Smooth Memoryless Time-Varying Control of a Class of Chained Nonholonomic Systems with Input Delays ⋆

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Abstract: This paper investigates the smooth, memoryless, time-varying control design for a class of chained nonholonomic systems, comprising a linear subsystem and a bilinear chained subsystem with input delays. For the linear subsystem, a smooth, memoryless, nonhomogeneous control law is proposed by leveraging certain properties of parametric Lyapunov equations, ensuring exponential convergence. The bilinear chained subsystem is then transformed into a linear time-varying system via a linear time-varying state transformation. By exploiting the structural properties of nonholonomic systems and employing the Razumikhin stability theorem, a smooth, linear, memoryless, time-varying controller is developed. The effectiveness of the proposed methods is demonstrated using a model of an underactuated axisymmetric spacecraft.

Keywords: Nonholonomic systems; linear time-varying systems; smooth memoryless time-varying feedback.

1. INTRODUCTION

Since nonholonomic systems were identified as distinct from general mechanical systems, their control challenges have remained a focal point of research. In 1983, Brockett introduced a necessary condition for the existence of smooth time-invariant feedback in nonlinear systems, now referred to as Brockett's necessary condition Brockett et al. (1983). Unfortunately, nonholonomic systems do not meet this condition. As a result, research on controlling nonholonomic systems has concentrated on smooth time-varying and non-smooth feedback methods. Notably, many nonholonomic systems, such as two-wheeled robots, can be transformed into chained nonholonomic systems, making them the most extensively studied class of these systems. For chained nonholonomic systems, various control strategies have been developed. These include smooth time-varying feedback approaches, such as sinusoid-based state feedback Murray and Sastry (1993) and linear time-varying state feedback Tian and Li (2000), as well as discontinuous time-invariant control laws, such as the σ-process method Astolfi (1996); Jiang (2000); Hong et al. (2005) and logic-based switching methods Hespanha and Morse (1999). This paper aims to further explore the control problem for chained nonholonomic systems.

Time delays are prevalent in numerous practical control systems, such as those involving long transmission lines in pneumatic systems, rolling mills, nuclear reactors, hydraulic systems, manufacturing processes, and digital control systems (see Cai and Sipahi (2024); Fridman (2014); Gao and Olgac (2016); Krstic (2010); Michiels and Niculescu (2007); Zhou et al. (2009); Zhou (2014)). The occurrence of time delays can result in substantial performance degradation or even system instability. Given that time-delay systems fall within the category of functional differential equations, which are inherently infinite-dimensional, their analysis and synthesis have garnered considerable research attention over the past three decades. Various methodologies have been devised to address the stabilization of time-delay systems. One effective approach is the infinite-dimensional predictor-based feedback method, which can handle arbitrarily large delays Karafyllis and Krstic (2013); Krstic (2010). By employing the input-state scaling technique, researchers have developed infinite-dimensional predictor feedback laws for input-delay chained nonholonomic systems Chen et al. (2018); Shang and Xie (2014). Additionally, studies have explored the control of stochastic nonholonomic systems in chained form with time-varying state delays Gao et al. (2012). Consensus-formation control problems for networked nonholonomic systems under communication

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delays have also been investigated Tang et al. (2014); Maghenem et al. (2020). Despite the theoretical advantages of infinite-dimensional predictor feedback laws, practical implementation remains challenging due to computational and numerical complexities (see Mondié and Michiels (2003); Van Assche et al. (1999); Zhou (2014)). Therefore, it is crucial to develop finite-dimensional controllers for general chained nonholonomic systems.

In this paper, we investigate the smooth memoryless time-varying control problem for a class of chained nonholonomic systems with input delays. The system can be decomposed into two components: a linear subsystem and a bilinear chained subsystem. For the linear subsystem, by leveraging specific properties of parametric Lyapunov equations, we propose a smooth memoryless nonhomogeneous control law that ensures exponential convergence. Through a linear time-varying state transformation, the bilinear chained subsystem is converted into a linear time-varying system. By exploiting the structural characteristics of nonholonomic systems and applying the Razumikhin stability theorem, we design a smooth memoryless time-varying feedback controller.

The main contributions of this paper are threefold:

(1) Firstly, unlike traditional chained nonholonomic systems studied in previous works such as Ge et al. (2003); Murray and Sastry (1993); Zhang et al. (2025, 2024); Zhou et al. (2023), which only consist of kinematic equations, the systems examined in this paper incorporate both kinematic and dynamic equations. Consequently, they are more general and better suited to practical applications.
(2) Secondly, unlike traditional predictor methods, such as those in Karafyllis and Krstic (2013); Krstic (2010), where controllers are infinite-dimensional and complex to implement, our proposed controllers are memoryless, making them simpler and easier to implement.
(3) Thirdly, unlike traditional discontinuous control methods (such as the σ-process method in Astolfi (1996); Ge et al. (2003); Hong et al. (2005); Zhou et al. (2023)), which may lead to singularity issues when initial conditions approach zero and cause chattering problems, the methods proposed in this paper are both globally effective and smooth.

