A Survey of Non-Probabilistic Uncertainty Treatment in Finite Element Analysis

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15th July 2003

Abstract

The objective of this paper is to critically review the emerging non-probabilistic approaches for uncertainty treatment in finite element analysis. The paper discusses general theoretical and practical aspects of both the interval and fuzzy finite element analysis. First, the applicability of the non-probabilistic concepts for numerical uncertainty analysis is discussed from a theoretical viewpoint. The necessary conditions for a useful application of the non-probabilistic concepts are determined, and are proven to be complementary rather than competitive to the classical probabilistic approach. The second part of the paper focuses on numerical aspects of the interval finite element method. It describes two principal strategies for the implementation, i.e. the anti-optimisation and the interval arithmetic approach, and gives a state-of-the-art of the interval finite element algorithms available from literature. It is shown how the application of the interval arithmetic approach to the classical finite element procedure can result in a severe overestimation of the uncertainty on the output, and the main sources of this conservatism are identified. A numerical example in the final part of the paper illustrates the capabilities of the different strategies on an eigenfrequency analysis of a built-up benchmark structure.

1 Introduction

The continuous exponential growth of computational capabilities of modern computers has resulted in the use of extensive numerical simulation techniques. In the area of finite element analysis, the most straightforward application of the increasing computational capabilities lies in the use of very detailed and hence large FE models. Simultaneously, this evolution has paved the way for a number of computationally intensive analysis techniques derived from the classical FE technique (non-linear analysis, multiphysics...). In this context, the non-deterministic FE analysis has become a very important issue in the search for valuable utilisation of the available computational resources. The probabilistic FE analysis [1, 2, 3] has gained a large popularity in this area. Recently, also non-probabilistic approaches for non-deterministic analysis are emerging. They can be mainly divided into two groups. The Interval FE (IFE) analysis is based on the interval concept for the description of non-deterministic model properties, and so far has been studied only on an academic level [4, 5, 6]. The Fuzzy FE (FFE) analysis is basically an extension of the IFE analysis, and has been studied in a number of specific research domains: static structural analysis [7, 8], dynamic analysis [9, 10, 11], geotechnical engineering [12, 13, 14], multi-body kinematics [15], steady-state analysis of rolling [16] and analysis of smart structures [17]. Recently, Rao et al. [18] used a procedure similar to FFEM to derive a fuzzy boundary element method. The numerical procedures used for all these implementations are strongly influenced by the specific properties of the analysed physical phenomenon. Furthermore, most of these only

consider academic examples with very limited size and complexity. So far, no effort has been undertaken to develop a general FFEM framework.

The growing interest for non-probabilistic methods for non-deterministic numerical analysis mainly originates from criticism on the credibility of probabilistic results [19]. Especially when extremely high reliabilities are analysed based on numerical models, design engineers often remain very sceptic regarding the trustworthiness of the numerical predictions. The recent development of the non-probabilistic approaches stems from the argumentation that this lack of credibility is always present in probabilistic analysis results, but generally remains unaccounted for. It is argued that the non-probabilistic concepts could be more appropriate to model certain types of non-deterministic information, resulting in a better representation of the simulated non-deterministic physical behaviour. Especially in early design stages when objective probabilistic information often is not available, non-probabilistic concepts are believed to be of great value. It is the aim of this paper to critically review this argumentation, and to study to what extent the non-probabilistic methods can be considered as useful alternatives to the existing probabilistic approach.

In section 2, this paper first studies the underlying theoretical arguments which cause the suspected lack of credibility of probabilistic methods. The study is founded on different possible interpretations of the information represented by non-deterministic properties, summarised in a structured way in section 2.1. Based on this summary, the issue whether or not the criticism on probabilistic methods is justifiable is addressed in section 2.2. Sections 2.3 and 2.4 further elaborate on the applicability of respectively the interval and fuzzy approach as alternative non-deterministic analysis techniques. Section 3 of this paper discusses the application of interval arithmetic procedures for the implementation of structural dynamic IFE analysis. It hereby focuses on the conservatism that is implicitly incorporated in the interval results. Finally, section 4 illustrates the IFE procedure on the eigenfrequency analysis of a beam structure.

2 Non-probabilistic concepts for non-deterministic numerical analysis

2.1 Different types of non-deterministic model properties in FEM

In literature, the use of the terminology for *uncertainty* and *variability* is not unambiguous. Different researchers apply the same terminology but the meaning attached to these is rather inconsistent. This paper applies the terminology proposed by Oberkampf [20]. It is briefly summarised here.

The term *variability* covers *the variation which is inherent to the modelled physical system or the environment under consideration.* Generally, this is described by a distributed quantity defined over a range of possible values. The exact value is known to be within this range, but it will vary from unit to unit or from time to time. Some literature refers to this variability as *aleatory uncertainty* or *irreducible uncertainty*, referring to the fact that even when all information on the particular property is available, the quantity cannot be deterministically determined. Typical examples of variability are:

- a property of a design which is subject to manufacturing tolerances
- environmental effects on a model (temperature, humidity, ...)
- properties of non-uniform materials

An *uncertainty* is *a potential deficiency in any phase or activity of the modelling process that is due to lack of knowledge.* The word *potential* stresses that the deficiency may or may not occur. This means that there may be no deficiency even though there is some lack of knowledge, i.e. when the numerical model of the phenomenon happens to be correct rather by chance than due to exact knowledge. This definition basically states that uncertainty

is caused by incomplete information resulting from either vagueness, nonspecificity or dissonance [21]. Vagueness characterises information that is imprecisely defined, unclear or indistinct. It is typically the result of human opinion on unknown quantities ("*the density of this material is around x*"). Nonspecificity refers to the availability of a number of different models that describe the same phenomenon. The larger the number of alternatives, the larger the nonspecificity. Dissonance refers to the existence of conflicting evidence of the described phenomenon, for instance when there is evidence that a quantity belongs to disjoint sets. Other authors refer to this uncertainty as *reducible*, *epistemic* or *subjective uncertainty*. Typical examples of uncertainties are:

- non-rigid models for boundary conditions
- models for material damping
- unpredictable model changes due to ageing, loading, ...

See Alvin et al. [22] for a comprehensive survey of sources of uncertainty and variability in general FE analysis.

The definitions of uncertainty and variability above are rather straightforward and comprehensible. However, they are not mutually exclusive, since a variability could be subject to lack of knowledge when information on its range or likelihood within the range is missing. This is for instance the case for every design dimension subject to tolerances, but without further specification of manufacturing process or supplier. The tolerances represent the bounds on the feasible domain, but there is no information on the likelihood of the possible values within these bounds. Consequently, because there is a lack of knowledge, such a variability is also an uncertainty. It is referred to here as an *uncertain variability*. Some vague knowledge may be available ("*the mean value is approximately x*") but also nonspecificity may play an important role in the uncertainty, for instance in choosing an appropriate model to describe a random quantity. Opposed to the uncertain variability, a *certain variability* refers to a variability the range and likelihood of which is exactly known.

On the other hand, it appears logical to state that every property in a numerical model corresponding to a physical quantity is a variability, since it will eventually have a range of possible values and a likelihood inside this range in the physical model. This argumentation implies that all uncertainties are also variabilities. In practice, however, the majority of model properties are implemented as constant deterministic values in the numerical model. Though they are subject to variation, the influence of their variability on the analysis result is considered to be negligible. Often, uncertainties refer to a possible lack of knowledge in these deterministic properties. This type of uncertainty is referred to as *invariable uncertainty*. The invariable uncertainties typically occur in model properties for model parts that are difficult to describe numerically, but considered constant in the final physical product (connections, damping, ...). Other examples are design properties which have negligible variability but which are not defined exactly in an early design stage. Figure 1 gives a graphical illustration of the proposed subdivision of the definitions for uncertainty and variability.

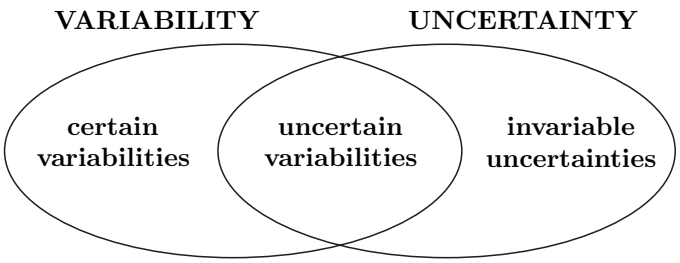

Figure 1: Classification of variabilities and uncertainties in numerical modelling

2.2 Argumentation on the credibility of probabilistic analysis

There are three main arguments generally used in literature to judge on the use of probabilistic techniques:

- Are probabilistic quantities always a realistic representation of incomplete information?
- Is a probabilistic analysis required for every non-deterministic problem?
- Are probabilistic methods time-efficient enough to be applied in every design stage?

In principle, only the first argument really addresses the credibility of probabilistic results. This argument is discussed here in the framework of the definitions of non-deterministic quantities from the previous section. The two other arguments are only briefly discussed.

