A novel act-and-wait control scheme for fixed-time stabilization of input-delay systems and assignment of the monodromy matrix*

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Abstract

An act-and-wait (AAW) fashioned control scheme is proposed for input delay systems, which is characterized by switches between different gains during the acting phases. Under very mild conditions, the novel control scheme allows to freely assign the monodromy matrix and to solve the fixed-time stabilization problem. In additionally, a variant is proposed where the control law is simplified by keeping the control signal constant after every switch, amendable for a sampled-data implementation. For multi-input systems it is also demonstrated how the number of switches per acting period in the basis scheme can be reduced to the dimension of the system, starting from the controllability indices set. The effectiveness of the proposed methods is verified on two simulation case-studies.

Keywords: Fixed-time stabilization, act-and-wait, periodic delayed feedback, monodromy matrix, controllability index.

1 Introduction

Throughout the paper, we develop spectrum based control design methods for the input delayed linear system

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \tag{1}$$

where $t \geq 0$, the pair $(A, B) \in (\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m})$ is controllable, $x(t)$ is the state vector and $u(t)$ is the input vector at time t, and $\tau > 0$ is a known constant input delay. Using finite-dimensional, instantaneous state feedback, the presence of the delay will cause the characteristic equation of the closed-loop system to be transcendental, with generically an infinite number of characteristic roots [6]. As a consequence, the problem of ensuring the spectrum be located in the open left half plane using a finite number of tunable controller gains parameters is inherently a reduced-order control design problem and very challenging.

Two complementary approaches have been proposed to address this challenge. The first one consists of using predictor feedback to compensate the input delay, combined with a control design for the corresponding model without delay, see, e.g., [7, 14, 26] and the references therein. The second approach, at the basis of the present paper, is the so-called act-and-wait (AAW) control approach, which was introduced by Stépán and Insperger in [9, 22]. This approach is an example of a periodic feedback strategy (for other periodic schemes and their application, we refer to [5,12,17,24]). In its simplest form, the AAW controller for (1) can be modeled as

$$u(t) = g(t)Kx(t), \tag{2}$$

where g is a T-periodic function defined by

$$g(t) = \begin{cases} 0, & 0 \leq t < t_{\mathrm{w}}, \\ 1, & t_{\mathrm{w}} \leq t < t_{\mathrm{w}} + t_{\mathrm{a}} = T. \end{cases} \tag{3}$$

From (2), we see that the control signal is switched periodically on and off, while during the open-loop phases, the evolution of the state is fully determined by an initial condition taken from $\mathbb{R}^n$. These properties ensure

*This work is supported in part by the National Natural Science Foundation of China under grant number 61773140 and 61903102, by the project G092721N of the Research Foundation-Flanders (FWO - Vlaanderen), and by the project C14/17/072 of the KU Leuven Research Council.

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that for $t_w \geq \tau$, the system's behavior can be characterized by a *finite-dimensional* transition operator, namely a monodromy matrix of dimensions $n \times n$ that match the number of controller gain parameters. It has been shown that the approach leads to wider stability regions in parameter spaces, compared to conventional continuous delayed feedback [10,13], such as the approaches proposed in [8,15,23]. However, a main restriction of AAW control from a design point of view is that the relationship between the monodromy matrix of the closed-loop system and the control gains is nonlinear, which makes tuning difficult, while there is in general no guarantee that the closed-loop system's spectrum is freely assignable. In the present paper, we propose a novel AAW control scheme which employs switches of the controller gain during the act phases. In this way, a linear relation can be established instead between the controller gains and the monodromy matrix, which will become arbitrarily assignable, as we will show in a constructive way.

We will pay special attention to providing solutions to the fixed-time stabilization problem. More precisely, we will address the following problem.

Problem 1 *Consider the linear system (1) and let $T > 0$ be a given positive constant. Design a controller $u(t)$ such that the corresponding closed-loop system is asymptotically stable and $x(t) = 0$ for all $t \geq T$.*

Also here, the AAW control strategy provides an elegant solution, by assigning the complete spectrum of the monodromy matrix to zero, which induces the desired deadbeat response. Already in [9] and [22], the authors pointed to this possibility using case-studies, and made verifications through numerical simulations. In [11], it was shown that the AAW control can always provide deadbeat control for first-order unstable system with any fixed value of the time-delay in the feedback loop and in [18], a proof was provided for all planar controllable systems. However, experiments with the optimization based tuning procedure described in [18] suggest that the problem is not always solvable. In all these works, the feedback control ensures that the monodromy matrix becomes nilpotent, but as a rule it will be nonzero. On the contrary, the novel AAW scheme allows to assign the whole monodromy matrix to the zero matrix. This positively impacts that the settling time T equals to τ plus one period (note that if the index of nilpotency of the monodromy matrix is equal to $\alpha \in \{1, \ldots, n\}$, then the settling time is given by τ plus α periods), while the above mentioned linear relation between the controller parameters and the monodromy matrix removes the restrictions on the applicability. We will further show that if the monodromy matrix is assigned to zero, the procedure to compute the gains simplifies considerably. We mention that, to solve this problem, the periodic control law of [25], which will be a main point of comparison in our simulations, employs a smooth time-varying controller gain. The authors of [21] extend the implicit Lyapunov function based method to time-delay systems. However, for this method, the settling time is generally related to the initial condition.

Finally, in the paper we will also discuss two extensions of the core results. First, we will show that for multi-input systems, the number of switches of the gain can be significantly reduced utilizing the concept of controllability index. Second, we will propose an adaptation of the control law, where the control input is kept constant between switching moments, which eases the implementation compared with the prototype and connects with the sampled-data setting. The modified control law will be positioned against the AAW control law for discrete-time systems proposed in [10], where the distinctive feature is once again spectrum assignment versus assignment of the full monodromy matrix.

The remainder of the paper is organized as follows. The main results are given in Section 2, where the piecewise-constant periodic feedback control law is presented in Section 2.1, and the number of switching is reduced in Section 2.2. Two simulation case-studies are presented in Section 3 to illustrate the effectiveness of the proposed theory. Some concluding remarks are given in Section 4.

Notation: the set of real numbers and the set of nonnegative integers are denoted by $\mathbb{R}$ and $\mathbb{N}$. For any matrix $A = [a_1, a_2, \ldots, a_m] \in \mathbb{R}^{n \times m}$, the notations A^{T}, A^{H} and $\mathrm{vec}(A)$ denote its transposition, conjugate transpose and the column-wise stacking of its elements in a vector,i.e., $[a_1^{\mathrm{T}}, a_2^{\mathrm{T}}, \ldots, a_n^{\mathrm{T}}]^{\mathrm{T}}$. The operation $\otimes$ denotes the Kronecker product between matrices.

2 Main Results

In this section, we first propose novel design schemes for the fixed-time stabilization of linear continuous time-delay systems, grounded in the AAW control concept, as well as procedures to arbitrarily assign the monodromy matrix. Next, we discuss a design scheme for multi-input systems with a reduced number of switches. We start with a preliminary result on which the construction of all control schemes rely. For sake of completeness, a proof is given in Appendix A1.

Lemma 1 *Let $h > 0$ be a given constant. Then, the pair $(\mathrm{e}^{A\frac{h}{n}}, B)$ is controllable if (A, B) is controllable and*

$$\mathrm{e}^{(\lambda_i-\lambda_j)\frac{h}{n}} \neq 1 \text{ whenever } \lambda_i \neq \lambda_j, \tag{4}$$

is satisfied.

As we assume (A, B) to be controllable, it follows from (4) that there *always* exists constants $h > 0$ such that the pair $(\mathrm{e}^{A\frac{h}{n}}, B)$ is controllable.