Notation: For a (symmetric) matrix M, let M^{T}, $\|M\|$, $\lambda_{\max}(M)$, and $\lambda_{\min}(M)$ denote its transpose, 2-norm, maximal eigenvalue, and minimal eigenvalue, respectively. In addition, $\mathrm{diag}\{A_1,\dots,A_n\}$ denotes a diagonal matrix whose ith diagonal element is A_i.

2. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Problem Formulation

Consider the following chained nonholonomic system with input delays

$$\begin{cases} \dot{X}_0 = A_m X_0 + b_m u_0\left(t-\tau_1\right), \\ \dot{X} = F(X_0)A_n X + b_n u(t-\tau_2), \end{cases} \tag{1}$$

where

$$F(X_0) = \left[\begin{array}{c|c|c} X_{02}I_{n_1-1} & \mathbf{0}_{(n_1-1)\times 1} & \mathbf{0}_{n_1\times n_2} \\ \hline \mathbf{0}_{1\times(n_1-1)} & 1 & \\ \hline \mathbf{0}_{n_2\times n_1} & & I_{n_2} \end{array}\right], \tag{2}$$

$$A_i = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \in \mathbf{R}^{i\times i}, \quad b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbf{R}^{i\times 1}, \tag{3}$$

$i = m, n$, $X_0 = [X_{01}, X_{02}, \dots, X_{0m}]^{\mathrm{T}} \in \mathbf{R}^m$, $X = [X_1^{\mathrm{T}}, X_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^n$, $n = n_1+n_2$, $X_1 = [X_{11}, X_{12}, \dots, X_{1n_1}]^{\mathrm{T}} \in \mathbf{R}^{n_1}$ and $X_2 = [X_{21}, X_{22}, \dots, X_{2n_2}]^{\mathrm{T}} \in \mathbf{R}^{n_2}$ are state vectors, $u_0 \in \mathbf{R}$ and $u \in \mathbf{R}$ are control inputs, τ_1 and τ_2 are two unknown nonnegative constants satisfying

$$0 \le \tau_1 \le D, \quad 0 \le \tau_2 \le D, \tag{4}$$

with a known constant D.

In this paper, we are interested in the following question.

Problem 1. We design smooth memoryless time-varying control laws for $u_0(t)$ and $u(t)$ such that, for any initial condition, the state and control of the closed-loop system converge to zero asymptotically.

2.2 Preliminaries

Consider the parametric Lyapunov equation (PLE)

$$\mathcal{A}^{\mathrm{T}}P + P\mathcal{A} - P\mathcal{B}\mathcal{B}^{\mathrm{T}}P = -\gamma P, \tag{5}$$

where $\gamma > 0$ is a constant. Let $(\mathcal{A}, \mathcal{B}) \in (\mathbf{R}^{N\times N}, \mathbf{R}^{N\times L})$ be controllable. It has been shown in Zhou (2014) that the PLE (5) has (unique) positive definite solution $P(\gamma)$ if and only

$$\gamma > -2 \min_{i=1,2,\dots,N} \left\{\mathrm{Re}\left(\lambda_i(\mathcal{A})\right)\right\}.$$

Next, we will recall some existing results for the PLE (5).

Lemma 1. Zhou (2014) Let $(A_m, b_m) \in (\mathbf{R}^{m\times m}, \mathbf{R}^{m\times 1})$ be defined as in (3) and $\gamma > 0$. Denote $P_0(\gamma)$ as the unique positive definite solution to (5) for $(\mathcal{A}, \mathcal{B}) = (A_m, b_m)$. The following properties hold:

(1) The corresponding positive definite matrix $P_0(\gamma)$ can be determined as follows
$$P_0(\gamma) = \gamma L_m P_m L_m, \tag{6}$$
where $L_m = L_m(\gamma) = \mathrm{diag}([\gamma^{m-1}, \gamma^{m-2}, \dots, \gamma, 1])$ and $P_m = P_0(1)$.
(2) The closed-loop system matrix $A_m - b_m b_m^{\mathrm{T}} P_0(\gamma)$ satisfies
$$\mathrm{Re}\left\{\lambda_i(A_m - b_m b_m^{\mathrm{T}} P_0(\gamma))\right\} = -\gamma, \quad i = 1, 2, \dots, m. \tag{7}$$
(3) The positive definite matrix $P_0(\gamma)$ satisfies the inequality
$$A_m^{\mathrm{T}} P_0(\gamma) A_m \le 3m^2\gamma^2 P_0(\gamma). \tag{8}$$
(4) The positive definite matrix $P_0(\gamma)$ satisfies
$$\mathrm{tr}(b_m^{\mathrm{T}} P_0(\gamma) b_m) = m\gamma. \tag{9}$$

Let $\alpha > 0$ be a constant. Denote

$$\begin{aligned} E =& [E_1, E_2, \dots, E_m]^{\mathrm{T}} \\ =& \left(\alpha I_m + A_m - b_m b_m^{\mathrm{T}} P_0(3\alpha)\mathrm{e}^{\alpha\tau_1}\right)^{-1} b_m \mathrm{e}^{\alpha\tau_1} \\ =& \frac{1}{\alpha^m\left(\mathrm{e}^{-\alpha\tau_1}-1\right) + (-1)^m 2^m \alpha^m} \end{aligned}$$

$$\times \left[(-1)^{m+1} \ (-1)^{m+2} \alpha \ \cdots \ (-1)^{2m} \alpha^{m-1} \right]. \quad (10)$$

