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Parametric Rational Model Order Reduction for Nonlinear Systems Subject to Periodic Inputs

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Giovanni Conni, Frank Naets, Karl Meerbergen
KU Leuven
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0 Outline

1. Introduction
2. psvAAA
3. Numerical tests
4. Conclusion

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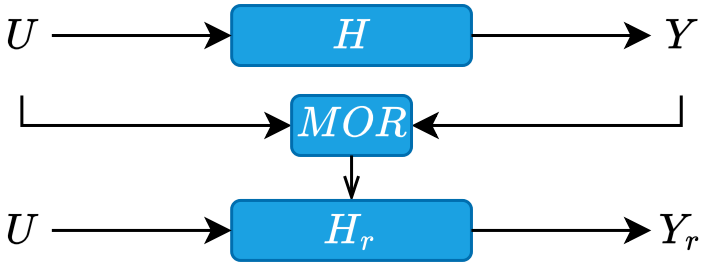
1 LTI system



$$\text{LTI SISO system} \Rightarrow \begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ y = C\boldsymbol{x} \end{cases}$$

- Large system, fast computations
- Black-box: no information on Σ, data-driven methods
- Rational-approximation based methods: Vector Fitting (Gustavsen et al. 1999), Loewner Framework (Mayo et al. 2007), AAA (Nakatsukasa et al. 2018)

1 Frequency domain MOR



$$Y = H \cdot U = C\left(s\mathbb{1} - A\right)^{-1} B \cdot U \quad \rightarrow \quad Y_r = H_r \cdot U = C_r\left(s\mathbb{1} - A_r\right)^{-1} B_r \cdot U$$
$$H : \mathbb{C} \rightarrow \mathbb{R}^n, \quad H_r : \mathbb{C} \rightarrow \mathbb{R}^m,\ m \ll n$$

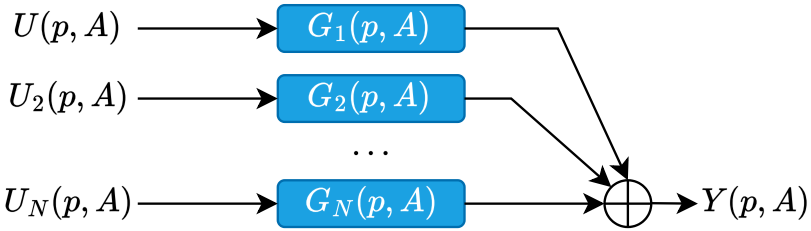
1 Nonlinear system



$$\text{Weakly nonlinear system} \Rightarrow \begin{cases} \dot{\boldsymbol{x}} = F(\boldsymbol{x}) + Bu \\ y = C\boldsymbol{x} \end{cases}$$

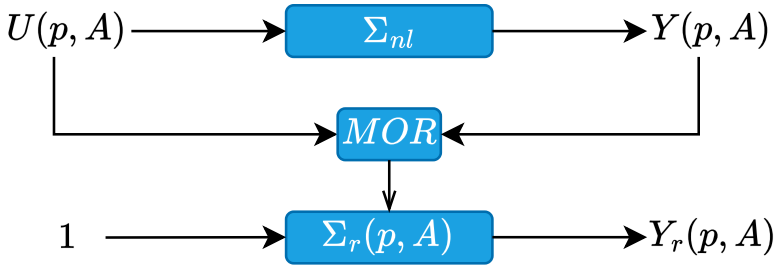
- No transfer function H, but functional series (Cheng et al. 2017)
- Volterra series:
  $Y(\omega) = \sum_{n=1}^{\infty} \frac{1/\sqrt{N}}{(2\pi)^{N-1}} \int_{\omega_1+\cdots+\omega_n=\omega} H_n(\omega_1, \ldots, \omega_n) \prod_{i=1}^{n} U(\omega_i) d\sigma$
- NOFRFs: $Y(\omega) \sum_{n=1}^{\infty} G_n(\omega) U_n(\omega)$, with
  $G_n(\omega) = \int_{\omega_1+\cdots+\omega_n=\omega} H_n(\omega_1,\ldots,\omega_n) \prod_{i=1}^{n} U(\omega_i) d\sigma \Big/ \int_{\omega_1+\cdots+\omega_n=\omega} \prod_{i=1}^{n} U(\omega_i) d\sigma$

1 NOFRFs



The input family depends on two parameters, A and p, such that its poles depend on one but not the other, e.g. periodic functions of frequency $p = 2\pi\omega_F$ and amplitude A.