2.1 Design with Piecewise-Constant Periodic Feedback Gains

Let $K(t)$ be a $2h$-periodic function defined by

$$K(t) = \begin{cases} 0, & t \in [2kh, (2k+1)h), \\ K_0, & t \in [(2k+1)h, (2k+1)h + \frac{1}{n}h), \\ K_1, & t \in [(2k+1)h + \frac{1}{n}h, (2k+1)h + \frac{2}{n}h), \\ \vdots & \\ K_{n-1}, & t \in [(2k+1)h + \frac{n-1}{n}h, 2(k+1)h), \end{cases} \tag{5}$$

where $k \in \mathbb{N}$, $t \geq 0$, $h \geq \tau$ is a given constant such that the pair $(\mathrm{e}^{A\frac{h}{n}}, B)$ is controllable, and $K_i \in \mathbb{R}^{m\times n}$, $i \in \{0, \ldots, n-1\}$, are undetermined matrices. We design a periodic state feedback controller

$$u(t) = K(t)x(t - (h - \tau)),\ t \in [0, \infty), \tag{6}$$

where $K(t)$ is defined in (5). The controller (6) is well-defined for $t \geq h - \tau$ since the state $x(t)$ is accessible for $t \geq 0$. For $0 \leq t < h - \tau$, it is clear that $[0, h - \tau) \subset [0, h)$ and thus $K(t) = 0$, according to (5). Therefore, the controller (6) is well-defined for all $t \geq 0$.

Consider the closed-loop system consisting of (1) and (6), which can be written as

$$\begin{aligned} \dot{x}(t) &= \begin{cases} Ax(t) + Bu(t-\tau), & t \in [0, \tau), \\ Ax(t) + BK\,(t-\tau)\,x(t-h), & t \in [\tau, \infty), \end{cases} \\ &= \begin{cases} Ax(t) + Bu(t-\tau), & t \in [0, \tau), \\ Ax(t), & t \in \mathcal{T}_k^{\mathrm{w}}, \\ Ax(t) + BK(t-\tau)x(t-h), & t \in \mathcal{T}_k^{\mathrm{a}}, \end{cases} \end{aligned} \tag{7}$$

where $k \in \mathbb{N}$, $\mathcal{T}_k^{\mathrm{w}} \triangleq [\tau + 2kh, \tau + (2k+1)h)$ denote waiting periods, and $\mathcal{T}_k^{\mathrm{a}} \triangleq \mathcal{T}_{k0}^{\mathrm{a}} \cup \mathcal{T}_{k1}^{\mathrm{a}} \cup \cdots \cup \mathcal{T}_{kn-1}^{\mathrm{a}}$ denote acting periods with $\mathcal{T}_{ki}^{\mathrm{a}} \triangleq [\tau + (2k+1)h + \frac{i}{n}h, \tau + (2k+1)h + \frac{i+1}{n}h)$, $i \in \{0, \ldots, n-1\}$.

In the interval $[0,\ \tau)$, the state variable x is fully determined by the initial conditions, $x(0)$ and $u_0 \equiv u(\theta),\ \theta \in [-\tau,\ 0)$. It follows from (7) that, for $t \in [\tau, \tau + h)$, we have

$$x(t) = \mathrm{e}^{A(t-\tau)}x(\tau),$$

while for $t \in [\tau + h, \tau + 2h)$, we have

$$\dot{x}(t) = Ax(t) + BK(t-\tau)x(t-h) = Ax(t) + BK\,(t-\tau)\,\mathrm{e}^{A(t-h-\tau)}x(\tau),$$

from which we obtain

$$\begin{aligned} x(\tau + 2h) =& \mathrm{e}^{Ah}x\,(\tau + h) + \int_{\tau+h}^{\tau+2h} \mathrm{e}^{A(\tau+2h-s)}BK\,(s-\tau)\,\mathrm{e}^{A(s-h-\tau)}\mathrm{d}s x\,(\tau) && (8) \\ =& \left(\mathrm{e}^{2Ah} + \int_{h}^{2h} \mathrm{e}^{A(2h-s)}BK\,(s)\,\mathrm{e}^{A(s-h)}\mathrm{d}s\right) x\,(\tau) && (9) \\ \triangleq& \mathcal{M}\,(h)\,x\,(\tau)\,, && (10) \end{aligned}$$

where $\mathcal{M}\,(h)$ is the monodromy matrix. We can express

$$\mathcal{M}\,(h) = \mathrm{e}^{2Ah} + \int_{h}^{2h} \mathrm{e}^{A(2h-s)}BK\,(s)\,\mathrm{e}^{A(s-h)}\mathrm{d}s$$

$$
\begin{aligned}
=&\mathrm{e}^{2Ah}+\sum_{i=0}^{n-1}\int_{h+\frac{i}{n}h}^{h+\frac{i+1}{n}h}\mathrm{e}^{A(2h-s)}BK\left(s\right)\mathrm{e}^{A(s-h)}\mathrm{d}s\\
=&\mathrm{e}^{2Ah}+\sum_{i=0}^{n-1}\int_{0}^{\frac{h}{n}}\mathrm{e}^{A\left(h-\left(\sigma+\frac{i}{n}h\right)\right)}BK\left(\sigma+h+\frac{i}{n}h\right)\mathrm{e}^{A\left(\sigma+\frac{i}{n}h\right)}\mathrm{d}\sigma\\
=&\mathrm{e}^{2Ah}+\sum_{i=0}^{n-1}\int_{0}^{\frac{h}{n}}\mathrm{e}^{A\left(h-\frac{i}{n}h-\sigma\right)}BK_i\mathrm{e}^{A\left(\sigma+\frac{i}{n}h\right)}\mathrm{d}\sigma\\
=&\mathrm{e}^{2Ah}+\int_{0}^{\frac{h}{n}}\mathrm{e}^{-A\sigma}\sum_{i=0}^{n-1}\left(\mathrm{e}^{A\left(h-\frac{i}{n}h\right)}BK_i\mathrm{e}^{A\frac{i}{n}h}\right)\mathrm{e}^{A\sigma}\mathrm{d}\sigma,
\end{aligned}
\tag{11}
$$

where we set $s-\left(h+\frac{i}{n}h\right)=\sigma$, and used (5), implying that

$$
K\left(\sigma+h+\frac{i}{n}h\right)=K_i,\ \forall\sigma\in\left[0,\frac{h}{n}\right),\ i\in\{0,\ldots,n-1\}.
$$

We are now ready to state the main results.

Theorem 1 *Let $h\geq\tau$ be a constant such that the pair $(\mathrm{e}^{A\frac{h}{n}},B)$ is controllable, and let Q be given by*

$$
Q=\left[\begin{array}{cccc} B & \mathrm{e}^{\frac{1}{n}Ah}B & \mathrm{e}^{\frac{2}{n}Ah}B & \cdots \end{array}\quad \mathrm{e}^{\frac{n-1}{n}Ah}B\right]. \tag{12}
$$

Then the controller given by (5)-(6) and

$$
K_i=F_i\mathrm{e}^{-A\frac{i}{n}h},\ i\in\{0,\ldots,n-1\}, \tag{13}
$$

where

$$
\left[\begin{array}{cccc} F_{n-1}^{\mathrm{T}} & F_{n-2}^{\mathrm{T}} & \cdots & F_0^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}=Q^{\mathrm{T}}\left(QQ^{\mathrm{T}}\right)^{-1}\mathrm{e}^{-\frac{1}{n}Ah}\,\Pi \tag{14}
$$

and

$$
\Pi=-\frac{n}{h}\,\mathrm{e}^{2Ah}, \tag{15}
$$

solves Problem 1 with settling time $T=\tau+2h$.