2.2.1 Probabilistic representation of incomplete information

For the study of the applicability of the probabilistic model, distinction between certain variabilities, uncertain variabilities and invariable uncertainties is necessary.

certain variabilities The numerical description using a probability density function (PDF) in this case is completely consistent with the definition as in section 2.1. The information on the range and the likelihood of a certain variability is unambiguously contained in the PDF. Therefore, the probabilistic analysis is the perfect tool for analysing models with well-defined variabilities.

uncertain variabilities The most straightforward probabilistic treatment of uncertain variabilities is to perform a number of analyses with different probabilistic descriptions, which should be chosen such that they are all consistent with the limited available information. A representation by a single random quantity is not sufficient. Engineering scientist FREUDENTHAL [23] who was one of the pioneers of probabilistic methods in engineering stated that *" ... ignorance of the cause of variation does not make such variation random."*. By this, he means that when crucial information on a variability is missing, it is not good practice to model it as a probabilistic quantity represented by a single random PDF. On the contrary, in this case it is mandatory to apply a number of different probabilistic models to examine the effect of the chosen PDF on the result. Especially when high reliabilities are studied, the impact of the chosen PDF on the numerical predictions can be extremely high.

invariable uncertainties Generally, invariable uncertainties are represented by random probabilistic quantities. A PDF is chosen which to the knowledge of the analyst represents best the uncertain nature of the quantity. The information contained in this random quantity does not represent a variation in the final product, since by definition, the invariable uncertainties are considered to be constant. The random quantity in this case represents the presumed likelihood that a model parameter will adopt a value. As such, the lack of knowledge is filled with subjective information provided by the analyst, expressed in the form of a PDF. This is sometimes referred to as a subjective PDF.

From this discussion, it is clear that the probabilistic concept can be applied for all types of non-deterministic model properties. In some cases however, the numerical description does not entirely rely on objective information. Whenever there is subjective information added to the input, it should also be acknowledged in the results. Furthermore, there is a very important difference in the use of the probabilistic concept for variabilities on the one hand, and invariable uncertainties on the other hand. The former represents an actual variation as defined in section 2.1, while the latter clearly may not be interpreted in this sense. Consequently, if both uncertainties and variabilities are present in a probabilistic analysis, it is imperative to distinguish between the different meanings attached to both, and to treat them appropriately during the numerical procedure (see also HOFFMAN et al. [24]).

2.2.2 Necessity of probabilistic analysis

When applying non-deterministic methods, it is very important for the analyst to clearly define the goal of the analysis. Ideally, an exact numerical reproduction of some non-deterministic physical behaviour is pursued. In most cases, however, non-deterministic analysis is merely a tool to enhance or optimise the expected physical behaviour of a design based on (limited) knowledge of external non-deterministic influences. While probabilistic analyses are applicable for this purpose, probabilistic information on the behaviour of a design is not always primordial. Other non-probabilistic techniques could give a valuable, maybe even additional insight into the non-deterministic nature of the simulated behaviour. Whether or not these techniques are valuable alternatives depends on the added value of the results.

2.2.3 Time-efficiency of probabilistic analysis

The criticism on the time-efficiency of probabilistic analysis refers to the popular implementation using a Monte Carlo simulation. This technique is rather time consuming as it uses a high number of deterministic calculations to simulate the probabilistic process. Therefore, its computational efficiency will always lag behind the efficiency of the corresponding deterministic analysis. However, taking into account the evolution of computational power nowadays, this issue becomes less important. Using efficient numerical techniques in a Monte Carlo simulation as well as the recent efforts to optimise the simulation procedure (e.g. see [25]) will further decrease the importance of this issue. Furthermore, since there are currently no commercial procedures available for non-probabilistic analysis of industrially sized non-deterministic models, it is impossible to make general statements concerning the relative efficiency of those alternatives.

2.2.4 Conclusion

Based on the discussion above, the probabilistic concept can be designated as a most valuable solution strategy when three conditions are met simultaneously: (1) trustworthy probabilistic information on the non-determinism is available, (2) a probabilistic description of the simulated non-deterministic behaviour is desired, and (3) the computational constraints are not too tight. Whether or not a non-probabilistic method could be a valuable alternative when one of these conditions is not satisfied depends correspondingly on (1) the degree of realism in the representation of the limited available information through the non-probabilistic concept, (2) the added value of the non-probabilistic results and (3) the computational cost to obtain them. All of these depend on the applied methodology, and should be analysed in the context of each alternative technique. In the general description of the non-probabilistic concepts in the following sections, this paper focuses only on issues (1) and (2). The computational cost of a non-probabilistic technique depends entirely on the underlying numerical algorithm, and therefore should be analysed by the developers of such procedures in order to give a clear indication of the capabilities of the developed methodology.

2.3 Interval analysis

2.3.1 Basic properties of the interval concept

Recent developments in interval arithmetics are mainly based on the work of Moore [26], who introduced interval vectors and matrices and the first non-trivial applications. By definition, an interval scalar consists of a single continuous domain in the domain of real numbers $\mathbb{R}$. The range is bounded by a lower and an upper bound. The domain of interval scalars defined over $\mathbb{R}$ is denoted by $\mathbb{IR}$. In order to facilitate the development of the mathematical background in the remainder of this work, this section first introduces a generalised notation for intervals and sets.

A general *interval scalar* is denoted by a boldface variable $\mathbf{x}$. The lower and upper bound of an interval scalar $\mathbf{x}$ are denoted by $\underline{x}$ respectively $\overline{x}$. A real closed interval scalar is defined as:

$$\mathbf{x} = \left\{ x \mid (x \in \mathbb{R}) \left(\underline{x} \leq x \leq \overline{x}\right) \right\} \tag{1}$$

An alternative notation for an interval $\mathbf{x}$ is $[\underline{x}, \overline{x}]$. The midpoint of the interval is defined as:

$$\check{\mathbf{x}} = \frac{\underline{x} + \overline{x}}{2} \tag{2}$$

The radius of an interval equals:

$$\bar{\mathbf{x}} = \frac{\overline{x} - \underline{x}}{2} \tag{3}$$

A straightforward extension of an interval scalar is a general *set scalar* denoted by $\langle x \rangle$. It consists of a number of disjoint interval scalars in $\mathbb{IR}$:

$$\langle x \rangle = \bigcup_{i=1\ldots n} \mathbf{x}_i \tag{4}$$

The *interval matrix* $[\mathbf{X}] \in \mathbb{IR}^{n \times m}$ describes the set of all matrices for which each matrix entry x_{ij} is contained within its corresponding interval scalar $\mathbf{x_{ij}}$:

$$[\mathbf{X}] = \left\{ [X] \mid x_{ij} \in \mathbf{x_{ij}} \right\} \tag{5}$$

An *interval vector* similarly is denoted by $\{\mathbf{x}\} \in \mathbb{IR}^n$. The definitions of equations (2) and (3) are easily extended to interval matrices and vectors by applying them on each entry in the vector or matrix.

According to the definition of equation (5), an interval matrix or vector includes any combination of the entries as long as they are within the bounds of their interval scalar. This means that they implicitly presume independence between all entries. Hence, an interval vector represents a hypercube, and therefore this representation cannot be used to give a precise description of a general convex set in a multidimensional space. This has important consequences for the interval FE analysis based on interval matrices and vectors, since it will cause the implicit introduction of conservatism into the analysis (see section 3).

The *set matrix* $\langle [X] \rangle$ describes the set of all possible matrices where each matrix entry x_{ij} is contained within its corresponding set scalar $\langle x_{ij} \rangle$:

$$\langle [X] \rangle = \left\{ [X] \mid x_{ij} \in \langle x_{ij} \rangle \right\} \tag{6}$$

Again, the matrix elements are implicitly considered independent.

2.3.2 Interval representation of incomplete information

The information represented by an interval object again depends on the type of modelled non-deterministic quantity.

certain variabilities For certain variabilities, the input interval objects are derived from the range of the corresponding input PDFs. Consequently, the result of an interval analysis only represents the actual range of the variable outcome of the analysis. The available information on the likelihood inside the range is lost, which in this case is an important disadvantage of the interval concept.

uncertain variabilities When upper and lower bounds of a non-deterministic property are well-defined, the interval model represents perfectly the available information. It is clear that the interval concept is most appropriate for this type of model properties. For uncertain variabilities without objective information on the actual range, a subjective interval has to be chosen. Still, in this case the interval concept is generally more appropriate than the probabilistic approach, since it requires less subjective information to be added to the analysis.

invariable uncertainties These always require a subjective interval definition. In this case, care should be taken not to interpret the interval quantity as the actual range in the physical product. It merely represents the values the analyst considers possible at the time the analysis is performed. Therefore, similar to the application of the probabilistic concept for invariable uncertainties, it is important to acknowledge the subjectivity in the result of the analysis. However, since the interval concept requires less subjective information to be added to the problem description, there is less room for misinterpretation of the results.

We can conclude that the PDF remains the most valuable concept for the representation of certain variabilities. The omission of a known PDF through the interval concept can only be justifiable when probabilistic information is not required, or the computational cost of the interval analysis is significantly lower. The interval concept is most valuable when dealing with uncertain variabilities or invariable uncertainties.

2.3.3 Basic concept of the IFE analysis

The goal of the IFE analysis is to obtain the maximal meaningful information on the possible outcome of the FE analysis when each model parameter is expressed either by an interval or by a crisp number. Numerically, this is equivalent to finding the minimal and maximal deterministic analysis results taking into account all possible models that lie within the interval uncertainty description. Considering the parameter vector $\{x\}$ defined to be contained within an interval vector $\{\mathbf{x}\}$ and the deterministic FE analysis generally represented by the function $f\big(\{x\}\big)$ applied on these parameters, the IFE procedure is numerically equivalent to finding the following solution set:

$$\big\langle\{y\}\big\rangle = \Big\{\{y\} \mid \Big(\{x\} \in \{\mathbf{x}\}\Big)\Big(\{y\} = f\big(\{x\}\big)\Big)\Big\} \tag{7}$$

with $\{y\}$ a general multidimensional result of the FE analysis. Generally, the components of the output vector are related to each other through the design parameters. Therefore, the solution set $\big\langle\{y\}\big\rangle$ can basically adopt any form in the output space. This makes it extremely difficult to calculate an exact description of the solution set. In most cases, however, only the individual ranges of some components of the result vector are really of interest. Therefore, research focuses on calculating an interval vector that approximates the exact solution set but neglects the interdependencies between the output vector components. This is referred to as a hypercubic approximation of the result. It describes a range for each vector component, but not all combinations of vector components within these ranges are part of the exact solution set. Generally, the smallest hypercube around the exact solution set is of most importance. Still, in early design stages, also conservative hypercubic approximations of the solution set can be very valuable. Figure 2 gives a two-dimensional illustration of an exact solution set and the corresponding approximate hypercubes.