1 Method



METHOD: build a parametric LTI ROM which approximates the outputs for a family of inputs

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2 psvAAA: barycentric form

$$H_{psv}(s;p,q) = \frac{\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\pi} w_{i,j}h_{i,j}(q)/(s-\sigma_i)(p-\pi_j)}{\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\pi} w_{i,j}/(s-\sigma_i)(p-\pi_j)}$$

- m_σ, m_π: reduction orders
- σ_i, π_i: interpolation points
- $h_{i,j}(q)$: functions defined s.t. $h_{i,j}(\rho_k) = H(\sigma_i;\pi_j,\rho_k)$
- $w_{i,j}$: barycentric weights

2 pAAA & svAAA

psvAAA combines:

- pAAA (Rodriguez et al. 2023)

$$H_p(s;p) = \frac{\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\pi} w_{i,j}h_{i,j}/(s-\sigma_i)(p-\pi_j)}{\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\pi} w_{i,j}/(s-\sigma_i)(p-\pi_j)}$$

- svAAA (Lietaert et al. 2022)

$$H_{sv}(s) = \frac{\sum_{i=1}^{m_\sigma} w_{i,j}\boldsymbol{h}_i/(s-\sigma_i)}{\sum_{i=1}^{m_\sigma} w_i/(s-\sigma_i)}, \quad [\boldsymbol{h}_i]_k = H(\sigma_i;\rho_k)$$

2 psvAAA: algorithm

Inputs: sampled s_i, p_i, q_i, $H(s_i; p_j, q_k)$, `mmax`, `tol`

- At step m, if `mmax` or `tol` are not met
  - find (σ_m, π_m) from

$$\arg\max_{s_i, p_j} \|H(s_i; p_j, q_k) - H_{psv}(s_i; p_j, q_k)\|_\infty$$

  - minimize linearized error

$$\|H(s;p,q) - H_{psv}(s;p,q)\|_2 \approx \left\| \left[\sum_{i=1}^{m_\sigma} \sum_{j=1}^{m_\pi} {}^{(H(s;p,q) - h_{i,j}(q))}/_{(s-\sigma_i)(p-\pi_j)} \right] w_{i,j} \right\|_2$$

- The functions $h_{i,j}(q)$ are found by interpolation

2 psvAAA: state-space form

$$E_r = \begin{bmatrix} 1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad A_r(p) = \begin{bmatrix} \sigma_1 & -\sigma_2 & & \\ \vdots & & \ddots & \\ \sigma_1 & & & -\sigma_{m_\sigma} \\ -\alpha_1(p) & -\alpha_2(p) & \cdots & -\alpha_{m_\sigma}(p) \end{bmatrix}$$

$$B_r = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad C_r^T(p,q) = \begin{bmatrix} \beta_1(p,q) \\ \vdots \\ \beta_{m_\sigma}(p,q) \end{bmatrix} \quad \alpha_i(p) = \sum_{j=1}^{m_\pi} \frac{w_{i,j}}{p-\pi_j} \quad \beta_i(p,q) = \sum_{j=1}^{m_\pi} \frac{w_{i,j} h_{i,j}(q)}{p-\pi_j}$$

$$H_{psv}(s;p,q) = C_r^T(p,q)\left(sE_r - A_r(p)\right)^{-1} B_r$$

2 NOFRFs

As we have said, we have $Y(\omega;\omega_F,A)\sum_{n=1}^{\infty}G_n(\omega;\omega_F,A)U_n(\omega;\omega_F,A)$, where ω_F modifies the system poles, while A does not:

- collect data $Y(\omega;\omega_F,A)$ and apply psvAAA to get

$$Y_{psv}(\omega;p,q)=\frac{\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\pi}w_{i,j}Y_{i,j}(A)/(\omega-\sigma_i)(\omega_F-\pi_j)}{\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\pi}w_{i,j}/(\omega-\sigma_i)(\omega_F-\pi_j)}$$