Proof. Using (13)-(14) and taking into account the controllability assumption, which induces that Q is of full rank, we obtain

$$
\sum_{i=0}^{n-1}\left(\mathrm{e}^{A\left(h-\frac{i}{n}h\right)}BK_i\mathrm{e}^{A\frac{i}{n}h}\right)=\sum_{i=0}^{n-1}\mathrm{e}^{A\left(\frac{n-i}{n}h\right)}BF_i=\sum_{j=1}^{n}\mathrm{e}^{A\frac{j}{n}h}BF_{n-j}=\mathrm{e}^{A\frac{h}{n}}Q\left[\begin{array}{cccc} F_{n-1}^{\mathrm{T}} & F_{n-2}^{\mathrm{T}} & \cdots & F_0^{\mathrm{T}} \end{array}\right]=\Pi.
$$

Substituting this result in (11) leads us to

$$
\mathcal{M}(h)=\mathrm{e}^{2Ah}+\int_{0}^{\frac{h}{n}}\mathrm{e}^{-A\sigma}\Pi\,\mathrm{e}^{A\sigma}\mathrm{d}\sigma. \tag{16}
$$

If we further employ (15), we obtain

$$
\mathcal{M}\left(h\right)=\mathrm{e}^{2Ah}-\frac{n}{h}\mathrm{e}^{2Ah}\int_{0}^{\frac{h}{n}}\mathrm{e}^{-A\sigma}\mathrm{e}^{A\sigma}\mathrm{d}\sigma=\mathrm{e}^{2Ah}-\mathrm{e}^{2Ah}=0
$$

and from (8) we get $x(\tau+2h)=0$. The latter equality implies that $x(t)=0$ for $t\geq\tau+2h$, inducing fixed-time stability with settling time $\tau+2h$. ■

The methodology behind the theorem relies on assigning the monodromy matrix to zero, rather than its spectrum only. In fact, by a suitable choice of matrix Π in (14), the monodromy matrix can be assigned arbitrarily. This is clarified in the following theorem.

Theorem 2 *Let $h \geq \tau$ be a constant such that the pair $(\mathrm{e}^{A\frac{h}{n}}, B)$ is controllable, let Q be given by (12), and let $\mathcal{M}_{\mathrm{des}} \in \mathbb{R}^{n\times n}$ be the desired monodromy matrix. Then matrix*

$$\mathcal{S} \triangleq \int_0^{\frac{h}{n}} \left(\mathrm{e}^{A^{\mathrm{T}}\sigma} \otimes \mathrm{e}^{-A\sigma}\right) \mathrm{d}\sigma$$

is invertible. Furthermore, the controller given by (5)-(6), where the gains are given by (13)-(14) and matrix Π is determined from

$$\mathrm{vec}(\Pi) = \mathcal{S}^{-1}\mathrm{vec}\left(\mathcal{M}_{\mathrm{des}} - \mathrm{e}^{2Ah}\right), \tag{17}$$

results in $\mathcal{M}(h) = \mathcal{M}_{\mathrm{des}}$.

Proof. We first prove the invertibility of $\mathcal{S}$. Let us decompose A in Jordan canonical form,

$$A = V^{-1}JV, \quad J = \mathrm{diag}(J_1, \ldots, J_p),$$

where $J_i \in \mathbb{R}^{n_i\times n_i}, i \in \{1, \ldots, p\}$, are block matrices of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{n_i\times n_i},$$

with $\sum_{i=1}^{p} n_i = n$. Then matrix $\mathcal{S}$ is invertible if and only if matrix

$$M \triangleq \int_0^{\frac{h}{n}} \left(\mathrm{e}^{J^{\mathrm{T}}\sigma} \otimes \mathrm{e}^{-J\sigma}\right) \mathrm{d}\sigma$$

is invertible. In view of the special structure of matrix J, the invertibility of matrix M is equivalent to the joint invertibility of each diagonal block

$$\int_0^{\frac{h}{n}} \mathrm{e}^{\lambda_i\sigma}\mathrm{e}^{-J\sigma}\mathrm{d}\sigma, \ i \in \{1, \ldots, p\},$$

which is an upper triangular matrix and is invertible if and only if the diagonal elements are nonzero. The latter take the following form,

$$\int_0^{\frac{h}{n}} \mathrm{e}^{(\lambda_i-\lambda_j)\sigma}\mathrm{d}\sigma = \begin{cases} \frac{h}{n}, & \lambda_i = \lambda_j, \\ \frac{\mathrm{e}^{(\lambda_i-\lambda_j)\frac{h}{n}}-1}{\lambda_i-\lambda_j}, & \text{otherwise.} \end{cases}$$

We can now invoke the controllability assumption on $(\mathrm{e}^{A\frac{h}{n}}, B)$ and Lemma 1.

Next, we address the assignment of the monodromy matrix. We observe that equation

$$\mathcal{S}\ \mathrm{vec}(\Pi) = \mathrm{vec}\left(\mathcal{M}_{\mathrm{des}} - \mathrm{e}^{2Ah}\right)$$

is equivalent to

$$\int_0^{\frac{h}{n}} \mathrm{e}^{-A\sigma}\Pi\ \mathrm{e}^{A\sigma}\mathrm{d}\sigma = \mathcal{M}_{\mathrm{des}} - \mathrm{e}^{2Ah}, \tag{18}$$

which is implied by the following property of the Kronecker product: $\mathrm{vec}\,(UXV) = (V^T \otimes U)\ \mathrm{vec}(X)$ for real valued matrices U, V, X of compatible dimensions. Recall from the proof of Theorem 1 that expressions (13)-(14) imply (16). By combining (16) with (18), the proof is completed. ■

Remark 1 *The introduction of parameter h in Theorems 1-2 provides an additional degree of freedom in the design. On the one hand, it can be used as a tuning factor that allows to determine the settling time and shape the response. On the other hand, it can be used for regularization purposes in case $(\mathrm{e}^{A\frac{h}{n}}, B)$ is not controllable or the corresponding controllability matrix is ill conditioned.*

Remark 2 *Inherently to the adopted AAW framework, the closed-loop system is running in an open-loop fashion during the waiting periods, since the feedback gain equals* 0 *for* $t \in [2kh, (2k+1)h)$. *In order to address this issue, in [25] a predictor-based AAW control method is proposed, as the compensation of the delay allows to add instantaneous feedback. More precisely the controller in [25] takes the form*

$$u(t) = K_{\mathrm{c}}\hat{x}(t) + K(t-\tau)\hat{x}(t-h),$$

where $\hat{x}(t)$ *denotes the prediction of* $x(t)$ *over a horizon of length* τ, $K(t)$ *is the AAW feedback gain, and* K_{c} *is the instantaneous gain such that* $A + BK_{\mathrm{c}}$ *is Hurwitz. At the price of a more complex control law, this scheme enables the closed-loop system to maintain an exponentially convergent trajectory during the waiting periods, and it also removes restriction on the period,* $h \geq \tau$, *under the premise of no saturation of the actuator. We point out that the design approaches inferred from Theorems 1-2 can be naturally extended towards such a predictor based framework.*

Particularly for large n, the control input signal (6) is relatively complicated, since between the state-jumps, the input signal is still a time-varying continuous-time signal. This may suggest to ease the implementation by keeping the input signal constant after every jump, which brings us to a sampled-data setting. More specifically, we now consider the periodic state feedback controller

$$u(t) = \begin{cases} 0, & t \in [2kh, (2k+1)h), \\ L_0 x((2k+1)h - (h-\tau)), & t \in [(2k+1)h, (2k+1)h + \frac{1}{n}h), \\ L_1 x((2k+1)h + \frac{1}{n}h - (h-\tau)), & t \in [(2k+1)h + \frac{1}{n}h, (2k+1)h + \frac{2}{n}h), \\ \vdots & \\ L_{n-1} x((2k+1)h + \frac{n-1}{n}h - (h-\tau)), & t \in [(2k+1)h + \frac{n-1}{n}h, 2(k+1)h), \end{cases} \tag{19}$$

where $k \in \mathbb{N}$, $t \geq 0$, $h \geq \tau$ is a given constant and $L_i \in \mathbb{R}^{m \times n}, i \in \{0, \ldots, n-1\}$, are matrices to be determined. Denoting by $\mathcal{M}_D$ the monodromy matrix corresponding to (1) and (19), we can state the following counter part of Theorems 1-2, whose proof can be found in Appendix A2.