2.3.4 Use and value of interval analysis

The application of the interval concept in numerical reliability studies is often referred to as *anti-optimisation*. This name stems from the fact that from all numerical models within the interval input boundaries, the one with the least favourable analysis result is the most interesting from reliability point of view. Finding this least favourable result is mathematically equivalent to performing a numerical optimisation aimed at the worst case result with respect to the input intervals.

The concept of anti-optimisation has been introduced as the basis for a non-probabilistic reliability framework [27, 28]. This requires an evolution from a reliability concept as *probability of failure* towards *range of acceptable behaviour*. This means that the design must assure that the performance remains within an acceptable domain, without specifying a likelihood of failure. Reliability then becomes a crisp criterion distinguishing between either

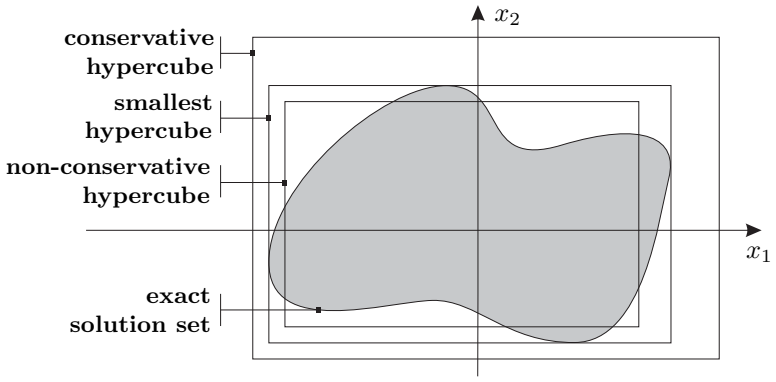

Figure 2: Hypercubic approximations of a continuous two-dimensional output set of an IFE analysis

acceptable or unacceptable designs. The most important benefit of the anti-optimisation concept is that it broadens the objectivity of reliability studies to uncertain variabilities with known range, because the interval model perfectly represents these uncertainties without the need for subjective input. For some cases, it can be shown that the anti-optimisation procedure results in the same choice of design parameters as a probabilistic analysis if the required reliability tends to one [29]. The anti-optimisation in this case proves to be far less expensive in computation time.

Another useful application of the interval concept lies in modelling invariable uncertainties. Though they are assumed to be constant, they could play an important role in decisions during design. During optimisation or general decision verification, the analyst may ask the question whether the defined ranges for the invariable uncertainties result in an allowable range for the behaviour, without really being interested in the probabilistic information. For instance, this can be very useful to analyse design tolerances. Pure probabilistic analysis in this case seems like an unnatural thing to do, since it requires information that is not available (probabilistic input) to produce information that is not requested (probabilistic output).

The numerical implementation of the anti-optimisation approach is subject to an important requirement. Since the result of the analysis is the source of a crisp decision between acceptable and unacceptable designs, approximate results should always be kept on the safe side of the exact result. This means that if approximate solution procedures are used in the numerical implementation, they should guarantee conservatism in their result. On the other hand, this conservatism should not be excessively high in order for the result to be of any practical value. See section 3 of this paper for a general overview of implementation techniques for the IFE procedure.

2.4 Fuzzy analysis

2.4.1 Basic properties of the fuzzy concept

The theory of fuzzy logic was introduced by Zadeh [30] in 1965, and has gained an increasing popularity during the last two decades. Its most important property is that it is capable of describing linguistic and, therefore, incomplete information through the use of fuzzy sets. A fuzzy set can be interpreted as an extension of a classical set. Where a classical set clearly distinguishes between members and non-members of the set, the fuzzy set introduces a degree of membership, represented by the *membership function*. This membership function describes the grade of membership to the fuzzy set for each element in the domain. The concept allows membership values different from zero and one. This enables the representation of a value that is only to a certain degree member of the set.

For a fuzzy set $\tilde{x}$, the membership function is defined as $\mu_{\tilde{x}}(x)$ for all x that belong to the domain X:

$$\tilde{x} = \left\{ \left(x, \mu_{\tilde{x}}(x)\right) \mid \left(x \in X\right)\left(\mu_{\tilde{x}}(x) \in [0,1]\right) \right\} \tag{8}$$

If $\mu_{\tilde{x}}(x) = 1$, x is definitely a member of the subset $\tilde{x}$. If $\mu_{\tilde{x}}(x) = 0$, x is definitely not a member of the subset $\tilde{x}$. For every x with $0 < \mu_{\tilde{x}}(x) < 1$, the membership is not certain. The most frequently applied shapes for the membership functions are the triangular and Gaussian shape.

The concept of fuzzy sets appears to be very useful to represent non-deterministic model properties since knowledge of uncertainties is commonly based on expert opinion. This has resulted in some applications of the fuzzy model in structural optimisation under uncertainty [31, 32], and recently initiated the development of the fuzzy finite element method for numerical analysis of non-deterministic models.

2.4.2 Fuzzy representation of incomplete information

Zadeh [33] extended the theory of fuzzy sets to a basis for reasoning with possibility. In this interpretation, the membership function is considered as a possibility distribution function, providing information on the values that the described quantity can adopt. More generally, the possibility is defined as *a subjective measure that expresses the degree to which the analyst considers that an event can occur.* It provides in a system of defining intermediate possibilities between strictly impossible and strictly possible events.

certain variabilities For a fuzzy representation of certain variabilities, the known PDF has to be converted to a compatible membership function. A number of methods have been developed for this purpose [34, 35]. The basic law for the conversion follows from the consistency principle, which states that the degree of possibility of an event is greater than or equal to its degree of probability. This principle implies the following rule for conversion of probabilistic into possibilistic distributions:

$$\int_B f_X(x)\, dx \;\leq\; \max_{x \in B} \left(\frac{\mu_{\tilde{x}}(x)}{\max \mu_{\tilde{x}}(x)} \right) \tag{9}$$

for any set B in the feasible domain. This means that even a completely known probabilistic quantity has an infinite number of possibilistic representations. Therefore, these conversion techniques always rely on some sort of subjective judgement.

uncertain variabilities For uncertain variabilities, the fuzzy concept can be used for a hybrid uncertainty model. It stems from an alternative interpretation of a possibility distribution introduced by Dubois and Prade [36] based on the *Evidence Theory* introduced by Shafer [37]. In this approach, a fuzzy number is used to represent a class of probability random quantities that have a cumulative distribution function (CDF) in between boundaries derived directly from the possibility distribution. The left boundary on the compatible CDFs coincides with the increasing branch of the fuzzy number. The right boundary coincides with the complement of the decreasing branch of the fuzzy number. Figure 3 clarifies this approach. In this concept, the possibilistic approach becomes a tool to simultaneously examine the effect of a set of different PDFs in a single analysis. While the ability of this concept to model classes of probabilistic data seems extremely powerful, it has only been applied very rarely in uncertainty analysis.

invariable uncertainties Finally, an invariable uncertainty requires a fuzzy set that represents the subjective expectation of the analyst. When the invariable uncertainty represents an open design decision subject to optimisation, the analyst can express his preference of the quantity through the possibility distribution. Still, when interpreting the results, reference to the chosen input membership functions is imperative.

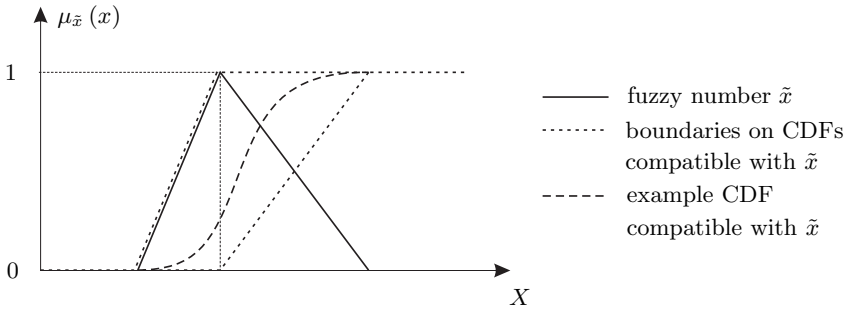

Figure 3: Possibility distribution of a fuzzy number and corresponding lower and upper boundaries for CDF compatible with the fuzzy number

Considering the explicit subjective nature of a fuzzy set, it is concluded that it is most useful to describe uncertainties. The more objective information becomes available on a non-deterministic model property, the less the fuzzy concept is appropriate to describe it. Furthermore, it is the author's opinion that forcing the application of fuzzy sets into the domain of certain variabilities through a conversion of PDFs as described above is rather irrational. Available objective probabilistic data is replaced by a subjective description, resulting in the loss of very valuable information.

