Dynamic interval analysis of FE models with uncertain substructures

H. De Gersem*, D. Moens†, W. Desmet, D. Vandepitte
K.U.Leuven, Department of Mechanical Engineering,
Celestijnenlaan 300 B, B-3001, Heverlee, Belgium
e-mail: **hilde.degersem@mech.kuleuven.be**

Abstract

The exponential growth of the computational capacity has, amongst other things, enabled the design engineer to include non-deterministic properties in the numerical simulation and validation of new designs. In this context, the interval concept has been introduced for the description of incomplete data, subjective knowledge and modelling uncertainties in a non-probabilistic manner. The use of the interval concept in the finite element context has led to the development of the interval finite element method for eigenfrequency and frequency response function analysis. For industrially sized models with a large number of uncertainties, the computation time of these methods can be considerable. A reduction in calculation time can be achieved by the substructuring of large models into substructures, which are then independently processed and reduced, and afterwards recombined.
The aim of the presented work is to combine the interval finite element method for dynamic analysis with the Craig-Bampton component mode synthesis substructuring technique, in which the static and dynamic behaviour of each substructure are represented by a set of component modes. Special attention is paid to the consequences of uncertain parameters on the numerical representation of an uncertain substructure. A numerical case study is presented to illustrate the concepts used in this paper.

1 Introduction

During the last decades, the exponential growth of the computational capacity has enabled the finite element (FE) analysis of very complex and detailed numerical models of new designs and structures. However, it is often very difficult to define a reliable finite element model for numerical simulation and validation, as some numerical properties are uncertain. For instance, manufacturing tolerances inevitably lead to geometrical uncertainties, and material characteristics may be subject to variability. Furthermore, it is often very difficult to model all physical phenomena, damping, boundary conditions, joints, etc. correctly. In an early design stage, some property values even may not yet be fixed. Due to the ever growing demands imposed on new products, it is of utmost importance for design engineers to investigate the influence of all relevant uncertainties on the static and/or dynamic behaviour of structures. Hence, non-deterministic approaches are gaining interest in design and development areas.

The extension of the deterministic finite element method towards uncertainty assessment is generally obtained by the probabilistic concept [1] in which all uncertain parameters are represented by probability density functions. Probabilistic methods are especially suitable in case of model variabilities for which information on both the range and the likelihood is available. However, the use of a probabilistic approach can lead to subjective results if the available statistical data on the uncertainties are limited and approximations and subjective information need to be included [2].

*Research Assistant of the Research Foundation - Flanders (FWO-Vlaanderen)
†Postdoctoral Fellow of the Research Foundation - Flanders (FWO-Vlaanderen)

The fuzzy set theory, introduced by Zadeh [3] in 1965, provides a concept for the description of linguistic or subjective knowledge and incomplete data in a non-probabilistic manner. The use of Zadeh's concept in the finite element context has led to the development of the fuzzy finite element method [4]. Its aim is to calculate the membership function of an output quantity, based on the fuzzy description of the uncertain input parameters. By using the α-level technique, the interval finite element method forms the core of the fuzzy procedure. Hence, non-probabilistic research is concentrated on the development of efficient and accurate interval methods.

For the interval frequency response function (FRF) calculation, a hybrid procedure has been developed and implemented by Moens [5, 6]. In this approach, a global optimisation step determines the correct intervals on the modal parameters, which are then combined to calculate the FRF upper and lower bounds with an interval arithmetic procedure. Special attention is paid to the limitation of the conservatism and the computational efficiency of the method [7]. Various numerical case studies prove that the interval and fuzzy finite element method can be useful tools to perform early design validation and optimisation [8].

For industrially sized models with a large number of uncertainties, the computation time of the interval and the fuzzy finite element method can be considerable. A reduction in calculation time can be achieved by the substructuring of large models into substructures, which are then independently processed and reduced [9]. Subsequently all reduced substructures are recombined to form the reduced structural system for which the desired output quantities are calculated. However, the introduction of uncertainties in a substructuring method requires a concept for the description of uncertainty on the substructure level. In this paper, the interval finite element method is combined with the Craig-Bampton component mode synthesis substructuring technique, in which the static and dynamic behaviour of each substructure are represented by a set of component modes.

Section 2 of this paper gives a short introduction on the use of interval methods for the eigenvalue analysis of FE models under uncertainty. Section 3 then describes the basic principles and the mathematical description of the Craig-Bampton component mode synthesis technique. In section 4, the above mentioned methods are combined for the interval eigenfrequency analysis of uncertain FE models with substructures. Different approaches to handle uncertainties in substructures are proposed. Special attention is paid to the accuracy and the numerical efficiency of the proposed approaches. In section 5, interval dynamic analyses are performed on a substructured FE model of the Garteur benchmark aircraft.

2 Interval Finite Element Method

In literature, a number of general interval solutions for the interval FE problem have been described. The most popular one used to be the interval arithmetic approach, in which all basic deterministic algebraic operations are replaced by their interval arithmetic counterparts. Although this approach is computationally very inexpensive, it is of little practical use as it tends to largely overestimate the interval outcome of each operation, accumulating in a huge amount of conservatism in the interval analysis result of realistic problems. Therefore, other approaches such as the vertex method [10], the transformation method [11, 12] and the global optimisation method [4] have been formulated for the solution of the interval FE problem. Each of these techniques has its specific disadvantages when applied to realistical engineering problems, such that none of them has been established as standard interval procedure, and the choice of technique depends on the type of the conducted FE analysis.