Clearly, there exists a sufficiently small positive constant $\alpha_* = \alpha_*(D)$, and positive constants $E_{2\max} = E_{2\max}(D) \in (0, \infty)$ and $E_{2\min} = E_{2\min}(D) \in (0, \infty)$, independent of τ_1, such that for all $\alpha \in (0, \alpha_*)$, $E_2 \in [E_{2\min}, E_{2\max}]$.

Next, we end this section with a lemma.

Lemma 2. Zhou (2014) (Jensen's Inequality) For any positive definite matrix $M > 0$, two scalars γ_2 and γ_1 with $\gamma_2 \geq \gamma_1$, and a vector valued function $\omega : [\gamma_1, \gamma_2] \to \mathbf{R}^n$ such that the integrals in the following are well-defined, then

$$\left(\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta) \mathrm{d}\beta \right) M \left(\int_{\gamma_1}^{\gamma_2} \omega(\beta) \mathrm{d}\beta \right) \leq (\gamma_2 - \gamma_1) \left(\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta) M \omega(\beta) \mathrm{d}\beta \right).$$

3. CONTROLLER DESIGN

3.1 Design of the Control Law for $u_0(t)$

For convenience, X_0-subsystem in (1) is rewritten as follows

$$\dot{X}_0 = A_m X_0 + b_m u_0 (t - \tau_1). \quad (11)$$

The control law $u_0(t)$ can be designed as follows.

Theorem 1. Let $\alpha > 0$ and $\gamma_0 = 3\alpha$ be two constant. Consider the following smooth time-varying nonhomogeneous feedback

$$u_0(t) = -b_m^{\mathrm{T}} P_0(\gamma_0) X_0(t) + \beta \mathrm{e}^{-\alpha t}, \quad (12)$$

where $\beta = -\frac{\alpha}{E_{2\max}}$. Then, there exists a sufficiently small positive constant $\alpha_* = \alpha_*(D)$, such that for any initial condition and $\alpha \in (0, \alpha_*)$, the state and control of closed-loop system consisting of (11) and (12) converge to zero exponentially. In particular, the state $X_{02}(t)$ can be expressed as

$$X_{02}(t) = \mathrm{e}^{-\alpha t} (Z_{02}(t) + \alpha \rho), \quad (13)$$

where $\rho = \frac{E_2}{E_{2\max}} \in \left[\frac{E_{2\min}}{E_{2\max}}, 1 \right]$ and $Z_{02}(t)$ is a time-varying function satisfying $\lim_{t \uparrow \infty} Z_{02}(t) = 0$ for any initial condition.

Proof. To simplify notations we let $P_0 = P_0(\gamma_0)$. The closed-loop system consisting of (11) and (12) can be expressed as

$$\dot{X}_0 = A_m X_0 - b_m b_m^{\mathrm{T}} P_0(\gamma_0) X_0 (t - \tau_1) + b_m \beta \mathrm{e}^{-\alpha(t-\tau_1)}. \quad (14)$$

Denote

$$Z_0 = \mathrm{e}^{\alpha t} X_0 + \varepsilon, \quad (15)$$

where $\varepsilon = E\beta$ with E being defined in (10). Then in view of (14), we can get

$$\begin{aligned} \dot{Z}_0 =& \alpha \mathrm{e}^{\alpha t} X_0 + \mathrm{e}^{\alpha t} \dot{X}_0 \\ =& \alpha Z_0 + \left(A_m - b_m b_m^{\mathrm{T}} P_0(\gamma_0) \mathrm{e}^{\alpha \tau_1} \right) Z_0 \\ &+ b_m b_m^{\mathrm{T}} P_0(\gamma_0) \mathrm{e}^{\alpha \tau_1} D_{\tau_1}(Z_0), \end{aligned} \quad (16)$$

where $D_{\tau_1}(Z_0) = Z_0(t) - Z_0(t - \tau_1)$ and we have used

$$\left(-\alpha I_m - A_m + b_m b_m^{\mathrm{T}} P_0(\gamma_0) \mathrm{e}^{\alpha \tau_1} \right) \varepsilon + b_m \beta \mathrm{e}^{\alpha \tau_1} = 0.$$