2.4.3 Basic concept of the FFE analysis

The principal goal of the FFE analysis is to obtain the membership function of the output quantities given the membership functions of all input quantities. It basically requires a concept to handle the combination of the input sets, i.e. a definition of a Cartesian product combining different fuzzy sets. This Cartesian product generally is defined as:

$$\mu_{\tilde{x}_1 \times \ldots \times \tilde{x}_n}(x_1, \ldots x_n) = \min\left(\mu_{\tilde{x}_1}(x_1), \ldots \mu_{\tilde{x}_n}(x_n)\right) \tag{10}$$

This definition states that the possibility of a combination of fuzzy events equals the minimum of the possibilities of all individual events.

Calculating the result of the FFE analysis further requires an arithmetic that handles the numerical evaluation of functions of fuzzy sets. A general concept follows directly from Zadeh's extension principle, which describes a general procedure for extending crisp mathematical procedures to fuzzy quantities. It states that the fuzzy output $\tilde{y}$ of the crisp function $f(x_1, x_2, \ldots, x_n)$ applied to n fuzzy numbers $\tilde{x}_i$ equals:

$$\begin{cases} \mu_{\tilde{y}}(y) = \sup\limits_{\substack{x_1, \ldots x_n \\ y = f(x_1, \ldots x_n)}} \left(\min\left(\mu_{\tilde{x}_1}(x_1), \ldots \mu_{\tilde{x}_n}(x_n)\right)\right) \\ \\ \mu_{\tilde{y}}(y) = \quad 0 \quad \text{if} \quad f^{-1}(y) = \emptyset \end{cases} \tag{11}$$

This definition implies that the membership value of the fuzzy result $\tilde{y}$ for a specific value y^* equals the largest among the membership values $\mu_{\tilde{x}_1 \times \ldots \tilde{x}_n}$ of all input combinations $(x_1, \ldots x_n)$ resulting in y^*. The input combinations which result in y^* are referred to as realisations of y^*. The possibilistic interpretation of the extension principle is that if a value y^* can be achieved for different combinations of the input quantities, it will adopt its degree of possibility from the realisation with the highest degree of possibility.

The definition of the extension principle as in equation (11) is not readily implementable. For each value y of the observed output domain, it requires the complete set of realisations in the input domain to derive the membership value. This is an extremely difficult task. An alternative approach consists of searching in the output domain for sets that have an equal

degree of membership. This is achieved by analysing the input domain on a specific level of membership α. At this level, the α-cuts of the input quantities are defined as:

$$\mathbf{x_{i}}_{\alpha} = \left\{ x_i \in X_i, \mu_{\tilde{x}_i}(x_i) \geq \alpha \right\} \tag{12}$$

This means that an α-cut is the interval resulting from intersecting the membership function at $\mu_{\tilde{x}_i}(x_i) = \alpha$ (see figure 4). After deriving the α-cuts of all input quantities at a specific level, a general interval analysis as described in section 2.3.3 is performed on these intervals:

$$\mathbf{y}_{\alpha} = \left\{ y \mid \Big(x_i \in \mathbf{x_{i}}_{\alpha}, \forall i \Big) \Big(y = f\big(\{x\}\big) \Big) \right\} \tag{13}$$

There are two important observations concerning the resulting output interval of this analysis:

- The output interval results from combining all possible input quantities with a membership value larger than or equal to α. Therefore, following the extension principle, all output values in the interval have a membership value which equals at least α.
- Values outside this interval cannot be obtained using exclusively input values with membership values larger than or equal to α. Therefore, all values outside the interval have a membership smaller than α.

These observations basically state that the obtained output interval is an intersection of the output membership function at the α-level, and consequently represents an α-cut of the output. This means that a discretised approximation of the output membership function can be obtained from repeating the α-level procedure at a number of levels. Figure 4 clarifies this procedure for the FFE analysis. As such, the IFE analysis has become the numerical core of the FFE analysis. Therefore, section 3 of this paper focuses entirely on the implementation of IFEM, which can be easily extended to FFEM using the α-cut strategy.

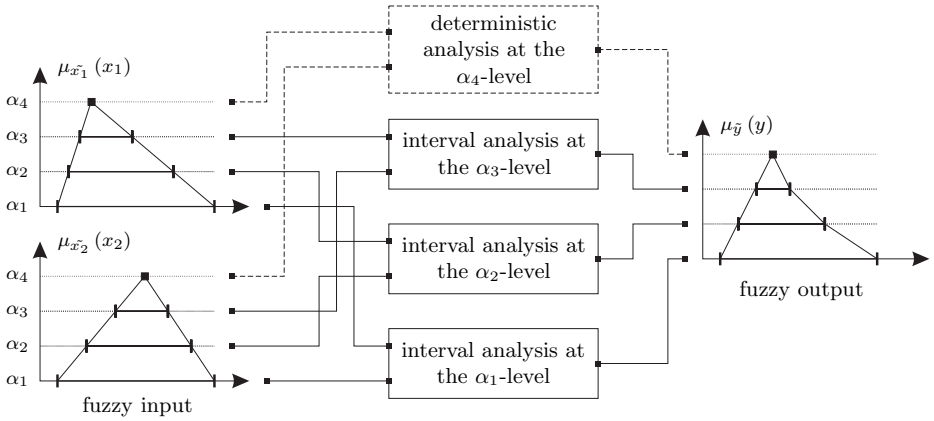

Figure 4: Scheme of the numerical procedure to perform a fuzzy FE analysis using 4 α-sublevels

2.4.4 Use and value of fuzzy analysis

The fuzzy concept can be applied to perform a possibilistic reliability analysis. This is based on the use of the membership function as limit CDFs as explained in section 2.4.2. It was shown by Ferrari et al. [38] that if the input membership functions represent boundaries on the CDFs of the input parameters, the membership function resulting from fuzzy analysis on this input forms reliable boundaries of the actual CDF of the result. Therefore, the fuzzy

result of a FFE analysis can be used to derive bounds on the probability of failure. TONON et al. [39] recently derived a fuzzy safety measure based on this approach.

A simple example illustrates the possible use of the outcome of a FFE analysis in a reliability framework. Suppose that a FFE analysis results in a membership function $\mu_{\tilde{\lambda}}(\lambda)$ representing a crucial eigenfrequency of a design as illustrated in figure 5. Suppose furthermore that a crisp criterion states that the design is acceptable if this eigenfrequency is kept below the value λ^*. The fuzzy result envelopes the exact CDF of the eigenfrequency. This means that the bounds on the probability that the eigenfrequency of the design lies below λ^* can be derived from the fuzzy result. The probability interval is obtained from taking the value of the envelope curves at λ^* as indicated in the figure by $\underline{P'_f}$ and $\overline{P'_f}$. The most conservative statement resulting from the analysis is that the probability of failure equals $(1 - \underline{P'_f})$ in the worst case.

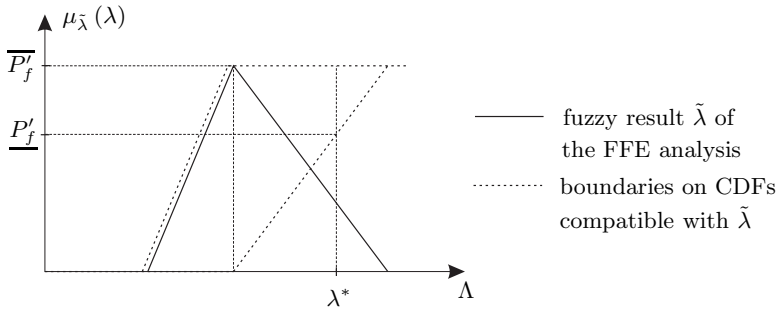

Figure 5: Example of the application of the fuzzy outcome of a FFE analysis to predict bounds on the probability of failure

Another useful application of the fuzzy concept is the possibilistic design optimisation, which enables an early optimisation of design parameters. From the α-cut strategy, it is clear that the fuzzy FE analysis is actually a large-scale sensitivity analysis of the combined effect of interval design variables and uncertainties on design requirements. It enables the analyst to calculate the α-level and corresponding design variable ranges for which the design meets the requirements according to the crisp interval reliability approach described in section 2.3.4. This is also referred to as *worst case optimisation* [40]. The analyst can control the analysis by defining the possibility distributions according to personal preference or practical limitations. A different possibility distribution for the design variables will yield a different possibility distribution of the analysis result, and consequently also different allowable ranges for the design variables. The design based on these alternative allowable ranges, however, is equally safe. In this context, the possibility distribution for a designer is rather a useful tool to control the allowable range for the uncertainties than an absolute quality measure.

2.5 Application of non-deterministic analysis in a design process

As principal conclusion of the discussion in the first part of this paper, we could state that the applicability of the different numerical non-deterministic concepts mainly depends on the amount of available objective information on the modelled properties. The application and value of the different concepts can be summarised analysing the necessity of non-deterministic techniques in a classical product design process:

- In an early design stage, the non-deterministic aspects are mostly uncertain rather than variable. Furthermore, in a general concept study, probabilistic aspects are seldom required. Therefore, the effect of uncertainties can be studied roughly without doing extensive reliability studies, based on the non-probabilistic approaches. An extremely useful application lies in the simultaneous study of different versions of a product based

on a single nominal design. Since the variations between different product versions are generally not of a probabilistic nature, the non-probabilistic approaches are of great value for this purpose.

- After concept approval, non-probabilistic analysis can be very valuable in the definition of design details, especially for the study of tolerances. Interval analysis enables an objective design validation based on the defined tolerances, while the fuzzy approach can be used to optimise them. Probabilistic analysis becomes more important when manufacturing process or material and component suppliers are specified, since those generally enable the use of objective probabilistic data.
- In a final stage of the design, generally more and more information becomes available, enabling a thorough objective probabilistic reliability analysis. Non-probabilistic techniques are less important in this stage, since they disregard the valuable probabilistic information.