In case of an *interval eigenfrequency analysis*, the aim is to calculate the minimum and maximum values of the eigenfrequencies of all considered modes, given the input uncertainty intervals. It can easily be shown that a vertex method, which takes all combinations of the boundaries of the input intervals into account, correctly predicts the intervals of output quantities if the problem is monotonic. However, in the application of structural dynamics, it is almost impossible to prove the property of monotonicity in a general way. Experience shows that many problems do exhibit indeed a monotonic behaviour, especially when global model parameters are uncertain. For parameters that affect only part of the structure, it is much harder to

predict the effect of a parameter change. In these cases, a vertex analysis can provide a first impression of the scatter on the eigenfrequency results, but the detection of the worst-case scenarios is not guaranteed. For those cases, it is necessary to apply a global optimisation procedure in search of the minimum and maximum eigenfrequency values. With $\langle \lambda_i \rangle_{\{\mathbf{x}\}}$ the interval on eigenvalue i due to the input uncertain parameter space $\{\mathbf{x}\}$, the global optimisation problem can be written as:

$$\langle \lambda_i \rangle_{\{\mathbf{x}\}} = \left[\min_{\{x\} \in \{\mathbf{x}\}} \left(\lambda_i \left(\{x\} \right) \right), \max_{\{x\} \in \{\mathbf{x}\}} \left(\lambda_i \left(\{x\} \right) \right) \right] \tag{1}$$

For each mode considered in the interval eigenfrequency analysis, a global minimisation and maximisation is required. As for each goal function evaluation a finite element eigenvalue analysis has to be performed, the interval method becomes very time-consuming for large numerical models. The use of a reduced model can substantially decrease the model evaluation time and hence increase the efficiency of the interval and fuzzy finite element methods drastically. Therefore, this paper aims at the introduction of a substructuring technique in the algorithm for interval eigenfrequency analysis.

3 Component Mode Synthesis

The aim of the well-known Component Mode Synthesis (CMS) method is to reduce the computational cost of large models and to enable a solution strategy in which individual components can be optimised without the need of the recalculation of the total structure. The CMS technique consists of three global steps. First the structure is divided into a residual structure and a number of components, also referred to as substructures or superelements. For each component, a set of component modes is calculated, which represent its static and dynamic behaviour. In the second step, these component modes are used to reduce the component model. In the third step, all reduced superelement models are combined with the non-reduced part of the structure to form the global reduced system, that is further used for global FE analyses. A short overview of these basic principles and steps of the component mode synthesis method is given in this section. A complete mathematical description of the CMS technique can be found in [9, 13, 14].

3.1 Calculation of component modes

For each component, all unconstrained degrees of freedom (dofs) are assigned to one of two specific sets. The set of boundary degrees of freedom (subscript t) contains all dofs that connect the considered component to the residual structure or to another component. The remaining dofs are assigned to the set of internal dofs (subscript o, also referred to as 'omitted' dofs). The stiffness and mass matrices of each component are assembled, and subsequently partitioned according to these sets [9]:

$$[K] = \begin{bmatrix} \overline{K_{tt}} & K_{to} \\ K_{ot} & K_{oo} \end{bmatrix} \quad , \quad [M] = \begin{bmatrix} \overline{M_{tt}} & M_{to} \\ M_{ot} & M_{oo} \end{bmatrix} \tag{2}$$

For each component, a set of component modes is calculated. In literature, several approaches using different types of component modes for static and dynamic reduction are described. The Craig-Bampton method [9] has been widely used and implemented in many finite element codes because the procedure for formulating the component modes is very straightforward, and because the method produces highly accurate results. The basic and most commonly applied version of the Craig-Bampton method uses constraint modes and fixed-interface normal modes to describe the static, respectively the dynamic behaviour of a component. Each constraint mode describes the static deformation of the component when a unit displacement (or rotation) is applied to one boundary degree of freedom, while the other dofs of the t-set are restrained. The static transformation matrix $[G_{ot}]$ with all constraint modes $\{\phi_i^C\}$ as columns, has the following mathematical description:

$$[G_{ot}] = \left[\{\phi_1^C\} \ \{\phi_2^C\} \dots \{\phi_t^C\} \right] = - \left[K_{oo} \right]^{-1} \left[K_{ot} \right] \tag{3}$$

The fixed-interface normal modes, which form the columns of the dynamic transformation matrix $[G_{oq}]$, are calculated from the eigenvalue analysis of the component with all boundary dofs fixed:

$$[K_{oo}]\{\phi_i\} = \lambda_i [M_{oo}]\{\phi_i\} \tag{4}$$

$$[G_{oq}] = \left[\begin{array}{cccc} \{\phi_1\} & \{\phi_2\} & \cdots & \{\phi_q\} \end{array} \right] \tag{5}$$

Each of these modes is assigned to a generalized degree of freedom (q-set). The accuracy of the dynamic reduction step is determined by the number of retained normal modes q. As the number of generalized degrees of freedom q needed for an accurate description of the dynamic behaviour of the component is usually several orders smaller than the number of internal dofs, the size of the component model is drastically reduced.