Proposition 1 *Let* $h \geq \tau$ *be a constant such that the pair* $(\mathrm{e}^{A\frac{h}{n}}, B)$ *is controllable, and let* Q_D *be of the form (12) with* B *being replaced with* $\mathcal{B} \triangleq \int_0^{\frac{h}{n}} \mathrm{e}^{-As} B \, \mathrm{d}s$. *Let* $\mathcal{M}_{\mathrm{des}} \in \mathbb{R}^{n \times n}$ *be the desired monodromy matrix. Then matrix* Q_D *is of full rank. Furthermore, the controller given by (19) and*

$$L_i = H_i \mathrm{e}^{-A\frac{i}{n}h}, \ i \in \{0, \ldots, n-1\},$$

where

$$\begin{bmatrix} H_{n-1}^{\mathrm{T}} & H_{n-2}^{\mathrm{T}} & \cdots & H_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = Q_D^{\mathrm{T}} \left(Q_D Q_D^{\mathrm{T}}\right)^{-1} \mathrm{e}^{-\frac{1}{n}Ah} \, \Pi_D \tag{20}$$

and

$$\Pi_D = \mathcal{M}_{\mathrm{des}} - \mathrm{e}^{2Ah}$$

results in $\mathcal{M}_D = \mathcal{M}_{\mathrm{des}}$. *Specifically, with the choice* $\Pi_D = -\mathrm{e}^{2Ah}$, *Problem 1 is solved with settling time* $T = \tau + 2h$.

Using piecewise constant inputs, the closed-loop system at sampling instants can be described by a discrete-time system, for which AAW strategies have also been developed in [10]. In this reference, it is shown that after the waiting period, acting during one sample interval is already sufficient to arbitrarily assign the spectrum of the monodromy matrix. Assigning all eigenvalues to zero will then result in a nilpotent monodromy matrix and the settling time will be τ plus α periods, with α the index of nilpotency, equal to n in the generic case. In contrast to this approach, Proposition 1 is based on assigning the full monodromy matrix, similarly to Theorems 1-2, and rendering this matrix zero results in a settling time of τ plus one period. This is at the price of multiple switches in the acting phase, showing the complementarity of both approaches.

2.2 Reducing the Number of Switches

In the methods underlying Theorems 1-2, the total number of switches of control signals is nm times in each control period, which is related to the composition of full row rank matrix Q. In fact, in addressing multi-input systems, we can further reduce the number of switches in each period. From an application perspective, this

might be beneficial in reducing the energy consumption caused by switching input signals. Now, considering that $(\mathrm{e}^{A\frac{h}{n}}, B) \in (\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m})$ is controllable, there exist a positive integer μ and a matrix

$$Q_\mu = \left[\begin{array}{cccccc} B & \mathrm{e}^{\frac{1}{n}Ah}B & \mathrm{e}^{\frac{2}{n}Ah}B & \cdots & \mathrm{e}^{\frac{\mu-1}{n}Ah}B \end{array} \right],$$

such that $\mathrm{rank}(Q_\mu) = n$ and $\mathrm{rank}(Q_{\mu-1}) < n$, where μ is the controllability index of system $(\mathrm{e}^{A\frac{h}{n}}, B)$. We partition matrix B as

$$B = \left[\begin{array}{cccc} b_1 & b_2 & \cdots & b_m \end{array} \right] \in \mathbb{R}^{n\times m}.$$

Without loss of generality, we assume that $\mathrm{rank}(B) = m$. Notice that matrix Q_μ can be rewritten as

$$Q_\mu = \left[\begin{array}{cccc|cccc|c|cccc} b_1 & \cdots & b_m & \mathrm{e}^{\frac{1}{n}Ah}b_1 & \cdots & \mathrm{e}^{\frac{1}{n}Ah}b_m & \cdots & \mathrm{e}^{\frac{\mu-1}{n}Ah}b_1 & \cdots & \mathrm{e}^{\frac{\mu-1}{n}Ah}b_m \end{array} \right].$$

We can then search for n linearly independent columns in Q_μ from left to right, and rearrange these selected columns as a square matrix of full rank,

$$Q_n = \left[\begin{array}{ccccc|ccccc|c|ccccc} b_1 & \mathrm{e}^{\frac{1}{n}Ah}b_1 & \cdots & \mathrm{e}^{\frac{\mu_1-1}{n}Ah}b_1 & & b_2 & \mathrm{e}^{\frac{1}{n}Ah}b_2 & \cdots & \mathrm{e}^{\frac{\mu_2-1}{n}Ah}b_2 & & \cdots & b_m & \mathrm{e}^{\frac{1}{n}Ah}b_m & \cdots & \mathrm{e}^{\frac{\mu_m-1}{n}Ah}b_m \end{array} \right], \tag{21}$$

in which $\{\mu_1, \mu_2, \ldots, \mu_m\}$ is the controllability indices set of the system $(\mathrm{e}^{A\frac{h}{n}}, B)$, with $\sum_{i=1}^m \mu_i = n$. Moreover, we let $\mu = \max\{\mu_1, \mu_2, \ldots, \mu_m\}$.

We consider linear system (1) with $u(t) \in \mathbb{R}^m$ and partition the input $u(t)$ as

$$u(t) = \left[\begin{array}{cccc} u_1(t) & u_2(t) & \cdots & u_m(t) \end{array} \right]^{\mathrm{T}}.$$

Then system (1) can be rewritten as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} b_i u_i(t-\tau). \tag{22}$$

We define a set of $2h$-periodic gain functions satisfying in the interval $[0,\ 2h)$,

$$k_i(t) = \begin{cases} 0, & t \in [0, h + \frac{n-\mu_i}{n}h), \\ k_{i,0}, & t \in [h + \frac{n-\mu_i}{n}h, h + \frac{n-\mu_i+1}{n}h), \\ & \vdots \\ k_{i,\mu_i-1}, & t \in [h + \frac{n-1}{n}h, 2h), \end{cases} \tag{23}$$

where $t \geq 0$, $h \geq \tau$ is a given constant such that the pair $(\mathrm{e}^{A\frac{h}{n}}, B)$ is controllable, and $k_{i,j} \in \mathbb{R}^{1\times n}$, $j \in \{0, \ldots, \mu_{i-1}\}$ $i \in \{1, \ldots, m\}$, are row vectors to be determined. It should be pointed out that the controller gain $k_i(t)$ is only switched μ_i times in each control period. Our goal is to design a set of periodic state feedback controllers

$$u_i(t) = k_i(t)x(t-(h-\tau)),\ t \in [0, \infty),\ i \in \{1, \ldots, m\}, \tag{24}$$

which are well-defined for all $t \geq 0$.

Similarly to the analysis in Section 2.1, the state $x(\tau + 2h)$ of the closed-loop system (22)-(24) can be expressed as

$$x(\tau+2h) = \left(\mathrm{e}^{2Ah} + \sum_{i=1}^{m} \int_h^{2h} \mathrm{e}^{A(2h-s)} b_i k_i(s)\, \mathrm{e}^{A(s-h)} \mathrm{d}s \right) x(\tau) \triangleq \tilde{\mathcal{M}}(h)\, x(\tau),$$

where $\tilde{\mathcal{M}}(h)$ is the monodromy matrix, which induces $x(\tau + (k+1)2h) = \tilde{\mathcal{M}}(h)x(\tau + k2h)$ for $k \geq 1$. We can now improve Theorems 1-2 in the following way.

Theorem 3 *Assume that* $\mathrm{rank}(B) = m > 1$. *Let* $h \geq \tau$ *be a constant such that the pair* $(\mathrm{e}^{A\frac{h}{n}}, B)$ *is controllable and let* Q_n *be given by (21). Then the controller given by (23)-(24) with*

$$k_{i,j} = f_{i,j}\mathrm{e}^{-A\frac{n-\mu_i+j}{n}h}, \quad j \in \{0,\ldots,\mu_{i-1}\}, \quad i \in \{1,\ldots,m\}, \tag{25}$$

where

$$\left[\begin{array}{ccc|c|ccc} f_{1,\mu_1-1}^{\mathrm{T}} & \cdots & f_{1,0}^{\mathrm{T}} & \cdots & f_{m,\mu_m-1}^{\mathrm{T}} & \cdots & f_{m,0}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}} = Q_n^{-1}\mathrm{e}^{-\frac{1}{n}Ah}\,\Pi, \tag{26}$$

solves Problem 1 with settling time $T = \tau + 2h$ *if matrix* Π *is chosen as in (15), while it assures* $\tilde{\mathcal{M}}(h) = \mathcal{M}_{\mathrm{des}}$ *if* Π *is selected according to (17). Futhermore, the total number of switches of the control inputs during the acting phase is given by* $\sum_{i=1}^{m}\mu_i = n$.