This overview clearly indicates that non-probabilistic methods are complementary rather than competitive to probabilistic methods, as also noted in a recent NASA publication [41].

3 Implementation of the Interval Finite Element Method

The remainder of this paper now focuses entirely on the numerical implementation of the IFE analysis. Section 3.1 first describes two general strategies for the implementation. It then focuses on the interval system matrix assembly and the interval solution phase respectively in section 3.2 and 3.3.

3.1 Numerical strategies for the IFEM implementation

3.1.1 The global optimisation strategy

In essence, calculating the smallest hypercube around the solution set expressed in equation (7) is equivalent to performing a global optimisation, aimed at the minimisation and maximisation of the components of the deterministic analysis results $\{y\}$. The deterministic FE analysis is the goal function of the optimisation and the uncertain parameters are the design variables. The interval vector in which the uncertain parameters are contained defines the constraints for the variables. The optimisation is performed independently on every element of the result vector $\{y\}$. Therefore, the solution set of equation (7) becomes an interval vector describing the hypercube around the exact solution:

$$\{\mathbf{y}\} = \left\{ \begin{array}{c} \mathbf{y_1} \\ \mathbf{y_2} \\ \vdots \\ \mathbf{y_n} \end{array} \right\} \tag{14}$$

with:

$$\underline{y}_i = \min_{\{x\} \in \{\mathbf{x}\}} f_i\big(\{x\}\big),\ i = 1 \ldots n \tag{15}$$

$$\overline{y}_i = \max_{\{x\} \in \{\mathbf{x}\}} f_i\big(\{x\}\big),\ i = 1 \ldots n \tag{16}$$

An efficient and robust optimisation algorithm is primordial for this solution strategy. Rao et al. [7] applied Powell's method to tackle the optimisation. Köylüoğlu [4] defined a linear programming solution for this purpose. The input interval vector defines the number of constraints and, therefore, strongly influences the performance of the procedure. Also, because of the required execution of the deterministic FE analysis in each goal function

evaluation, the optimisation approach is numerically expensive. Therefore, this approach is best suited for rather small FE models with a limited number of input uncertainties. These limitations of the global optimisation strategy have initiated the development of the alternative interval arithmetic strategy.

3.1.2 The interval arithmetic strategy

The interval arithmetic approach is the analytical counterpart of the global optimisation strategy. It consists of translating the complete deterministic numerical FE procedure to an interval procedure using the arithmetic operations for addition, subtraction, multiplication and division of interval scalars. The outline of the interval procedure corresponds completely to the deterministic FE analysis. The main difference is that each substep of the interval algorithm calculates the range of the intermediate subfunction instead of the deterministic result. Application of this principle on the complete algorithm results in the range of the output of the analysis.

There is an important drawback for this method. The inclusion property for ranges of nested functions states that an arithmetic interval operation introduces conservatism in its result if it neglects correlation that exists between the operands. A simple example illustrates this. Consider the function:

$$f(x) = x^2 - x \tag{17}$$

applied on the interval $\mathbf{x} = [0, 1]$. Applying interval arithmetic, both terms are considered independently. This results in the interval solution $[-1, 1]$. However, the exact range of the function equals $\left[-\frac{1}{4}, 0\right]$.

In an automatic computer procedure as for the IFE analysis, this phenomenon cannot be avoided because it is impossible to keep track of the relationships between all intermediate results of the algorithm. Consequently, each interval substep results in an enclosure of the exact substep range. Therefore, also the final result is a conservative approximation of the exact range of the FE analysis. Generally, the degree of conservatism is unknown. It possibly can be too high to be useful for practical applications.

The implementation of the interval arithmetic IFE approach consists of two parts:

1. The translation of the input intervals to an interval system description in the form of interval system matrices. These are obtained by translating the deterministic assembly procedure to interval analysis and results in the interval system matrices.

2. The approximation of the solution of the analysis expressed as an interval problem using the interval system matrices. For structural dynamic analysis, these are denoted by $[\mathbf{K}], [\mathbf{M}]$ and $[\mathbf{C}]$. The exact solution set then becomes:

$$\left\{ \{y\} \mid \Big([K] \in [\mathbf{K}]\Big)([M] \in [\mathbf{M}]\Big)([C] \in [\mathbf{C}]\Big)\Big(\{y\} = f\big([K], [M], [C]\big)\Big) \right\} \tag{18}$$

   with f the function representing the calculation of the analysis result based on the system matrices.

While nearly all literature on IFE and FFE is based on the solution phase, the interval matrix assembly phase has never been analysed thoroughly. Therefore, section 3.2 gives a description of the interval matrix assembly, focussing on the conservatism introduced by the interval translation. Still, the applicability of the interval arithmetic strategy mainly depends on the availability of a calculation procedure for the hypercubic approximation of the solution set of equation (18). Section 3.3 gives a state-of-the-art of the numerical procedures applicable for this purpose.

3.2 Interval arithmetic IFE matrix assembly

3.2.1 IFEM element matrices

The calculation of an element stiffness or mass matrix depends on the complexity of the element type. For 1D elements, the analytical expressions of the element matrices are generally available in reference to the element co-ordinate system. The corresponding interval element matrices simply result from substituting the non-deterministic interval properties directly into this analytical expression. Whether the interval entries for 1D elements describe the exact range or are conservative approximations depends on the type of the interval properties. Basically, only interval properties that have multiple occurrences in the calculation of one global matrix entry cause an overestimation in the corresponding entry of the interval matrix.

Interval material properties generally have no multiple occurrences in a single global matrix entry calculation. Therefore, these do not introduce conservatism.

Interval geometry properties on the other hand form a major source of conservatism. For instance, when the global geometry of the model is uncertain, defined by interval scalars for the uncertain element's nodal co-ordinates, there exists a correlation between the orientation angle and the size of an element. The interval matrix rotation neglects this correlation, since it considers the rotation matrix and the local element matrix as independent entities.

For 2D and 3D elements, element matrices are generally calculated by applying a numerical integration scheme. The deterministic Gauss-Legendre numerical integration is based on the evaluation of the integrand in a predefined number of Gauss points in the element's natural co-ordinate system. For 3D elements, the stiffness matrix yields:

$$[K^e] \simeq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} H_i H_j H_k \left[f_K\left(\xi_i, \eta_j, \zeta_k\right)\right] \tag{19}$$

with

$$\left[f_K\left(\xi, \eta, \zeta\right)\right] = \left[B\left(\xi, \eta, \zeta\right)\right]^T [D] \left[B\left(\xi, \eta, \zeta\right)\right] \det\left[J\left(\xi, \eta, \zeta\right)\right] \tag{20}$$

with (ξ_i, η_j, ζ_k) the evaluation co-ordinates in the natural co-ordinate system, H_i,H_j and H_k the corresponding weights from the Gauss-Legendre integration scheme and $[J]$ the Jacobian describing the transformation between the element's natural co-ordinate system and the global co-ordinate system. The calculation of the interval element matrix requires the translation of this numerical integration scheme to interval arithmetic. Again, the effect of interval properties depends on the type of the non-deterministic property.

Interval material properties are implemented by substituting the corresponding interval objects directly in the integrand evaluation at the Gauss points. These properties generally have no multiple occurrences, and therefore do not introduce conservatism.

Interval geometry properties affect both the Jacobian and the $[B]$ matrix in the integrand. The calculation of $[\mathbf{B}]$ and $\det[\mathbf{J}]$ in the Gauss points is based on the translation of the deterministic matrix inversion and determinant calculation schemes to interval arithmetic. These calculation schemes generally consist of a large number of operations on the matrix entries. The interval translation therefore causes a high degree of conservatism, since the Jacobian matrix entries are strongly coupled. Furthermore, multiplying the interval matrices as in equation (20) again unlinks the $[\mathbf{B}]$ matrix entries from $\det[\mathbf{J}]$ while all are based on the common uncertain geometry of the element.

Another source of conservatism results from the weighted averaging of the integrand intervals obtained at the Gauss points. The interval averaging considers the contribution of each Gauss point independently, while in reality these are coupled through the uncertainties used to describe the integrand.

3.2.2 IFEM system matrices assembly

The total interval system matrices are obtained from assembly of the interval element matrices. The interval counterpart of the deterministic assembly phase is very simple, since it consists of the addition of interval scalars for the entries of the interval element matrices. Still, this phase combines independently the interval element matrices. It unlinks as such the physical properties that different elements may have in common. This is easily illustrated taking the example of an uncertain Young's modulus defined globally for all elements. The result of independent addition of interval stiffness matrix entries from different elements implicitly encloses the result obtained from using different values for the Young's modulus in each element. This does not comply with the realistic interpretation of the uncertainty, where the Young's modulus is uncertain, but equal for all elements. Therefore, this phase increases the degree of conservatism for all total system matrix entries that result from the addition of element matrix entries with common uncertain properties.

Another source of conservatism arises at this point. It results from the fact that an interval matrix covers all possible combinations of its entries within their prescribed bounds, as stated in equation (5). As such, the interval matrix neutralises every possible dependency between the entries of the matrix. Consequently, the interval matrices generally loose important specific matrix properties as symmetry or positive-definiteness since not all matrices within the defined set result from physical models. Hence, the algorithms that rely on these properties are not necessarily extendable to the domain of interval analysis. This phenomenon is very likely to occur, since different matrix entries are generally based on common physical properties. For instance, for a mass matrix it is obvious that the entries in an element matrix are coupled through the density and geometrical properties of the element. Combining different values for material density in the different entries of a single element's mass matrix is artificial, yet implicitly enabled by the interval system matrix.