3.2 Component reduction

In the second step, the set of component modes is used for the transformation $[\Gamma]$ of the FE model from the set of physical dofs into the set of reduced dofs:

$$\left\{ \begin{array}{c} x_t \\ x_o \end{array} \right\} = [\Gamma] \left\{ \begin{array}{c} x_t \\ q \end{array} \right\} = \left[\begin{array}{cc} [I] & [0] \\ [G_{ot}] & [G_{oq}] \end{array} \right] \left\{ \begin{array}{c} x_t \\ q \end{array} \right\} \tag{6}$$

Using this transformation, the component stiffness and mass matrix are reduced to form the reduced superelement matrices [9, 14]:

$$[K]_{reduced} = [\Gamma]^T [K][\Gamma] = \left[\begin{array}{cc} K_{tt} & 0 \\ 0 & K_{qq} \end{array} \right] \quad , \quad [M]_{reduced} = [\Gamma]^T [M][\Gamma] = \left[\begin{array}{cc} M_{tt} & M_{tq} \\ M_{qt} & M_{qq} \end{array} \right] \tag{7}$$

with:

$$[K_{tt}] = \left[\overline{K_{tt}}\right] + [K_{to}][G_{ot}] \tag{8}$$

$$[K_{qq}] = [G_{oq}]^T [K_{oo}][G_{oq}] \tag{9}$$

$$[M_{tt}] = \left[\overline{M_{tt}}\right] + [M_{to}][G_{ot}] + [G_{ot}]^T [M_{ot}] + [G_{ot}]^T [M_{oo}][G_{ot}] \tag{10}$$

$$[M_{tq}] = [M_{to}][G_{oq}] + [G_{ot}]^T [M_{oo}][G_{oq}] \tag{11}$$

$$[M_{qt}] = [M_{tq}]^T \tag{12}$$

$$[M_{qq}] = [G_{oq}]^T [M_{oo}][G_{oq}] \tag{13}$$

These reduced matrices represent the properties of the superelement as seen at its interface with adjacent structures. Note that no coupling terms between the t-set and the q-set exist in the reduced stiffness matrix, due to the definition of the static transformation matrix. In case of mass normalised normal modes, the matrices $[K_{qq}]$ and $[M_{qq}]$ simplify to respectively a diagonal matrix with the eigenvalues and a unity matrix:

$$[K_{qq}] = [\Lambda_{qq}] \quad , \quad [M_{qq}] = [I_{qq}] \tag{14}$$

3.3 Assembly of the reduced structural system

In a last step, the reduced stiffness and mass matrices of all components are assembled with the non-reduced residual structure, to form the reduced stiffness and reduced mass matrix of the complete structural system. These can then be used to perform finite element analyses (e.g. an eigenfrequency or frequency response function analysis) on the global structure. Data recovery for each superelement is performed by expanding the solution at the attachment points, using the same transformation matrices that were used to perform the original reduction on the superelement.

4 Uncertainty in substructured FE models

For the calculation of the eigenvalue intervals of an uncertain structure with the global optimisation strategy, multiple eigenvalue problems have to be solved (cf. section 2). Hence, the interval finite element method for dynamic analysis becomes very time-consuming for large systems. In this paper, the IFEM is combined with the component mode synthesis method, in order to reduce the calculation time. The uncertain parameters affecting the structural behaviour of a substructured finite element model can be located either in the non-reduced part of the structure, or in one or more superelements. Sections 4.1 and 4.2 describe the combination of the interval FE analysis with the substructuring technique for these two uncertainty location cases.

4.1 Uncertainties in the residual structure

In case that all uncertain parameters are included in the residual structure only, all components can be reduced deterministically, as described in equations (8)-(13). Hence, only the residual structure needs to be recalculated during an interval or fuzzy analysis of the global structure, while the reduced component matrices remain unchanged. In this case, the use of a substructuring technique does not change the implementation of the interval or fuzzy analysis. Numerical examples have shown that the CMS technique enhances the efficiency of the interval eigenvalue and frequency response function analysis, without compromising the accuracy of the interval results [15]. However, this approach limits the numerical reduction and hence the gain in calculation time that can be achieved, e.g. when large parts of the structure are affected by global uncertainties.

4.2 Uncertainties on the superelement level

Uncertainties in a superelement of a reduced model affect each step of the reduction procedure: the uncertainties generally affect the constraint modes, the normal modes and corresponding eigenfrequencies, the static and dynamic transformation matrices, and the reduced stiffness and mass matrices. This paper describes three approaches for the implementation of this reduction step: repeated component reduction, approximative component reduction based on component eigenvalue ranges, and non-reduced component matrices updating.

Repeated component reduction A first possible approach to handle uncertainties on the superelement level during an interval eigenfrequency analysis, is to recalculate the component modes, static and dynamic transformation matrices, and reduced matrices of each superelement affected by uncertain parameters, during each evaluation of the goal function of the global optimisation procedure. As in each iteration step all superelements are correctly reduced, this approach gives the correct eigenvalue intervals and can hence be used as reference for approximative methods. Although the implementation of this approach is straightforward, it limits the efficiency of the substructuring technique as for each eigenvalue analysis of the global structure, one or more superelement reduction steps have to be repeated.