Proof. Using (25)-(26), we can expresss

$$\begin{aligned}
\tilde{\mathcal{M}}(h) =& \mathrm{e}^{2Ah} + \sum_{i=1}^{m}\sum_{j=0}^{n-1}\int_{h+\frac{j}{n}h}^{h+\frac{j+1}{n}h}\mathrm{e}^{A(2h-s)}b_i k_i(s)\,\mathrm{e}^{A(s-h)}\mathrm{d}s \\
=& \mathrm{e}^{2Ah} + \sum_{i=1}^{m}\sum_{j=0}^{n-1}\int_{0}^{\frac{h}{n}}\mathrm{e}^{A\left(h-\left(\sigma+\frac{j}{n}h\right)\right)}b_i k_i(\sigma+h+\frac{j}{n}h)\mathrm{e}^{A\left(\sigma+\frac{j}{n}h\right)}\mathrm{d}\sigma \\
=& \mathrm{e}^{2Ah} + \sum_{i=1}^{m}\sum_{j=n-\mu_i}^{n-1}\int_{0}^{\frac{h}{n}}\mathrm{e}^{A\left(h-\frac{j}{n}h-\sigma\right)}b_i k_{i,j-n+\mu_i}\mathrm{e}^{A\left(\sigma+\frac{j}{n}h\right)}\mathrm{d}\sigma \\
=& \mathrm{e}^{2Ah} + \int_{0}^{\frac{h}{n}}\mathrm{e}^{-A\sigma}\sum_{i=1}^{m}\sum_{j=n-\mu_i}^{n-1}\left(\mathrm{e}^{A\left(h-\frac{j}{n}h\right)}b_i k_{i,j-n+\mu_i}\mathrm{e}^{A\frac{j}{n}h}\right)\mathrm{e}^{A\sigma}\mathrm{d}\sigma \\
=& \mathrm{e}^{2Ah} + \int_{0}^{\frac{h}{n}}\mathrm{e}^{-A\sigma}\mathrm{e}^{\frac{1}{n}Ah}Q_n\left[\begin{array}{ccc|c|ccc} f_{1,\mu_1-1}^{\mathrm{T}} & \cdots & f_{1,0}^{\mathrm{T}} & \cdots & f_{m,\mu_m-1}^{\mathrm{T}} & \cdots & f_{m,0}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}\mathrm{e}^{A\sigma}\mathrm{d}\sigma \\
=& \mathrm{e}^{2Ah} + \int_{0}^{\frac{h}{n}}\mathrm{e}^{-A\sigma}\Pi\,\mathrm{e}^{A\sigma}\mathrm{d}\sigma,
\end{aligned}$$

which takes exactly the same form as (16). From this point on, the proofs of Theorems 1-2 can be followed. ■

Remark 3 *The total number of times per period where a control signal is switched is equal to* n, *which is significantly smaller than the* mn *times required by the controllers form Theorems 1-2. However, the price to pay for reducing the number of switching times is an increase in the amplitude of the control signal, induced by a shortening of the acting time in each control input channel, as apparent from (23). This phenomenon will occur in the numerical simulations.*

Remark 4 *The number of gain vectors in control law (24) is given by* $\sum_{i=1}^{m}\mu_i = n$, *and the number of scalar gain parameters becomes* n^2. *At the same time, the assigned monodromy matrix has dimension* $n \times n$. *This observations gives a strong indication that the number of switching times using the adopted approach is minimal.*

Finally, similarly as done in Proposition 1 for Theorems 1-2, the results of Theorem 3 can be adapted towards a sampled-data setting, where the input signals are kept piece-wise constant. This extension is however omitted for conciseness of the presentation.

3 Two Illustrative Examples

We consider two numerical examples where we assign the monodromy matrix to zero and, hence, provide a constructive solution to Problem 1.

3.1 A Scalar Linear System

This example is borrowed from [25]. We consider linear system

$$\dot{x}(t) = Ax(t) + Bu(t-\tau)$$

with $n = m = 1$, $B = 1$ and distinguish between two cases.

First, we consider the case where $A > 0$. It is shown in [25] that the system is exponentially stabilizable with instantaneous feedback

$$u(t) = Kx(t) \tag{27}$$

if and only if $\tau < 1/A$. On the contrary, according to Theorem 1 the system can always be fixed-time stabilized by the picecwise-constant periodic feedback (6) with settling time $T = \tau + 2h$, with $h \geq \tau$. As a comparative experiment, we use the same data as in [25], where $A = 1$, $\tau = 2 > 1/A$ and $h = \tau$. Then

$$u(t) = K(t)x(t),\ t \in [0, \infty), \tag{28}$$

where $K(t)$ is designed as (5), specifically,

$$K(t) = \begin{cases} 0, & t \in [4k, 4k+2), \\ -\frac{1}{2}\mathrm{e}^2, & t \in [4k+2, 4(k+1)), \end{cases} \quad k \in \mathbb{N}.$$

For the simulation, we use the same initial conditions as in [25], where $x(0) = 10$ and $u(\theta) = 0$, $\forall \theta \in [-\tau, 0)$. We conducted simulation experiments according to Theorem 1 and the method in [25], the latter relying on smooth periodic feedback where

$$u(t) = -K_{(A,h)}(t)x(t),\ t \geq 0$$

with $K_{(A,h)}(t) = 0, t \in [4k, 4k+2]$ and

$$K_{(A,h)}(t) = \frac{A\mathrm{e}^{A(5h-2t)}\left(\mathrm{e}^{A(t-h)} - 1\right)^2\left(\mathrm{e}^{A(t-2h)} - 1\right)^2}{Ah - 3\mathrm{e}^{2Ah} + 4Ah\mathrm{e}^{Ah} + Ah\mathrm{e}^{2Ah} + 3}, t \in [4k+2, 4(k+1)], k \in \mathbb{N}.$$

The results are shown in Figure 1, from which we clearly see that the method in Theorem 1 can also drive the state to zero at $t = 3\tau = 6$s by a simpler controller, however, the states and inputs under these two controllers suffer large overshoots, which is reasonable since both closed-loop systems actually operate in an open-loop manner in the first 4s, inherently to the act-and-wait strategy. To address this issue, the results of the present paper can be trivially combined with a predictor, as mentioned in Remark 2.

Next, we assume $A = 0$. We also choose $\tau = 2$, $h = \tau$, but the initial conditions $x(0) = 10$ and $u(\theta) = 10$, $\forall \theta \in [-\tau, 0)$, which are in accordance with these in [25]. In this case, the picecwise-constant periodic feedback takes the same form of (28) with

$$K(t) = \begin{cases} 0, & t \in [4k, 4k+2), \\ -\frac{1}{2}, & t \in [4k+2, 4(k+1)), \end{cases} \quad k \in \mathbb{N},$$

and the smooth periodic feedback gain is given by $K_{(A,h)}(t) = 0, t \in [4k, 4k+2]$ and

$$K(t) = -\frac{6\mathrm{e}^3 \ln\left(1 - \mathrm{e}^{h-3}\right)\left(\mathrm{e}^{t-h} - 1\right)^2\left(\mathrm{e}^{t-2h} - 1\right)^2}{6h + \sinh(2h) - 8\sinh(h)}, t \in [4k+2, 4(k+1)], k \in \mathbb{N}.$$

The simulation results corresponding to these two controller are shown in Figure 2, from which we observe that both controllers lead to fixed-time stability with settling time 6s. However, due to the differences in the controller design, the amplitude of the control input under Theorem 1 is significantly lower than in the other case. Here we point out that in this case the integral from 2 till 4 of the input is the same for both controllers, which explains why the smooth control law exhibits a larger overshoot.