- the impulse response of $\Sigma_r(\omega_F,A)$ (parametric LTI) approximates $y(t;\omega_F,A)$

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3 Synthetic example

To test the method define

- inspired by the Duffing oscillator response to sinusoidal inputs (Peng et al. 2007),

$$G_1(\imath\omega) = \frac{1}{-m\omega^2 + 2c\imath\omega + k}, \quad G_2(\imath\omega) = -k_2 G_1^2(\imath\omega) G_1(2\imath\omega)$$

- from $u(t; A, \omega_F) = Ae^{-t}\cos(\omega_F t)\theta(t)$, with $\theta(t)$ the Heaviside function

$$U_1(\imath\omega; A, \omega_F) = \frac{A/2}{\imath\omega - (-1 + \imath\omega_F)} + \frac{A/2}{\imath\omega - (-1 - \imath\omega_F)}$$

$$U_2(\imath\omega; A, \omega_F) = \frac{A^2/4}{\imath\omega - (-2 + 2\imath\omega_F)} + \frac{A^2/4}{\imath\omega - (-2 - 2\imath\omega_F)} + \frac{A^2/2}{\imath\omega + 2}$$

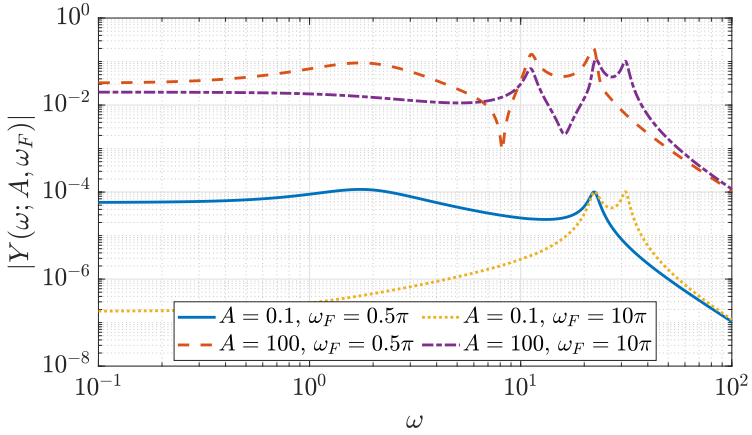
3 Simulation parameters

- $[s_i]_{i=1}^{N} \in \{0\} \cup [0.1\imath, 100\imath]$, $N = 100$
- $[\omega_{Fi}]_{i=1}^{P} \in [0.5\pi, 10\pi]$, $P = 20$
- $[A_i]_{i=1}^{Q} \in [0.1, 100]$, $Q = 10$
- for G_i, $m = 1$, $c = 1$, $k = 50$, and $k_2 = 10^3$
- for the first application, `mmax` $= 100$, `tol` $= 10^{-10}$
- for psvAAA, `mmax` $= 100$, `tol` $= 10^{-8}$

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3 Y

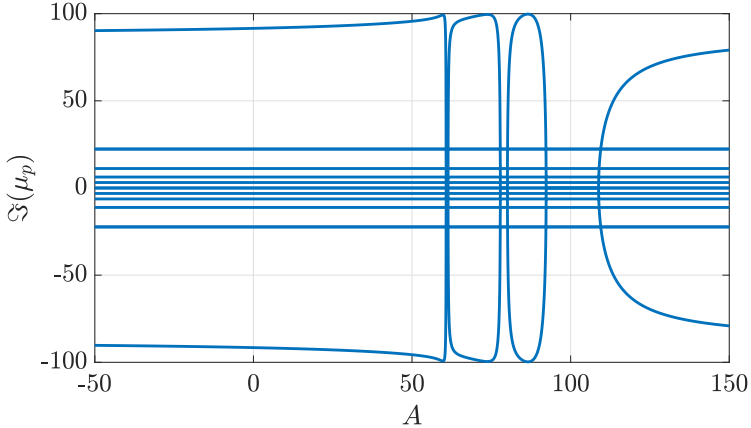
The function for different parameter choices