Approximative component reduction based on component eigenvalue ranges Another approach to handle uncertainties on the superelement level, is to approximate the description of an uncertain superelement, such that it can represent the effect of the uncertain parameters on the static and dynamic behaviour adequately, and no component recalculation has to be performed during a global interval or fuzzy analysis of the total reduced structural system.
In literature, the approximation of uncertain superelements is most often carried out by assuming that only the eigenvalues of the superelements are affected by the uncertain parameters, and that the component modes remain unaffected [16, 17, 18]. For the Craig-Bampton method with mass normalized modes, this means that only the reduced matrix $[K_{qq}]$ is affected, while all other reduced matrices remain constant, as can be seen

in equations (8)-(14). Therefore, in the reduction step of the algorithm, a separate interval eigenfrequency analysis is performed on the superelement in order to calculate the superelement eigenvalue ranges:

$$\langle\, [\Lambda_{qq}] \,\rangle_{\{\mathbf{x}\}} = \left[\min_{\{x\}\in\{\mathbf{x}\}} [\Lambda_{qq}(\{x\})] \,,\, \max_{\{x\}\in\{\mathbf{x}\}} [\Lambda_{qq}(\{x\})] \right] \tag{15}$$

After this preliminary interval eigenvalue analysis on the uncertain superelement, the reduced component matrix $[K_{qq}(\{x\})]$, containing the uncertainties through the calculated eigenvalue ranges, is included in the global reduced FE model. Hence, the uncertainty originally defined on the superelement is translated towards uncertainty affecting directly the reduced structure, and no further superelement calculations are required during all global interval analyses performed on the total reduced structure, resulting in a decrease of calculation time for these global uncertainty analyses.

Non-reduced component matrices updating The assumption that uncertainties in a superelement only affect its eigenvalue ranges while the component modes and the other reduced matrices remain constant, might be too severe for general cases, e.g. when local geometrical uncertainties affect the component modes, or large stiffness and/or mass changes are present. Therefore, the method using only the eigenvalue ranges of the uncertain superelement might underestimate the real scatter on the output quantities calculated with the approximated reduced model.

Another approach for approximate component reduction is to update the non-reduced substructure matrices and recalculate the reduced matrices, in each iteration step of a global optimisation procedure on the reduced model. Initially, this can be done with the static and dynamic superelement transformation matrices kept constant. This leads to a large reduction in calculation time, as on the superelement level only the system matrix assembly has to be performed, and no time-consuming matrix inversions or substructure eigenvalue analyses has to be performed. In case a large influence of the uncertain parameters on the constraint and/or normal modes is expected, a first order Taylor series expansion of the transformation matrices can be added [15]. Mathematically, the approximative method comes down to the recalculation of the matrices in equations (7) - (13). However, care should te taken when calculating the reduced stiffness matrix, as coupling terms $[K_{tq}]$ similar to the mass coupling terms $[M_{tq}]$ arise due to the fact that the approximated static transformation matrix $[G_{ot}]$ does not necessarily equal the term $-[K_{oo}]^{-1}[K_{ot}]$ with updated system matrices.

4.3 Computational efficiency

The computational efficiency of the methods described in sections 4.1 and 4.2 depends strongly on the substructured FE model, the type of the uncertainties and the uncertainty locations, as each of the methods has its specific advantages and disadvantages. An analysis engineer has to make a choice based on considerations regarding the size of the residual structure, the number of uncertain superelements, the type of the uncertainties (global vs. local), and the size of the uncertainty intervals.

The case in which all uncertain parameters are located in the residual structure only, has as advantage that no superelement recalculations have to be performed. This can be very interesting and efficient as for each superelement reduction a stiffness matrix inverse and a normal modes analysis have to be performed in order to determine the static respectively the dynamic transformation matrix (cf. equations (3)-(5)). Another advantage is that the use of a substructured FE model does not influence the implementation of the interval and fuzzy finite element method. On the other hand, in case of uncertainties affecting large parts of the model, the residual structure contains an unrealistically large number of degrees of freedom, such that the reduced FE model is almost not workable for interval or fuzzy analyses. Therefore, this method is only useful for superelement FE models with a small number of local uncertainties, e.g. joints with uncertain properties, affecting only a small residual structure.

In the case in which uncertainties are allowed on the superelement level, the uncertainties do not limit the numerical reduction that can be achieved. On the other hand, the uncertainties affecting a superelement

have to be translated towards the reduced component matrices. The first approach, the repeated component reduction, is the most straightforward and the most accurate method, but has as drawback that it can be time-consuming in case of large uncertain superelements, as in each goal function evaluation of a global optimisation on the global reduced system, a matrix inverse and a normal modes analysis on the particular superelement have to be performed to recalculate the transformation matrices $[G_{ot}]$ and $[G_{oq}]$.
Approaches 2 and 3 are numerically more efficient, but only approximate the interval results of the global reduced system. The approximative component reduction based on component eigenvalue ranges is especially suitable for global superelement uncertainties and/or small uncertainty intervals, as only the influence of the uncertainties on the component eigenvalues is taken into account, thereby neglecting the influence on the component modes. This method requires the calculation of the eigenvalue ranges of the uncertain superelement(s).
The third approach, in which the non-reduced component matrices are updated in each iteration step of the global optimisation procedure on the global reduced system, does not need extra eigenvalue analyses on the component level. As the substructure system matrix assembly has only a minor numerical cost with respect to system matrix inversions and eigenvalue analyses, a large decrease in calculation time can be achieved.