3.2 The Spacecraft Rendezvous Control System

Consider the spacecraft rendezvous control system [3],

$$\begin{cases} \ddot{x}_1 = 2\omega_0\dot{x}_2 + \omega_0^2 J_1 - J_2\xi J_1 + u_1(t-\tau), \\ \ddot{x}_2 = -2\omega_0\dot{x}_1 + \omega_0^2 x_2 - J_2\xi x_2 + u_2(t-\tau), \\ \ddot{x}_3 = -J_2\xi x_2 + u_3(t-\tau), \end{cases} \tag{29}$$

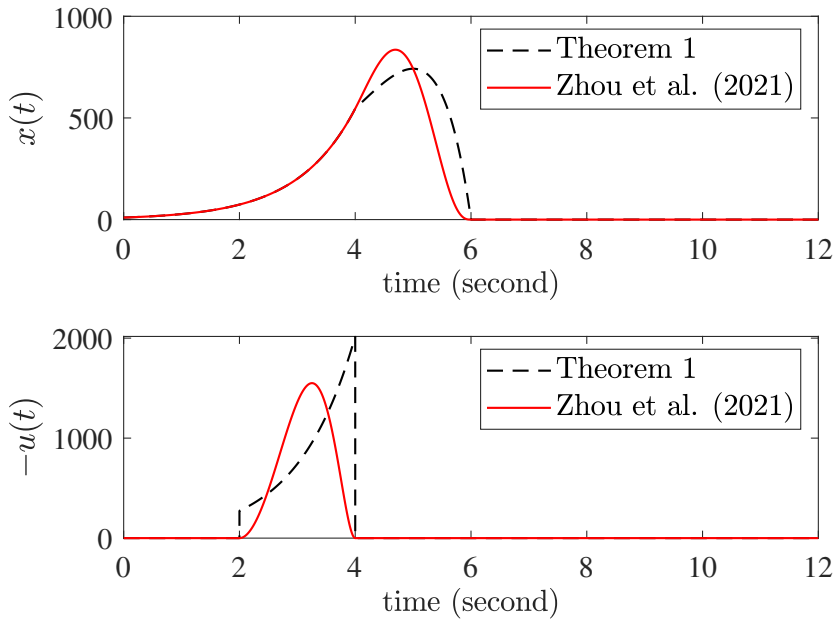


Figure 1: The state and control signal for system (1) under different controllers with $A = 1$, $B = 1$, and $\tau = 2$.

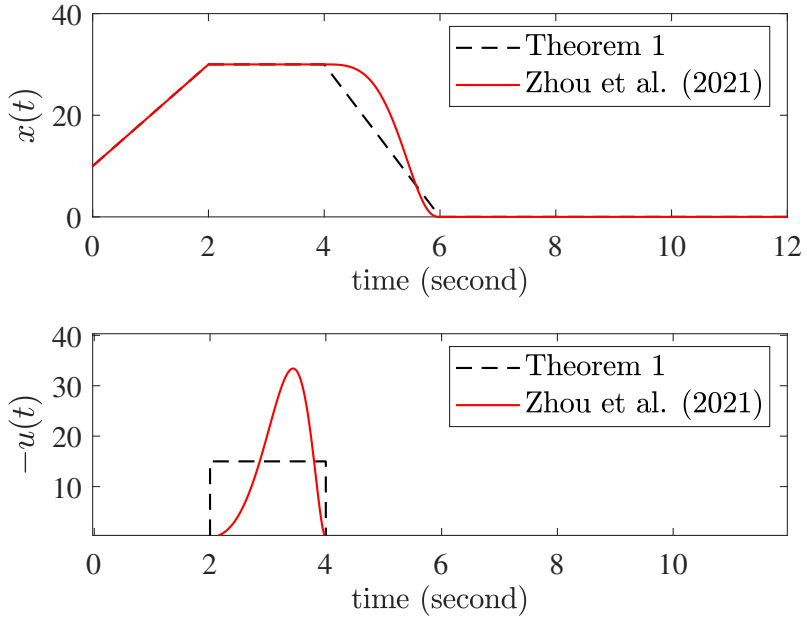


Figure 2: The state and control signal for system (1) under different controllers with $A = 0$, $B = 1$, and $\tau = 2$.

where (x_1, x_2, x_3) is the relative distance between the target and the chaser spacecraft, ξ is the gravitational parameter, J_0 is the radius of the target spacecraft orbit, ω_0 is the orbit rate of the target orbit, $u = [u_1, u_2, u_3]^{\mathrm{T}}$ is the normalized acceleration vector due to thrust forces on the chaser, and τ is the input delay. The linearized equations takes the form of (1) with $(n, m) = (6, 3)$, $x = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3]^{\mathrm{T}}$ and

$$A = \begin{bmatrix} 0 & I_3 \\ \omega_0^2 A_{21} & 2\omega_0 A_{22} \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_3 \end{bmatrix},$$

with

$$A_{21} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The fixed-time stabilization problem for the spacecraft rendezvous system without input delay has been investigated in [27, 28], in which the authors proposed time-varying high-gain feedback schemes, grounded in solving parametric Lyapunov equations.

For the simulations, we use the same system and initial data as in [27, 28]. Let the target spacecraft be on a geosynchronous orbit with orbit rate $\omega_0 = 7.2722 \times 10^{-5}$rad/s, $x(0) = [-1000, 1000, 1000, 2, -2, 2]^{\mathrm{T}}$, $u(\theta) = 0, \theta \in [-\tau, 0)$, and $\tau = 20$s. We first conduct a simulation on system (1) under the action of controller (6) with $h = 200$s. According to Theorem 1, the state of the closed-loop system converges to zero at $T = 420$s, which is visualized in Figure 3. The state of the closed-loop system also exhibits a relatively large overshoot, due to the open-loop phase during the first 220s. We also conducted an additional experiment, in which the system operates under the sampled-data type control law (19). The corresponding state and input trajectories, visualized in Figure 4, are qualitatively similar.

As mentioned in Remark 1, we can choose different h to make the trade-off between setting time and the shape of the system. To show this, we choose $h = \tau = 20$s, and keep the other parameter unchanged. According to Theorem 1, the state of the closed-loop system converges to zero at $T = 60$s, which is visualized in Figure 5. However, in this case, the state variables fluctuate significantly, and the required input signal is much larger, which is not beneficial in practical engineering.

In addition, we conducted a simulation of system (1) under controller (24). It can be easily verified that the controllability index of the system is 2, and the associated controllability index set is $\{\mu_1 = 2,\ \mu_2 = 2,\ \mu_3 = 2\}$. We assume $h = 200$s. Then it can be noticed in Figure 6 that the state of the closed-loop system consisting of (22) and (24) converges to zero at $T = 420$s and the total number of switching of input signals has been reduced to $n = 6$, in accordance with Theorem 3. However, the amplitude of the control input exhibits a large fluctuation, and the system state also experiences a larger overshoot, which might be undesirable in engineering applications.

4 Concluding Remarks

The inclusion of multiple switches during the acting phase led to novel AAW control schemes for systems with input delay, exhibiting a linear relation between the controller gain parameters and the monodromy matrix. This property enabled an arbitrary assignment of the monodromy matrix and a constructive solution to the fixed time-stabilization problems. For the latter problem, the proposed approach can be easily extended to construct predictor-based AAW schemes, having more flexibility in shaping the response of the system.

Since the methods proposed in this paper require an accurate model, it is meaningful to study extensions towards linear time-delay systems with unknown time-delays and/or parametric uncertainties. As the adopted approach is spectrum based, it is natural to address stability robustness by computing the pseudospectral radius for an a priori defined bounded set of perturbations [2]. This is however a very challenging problem. First, the dependence of the monodromy matrix on matrix A is nonlinear. Second, this operator may undergo major structural changes under delay perturbations. To see the latter, consider the case where $h = \tau$. If the actual system delay is not equal to delay τ in the model but to $\tau + \delta\tau$, then for $\delta\tau > 0$ the monodromy operator in general becomes infinite-dimensional. For $\delta\tau < 0$, on the contrary, it remains finite-dimensional but its structure changes, in particular when $k(\tau+\delta\tau) = \tau$ with $k \geq 1$, as for the corresponding delay perturbations the number of actual delay intervals per period in which the system operates in closed-loop mode changes. For the first problem, the *linearization* approach of [2] might be a promising step towards a solution. To handle also delay robustness, this approach could possibly be combined with a variational formulation of a directional derivative of an eigenvalue with respect to the delay, associated with the so-called characteristic matrix (see [16] for such a variational formulation in the context of stability optimization).