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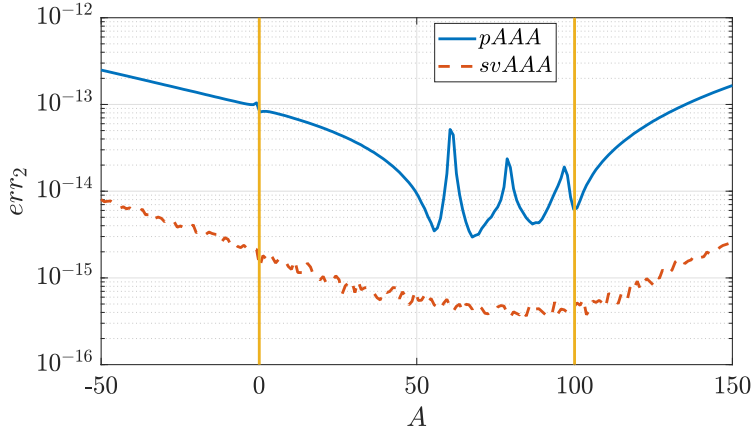
3 pAAA vs svAAA

Fixing $\omega_F = \pi$, we have $Y(\omega; A)$: the poles are constant, pAAA does not work



3 pAAA vs svAAA

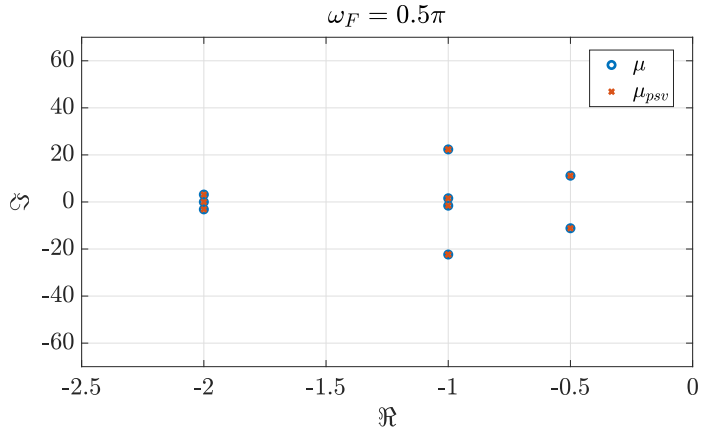
At the same order, svAAA is more accurate



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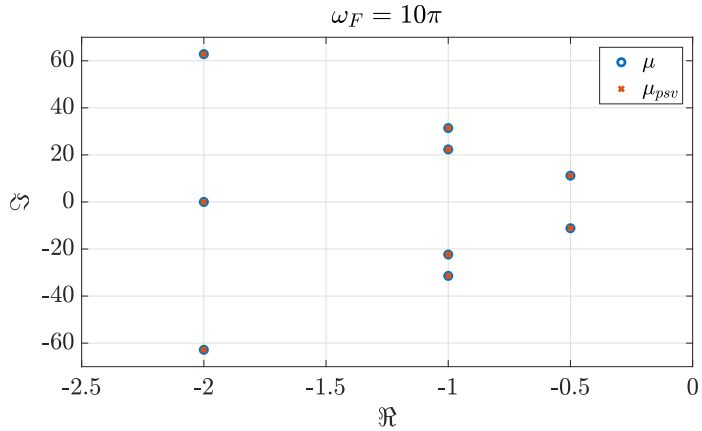
3 psvAAA poles