Choose the Lyapunov function

$$V_0(Z_0) = Z_0^{\mathrm{T}} P_0 Z_0,$$

whose time-derivative along the trajectory of closed-loop system (16) can be written as

$$\begin{aligned} &\dot{V}_0(Z_0) \leq 2\alpha Z_0^{\mathrm{T}} P_0 Z_0 + Z_0^{\mathrm{T}} \left(A_m^{\mathrm{T}} P_0 + P_0 A_m \right) Z_0 \\ &- 2\mathrm{e}^{\alpha \tau_1} Z_0^{\mathrm{T}} P_0 b_m b_m^{\mathrm{T}} P_0 Z_0 \\ &+ 2\mathrm{e}^{\alpha \tau_1} Z_0^{\mathrm{T}} P_0 b_m b_m^{\mathrm{T}} P_0 D_{\tau_1}(Z_0) \\ &\leq 2\alpha Z_0^{\mathrm{T}} P_0 Z_0 + Z_0^{\mathrm{T}} \left(A_m^{\mathrm{T}} P_0 + P_0 A_m - P_0 b_m b_m^{\mathrm{T}} P_0 \right) Z_0 \\ &- \mathrm{e}^{\alpha \tau_1} Z_0^{\mathrm{T}} P_0 b_m b_m^{\mathrm{T}} P_0 Z_0 + 2\mathrm{e}^{\alpha \tau_1} Z_0^{\mathrm{T}} P_0 b_m b_m^{\mathrm{T}} P_0 D_{\tau_1}(Z_0) \\ &\leq 2\alpha Z_0^{\mathrm{T}} P_0 Z_0 - \gamma_0 Z_0^{\mathrm{T}} P_0 Z_0 \\ &+ \mathrm{e}^{\alpha \tau_1} D_{\tau_1}^{\mathrm{T}}(Z_0) P_0 b_m b_m^{\mathrm{T}} P_0 D_{\tau_1}(Z_0) \\ &\leq 2\alpha Z_0^{\mathrm{T}} P_0 Z_0 - \gamma_0 Z_0^{\mathrm{T}} P_0 Z_0 + m\gamma_0 \mathrm{e}^{\alpha \tau_1} D_{\tau_1}^{\mathrm{T}}(Z_0) P_0 D_{\tau_1}(Z_0), \end{aligned} \quad (17)$$

where we have used (5) with $(\mathcal{A}, \mathcal{B}) = (A_m, b_m)$, Young's inequality, and (9).

Let $V_0(Z_0(t + \upsilon)) < pV_0(Z_0(t))$, $\upsilon \in [-2D, 0]$, $t \in [0, \infty)$, where $p > 1$ is any constant. Integrating both sides of (16) gives

$$\begin{aligned} &D_{\tau_1}(Z_0) = Z_0(t) - Z_0(t - \tau_1) \\ &= \int_{t-\tau_1}^{t} \mathrm{d}Z_0(s) \\ &= \int_{t-\tau_1}^{t} (\alpha Z_0(s) + A_m Z_0(s) - b_m b_m^{\mathrm{T}} P_0 \mathrm{e}^{\alpha \tau_1} Z_0(s - \tau_1)) \mathrm{d}s. \end{aligned}$$

From Jensen's inequality (Lemma 2), (4), (8), and (9), it follows that

$$\begin{aligned} &D_{\tau_1}^{\mathrm{T}}(Z_0) P_0 D_{\tau_1}(Z_0) \\ \leq& 3\alpha^2 \left(\int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s) \mathrm{d}s \right) P_0 \left(\int_{t-\tau_1}^{t} Z_0(s) \mathrm{d}s \right) \\ &+ 3 \left(\int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s) A_m^{\mathrm{T}} \mathrm{d}s \right) P_0 \left(\int_{t-\tau_1}^{t} A_m Z_0(s) \mathrm{d}s \right) \\ &+ 3\mathrm{e}^{2\alpha \tau_1} \left(\int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s - \tau_1) P_0 b_m b_m^{\mathrm{T}} \mathrm{d}s \right) \\ &\times P_0 \left(\int_{t-\tau_1}^{t} b_m b_m^{\mathrm{T}} P_0 Z_0(s - \tau_1) \mathrm{d}s \right) \\ \leq& 3\alpha^2 \tau_1 \int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s) P_0 Z_0(s) \mathrm{d}s \\ &+ 3\tau_1 \int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s) A_m^{\mathrm{T}} P_0 A_m Z_0(s) \mathrm{d}s \\ &+ 3\tau_1 \mathrm{e}^{2\alpha \tau_1} \int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s - \tau_1) P_0 b_m b_m^{\mathrm{T}} \\ &\times P_0 b_m b_m^{\mathrm{T}} P_0 Z_0(s - \tau_1) \mathrm{d}s \\ \leq& 3\alpha^2 \tau_1 \int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s) P_0 Z_0(s) \mathrm{d}s \\ &+ 9m^2 \gamma_0^2 \tau_1 \int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s) P_0 Z_0(s) \mathrm{d}s \\ &+ 3\tau_1 \mathrm{e}^{2\alpha \tau_1} m^2 \gamma_0^2 \int_{t-\tau_1}^{t} Z_0^{\mathrm{T}}(s - \tau_1) P_0 Z_0(s - \tau_1) \mathrm{d}s \\ \leq& 3\alpha^2 D^2 p V_0(Z_0) + \left(9 + 3\mathrm{e}^{2\alpha D} \right) m^2 \gamma_0^2 D^2 p V_0(Z_0). \end{aligned}$$

