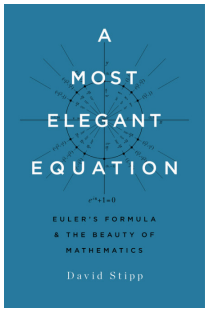
A Most Elegant Equation by *David Stipp*. Basic Books, 2017, isbn 978-0-4650-9377-9, 240 p.

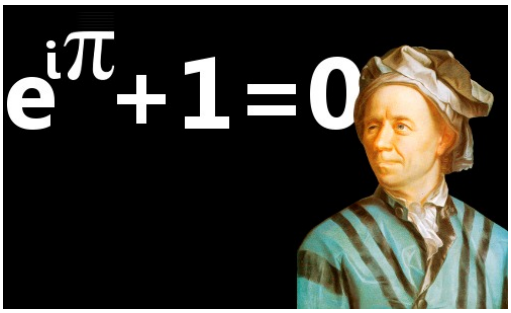




David Stipp

David Stipp is a freelance science writer and with this book he delivers an eulogy of the Euler identity $e^{i\pi} + 1 = 0$ which is considered to be the most beautiful formula of mathematics. This was already done by Paul Nahin in *Dr. Euler's Fabulous Formula* (2006) but here the story is told at the level of beginning secondary school children.

There is of course an interesting story to tell. The equation has five fundamental mathematical constants: $\pi, i, 0, 1, \mathrm{e}$, and books have been written about each of them. Moreover Leonhard Euler was a very colourful personality to tell stories about, an open door for a science writers such as Stipp who very generously declares his admiration for the man and his formula.



First thing to do is explain what e is. He does this using the Bernouilli approach with the formula for compound interest $\mathrm{e} = \lim_{n\to\infty}(1 + 1/n)^n$. Next the notion of a function is introduced. In particular e^x, the only function that is equal to its own derivative. Therefore he introduces a derivative as an instantaneous change leading to a discussion with a police officer who used a speed gun and is giving a speeding ticked. Since e is irrational, this leads to an excursion about the notion of infinity.

The second fascinating constant to consider is π, a number that people associate with the circle, but that pops up in many other computations that are seemingly totally unrelated to circles (e.g. $\sum_{k=0}^{\infty}(-1)^k/(2k+1) = \pi/4$, called Leibnitz series and the Basel problem: $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, as proved by Euler, etc.). To show this, Stipp also needs to discuss convergence of infinite sums.

The imaginary unit i is really provocative. Where a negative number could still correspond to for example a debt if the positive values represent an asset, there is not some real quantity imaginable that corresponds to the square root of a negative number. On the other hand it turned out to be very useful in computations. It popped up like a local glitch while solving a cubic equation and dissolved like an hallucination in the next steps to give eventually a real (and correct) result.

Time to give more details about the master himself. Stipp sketches a lovely picture of Euler as described by Dieudonné Thiébault who met him in mid-life and describes him as writing his immortal works with a child on his knees and a cat on his back. A social as well as a family man who loved joking, but who met a lot of misery in his life. Only two of his 13 children survived him, he lost eye sight in one eye early in life and he was almost completely blind when he lost his wife. He still did all the computations in his head and continued to be even more productive since. The Prussian king Frederick hired him to help develop the Berlin Academy of Sciences, but in a letter to Voltaire he called Euler the "great cyclops". When Euler left for Russia to work for Catherine II at the St. Petersburg Academy, the king wrote that the "one-eyed monster was replaced by another who had two eyes" (which was Joseph-Louis Lagrange).

Euler fits in a list of the greatest of all times. Stipp compares the characters of two other great names of the Enlightenment that were in many ways the opposite of Euler's: Newton (who was a "shy, prickly loner who held grudges", "a tyrannical president of the Royal Society", and who had a fight with anyone criticizing his work like Hooke and Leibnitz) and Gauss (who disliked teaching, had few friends, alienated his son Daniel, was reluctant to publish his work, and humiliated János Bolyai who had discovered non-Euclidean geometry "praising his work would be like prising myself").