3.2.3 Discussion

The major advantage of the interval arithmetic strategy is its simplicity. The element and system interval matrix assembly proves to be a series of simple interval operations on the interval properties. This is rather easy to implement. From a numerical viewpoint, the basic interval operations consist of the corresponding deterministic operations on some combination of the lower and upper bounds of the operands. An interval matrix assembly requires roughly between two and four times the computational effort of the corresponding deterministic matrix assembly.

The major drawback of this method is its repeated vulnerability to conservatism. In the assembly phase, there are three major sources of conservatism:

- neglecting the correlation between the terms that constitute the integrand in the numerical integration for the element matrix computation
- neglecting the correlation between element matrix entries of different elements during the assembly
- neglecting the internal correlation between total system matrix entries

Each of these is caused by wrongly unlinking intermediate variables of the algorithm that are in reality coupled through common uncertain physical properties. The impact of each of these on the final analysis result depends strongly on the type of analysis and the nature of the uncertainties:

- For the first source, the amount of conservatism depends strongly on the type and complexity of the element. Using simple elements and avoiding geometry uncertainties is the recipe to neutralise all conservatism in this phase of the algorithm. However, in particular the latter is inherent to the problem description and, therefore, not to be decided for by the analyst.

- The second source is present if there are uncertainties common to a group of elements. While theoretically possible, it seems very unlikely that in a realistic analysis each element has its own independent uncertainty description, which makes this source of conservatism present in nearly all analyses.

- The third source of conservatism cannot be avoided, since the entries of a system matrix are mutually always closely related.

During the solution phase, even a fourth important source of conservatism arises. It stems from neglecting the correlation between the different system matrices during the solution, which is common practice in current literature. Especially when geometrical uncertainties are present, this correlation can be very high.

Table 1 gives an overview of all the sources of conservatism which are to some extent controllable by the analyst. It could serve as a guideline for controlling the amount of conservatism for interval uncertainty modelling. Though the guidelines of table 1 may serve well for some applications with limited complexity, it is the author's opinion that the approach based on interval system matrices needs special attention when it is used for implementation in a software environment intended for general uncertainty analysis. Especially for large models, the conservatism in the interval matrices increases to a very high level through the unlinking of elements and internal system matrix entries. Depending on the effect of the intended solution algorithm, the conservatism of the final result will increase to an unacceptable level. Therefore, in order for the result to be used for design validation or optimisation, a thorough verification of the conservatism is advisable. How this can be done should be a principal part of the future research on IFE analysis.

analysis phase	source of conservatism
1D element matrices	geometrical uncertainties
2D & 3D element matrices	geometrical uncertainties
system matrices assembly	common uncertainties between elements
analysis phase	algorithm applied on system matrices

Table 1: Controllable sources of conservatism during an IFE analysis

3.3 Interval arithmetic IFE solution phase

3.3.1 Linear equilibrium IFE analysis

For the vast majority of FEM problems, the second phase of the interval analysis consists of an equilibrium or steady-state problem. This is numerically equivalent to a matrix equation that requires the solution of a system of equations. The corresponding interval problem yields:

$$\langle \{y\} \rangle = \left\{ \{y\} \mid \Big([A] \in [\mathbf{A}] \Big) \Big(\{b\} \in \{\mathbf{b}\} \Big) \Big([A]\,\{y\} = \{b\} \Big) \right\} \tag{21}$$

with $\{\mathbf{b}\}$ the interval vector representing the uncertain generalised loading of the considered model. This solution set contains all vectors $\{y\}$ which are a solution of the matrix equation $[A]\,\{y\} = \{b\}$ with $[A]$ and $\{b\}$ ranging respectively over the interval objects $[\mathbf{A}]$ and $\{\mathbf{b}\}$. From the world of interval arithmetic, this solution set is referred to as the *united solution set* [42] and denoted by $\Sigma_{\exists\exists}\big([\mathbf{A}], \{\mathbf{b}\}\big)$. Though other definitions of solution sets corresponding to similar interval problems exist [43], the united solution set is considered to be the most meaningful in designing under uncertainty.

Before reviewing the procedures to calculate an approximate hypercube for the united solution set, it is important to gain knowledge of the general properties of this solution set. In general, the components of the exact united solution set are related because they result

from common interval coefficients in the equations. The exact solution set was described analytically by the Oettli-Prager [44] lemma:

$$\{y\} \in \Sigma_{\exists\exists}\Big([\mathbf{A}], \{\mathbf{b}\}\Big) \Leftrightarrow \Big|\big[\check{\mathbf{A}}\big]\,\{y\} - \{\check{\mathbf{b}}\}\Big| \leq \big[\bar{\mathbf{A}}\big]\Big|\{y\}\Big| + \{\bar{\mathbf{b}}\} \tag{22}$$

with $\Big|\{y\}\Big|$ containing the absolute values of each component of the vector $\{y\}$.

In order to illustrate the complex form of the united solution set, a simple example is analysed. Consider the following interval system of equations:

$$\begin{bmatrix} 3.5 & [0,2] & [0,2] \\ [0,2] & 3.5 & [0,2] \\ [0,2] & [0,2] & 3.5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{Bmatrix} \tag{23}$$

For this problem, the description of the exact united solution set as defined in equation (21) is possible using the Oettli-Prager lemma. All combinations of y_1, y_2 and y_3 that satisfy the inequality in the right-hand side of equation (22) are inside the united solution set. It can be proven that this inequality defines a polyhedron in the output space, as shown in figure 6 for this specific case.

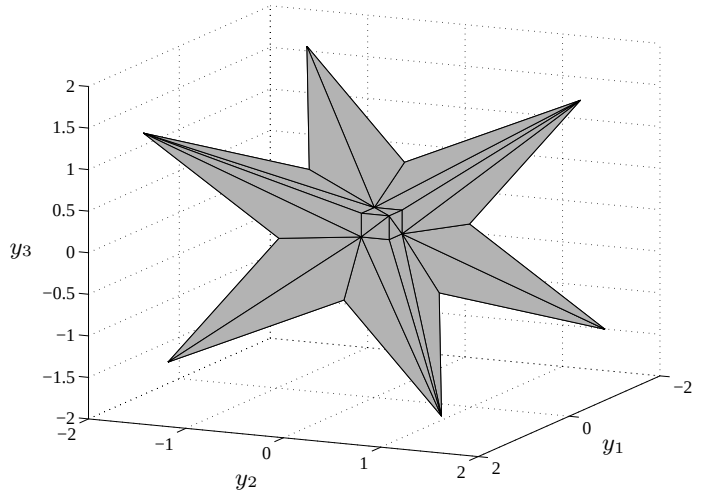

Figure 6: A non-convex polyhedron as solution set of a system of three interval equations

This figure clearly illustrates the complex nature of the united solution set. It indicates that it will be very difficult to describe the united solution set based on the Oettli-Prager lemma for realistically sized systems, as shown by Neumaier [45]. Furthermore, it has been shown in literature that any iterative solution scheme which pursues this exact solution set is extremely hard even for small academic problems. Therefore, current research in this domain focuses on the development of efficient calculation procedures of hypercubic approximations of the exact solution set. The following section gives an overview of the methods currently available in literature.

3.3.2 Numerical procedures for linear equilibrium IFE analysis

Basically, any deterministic procedure that solves a system of equations is a candidate for step-by-step translation to the interval approach. One has to keep in mind, however, that the conservatism of the solution increases with every interval operation. Therefore, the amount of conservatism depends strongly on the chosen algorithm, as clearly demonstrated by Hanss [46] comparing the result of the interval equivalent of an FE analysis using the direct matrix inversion and decomposition technique.

One of the first efforts to give a hypercubic approximation of the united solution set of a system of interval equations is given by Alefeld et al. [47], who applied the interval equivalent of the standard Gaussian elimination scheme. Neumaier [45] formalised and extended the Gaussian approach by preconditioning the interval matrices. Due to the large number of required interval operations, this method is subject to extreme conservatism as far as realistic FE models are concerned. Rao et al. [48] proposed a strategy to limit the conservatism in the Gaussian approach, by placing a maximum on the width of the intervals in each substep of the algorithm. The truncation threshold is determined based on the width of the input variables. While this is a simple approach, it is a rather artificial and unsafe method to reduce conservatism.

More recently, Shary [42, 49] introduced an algebraic approach for the hypercubic approximation. His approach, however, poses strong restrictions on the properties of the interval matrix. These properties are seldom met by a generalised FE stiffness matrix. Over the last ten years, an iterative research spread over different research groups around the world has led to the Hansen-Bliek-Rohn-Ning-Kearfott procedure. The procedure is based on an unexpected theoretical result by Hansen [50] who gave explicit formulas for the outer interval estimates of the united solution set. The procedure was then modified by Rohn [51], Ning and Kearfott [52]. Finally, Neumaier [53] gave a substantiation of the procedure that provides an additional insight into its nature. Neumaier also pointed out how to render the algorithm rigorous in finite precision computer arithmetic. These recent developments made the algorithm a possible candidate for application in the IFE procedure. However, it has not yet been implemented and tested for realistic FE analysis in its current optimised form.

The remainder of this section gives a brief conceptual overview of the successfully implemented solution techniques for the hypercubic approximation of the united solution set available in recent literature. See also Rao et al. [48] and McWilliam [54] for an elaborated comparison and numerical examples of some strategies.