Substituting this expression in (17) yields

$$\begin{aligned} &\dot{V}_0(Z_0) \\ &\leq -\left(\gamma_0 - 2\alpha - 3\mathrm{e}^{\alpha D} \alpha^2 D^2 p m \gamma_0 - \varphi_1 \mathrm{e}^{\alpha D} \gamma_0^3 \right) Z_0^{\mathrm{T}} P_0 Z_0 \end{aligned}$$

$$= -\alpha\left(1 - 3\alpha \mathrm{e}^{\alpha D} D^2 pm\gamma_0 - 27\mathrm{e}^{\alpha D}\varphi_1\alpha^2\right) Z_0^{\mathrm{T}} P_0 Z_0,$$

where $\varphi_1 = \left(9 + 3\mathrm{e}^{2\alpha D}\right) m^3 D^2 p$ and $\gamma_0 = 3\alpha$. Clearly, there exists a sufficiently small positive constant $\alpha_* = \alpha_*(D)$ such that, for all $\alpha \in (0, \alpha_*)$,

$$\dot{V}_0(Z_0(t)) \leq -\alpha\varepsilon_0 V_0(Z_0(t)),$$

where ε_0 is a positive constant. Then the global stability of the transformed closed-loop system (16) is guaranteed by the Razumikhin stability theorem, namely,

$$\lim_{t\to\infty} Z_0(t) = 0.$$

Therefore, in view of (15), we can rewrite X_0 as

$$X_0(t) = \mathrm{e}^{-\alpha t}\left(Z_0(t) - \varepsilon\right),$$

where $\varepsilon = E\beta$ and thus

$$X_{02}(t) = \mathrm{e}^{-\alpha t}\left(Z_{02}(t) - \beta E_2\right),$$

which proves (13). The proof is finished by noting that $u_0(t)$ also converges to zero exponentially. ■

3.2 Design of the Control Law for $u(t)$

Consider the linear time-varying state transformation

$$z(t) = T(t)\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}, \tag{18}$$

where

$$T(t) = \left[\begin{array}{c|c} L_{n_1}(\mathrm{e}^{\alpha t}) & \mathbf{0}_{n_1\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & \frac{1}{\alpha^{n_2}} L_{n_2}(\alpha) \end{array}\right], \tag{19}$$

$$L_n(\gamma) = \mathrm{diag}([\gamma^{n-1}, \gamma^{n-2}, \ldots, \gamma, 1]). \tag{20}$$

By using (18), (1) can be rewritten as

$$\begin{aligned}\dot{z} &= \frac{\mathrm{d}T(t)}{\mathrm{d}t} X + T(t)\left(F(X_0)A_n X + b_n u(t-\tau_2)\right) \\ &= \frac{\mathrm{d}T(t)}{\mathrm{d}t} T^{-1}(t) z + F(X_0) T(t) A_n T^{-1}(t) z + \frac{1}{\alpha^{n_2}} b_n u(t-\tau_2),\end{aligned}$$

substituting (13) into which yields

$$\dot{z} = \alpha\rho A_n z + \alpha\mathcal{A}_2 z + Z_{02}(t)\mathcal{A}_3 z + \frac{1}{\alpha^{n_2}} b_n u\left(t-\tau_2\right), \tag{21}$$

where

$$\mathcal{A}_2 = \left[\begin{array}{c|c} \mathrm{diag}([n_1-1, n_1-2, \ldots, 1, 0]) & \mathbf{0}_{n_1\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & \mathbf{0}_{n_2\times n_2} \end{array}\right],$$

and

$$\mathcal{A}_3 = \left[\begin{array}{c|c|c} \mathbf{0}_{(n_1-1)\times 1} & I_{(n_1-1)\times(n_1-1)} & \\ \hline \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(n_1-1)} & \mathbf{0}_{n_1\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & & \mathbf{0}_{n_2\times n_2} \end{array}\right].$$

Next, we will design state feedback controller for such linear time-varying system.

The solution to Problem 1 can be stated as follows.

Theorem 2. Let $\gamma > 0$ be a constant. Then there exists a sufficiently small positive constant $\alpha_* = \alpha_*(D)$ such that, for any initial condition, the states X_0, X and controls u_0, u of the closed-loop system consisting of (1), smooth time-varying controllers (12) and

$$u\left(t\right) = -\alpha^{n_2+1} b_n^{\mathrm{T}} P(\gamma) T(t) X\left(t\right), \quad \alpha \in (0, \alpha_*), \tag{22}$$

with $P(\gamma)$ being the unique positive solution to (5) with $(\mathcal{A}, \mathcal{B}) = (A_n, b_n)$ and $T(t)$ being defined in (19), converge to zero asymptotically.

Proof. Let $P = P(\gamma)$. The closed-loop system consisting of (21) and (22) can be expressed as

$$\dot{z} = \alpha\rho A_n z + \alpha\mathcal{A}_2 z + Z_{02}(t)\mathcal{A}_3 z - \alpha b_n b_n^{\mathrm{T}} P z(t-\tau_2), \tag{23}$$

which can be also re-written as

$$\dot{z} = \alpha\rho A_n z + \alpha\mathcal{A}_2 z - \alpha b_n b_n^{\mathrm{T}} P z + Z_{02}(t)\mathcal{A}_3 z + \alpha b_n b_n^{\mathrm{T}} P D_{\tau_2}(z), \tag{24}$$

where $D_{\tau_2}\left(z\right) = D_{\tau_2}\left(z(t)\right) = z(t) - z(t-\tau_2)$.