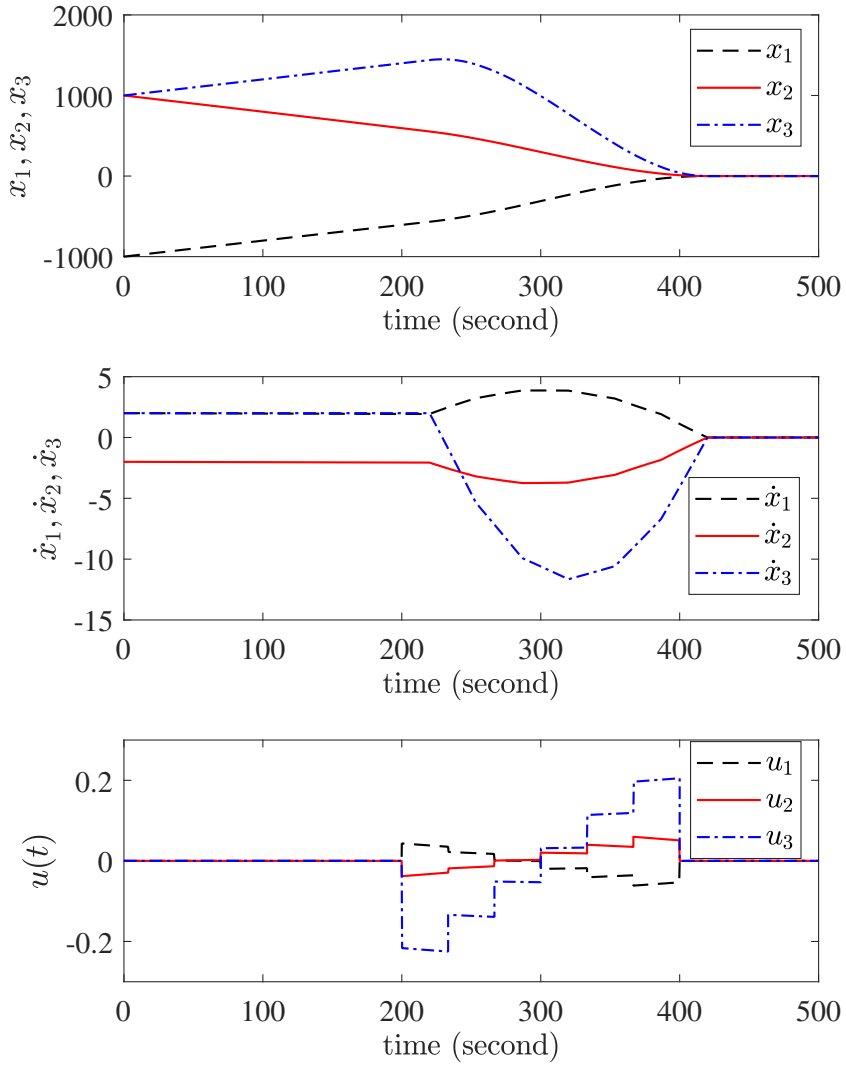


Figure 3: The state and control signals for the spacecraft rendezvous system under controller (6) with $h = 200$s.

Appendix

A1: Proof of Lemma 1

Under condition (4), there is a one-to-one mapping between the spectra of A and the matrix exponential. Left and right eigenvectors (and the whole Jordan canonical systems) are preserved. The assertion can be rephrased as

$$\operatorname{rank}\left[\begin{array}{cc} zI - \mathrm{e}^{A\frac{h}{n}} & B \end{array}\right] = n, \ \forall z \in \mathbb{C}$$

and we prove it by contradiction. If it is not satisfied, there exist $\hat{z} \in \mathbb{C}$ and $u \in \mathbb{C}^n$ such that

$$u^{\mathrm{H}}\left[\begin{array}{cc} \hat{z}I - \mathrm{e}^{A\frac{h}{n}} & B \end{array}\right] = 0,$$

hence, u is a left eigenvector of $\mathrm{e}^{A\frac{h}{n}}$ corresponding to eigenvalue $\hat{z}$ that satisfied $u^{\mathrm{H}}B$=0. By the one-to-one correspondence of eigenvalues and eigenvectors, $\hat{z}$ can be written in the form $\mathrm{e}^{\hat{\lambda}\frac{h}{n}}$, with $\hat{\lambda}$ an eigenvalue of A,

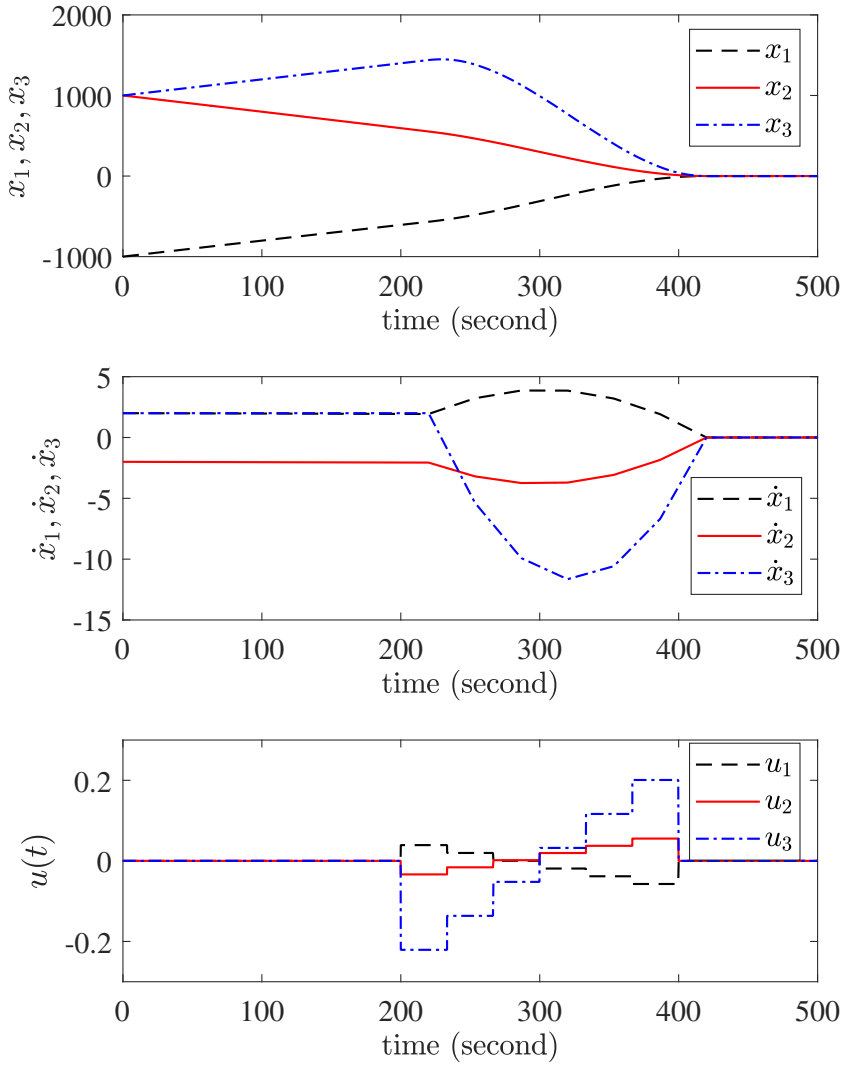


Figure 4: The state and control signals for the spacecraft rendezvous system under controller (19) with $h = 200$s.

and the corresponding left eigenvector is u. Thus we have

$$u^{\mathrm{H}} \left[\begin{array}{cc} \hat{\lambda} I - A & B \end{array} \right] = 0,$$

which violates the controllability of (A, B). Then the proof is completed.

A2: Proof of Proposition 1

In the context of sampled-data systems, it is well known that the pair $(\mathrm{e}^{A\frac{h}{n}}, \mathcal{B})$ is controllable if and only if condition (4) is satisfied. Therefore, the assumed controllability of $(\mathrm{e}^{A\frac{h}{n}}, B)$ implies the controllability of $(\mathrm{e}^{A\frac{h}{n}}, \mathcal{B})$. As a consequence, matrix Q_D is of full row rank.