5 Case study: Garteur benchmark aircraft

5.1 Problem description

The simplified aircraft model of the Garteur benchmark problem is a common testbed designed and manufactured by the Garteur Action Group to evaluate the efficiency and reliability of different ground vibration test techniques [19]. It has also been used extensively for testing model updating methods and model error localisation procedures.
The structure consists of a small-scale aluminium aircraft model with a length of 1.5 m, a wing span of 2 m and a mass of 44 kg. The fuselage of the aircraft consists of a rectangular plate with a thickness of 50 mm. The tail of 10 mm thickness is connected rigidly to the fuselage. The wings are connected to the fuselage through an intermediate steel plate. Wingtips are connected rigidly at both ends of the wings. Both the wings and wingtips are rectangular plates with a thickness of 10 mm. A visco-elastic layer is glued onto a part of the wings. Three concentrated masses are present: one on each wingtip, and one on the fuselage. The FE model, as illustrated in figure 1, contains almost 20000 degrees of freedom.

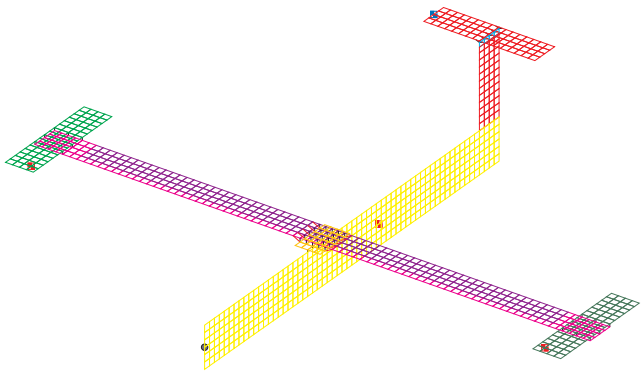


Figure 1: Finite element model of the Garteur benchmark aircraft

The Garteur aircraft model contains some inherent uncertainties, due to a lack of knowledge on the physical model as well as due to uncertainty on the modelling level. The following three uncertainties are considered during the dynamic analyses of the structure described in this paper:

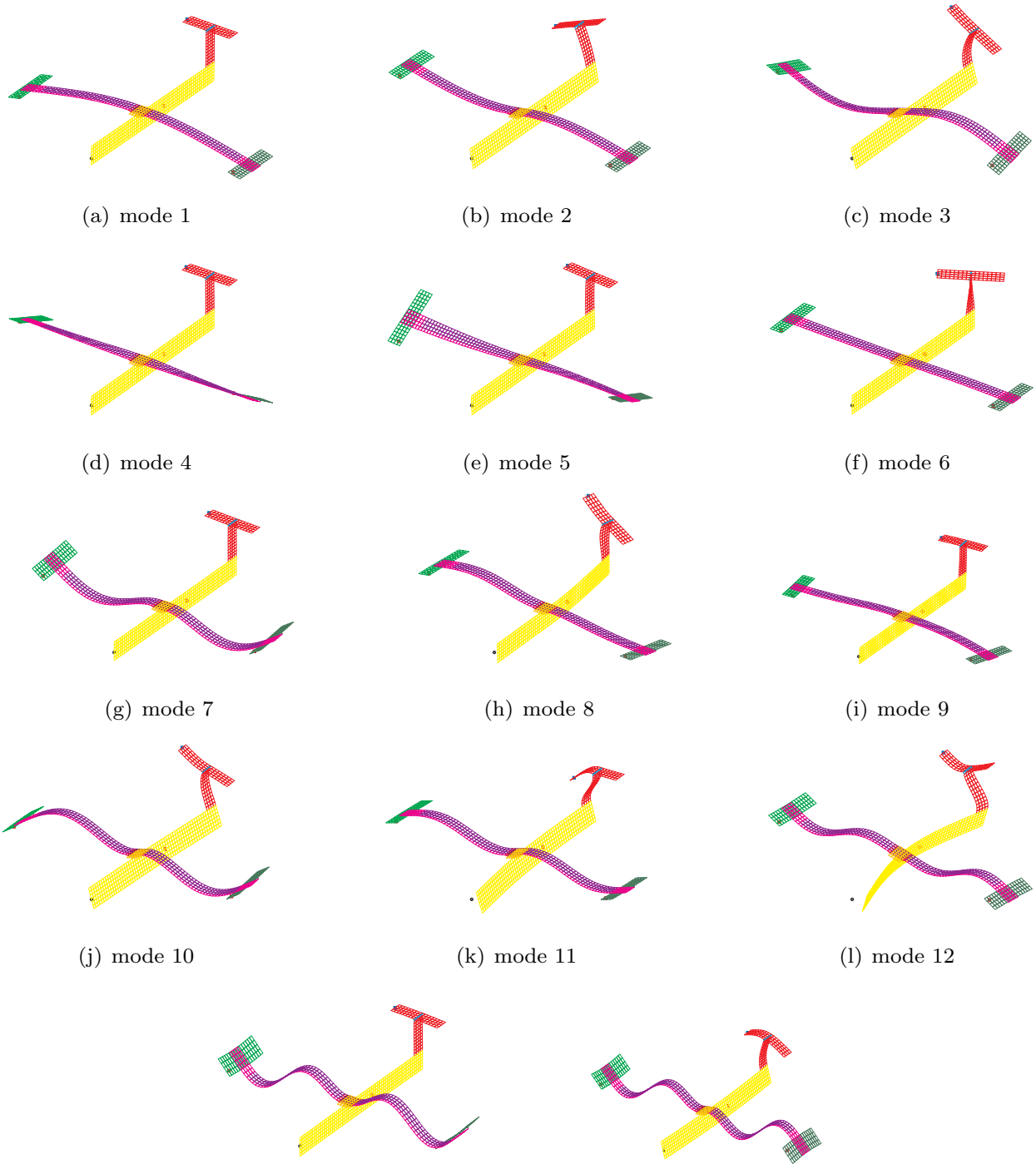
- The thickness of the horizontal and vertical part of the tail are considered independently as uncertain parameters. The uncertainty on these thicknesses ranges between 9 and 11 mm.
- For the determination of the accuracy of updated models of the Garteur benchmark aircraft, small structural modifications were designed and tested by the Garteur Action Group [20]. One of the modifications consisted of the addition of a concentrated mass of 0.92 kg on the left edge of the horizontal part of the tail. An uncertainty interval between 0.84 and 1.99 kg has been considered in the performed interval eigenfrequency analyses.