Choose the Lyapunov function

$$V\left(z\right) = z^{\mathrm{T}} P(\gamma) z,$$

whose time-derivative along the trajectory of closed-loop system (24) can be calculated as

$$\begin{aligned}\dot{V}\left(z\right) \leq & -\alpha\rho\gamma z^{\mathrm{T}} P z - \alpha z^{\mathrm{T}} P b_n b_n^{\mathrm{T}} P z + 2\alpha z^{\mathrm{T}} P\mathcal{A}_2 z \\ & + 2\alpha z^{\mathrm{T}} P b_n b_n^{\mathrm{T}} P D_{\tau_2}\left(z\right) + 2Z_{02}(t) z^{\mathrm{T}} P\mathcal{A}_3 z \\ \leq & -\alpha\rho\gamma z^{\mathrm{T}} P z - \alpha z^{\mathrm{T}} P b_n b_n^{\mathrm{T}} P z \\ & + \alpha z^{\mathrm{T}}\mathcal{A}_2 P\mathcal{A}_2 z + \alpha z^{\mathrm{T}} P z \\ & + 2\alpha z^{\mathrm{T}} P b_n b_n^{\mathrm{T}} P D_{\tau_2}\left(z\right) + 2Z_{02}(t) z^{\mathrm{T}} P\mathcal{A}_3 z \\ \leq & -\alpha\rho\gamma z^{\mathrm{T}} P z + \alpha D_{\tau_2}^{\mathrm{T}}\left(z\right) P b_n b_n^{\mathrm{T}} P D_{\tau_2}\left(z\right) \\ & + \alpha\gamma z^{\mathrm{T}}\mathcal{A}_2 L_n P_n L_n\mathcal{A}_2 z + \alpha z^{\mathrm{T}} P z \\ & + \left|Z_{02}(t)\right| z^{\mathrm{T}} P z + \left|Z_{02}(t)\right| z^{\mathrm{T}}\mathcal{A}_3^{\mathrm{T}} P\mathcal{A}_3 z \\ \leq & -\alpha\rho\gamma z^{\mathrm{T}} P z + \alpha\lambda_A D_{\tau_2}^{\mathrm{T}}\left(z\right) P D_{\tau_2}\left(z\right) \\ & + \alpha\frac{\lambda_{\max}(\mathcal{A}_2 P_n\mathcal{A}_2)}{\lambda_{\min}(P_n)} z^{\mathrm{T}} P z + \alpha z^{\mathrm{T}} P z \\ & + \left|Z_{02}(t)\right| z^{\mathrm{T}} P z + \left|Z_{02}(t)\right| \frac{\lambda_{\max}(\mathcal{A}_3^{\mathrm{T}} P\mathcal{A}_3)}{\lambda_{\min}(P)} z^{\mathrm{T}} P z,\end{aligned} \tag{25}$$

where $\lambda_A = b_n^{\mathrm{T}} P b_n$, we have used (6) and (5) with $(\mathcal{A}, \mathcal{B}) = (A_n, b_n)$, and Young's inequality. Integrating both the left and right sides of (23) yields

$$\begin{aligned} & D_{\tau_2}\left(z(t)\right) = \int_{t-\tau_2}^{t} \mathrm{d}z(s) \\ = & \int_{t-\tau_2}^{t} \left(\alpha\rho A_n z(s) + \alpha\mathcal{A}_2 z(s) - \alpha b_n b_n^{\mathrm{T}} P z\left(s-\tau_2\right)\right. \\ & \left. + Z_{02}(s)\mathcal{A}_3 z\left(s\right)\right)\mathrm{d}s.\end{aligned}$$

By using Jensen's inequality (Lemma 2) and (6), we can get

$$\begin{aligned} & D_{\tau_2}^{\mathrm{T}}(z) P D_{\tau_2}(z) \\ \leq & 4\alpha^2\tau_2\rho^2 \int_{t-\tau_2}^{t} z^{\mathrm{T}}(s) A_n^{\mathrm{T}} P A_n z(s)\mathrm{d}s \\ & + 4\alpha^2\tau_2 \int_{t-\tau_2}^{t} z^{\mathrm{T}}(s)\mathcal{A}_2^{\mathrm{T}} P\mathcal{A}_2 z(s)\mathrm{d}s \\ & + 4\alpha^2\tau_2 \int_{t-\tau_2}^{t} z^{\mathrm{T}}(s-\tau_2) P b_n b_n^{\mathrm{T}} P b_n b_n^{\mathrm{T}} P z(s-\tau_2)\mathrm{d}s \\ & + 4\tau_2 \int_{t-\tau_2}^{t} Z_{02}^2(s) z^{\mathrm{T}}(s)\mathcal{A}_3^{\mathrm{T}} P\mathcal{A}_3 z(s)\mathrm{d}s \\ \leq & 4\alpha^2 D\lambda(\gamma) \int_{t-\tau_2}^{t} z^{\mathrm{T}}(s) P z(s)\mathrm{d}s \\ & + 4\alpha^2 D\frac{\lambda_{\max}(\mathcal{A}_2 P_n\mathcal{A}_2)}{\lambda_{\min}(P_n)} \int_{t-\tau_2}^{t} z^{\mathrm{T}}(s) P z(s)\mathrm{d}s \\ & + 4\alpha^2\lambda_A^2 D \int_{t-\tau_2}^{t} z^{\mathrm{T}}(s-\tau_2) P z(s-\tau_2)\mathrm{d}s\end{aligned}$$