After 11 iterations, convergence is met with $m_\sigma = 14$ and $m_\pi = 6$



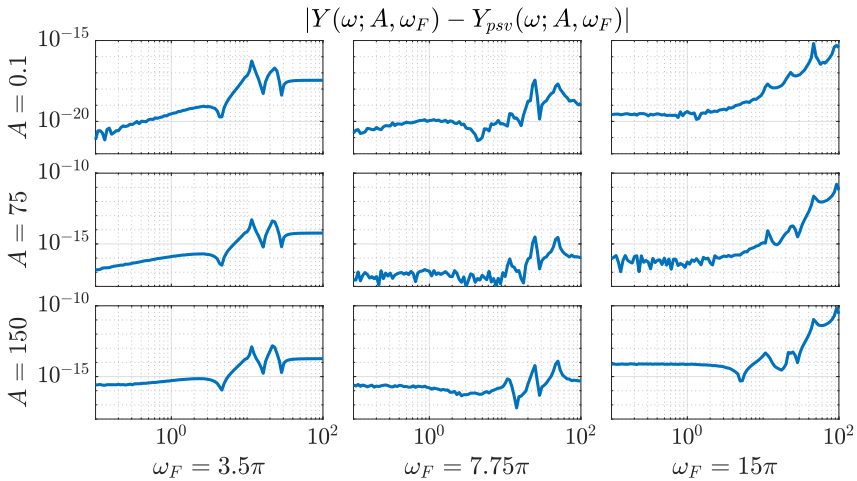
3 psvAAA poles

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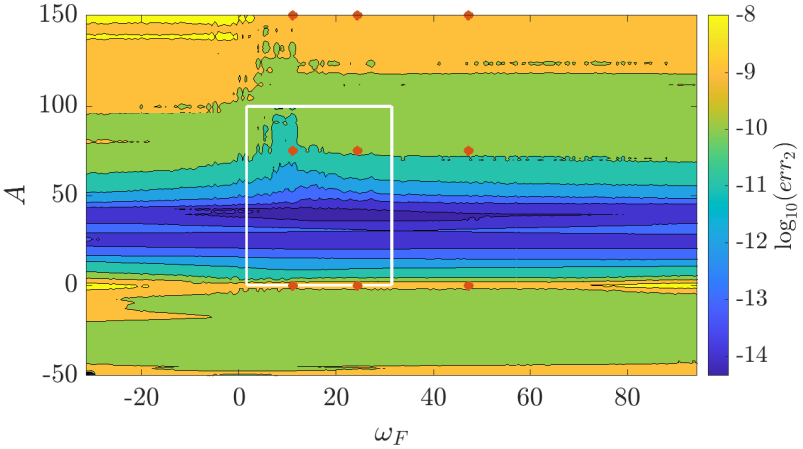
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3 psvAAA error



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3 psvAAA error



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3 Duffing oscillator

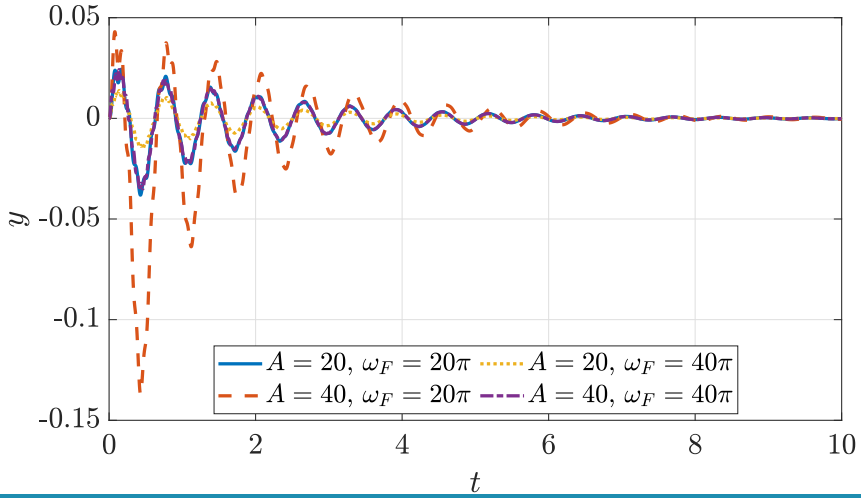
In this case we consider the output of the Duffing oscillator