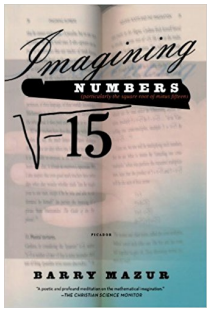
With the remaining constants 0 and 1, Stipp is a bit more poetical. The constants may be less glamorous than e, π or i, but they are not less important. For 1, Stipp quotes Alex Bellos who describes it as "independent, strong, honest, brave, straightforward, pioneering and lonely". Zero is "secretly peculiar", "as diaphanous as a fairy's wing, yet it is as powerful as a black hole".

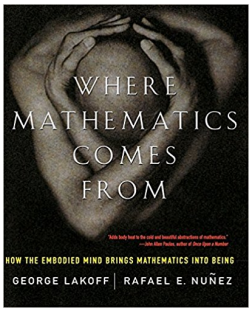
So, what could be expected from $e^{i\pi}$, a transcendental number raised to another imaginary transcendental number? The flabbergasting fact about Euler's formula is that this evaluates to the relatively simple

-1. To convince a mathematical novice in his early teens of this fact, Stipp again needs to introduce some extra mathematics. To start with, since he already defined a function, he needs to connect the sine and cosine, (defined as the ratios of the length of sides in a rectangular triangle) to a function mapping an angle $\theta \in [0, 90°)$ to two real numbers $\sin\theta$ and $\cos\theta$ in $[-1, 1]$. To achieve this he introduces the goniometric circle and the Cartesian plane, which he does very patiently and very elementary. The formulas of De Moivre $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ and the Taylor series of the trigonometric functions are the final elements that allow to prove Euler's formula. The geometric representation of complex numbers as vectors or points in the plane can visualize complex arithmetic and interpret a multiplication with $e^{i\theta}$ as a rotation, and hence associate complex analysis with oscillating phenomena. The latter made electrical engineers early adaptors of complex numbers to analyse their AC circuits.



But most of all, Euler's formula is considered to be a (if not the most) beautiful formula in mathematics. This triggers Stipp to investigate what other authors have said in this regard and to ponder on what exactly is meant by (mathematical) beauty and on the Platonic versus the nominalist views on mathematics. He considers himself to be a "quasi-platonist". He believes that mathematics exists outside the human brain and has to be discovered, but he is not fanatical about it. I think this is a sensible thing to do. There is beauty out there that we can *discover*, and at other days we elaborate on the accumulated mathematical background and actually *create* theorems that are not obtained by observation. This is in line with Barry Mazur who describes this ambivalence in his book *Imagining Numbers*. He gives arguments to support his praise of the formula and he defends his views against other opinions that attack Platonism like George Lakoff, Rafael E. Núñez in *Where Mathematics Comes From*.





In an appendix, Stipp describes the original proof that Euler gave of his formula. It has been criticised for its dubious way he dealt with infinitesimals, but on the other hand he was excellent in explaining his way of thinking that led to the result. Nowadays only a polished paper is published, thereby loosing insight into the author's meandering path that eventually gave the result.

The last part (Platonism and appendix) may be less of interest to a 12-year-old, but the style of the whole book is fresh, witty and sparking with a lot of humour and hooking up with the modern media-world and language that the targeted readership is familiar with. To give just one example (there are a 'gazillion' of other examples) the square root of a negative number was considered in Cardano's time to be "perverse gobbledygook".

Some books on related material:

P.J. Nahin, *Dr. Euler's Fabulous Formula: Cures Many Mathematical Ills*. Princeton U. Press, 2017.
E. Maor, *e: The Story of a Number*. Princeton U. Press, 2015.
P.J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$. Princeton U. Press, 2016.
R. Kaplan, *The Nothing that Is: A Natural History of Zero*. Oxford U. Press, 2000.
J. Havil, *The Irrationals. A Story of the Numbers You Can't Count On*. Princeton U. Press, 2012.
R.S. Calinger, *Leonhard Euler: Mathematical Genius in the Enlightenment*, Princeton U. Press, 2015.
E.A. Fellman, *Leonhard Euler*, Birkhäuser Verlag, 2007.
B. Mazur, *Imagining Numbers (Particularly the Square Root of Minus Fifteen)*, Picador, 2004.
G. Lakoff, R.E. Núñez, *Where Mathematics Comes From*. Basic Books, 2001.

Adhemar Bultheel