$$+4D\frac{\lambda_{\max}(\mathcal{A}_3^{\mathrm{T}}P\mathcal{A}_3)}{\lambda_{\min}(P)}\int_{t-\tau_2}^{t}Z_{02}^2(s)z^{\mathrm{T}}(s)Pz(s)\mathrm{d}s, \quad (26)$$

where $\lambda(\gamma)=\lambda_{\max}(A_n^{\mathrm{T}}PA_n)/\lambda_{\min}(P)$. Since $\lim_{t\uparrow\infty}Z_{02}^2(t)=0$, we have

$$\int_{t-\tau_2}^{t}Z_{02}^2(s)z^{\mathrm{T}}(s)Pz(s)\mathrm{d}s \le \max_{s\in[0,\tau_2]}Z_{02}^2(t-s)\int_{t-\tau_2}^{t}z^{\mathrm{T}}(s)Pz(s)\mathrm{d}s.$$

Let $p>1$ be any constant,

$$\alpha_*=\frac{1}{2}\sqrt{\frac{E_{2\min}\gamma}{\lambda_A E_{2\max}\pi_0}},$$

$$\pi_0=4D^2\lambda(\gamma)p+4D^2\frac{\lambda_{\max}(\mathcal{A}_2P_n\mathcal{A}_2)}{\lambda_{\min}(P_n)}p+4\lambda_A^2D^2p,$$

$$\vartheta(t)=\frac{1}{\alpha}|Z_{02}(t)| + \left(4\max_{s\in[0,\tau_2]}Z_{02}^2(t-s)D^2p\lambda_A+\frac{1}{\alpha}|Z_{02}(t)|\right)\frac{\lambda_{\max}(\mathcal{A}_3^{\mathrm{T}}P\mathcal{A}_3)}{\lambda_{\min}(P)},$$

$\alpha\in(0,\alpha_*)$, and t_0 be the smallest positive constant such that $\vartheta(t)\le\frac{E_{2\min}\gamma}{4E_{2\max}}$ for all $t\ge t_0$. Let $V(z(t+\upsilon))<pV(z(t))$, $\upsilon\in[-2D,0]$, $t\in[t_0,\infty)$. Then (26) can be calculated as

$$D_{\tau_2}^{\mathrm{T}}(z(t))PD_{\tau_2}(z(t)) \le \alpha^2\pi_0V(z)+4\max_{s\in[0,\tau_2]}Z_{02}^2(t-s)D^2p\frac{\lambda_{\max}(\mathcal{A}_3^{\mathrm{T}}P\mathcal{A}_3)}{\lambda_{\min}(P)}V(z). \quad (27)$$

Therefore, (25) can be continued as

$$\begin{aligned}\dot{V}(z)\le&-\alpha\frac{E_{2\min}}{E_{2\max}}\gamma z^{\mathrm{T}}Pz+\alpha^3\lambda_A\pi_0V(z)\\&+4\alpha\max_{s\in[0,\tau_2]}Z_{02}^2(t-s)D^2p\lambda_A\frac{\lambda_{\max}(\mathcal{A}_3^{\mathrm{T}}P\mathcal{A}_3)}{\lambda_{\min}(P)}V(z)\\&+|Z_{02}(t)|z^{\mathrm{T}}Pz+|Z_{02}(t)|\frac{\lambda_{\max}(\mathcal{A}_3P\mathcal{A}_3)}{\lambda_{\min}(P)}z^{\mathrm{T}}Pz\\=&-\alpha\left(\frac{E_{2\min}}{E_{2\max}}\gamma-\alpha^2\lambda_A\pi_0-\vartheta(t)\right)V(z).\end{aligned}$$

Since $\vartheta(t)\le\frac{E_{2\min}\gamma}{4E_{2\max}}$ and $\alpha^2\lambda_A\pi_0\le\frac{E_{2\min}\gamma}{4E_{2\max}}$ for all $\alpha\in(0,\alpha_*)$, we have,

$$\dot{V}(z(t))\le-\frac{1}{2}\alpha\frac{E_{2\min}}{E_{2\max}}\gamma V(z(t)), \quad t\in[t_0,\infty).$$

Then the global stability of the transformed closed-loop system (23) is guaranteed by the Razumikhin stability theorem, namely, $\lim_{t\uparrow\infty}z(t)=0$. Thus, in light of (18), we can get