A reduced model of the Garteur aircraft has been assembled, with the tail as uncertain superelement, and four deterministic superelements: the fuselage, the wings attached to the intermediate steel plate, and both wingtips. The reduced model contains only 99 degrees of freedom (versus 19898 dofs in the full model): the physical dofs of the boundary between the fuselage and the tail, and the generalized coordinates of these superelements. Table 1 gives an overview of the FE model, with indication of the number of degrees of freedom in the residual structure and the superelements. The terminology and notation used in this table are compatible with the FE software MSC.Nastran [14]. For the residual structure, a distinction is made between the degrees of freedom that are constrained during the analyses, and degrees of freedom that are free to displace/rotate. Also the size of each superelement is indicated, before reduction as well as after reduction. Before reduction, the degrees of freedom are subdivided between boundary and internal degrees of freedom. After reduction, the dofs are subdivided according to the t-set (boundary dofs, connected to the constraint modes) and the q-set (generalized dofs, connected to the component normal modes).

Table 2 gives a comparison of the first 14 eigenfrequencies of the full (i.e. non-reduced) model versus the reduced model. Despite the very small number of dofs in the residual structure of the reduced model, the eigenfrequencies are very accurately predicted. Figure 2 shows the mode shapes of the first 14 modes. Modes 1, 2, 7, 10, 13 and 14 are mainly the first six bending modes of the wings. The first, third and fifth bending modes are symmetric, with both wings vibrating in phase while the remainder of the structure hardly moves. The second, fourth and sixth bending modes are antisymmetric, with both wings vibrating in anti-phase. In these modes, the fuselage and the tail tilt around the x-axis. Especially in the second mode shape (first antisymmetric bending of the wings), the tail moves strongly and is in anti-phase with the wings. Mode 3 has a mode shape that combines different movements: the second bending of the wings with a small wing torsion, combined with a pronounced bending of the vertical part of the tail, in phase with the wings. Modes 4 and 5 are torsion modes (in phase, respectively anti-phase) of the wings around the y-direction, while the remainder of the structure stands more or less still. Mode 6 consists of the local torsion of the tail with respect to the fuselage. Modes 8 and 9 are bending modes of the wings in the horizontal xy-plane, with the wings in anti-phase in mode 8 (combined with a bending of the tail), and the wings in phase in mode 9 (with much smaller deformations in the tail). Modes 11 and 12 are again modes that combine different movements of the wings and the tail combined with the first bending of the fuselage. Mainly in mode 11, the asymmetric bending of the horizontal part of the tail, due to the concentrated mass, is profoundly present.

5.2 Interval eigenvalue analysis

An interval eigenfrequency analysis with the three above mentioned uncertain parameters has been performed on the reduced model of the Garteur benchmark aircraft. Table 3 gives the resulting eigenfrequency intervals of the first 14 modes, as calculated using the three approaches described in section 4.2. The interval results of the repeated component reduction approach are used as reference, as the uncertain superelement is correctly re-reduced in each step of the global optimisation procedure applied on each of the 14 eigenfrequencies of the Garteur aircraft.
Comparing the interval eigenfrequency results with the mode shapes described in section 5.1, it is clear that the uncertain parameters strongly affect the torsion of the tail (mode 6), and modes 2, 3 and 11 in which the deformation of the tail is quite profound. On the other hand, the symmetric bending modes of the wings are hardly affected, as in these mode shapes the tail and fuselage barely deform.
In the second and third approach, the combined physical uncertainty of the three parameters is transferred



