifurcation of the sub-critical type occurs in our example. Instability of the network equilibrium results in an unstable limit cycle which is connected to a stable oscillatory regime of relaxation type located out of the neighbourhood of the bifurcation point. This regime is observed in Fig. 3. Moreover, this regime exists even before the bifurcation and can be analysed by the DMHB method. It can be observed that the DC components and amplitudes of the signals are the same for each node i, where $i = 1,2,3$. The approximation using only the first harmonic cannot be very accurate in terms of predicting the amplitudes and DC components due to the relaxation type of oscillations.

The oscillatory profile predicted by the DMHB method is presented in Table I. In order to measure the accuracy

TABLE I

OSCILLATORY PROFILE PREDICTED BY THE DMHB METHOD

Node	ω	q_{0i}	α_i	ϕ_i
1	0.4542	0.707	1.884	0
2	0.4542	0.707	1.884	120
3	0.4542	0.707	1.884	240

of the method we provide the residual norm of objective function (24) and compare the DC components, amplitudes and frequency with the ones that are extracted from the DDE simulation as follows,

$$\Delta q_0 = 100||q_0 - q_{0s}||/||q_{0s}||;$$
$$\Delta \alpha = 100||\alpha - \alpha_s||/||\alpha_s||;$$
$$\Delta \omega = 100(\omega - \omega_s)/\omega_s,$$

where q_{0s}, α_s and ω_s are obtained from numerical simulation. Note that all values in Table II are expressed in percentages. The DMHB method accurately predicts the frequency, amplitudes, and phases, but not the DC components in the case-study.

TABLE II

COMPARISON OF THE DMHB METHOD AND DDE SIMULATION

Residual	$\Delta\omega$	Δq_{0i}	$\Delta\alpha_i$
2.94e-13	2.25	26.04	4.95

VII. CONCLUSION

In this work, we presented an approach aiming at prediction of periodic solutions in a network of Lur'e systems (e.g. FHN model neurons) which are interconnected via delayed linear coupling. We extended the Multivariable Harmonic Balance for the case where the coupled nonlinear systems are described by DDEs, which allowed to detect nonharmonic (relaxation) oscillations in the presented case-study. Moreover, these relaxation oscillations are seen even before the bifurcation occurs in the network, suggesting the bifurcation is sub-critical. Additional bifurcation analysis is required to have a full picture of this phenomenon. Moreover, the fact that we observe the relaxation oscillations before the bifurcation (even for small coupling strength for some initial conditions) may lead to existence of a hidden attractor [30]. Despite a substantial difference in the offset component which is caused by higher harmonics, the remaining components of the oscillatory profile are accurately predicted. Taking into account that the studied network exhibits oscillations of relaxation type, we can conclude the MHB for DDEs gives an accurate approximation of periodic solutions. That implies that the MHB analysis of DDEs can be used as as a first step towards the synthesis of networks exhibiting desired behavior.

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