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}T^{-1}(t)z(t)=0.$$

Finally, we end this proof by noting that $\lim_{t\uparrow\infty}u(t)=0$. ■

4. NUMERICAL EXAMPLE

Consider the kinematics and dynamics of an underactuated axisymmetric spacecraft model proposed in Morin and Samson (1997); Zhou and Zhang (2024),

$$\begin{cases}\dot{R}=S(\omega)R,\\ J_\Delta\dot{\omega}=S(\omega)J_\Delta R+[T_1(t),T_2(t),0]^{\mathrm{T}},\end{cases} \quad (28)$$

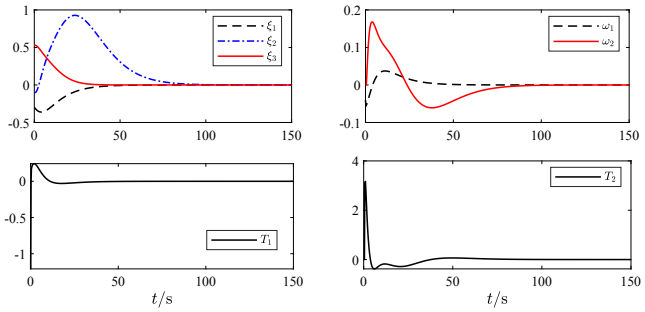

Fig. 1. The state responses for system (29)

where R is the rotation matrix, $\omega=[\omega_1,\omega_2,\omega_3]^{\mathrm{T}}$ is the angular velocity vector, $T_1(t)$ and $T_2(t)$ are the control torques, $J_\Delta=\mathrm{diag}\{J,J,J\}$ is the diagonal matrix of the principal moments of inertia, and

$$S(\omega)=\begin{bmatrix}0&\omega_3&-\omega_2\\-\omega_3&0&\omega_1\\\omega_2&-\omega_1&0\end{bmatrix}.$$

By using the Rodrigues parameter $\xi=[\xi_1,\xi_2,\xi_3]^{\mathrm{T}}$ and taking into account the input delays τ_1 and τ_2, (28) can be expressed as

$$\begin{cases}\dot{\xi}_1=\frac{1}{2}\omega_1,\\ \dot{\xi}_2=\frac{1}{2}\omega_2,\\ \dot{\xi}_3=\frac{1}{2}\xi_1\omega_2-\frac{1}{2}\xi_2\omega_1,\\ \dot{\omega}_1=\frac{T_1(t-\tau_1)}{J},\\ \dot{\omega}_2=\frac{T_2(t-\tau_2)}{J},\end{cases} \quad (29)$$

which, together with

$$X_{01}=\xi_1,\quad X_{12}=-2\xi_2,\quad X_{11}=\xi_3-\xi_1\xi_2,$$
$$X_{02}=\frac{1}{2}\omega_1,\quad X_{21}=-\omega_2,$$

can be rewritten as

$$\begin{cases}\dot{X}_{01}=X_{02},\\ \dot{X}_{02}=u_0(t-\tau_1),\\ \dot{X}_{11}=X_{02}X_{12},\\ \dot{X}_{12}=X_{21},\\ \dot{X}_{21}=u(t-\tau_2),\end{cases}$$

where $u_0(t-\tau_1)=T_1(t-\tau_1)/(2J)$ and $u(t-\tau_2)=-T_2(t-\tau_2)/(2J)$. According to (10), we can calculate that

$$\begin{aligned}E&=\left(\alpha I_2+A_2-b_2b_2^{\mathrm{T}}P(3\alpha)\mathrm{e}^{\alpha\tau_1}\right)^{-1}b_2\mathrm{e}^{\alpha\tau_1}\\&=\frac{\mathrm{e}^{\alpha\tau_1}}{3\alpha^2\mathrm{e}^{\alpha\tau_1}+\alpha^2}\begin{bmatrix}-1\\\alpha\end{bmatrix}.\end{aligned}$$

Clearly, $E_{2\max}=\frac{1}{3\alpha+\alpha\mathrm{e}^{-\alpha D}}$ and $E_{2\min}=\frac{1}{4\alpha}$, and we choose the parameter $\beta=-\alpha/E_{2\max}$ as in Theorem 1. For simulation, take the initial condition $[X_{01}(0),X_{02}(0)]=[-0.3,-0.02]$ and $[X_{11}(0),X_{12}(0),X_{21}(0)]=[0.5,0.2,0]$, time delays $\tau_1=0.2,\tau_2=0.5$, $D=0.5$, principal moments of inertia $J=15\mathrm{kg}\cdot\mathrm{m}^2$, parameters $\gamma=1$ and $\alpha=0.1$. The states are plotted in Fig. 1. From above figure, we can observer that the proposed method can indeed drive the states to zero.

5. CONCLUSION

This paper provided a constructive solution to the smooth memoryless time-varying control problem for a class of chained nonholonomic systems consisting of a linear subsystem, a bilinear subsystem, and an integrator chain subsystem with input delays. As the approach is time-domain based, it has potential for extension towards bounded time-varying input delays.

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