(a) mode 1 (b) mode 2 (c) mode 3

(d) mode 4 (e) mode 5 (f) mode 6

(g) mode 7 (h) mode 8 (i) mode 9

(j) mode 10 (k) mode 11 (l) mode 12

(m) mode 13 (n) mode 14

Figure 2: Mode shapes of the first 14 modes of the Garteur benchmark aircraft

	full model	reduced model
number of dofs in the residual structure	19896 dofs - constrained: 3542 - free: 16354	99 dofs - constrained: 9 - free: 90
uncertain superelements	/	superelement TAIL - before reduction: 2544 - after reduction: 64 boundary: 54 t-set: 54 internal: 2490 q-set: 10
deterministic superelements	/	superelement LEFT WINGTIP - before reduction: 1446 - after reduction: 60 boundary: 48 t-set: 48 internal: 1398 q-set: 12 superelement RIGHT WINGTIP - before reduction: 1446 - after reduction: 60 boundary: 48 t-set: 48 internal: 1398 q-set: 12 superelement WINGS - before reduction: 7356 - after reduction: 66 boundary: 42 t-set: 42 internal: 7314 q-set: 24 superelement FUSELAGE - before reduction: 7344 - after reduction: 89 boundary: 54 t-set: 54 internal: 7290 q-set: 35

Table 1: Overview of the full and the reduced FE models and their degrees of freedom

mode no.	full model	reduced model
1	5.8075	5.8075
2	12.989	12.989
3	30.800	30.802
4	33.220	33.220
5	33.373	33.374
6	37.940	37.940
7	46.645	46.645
8	47.940	47.943
9	54.069	54.070
10	56.063	56.072
11	77.244	77.276
12	114.53	114.57
13	137.76	137.76
14	146.21	146.28

Table 2: Eigenfrequency values [Hz] for the full model and the reduced model, for the deterministic case

to the representation of the uncertain superelement by its reduced matrices. For the approach using only the component eigenvalue ranges, an underestimation of the eigenfrequency interval width is detected for modes 1, 2 and 4, while the other intervals are overestimated. Using the third approach, a significant improvement of the results is detected for modes 1, 2, 3, 5, 8 and 11. Only the width of the fourth interval is underestimated. A look at the approximated reduced matrices $[M_{qq}]$ during the optimisation procedures has revealed that the fixed-interface normal mode shapes of the tail do not change excessively due to the uncertain parameters, as the diagonal terms of the should-be unity matrix only range between 0.9 and 1.1, with negligible off-diagonal terms.

	repeated component reduction		approximate component reduction, based on substr eigenvalue ranges		non-reduced component matrices updating	
mode no.	frequency interval	interval width (%)	frequency interval	interval width (%)	frequency interval	interval width (%)
1	5.8070 - 5.8080	0.02	5.8075 - 5.8075	0.00	5.8070 - 5.8080	0.02
2	12.075 - 13.717	12.6	12.192 - 13.531	10.3	12.077 - 13.718	12.6
3	28.159 - 32.802	15.1	27.797 - 32.925	16.6	28.160 - 32.807	15.1
4	31.478 - 33.225	5.26	31.479 - 33.224	5.25	31.526 - 33.225	5.11
5	33.237 - 34.116	2.64	33.236 - 34.493	3.77	33.237 - 34.117	2.64
6	33.310 - 44.482	29.4	33.303 - 44.437	29.4	33.311 - 44.527	29.6
7	46.284 - 46.650	0.78	46.240 - 46.652	0.88	46.288 - 46.650	0.78
8	46.689 - 49.256	5.36	46.680 - 49.592	6.07	46.689 - 49.263	5.37
9	53.950 - 54.165	0.40	53.937 - 54.186	0.46	53.951 - 54.167	0.40
10	54.340 - 58.054	6.62	54.296 - 58.207	6.97	54.342 - 58.056	6.63
11	71.002 - 83.708	16.4	70.727 - 83.892	17.0	71.011 - 83.716	16.4
12	112.26 - 117.95	4.97	111.91 - 118.38	5.65	112.26 - 117.96	4.97
13	137.75 - 137.76	0.00	137.75 - 137.77	0.01	137.75 - 137.76	0.01
14	144.59 - 147.73	2.14	144.49 - 147.99	2.39	144.61 - 147.73	2.14

Table 3: Comparison of the interval eigenfrequency results [Hz] of the reduced model, with the three approaches for superelement uncertainty handling

5.3 Computational efficiency

A comparison of the three approaches for handling uncertainty on the superelement level is made considering a vertex analysis on the presented case study. Taking into account the three uncertain parameters, a full vertex analysis requires 8 FE runs. For a deterministic eigenfrequency analysis with superelements, the following steps have to be performed:

1. superelement reduction:
   a. superelement system matrix assembly
   b. calculation of static transformation matrix $[G_{ot}]$
      tail: inversion of 2005 by 2005 stiffness matrix $[K_{oo}]$
   c. calculation of dynamic transformation matrix $[G_{oq}]$
      tail: eigenvalue problem with dimension 2005 by 2005
   d. calculation of reduced matrices
2. assembly of the residual structure, based on the reduced matrices of all superelements
3. eigenvalue analysis on the residual structure (dimension 90 by 90)

The repeated component reduction approach requires in each vertex analysis the execution of each of these steps, including the inversion of the superelement stiffness matrix (step 1b) and the eigenvalue analysis (step 1c) on the tail, followed by the assembly and small eigenvalue analysis on the residual structure. Using the approximative component reduction based on the component eigenvalue ranges, a vertex analysis is performed on each uncertain superelement seperately, in order to determine its eigenvalue intervals (step 1c). These are then directly introduced in the residual structure, hence no recalculation of the static transformation matrix or reduced matrices is needed (steps 1b and 1d), which results in a significant reduction of calculation time. However, the solution of the superelement eigenvalue problem on each vertex can be time-consuming, as the computation time is roughly proportional with the square of the number of degrees of freedom. In the third approach, only the superelement system matrix assembly and the reduced matrix calculation is redone, hence a significant additional reduction in computation time is achieved, as the time-consuming steps 1b and 1c are omitted.