The vertex method

Dong et al. [55] first introduced the vertex method. This method approximates the range of the result of a numerical procedure by introducing all possible combinations of the boundary values of the input intervals into the analysis. For N input intervals, there are 2^N vertices for which the analysis has to be performed. These vertices are denoted by $\{c_j\}, j = 1, \ldots 2^N$. Each of these represents one unique combination of lower and upper bounds on the N input intervals. The approximate analysis range is deduced from the extreme values of the set of results for these vertices:

$$\{\mathbf{y}\} \approx \left[\min_j f\big(\{c_j\}\big), \max_j f\big(\{c_j\}\big)\right] \tag{24}$$

Despite its simplicity, this method has some important disadvantages. It is clear from equation (24) that the computational cost increases exponentially with the number of input intervals. This limits the applicability of the vertex method to rather small systems, or systems with very few interval entries in the system matrices. The main disadvantage of this method, however, is that it cannot identify local optima of the analysis function which are not on the vertex of the input space. It only results in the smallest hypercube if the analysis function is monotonic over the considered input range. This is a strong condition that is difficult to verify for FE analysis because of the complicated relation of analysis output to physical input uncertainties. The approximation obtained when monotonicity is not guaranteed is not necessarily conservative as discussed in [56]. This fact reduces the validity of this method for design validation purposes. Recently, Hanss [57] suggested an extension of the classical vertex procedure in order to deal with this problem of non-monotonicity.

A design of experiments based approach

Rao et al. [58] introduced an algorithm that searches the extreme vertex results without

the necessity to solve the matrix equation in all vertices. He considers each equation of the system individually and uses a search-based approach, starting from the solution of the crisp equation in the midpoint of the interval system. The lower and upper bounds of the right-hand term of the ith equation $\underline{b_i}$ and $\overline{b_i}$ are taken as the two target values for a search resulting in $\{\underline{x_i}\}$ and $\{\overline{x_i}\}$ respectively:

$$\sum_{j=1\ldots n} \mathbf{a_{ij}}\, \underline{x_{ij}} \;=\; \underline{b_i} \tag{25}$$

$$\sum_{j=1\ldots n} \mathbf{a_{ij}}\, \overline{x_{ij}} \;=\; \overline{b_i} \tag{26}$$

In order to perform this search, a TAGUCHI approach is adopted from the world of design of experiments. Once the local solutions are found for all equations, the global solution is determined as the intersection of the individual local solutions:

$$\{\mathbf{x}\} = \bigcap_{i=1,\ldots n} \{\mathbf{x_i}\} \tag{27}$$

The numerical results show that this method is far less expensive than the vertex method [58]. Still, it focuses on the vertex solution, and therefore, has only limited value in the field of numerical reliability analysis of uncertain structures.

RUMP**'s inclusion method**

Another method is based on the inclusion method of RUMP [59], which states that the calculation of a hypercubic approximation of the united solution set $\Sigma_{\exists\exists}\big([\mathbf{A}], \{\mathbf{b}\}\big)$ is equivalent to the solution of:

$$\{\mathbf{y}^*\} = [\mathbf{G}]\{\mathbf{y}^*\} + \{\mathbf{g}\} \tag{28}$$

with

$$[\mathbf{G}] = [I] - [R]\,[\mathbf{A}] \tag{29}$$

$$\{\mathbf{g}\} = [R]\,\big(\{\mathbf{b}\} - [\mathbf{A}]\{\mathbf{y_0}\}\big) \tag{30}$$

$$\{\mathbf{y}\} = \{\mathbf{y_0}\} + \{\mathbf{y}^*\} \tag{31}$$

The matrix $[R]$ is an arbitrary non-singular matrix and $\{y_0\}$ a start vector. The algorithm for solving equation (28) is an iterative procedure that generally leads to sharp bounds of the united solution set quite fast. It has been successfully implemented by DESSOMBZ et al. [6] in the framework of a structural dynamic IFE procedure. The algorithm, however, needs to be adapted in order to take internal relationships between matrix entries into account, and, therefore, needs some extra research before it can be generally applied in IFE analysis procedures.

The interval perturbation analysis

In this approach, the interval matrix equation is rewritten using an alternative perturbation notation for an interval object:

$$[\mathbf{X}] = \big[\check{\mathbf{X}}\big] + [\mathbf{\Delta X}] \tag{32}$$

with

$$[\mathbf{\Delta X}] = \frac{\Big[-\big[\bar{\mathbf{X}}\big], \big[\bar{\mathbf{X}}\big]\Big]}{2} \tag{33}$$

Introducing this notation in the system of interval equations yields:

$$\Big(\big[\check{\mathbf{A}}\big] + [\mathbf{\Delta A}]\Big)\Big(\{\check{\mathbf{y}}\} + \{\mathbf{\Delta y}\}\Big) = \Big(\{\check{\mathbf{b}}\} + \{\mathbf{\Delta b}\}\Big) \tag{34}$$

After substituting the inverse of $[\mathbf{A}]$ and neglecting second-order terms, the solution is given by:

$$\{\mathbf{\Delta y}\} = \big[\check{\mathbf{A}}\big]^{-1}\{\mathbf{\Delta b}\} - \big[\check{\mathbf{A}}\big]^{-1}[\mathbf{\Delta A}]\{\check{\mathbf{y}}\} \tag{35}$$

This approach was applied for the implementation of IFEM by Qiu [60]. It does not necessarily yield a conservative approximation, as the effect of neglecting the higher order terms is unpredictable. Furthermore, in order to safely remove the higher order terms, this method is only applicable to analyses with small intervals. An improvement of this method was introduced by the same author in the form of the subinterval perturbation method which divides large input intervals into smaller subintervals, and combines the results of all interval analyses on the corresponding subproblems. Lallemand [61] applied this method to solve the interval eigenvalue analysis. Later, Cherki [8] applied this method in the analysis of structures with uncertain boundary conditions.

It appears that up to now, there is no generally accepted optimal procedure to approximate the united solution set. Different methods are being used, which all have very specific consequences for the analysis. So far, none of the methods has been thoroughly validated using realistic industrially sized problems.

3.3.3 IFE eigenvalue analysis

The deterministic procedure of the FE eigenvalue analysis consists of the assembly of the system stiffness and mass matrices $[K]$ and $[M]$, after which the deterministic eigenvalue λ_i satisfies the equation:

$$[K]\{\phi_i\} = \lambda_i [M]\{\phi_i\} \tag{36}$$

with $\{\phi_i\}$ the corresponding eigenvector. The equivalent interval arithmetic IFE procedure requires the calculation of the solution set:

$$\left\{\lambda_i \mid \Big([K] \in [\mathbf{K}]\Big)\Big([M] \in [\mathbf{M}]\Big)\Big([K]\{\phi_i\} = \lambda_i [M]\{\phi_i\}\Big)\right\} \tag{37}$$

with $[K]$ and $[M]$ incorporating implicitly the dependency of the system matrices on the input parameters. It can be shown that, assuming independent interval system matrices, the bounds of this exact solution set are achieved for vertex matrix combinations [62]. This means that the exact solution of the interval eigenvalue problem can be found. Therefore, the solution phase introduces only conservatism through the artificial independence of the stiffness and mass matrix. The exact amount of this conservatism is entirely problem-dependent. Additionally, there can be a substantial amount of conservatism in the result caused by the interval system matrices assembly, as discussed in the previous section.

Some algorithms have been developed which efficiently calculate the exact vertex solution of the interval eigenvalue problem. Chen et al. [62] introduced a non-iterative procedure based on the Rayleigh quotient, which states that the lower and upper bound on the ith eigenvalue follow directly from two deterministic eigenvalue problems:

$$\left(\left[\check{\mathbf{K}}\right] + \left[S^i\right]\left[\bar{\mathbf{K}}\right]\left[S^i\right]\right)\{\overline{\phi_i}\} = \overline{\lambda_i}\left(\left[\check{\mathbf{M}}\right] - \left[S^i\right]\left[\bar{\mathbf{M}}\right]\left[S^i\right]\right)\{\overline{\phi_i}\} \tag{38}$$

$$\left(\left[\check{\mathbf{K}}\right] - \left[S^i\right]\left[\bar{\mathbf{K}}\right]\left[S^i\right]\right)\{\underline{\phi_i}\} = \underline{\lambda_i}\left(\left[\check{\mathbf{M}}\right] + \left[S^i\right]\left[\bar{\mathbf{M}}\right]\left[S^i\right]\right)\{\underline{\phi_i}\} \tag{39}$$

with $\left[S^i\right] = \mathrm{diag}\left(\mathrm{sgn}(\phi_i^1), \ldots \mathrm{sgn}(\phi_i^n)\right)$ and $\{\phi_i\}$ the ith eigenvector from the deterministic analysis at the midpoints of the matrices. This method requires all the components of the eigenvector to have a constant sign over the considered domain and does not allow the occurrence of an eigenfrequency cross-over in the input parameter space. An enhanced methodology was developed by El-Gebeily et al. [63]. It provides a solution for the original problem with an extra restriction of symmetry on the considered system matrices:

$$\left\{\lambda_i \mid \Big([K_s] \in [\mathbf{K}]\Big)\Big([M_s] \in [\mathbf{M}]\Big)\Big([K_s]\{\phi_i\} = \lambda_i [M_s]\{\phi_i\}\Big)\right\} \tag{40}$$

with K_s and M_s symmetric. The most important effect of this extra restriction is that it intrinsically removes the conservatism resulting from allowing artificial non-symmetric system matrices. The numerical procedure is based on the interval translation of the deterministic Sturm sequence. It proves to be an efficient iterative algorithm. Unfortunately, it is limited to tridiagonal system matrices. This makes it only applicable for specific cases.