$$m\ddot{x} + c\dot{x} + kx + k_2x^2 = u(t; \omega_F, A)$$

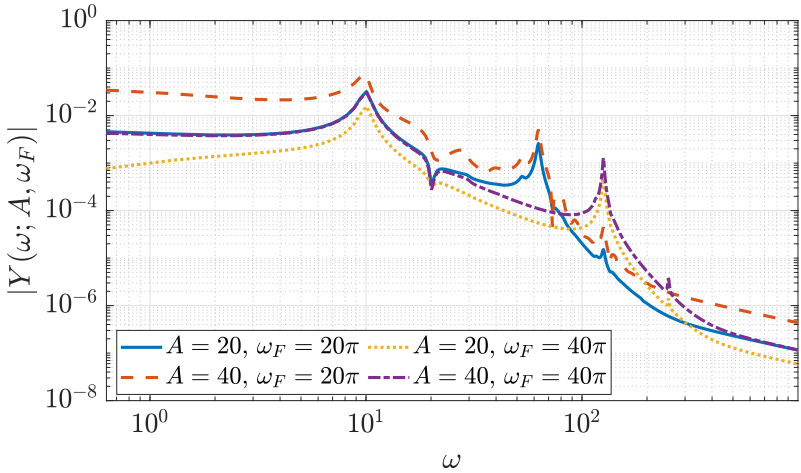
, with $u(t; \omega_F, A) = Ae^{-t}\sin(\omega_F t)$. The parameters used for the simulation are:

- $m = 1$, $c = 1$, $k = 100$, and $k_2 = 10$
- $Nt = 2 \cdot 10^4$ time samples, $dt = 5 \cdot 10^{-4}$ time step
- $[\omega_F]_{i=1}^{P} \in [20\pi, 40\pi]$, $P = 11$
- $[A_i]_{i=1}^{Q} \in [20, 40]$, $Q = 11$
- for psvAAA, $\texttt{mmax} = 100$, $\texttt{tol} = 5 \cdot 10^{-3}$

3 Time-domain data

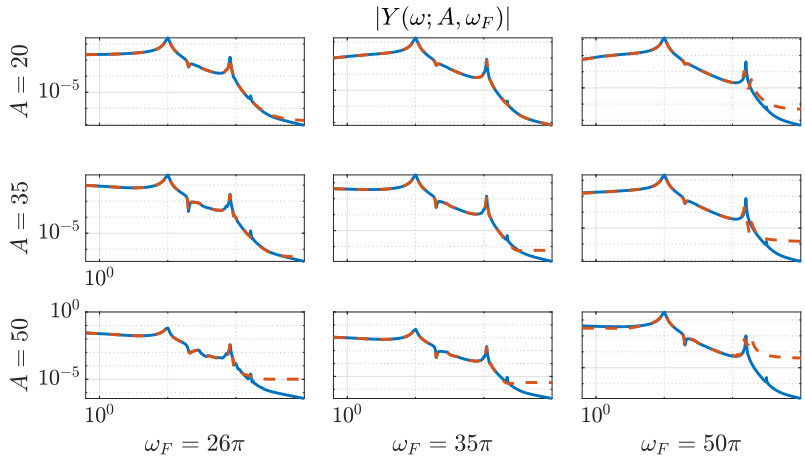


3 Frequency-domain data

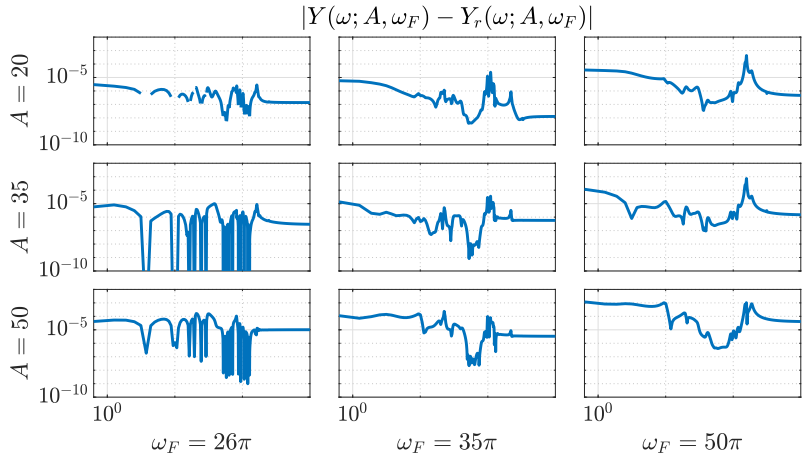


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3 psvAAA approximation



3 psvAAA error



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4 Summing up

- We have developed a new combination of methods for tackling the approximation of nonlinear systems
- By limiting the focus on families of input functions, we are able to define a parametric LTI system which approximates the nonlinear output
- The method has been proven effective on a synthetic case and on a Duffing oscillator

Thank you for your attention!

5 Bibliography I

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