Similarly to the analysis in Section 2.1, the state $x(\tau + 2h)$ of the closed-loop system consisting of (1) and (19) can be expressed as

$$x(\tau + 2h) = \mathcal{M}_D(h)\, x(\tau)$$

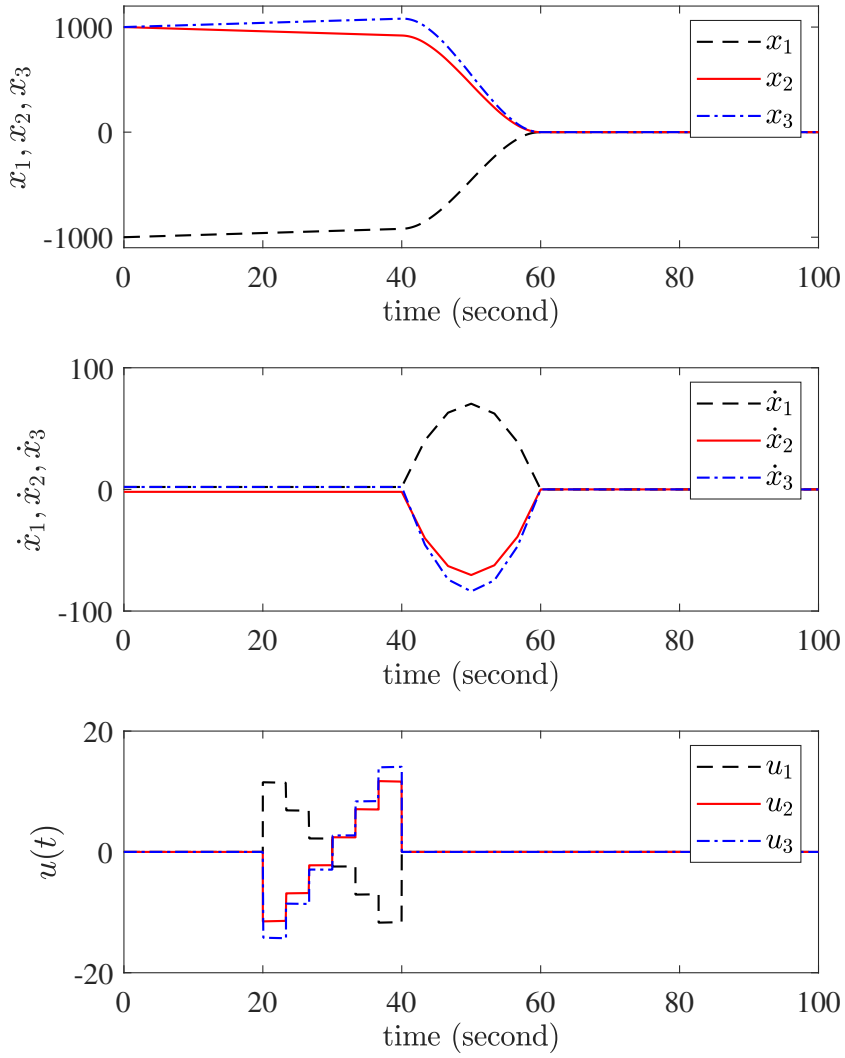


Figure 5: The state and control signals for the spacecraft rendezvous system under controller (6) with $h = 20$s.

$$
\begin{aligned}
&= \left(\mathrm{e}^{2Ah} + \left(\int_0^{\frac{h}{n}} \mathrm{e}^{-As} \mathrm{d}s \right) \sum_{i=0}^{n-1} \mathrm{e}^{A(\frac{n-i}{n}h)} B L_i \mathrm{e}^{A\frac{i}{n}h} \right) x(\tau) \\
&= \left(\mathrm{e}^{2Ah} + \mathrm{e}^{A\frac{h}{n}} \begin{bmatrix} \mathcal{B} & \mathrm{e}^{\frac{h}{n}A}\mathcal{B} & \cdots & \mathrm{e}^{\frac{n-1}{n}Ah}\mathcal{B} \end{bmatrix} \begin{bmatrix} L_{n-1}\mathrm{e}^{\frac{n-1}{n}Ah} \\ L_{n-2}\mathrm{e}^{\frac{n-2}{n}Ah} \\ \vdots \\ L_0 \end{bmatrix} \right) x(\tau) \\
&= \left(\mathrm{e}^{2Ah} + \mathrm{e}^{A\frac{h}{n}} Q_D H \right) x(\tau),
\end{aligned}
$$

where

$$
H = \begin{bmatrix} H_{n-1}^{\mathrm{T}} & H_{n-2}^{\mathrm{T}} & \cdots & H_0^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.
$$

Substituting (20) results in $\mathcal{M}_D = \mathrm{e}^{2Ah} + \Pi_D$. The assertions of the proposition directly follow.

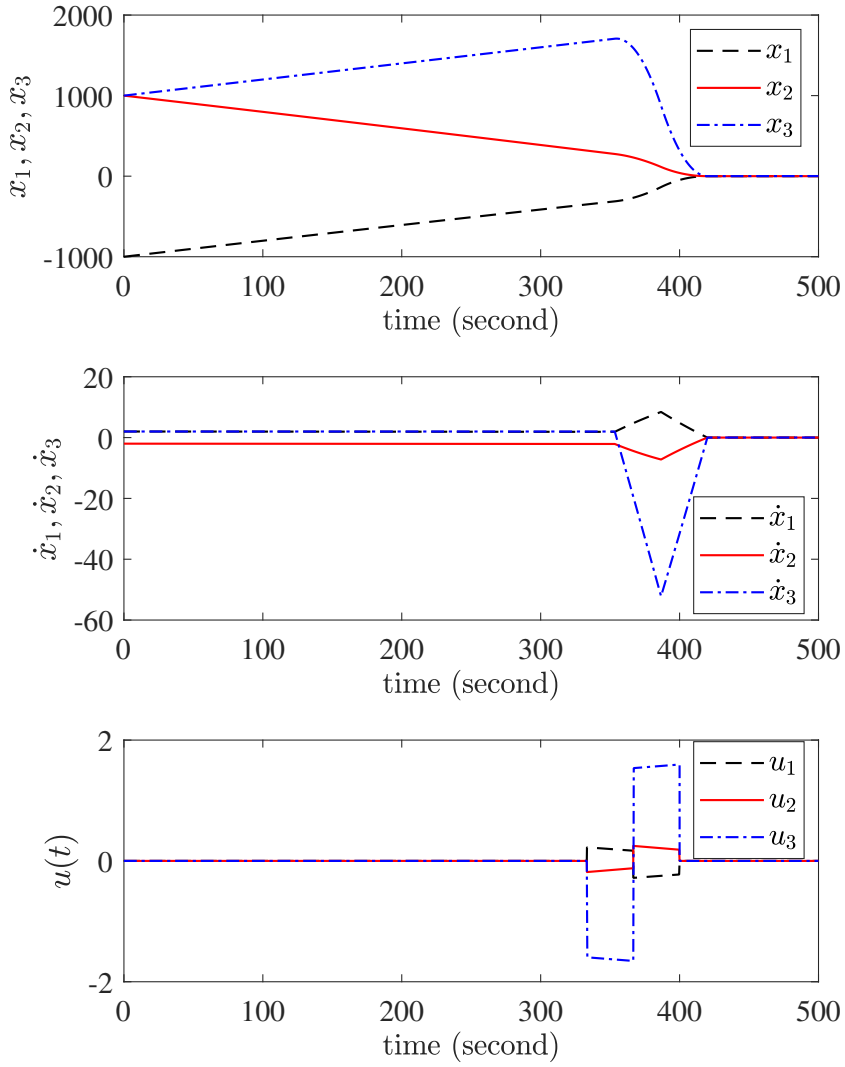


Figure 6: The state and control signals for the spacecraft rendezvous system under controller (24) with $h = 200$s.

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