3.4 Hybrid IFE analysis

In order to extend the applicability of IFEM, a general remedy to excessive conservatism is introduced. It is a hybrid procedure, consisting of both a global optimisation and an interval arithmetic part. In the first part, an optimisation is applied to calculate the interval result at some intermediate step of the total algorithm. In the second part, the interval analysis is performed on these intermediate results. This method has two major advantages:

- because of the global optimisation, all conservatism prior to the optimised intermediate result is neutralised
- the performance of the optimisation step is controllable by adequately choosing the level on which to perform it

This approach has been successfully applied in an IFE procedure for the calculation of interval FRFs [11]. In the first part of this procedure, the optimisation is used to translate the interval properties defined on the FE model to the exact interval modal stiffness and mass parameters of the structure. The calculation of the envelope FRFs in the second part is done by applying the interval arithmetic equivalent of the modal superposition procedure on these interval modal parameters. This procedure neutralises all conservatism in the matrix assembly phase, since it directly uses the modal parameters as goal functions in the optimisation part. The final envelope FRFs have been proven to contain only a very limited amount of conservatism.

4 Numerical example

Both the optimisation and interval arithmetic approach are now demonstrated on an IFE eigenvalue analysis. First, the reference model and the implementation of both approaches are briefly discussed. Then, the results for a simple 2D structure are compared.

4.1 Reference model

The deterministic reference model illustrated in figure 7 is a two dimensional beam model with 27 DOFs. It has 10 resonance frequencies between 20 and 60 Hz. It was originally defined as a benchmark problem for modal updating procedures, and has an artificial complexity [64]. It was designed such that it is extremely vulnerable to eigenfrequency cross-over. Table 2 and table 3 give the elemental and nodal input data for the reference model.

element number	density $10^5 kg/m^3$	area $10^{-4}m^2$	moment of inertia $10^{-6}m^4$
1	0.5384	7.4283	369.5200
2	1.0510	5.7103	5.2872
3	1.4010	6.8425	7.1377
4	1.0770	9.2854	527.8800
5	1.0510	5.7103	5.2872
6	1.4010	4.5682	6.6090
7	5.0450	0.1487	1.9457
8	10.0900	0.1487	2.8198
9	0.0000	0.0857	0.4640

Table 2: Elemental properties of the reference model

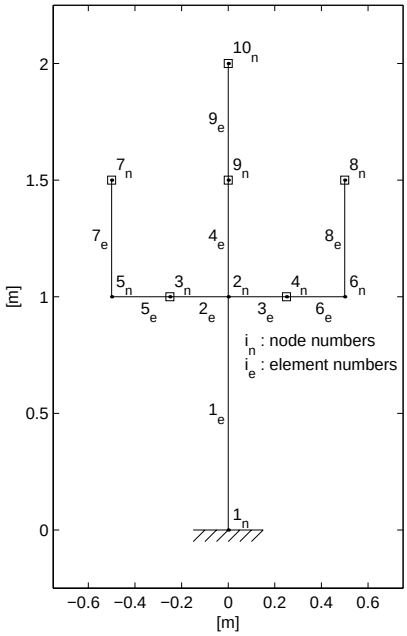

Figure 7: Reference model for the comparison of the global optimisation and interval arithmetic approach for the eigenvalue IFE analysis

nodal point number	mass kg	rotational mass kgm^2
3	180.0	42.0
4	165.0	42.0
7	14.0	1.8
8	39.0	4.2
9	360.0	37.5
10	12.0	1.0

Table 3: Nodal properties of the reference model

4.2 Implementation of the IFE eigenvalue analysis

The IFE procedures are implemented in the MATLAB environment. The implementation of the FE model is based on two functions that assemble the global stiffness and mass matrix as a function of the input parameters. In the interval arithmetic procedure, these functions are used to calculate the interval system matrices. The solution step of the interval arithmetic procedure uses the method of CHEN given in equations (38) and (39).

The application of the global optimisation strategy requires the implementation of the function $\Lambda_i^x(\{x\})$:

$$\lambda_i = \Lambda_i^x(\{x\}) \tag{41}$$

The bounds on the ith eigenvalue then can be obtained through a global optimisation:

$$\lambda_{\mathbf{i}} = \left[\min_{\{x\}\in\{\mathbf{x}\}} \left(\Lambda_i^x(\{x\})\right), \max_{\{x\}\in\{\mathbf{x}\}} \left(\Lambda_i^x(\{x\})\right) \right] \tag{42}$$

The optimisation procedure uses the matrix assembly functions inside these goal function evaluations. They constitute the structure on which the eigenvalue analysis is performed at each iteration point. The optimisation itself is implemented using a sequential quadratic

programming (SQP) algorithm. In order to enhance the numerical performance of the optimisation, the analytical derivatives of the eigenvalues are implemented as direct functions. In that case, a subspace trust region method based on the interior-reflective NEWTON method is applied.

4.3 Comparison of the optimisation and interval arithmetic approach

The model has three uncertain parameters: the length of the upper element (element 9), the lumped mass localised in node 3 and the cross section area of element 6. Table 4 gives the interval properties for the analysed models.

property	interval
length beam 9 $[m]$	$[0.475, 0.525]$
lumped mass node 3 $[kg]$	$[170, 190]$
area element 6 $[m^2]$	$[4 \times 10^{-4}, 5 \times 10^{-4}]$

Table 4: Interval properties in the interval eigenfrequency problem solved with the interval arithmetic approach

Figure 8 compares the eigenfrequency ranges resulting from the interval arithmetic approach with those resulting from the optimisation strategy. This figure clearly indicates that the eigenfrequency ranges resulting from the interval arithmetic approach are very conservative approximations of the exact ranges that are found with the optimisation strategy. The apparent advantage that the interval arithmetic approach is a fast analytical procedure in this case is non-existent in view of the extreme conservatism.

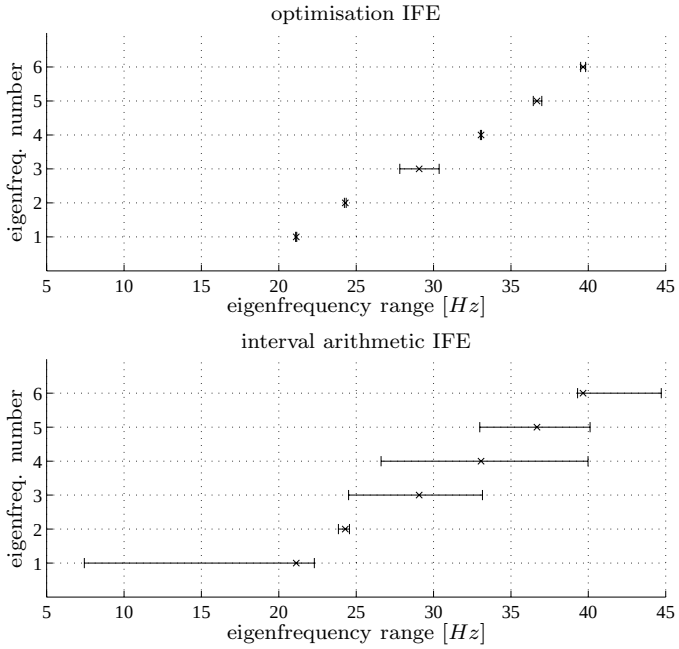

Figure 8: Eigenfrequency ranges of the first six modes using the optimisation and the interval arithmetic approach

5 Conclusion

The emerging non-probabilistic approaches are redefining the landscape for non-deterministic FE analysis. It is the aim of this paper to give insight into the possible useful applications of these approaches, referring to the generally accepted and widely adopted probabilistic approach.

A study of the credibility of probabilistic analysis based on different types of non-deterministic model properties shows that it remains by far the most interesting approach to tackle problems that are subject to complete and objective probabilistic influences. For these problems, the non-probabilistic approaches all disregard probabilistic information, and therefore represent less information than is objectively available. However, in the presence of uncertain quantities that require subjective information in order to be described numerically, the interval and fuzzy approach become increasingly interesting. The fuzzy concept is shown to be a perfect tool for an analyst who wants to study the effect of uncertainties of which he has expert knowledge or which he wants to optimise in the design. This makes the non-probabilistic approaches most valuable in early design stages, whereas the probabilistic approach remains indispensable in later stages. This leads to the conclusion that the non-probabilistic approaches should be regarded as complementary rather than competitive to the probabilistic approach.

This paper further shows that the popular interval arithmetic implementation of the IFE procedure is extremely vulnerable to conservatism, especially during the interval system matrices assembly phase. This conservatism causes the IFE result to be a severe overestimation of the exact range of the FE analysis, as shown in the numerical example. A hybrid solution procedure has been proposed to reduce the excessive conservatism in IFE procedures. Still, there is very little effort in literature to remedy, study or even acknowledge the sources of conservatism inherent to the interval arithmetic procedure. Nearly all current research in this area starts from the interval system matrix formulation of the problem, focussing entirely on the approximation of the united solution set. While this is one of the greatest numerical challenges IFE research is facing, one fact remains too often covered in silence: even if the exact solution of the united solution set can be obtained, there remains a substantial amount of conservatism with respect to the original problem due to the conservatism in the matrix assembly phase. Next to the numerical efficiency, limiting this conservatism will be a key feature in the success of any numerical IFE procedure.

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