6 Conclusions

This paper uses the non-probabilistic interval finite element method for the dynamic analysis of structures affected by uncertain parameters. A reduction of calculation time of these uncertainty analyses is obtained by the application of a substructuring technique, i.e. the Craig-Bampton component mode synthesis technique. This work focusses on the influence of uncertain parameters on the numerical representation of superelements using component modes and reduced component matrices. Two approximative methods for uncertain component reduction are proposed and compared with respect to accuracy and numerical efficiency. The method that considers only the influence of the uncertain parameters on the eigenvalue ranges of uncertain superelements, is numerically more efficient than a repeated component reduction approach, and gives accurate results in case of small uncertainties. However, the accuracy of the interval results cannot be guaranteed in case of larger uncertainties in stiffness or mass matrices, and in case that the component modes are influenced by the uncertain parameters. The approach in which only the non-reduced component matrices of an uncertain superelement are updated, is computationally very efficient, and is able to take large uncertainties in stiffness and mass matrices into account. In case of large influence on the component mode shapes, an approximation of the static and/or dynamic transformation matrices might be appropriate.

References

[1] A. Haldar and S. Mahadevan. *Reliability Assessment using Stochastic Finite Element Analysis*. John Wiley & Sons, Inc., New York, first edition, 2000.

[2] I. Elishakoff. Essay on uncertainties in elastic and viscoelastic structures: from A.M. Freudenthal's criticism to modern convex modelling. *Computers & Structures*, 56(2):871–895, 1995.

[3] L. Zadeh. Fuzzy sets. *Information & Control*, 8(3):338–353, 1965.

[4] S.S. Rao and P. Sawyer. Fuzzy finite element approach for the analysis of imprecisely defined systems. *AIAA Journal*, 33(12):2364–2370, 1995.

[5] D. Moens and D. Vandepitte. An interval finite element approach for the calculation of envelope frequency response functions. *International Journal for Numerical Methods in Engineering*, 61(14):2480–2507, 2004.

[6] D. Moens, H. De Gersem, and D. Vandepitte. Frequency response function analysis of structures with fuzzy modal damping parameters. In *Proceedings of the 7th AIAA Non-Deterministic Approaches Forum*, Austin, 2005.

[7] D. Moens and D. Vandepitte. A fuzzy finite element procedure for the calculation of uncertain frequency response functions of damped structures: Part 1 - procedure. *Journal of Sound and Vibration*, 288(3):431–462, 2005.

[8] H. De Gersem, D. Moens, W. Desmet, and D. Vandepitte. A fuzzy finite element procedure for the calculation of uncertain frequency response functions of damped structures: Part 2 - numerical case studies. *Journal of Sound and Vibration*, 288(3):463–486, 2005.

[9] R.R. Craig and M.C.C. Bampton. Coupling of substructures for dynamic analyses. *AIAA Journal*, 6(7):1313–1319, 1968.

[10] W. Dong and H. Shah. Vertex method for computing functions of fuzzy variables. *Fuzzy Sets and Systems*, 24(1):65–78, 1987.

[11] M. Hanss. The transformation method for the simulation and analysis of systems with uncertain parameters. *Fuzzy Sets and Systems*, 130(3):277–289, 2002.

[12] S. Donders, D. Vandepitte, J. Van de Peer, and W. Desmet. Assessment of uncertainty on structural dynamic responses with the short transformation method. *Journal of Sound and Vibration*, 288(3):523–549, 2005.

[13] R.R. Craig Jr. A review of time-domain and frequency-domain component mode synthesis methods. *Journal of Modal Analysis*, 2(2):59–72, 1987.

[14] The MacNeal-Schwendler Corporation, Los Angeles. *MSC.Nastran superelement user's guide*, 2001.

[15] H. De Gersem, D. Moens, W. Desmet, and D. Vandepitte. Non-probabilistic uncertainty assessment in finite element models with superelements. In *Proceedings of the 8th AIAA Non-Deterministic Approaches Conference*, Newport, 2006.

[16] B.R. Mace and P.J. Shorter. A local modal/perturbational method for estimating frequency response statistics of built-up structures with uncertain properties. *Journal of Sound and Vibration*, 242(5):793–811, 2001.

[17] B. Lallemand, A. Cherki, T. Tison, and P. Level. Fuzzy modal finite element analysis of structures with imprecise material properties. *Journal of Sound and Vibration*, 220(2):353–364, 1999.

[18] B. Van den Nieuwenhof. *Stochastic finite elements for elastodynamics: random field and shape uncertainty modelling using direct and modal perturbation-based approaches*. PhD thesis, Université Catholique de Louvain, Louvain-La-Neuve, Belgium, 2003.

[19] M. Degener and M. Hermes. Ground vibration test and finite element analysis of the garteur SM-AG19 testbed. Technical Report IB 232-96 J 08, DLR - German Aerospace Research Establishment, Institute for Aeroelasticity, Oct 1996.

[20] M. Friswell. New measurements on the garteur testbed. Technical report, University of Wales Swansea, UK, Dec 2000.