NUMERICAL EXPERIMENTS FOR APPROXIMATION OF LINEAR SYSTEMS USING THE METHOD OF A.A.K.

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THE PROBLEM:

$$x_{k+1} = A x_k + B u_k$$
$$y_k = C x_k + D u_k \quad \rightarrow \quad H(z) = C(zI - A)^{-1} B + \underbrace{D}_{0}$$

$r \times m$

n : McMillan degree

MINIMAL NORM APPROXIMATION (MNA)

! $H(z)$ McMillan deg n ; $k < n$

? $\tilde{H}(z)$ " " $\leq k$.

s.t.

$\| H(z) - \tilde{H}(z) \|_H$ minimal

MINIMAL DEGREE APPROXIMATION (MDA)

! $H(z)$ McMillan deg. n ; $\rho > 0$

? $\tilde{H}(z)$ s.t.

$\| H(z) - \tilde{H}(z) \|_H \leq \rho$

McMillan deg. $\tilde{H}(z)$ minimal

HANKEL NORM

$H(z) = C(zI - A)^{-1} B$ stable $(|\lambda_i(A)| < 1)$

$$\| H(z) \|_H := \| \Gamma[H(z)] \|_2 = \sigma_{max}(\Gamma[H(z)])$$

$$\Gamma[H(z)] = \begin{bmatrix} CB & CAB & CA^2B & \cdots \\ CAB & CA^2B & \vdots & \cdots \\ CA^2B & \vdots & & \\ \vdots & & & \end{bmatrix}$$

if not stable : $\| H(z) \|_H = \| [H(z)]_- \|_H$

$$T(z) := \frac{N(z)\hat{P}(z) - \rho\hat{a}^*(z)Q(z)}{a(z)}$$

$$W(z) := \frac{N^*(z)\hat{Q}(z) - \rho\hat{a}(z)P(z)}{a^*(z)}$$

□

WHAT IF $\rho \in (\sigma_{k+1}, \sigma_k)$ → EXTENSION!

$$\underline{H}(z) = H_0 z^{-1} + H(z) z^{-1}$$

THM 2 (EXTENSION THM)

$\exists!\ H(z)$, $\sigma(\Gamma[H(z)])$: $\sigma_1 \geq \sigma_2 \geq \cdots$; $\rho \in (\sigma_{k+1}, \sigma_k)$

$\exists\ H_0$ s.t. $\sigma(\Gamma[\underline{H}(z)])$: $\underline{\sigma}_1 \geq \cdots \geq \underline{\sigma}_k > \rho = \underline{\sigma}_{k+1} = \cdots = \underline{\sigma}_{k+p} > \underline{\sigma}_{k+p+1}$

at most p values in (σ_{k+1}, σ_k) for which H_0 doesn't exist!

□

THM 3

$\exists!\ H(z)$, $\rho \in (\sigma_{k+1}, \sigma_k)$; $\sigma_1 \geq \sigma_2 \geq \cdots$

→ $\underline{H}(z)$ (THM 2)

→ $\underline{T}(z)$, $\underline{P}(z)$, $\underline{Q}(z)$, $\underline{W}(z)$ (THM 1)

→ $H_{MDA}(z) = [z\,\underline{T}(z)\,\hat{\underline{P}}^{-1}(z)]_- = [\hat{\underline{Q}}^{*-1}(z)\,\underline{W}^*(z)\,z]_-$

$\|H(z) - H_{MDA}(z)\|_H \leq \rho$

McMillan deg. $H_{MDA}(z) = k$

□

If $H(z)$ not square: - add 0-rows/columns $H \to \bar{H}$

- construct $\bar{H}_{MDA}$
- drop rows/columns $\bar{H}_{MDA} \to H_{MDA}$

ALGORITHM

1. WRITE $H(z)$ AS $\frac{1}{a(z)} N(z)$ $\quad a(z)$ monic of deg N, $N(z)$ $p \times p$ of deg $\leq N-1$

2. SOLVE POLYN. MATRIX EQ.

$$V(z)\, Z(z) = 0$$

! $$V(z) = \begin{bmatrix} a(z) I_p & -N(z) & \rho \hat{a}^*(z) I_p & 0 \\ 0 & \rho a(z) I_p & -N^*(z) & \hat{a}^*(z) I_p \end{bmatrix} \in \mathbb{C}^{2p \times 4p}[z]$$

? $$Z(z) = \begin{bmatrix} R'(z) & R''(z) \\ \underline{\hat{P}}'(z) & \underline{\hat{P}}''(z) \\ \underline{Q}'(z) & \underline{Q}''(z) \\ \hat{S}'(z) & \hat{S}''(z) \end{bmatrix} \in \mathbb{C}^{4p \times 2p}[z]$$

3. SOLVE RICCATI EQ.

$$R_0' - R_0'' H_0^* + H_0 \hat{P}_0' - H_0 \hat{P}_0'' H_0^* = 0$$

$$[\, R_0'^* = R_0' \;;\; (\underline{\hat{P}}_0'')^* = \underline{\hat{P}}_0'' \;;\; (\underline{\hat{P}}_0')^* = -R_0'' \,]$$

4. $$H_{MDA}(z) = [\, (R'(z) - R''(z) H_0^*)\, (\underline{\hat{P}}'(z) - \underline{\hat{P}}''(z) H_0^*)^{-1} \,]_-$$

$$= [\, (\hat{Q}'^*(z) - H_0 \hat{Q}''^*(z))^{-1} (S'^*(z) - H_0 S''^*(z)) \,]_-$$

THEORY (KUNG, LIN, IEEE AC 1981)

$$F(z) = \sum_{-\infty}^{\infty} F_k z^k \quad \in L_2^{2\times m}$$

$$F^*(z) = \sum_{-\infty}^{\infty} F_k^* z^{-k} \qquad .^* \text{ compl. conj. transp.}$$

$$\hat{F}(z) = z^n F(z^{-1}) \qquad \text{for polynom. of deg. } n.$$

$$H(z) = \sum_{1}^{\infty} H_k z^{-k} \qquad \text{stable tf. fct.} \quad H_k \in \mathbb{C}^{p\times p}$$

$$\Gamma[H(z)] = [H_{i+j}]$$

$$= \sum_{1}^{\infty} \sigma_i \zeta^{(i)} \eta^{(i)*} = U \Sigma V^*$$

$$U := [\zeta^{(1)} | \zeta^{(2)} | \cdots] \qquad V := [\eta^{(1)} | \eta^{(2)} | \cdots]$$

$$\Sigma = \operatorname{diag}[\sigma_1, \sigma_2, \cdots]$$

$$\zeta^{(i)}(z) = \sum_{k=1}^{\infty} \zeta_k^{(i)} z^{-k} \qquad \eta^{(i)}(z) = \sum_{k=1}^{\infty} \eta_k^{(i)} z^{-k}$$

THM 1 pxp system ; ρ = s.v. of $\Gamma[H(z)]$ of multipl. p.

$$H(z) = \frac{1}{a(z)} N(z) \quad : \quad a(z) \text{ monic, deg. } N$$
$$N(z) \quad p\times p \text{ , deg} \le N-1$$

$$\Gamma[H(z)] = \sum_{1}^{\infty} \sigma_k \zeta^{(k)} \eta^{(k)*} \quad : \sigma_k > \rho = \sigma_{k+1} = \cdots = \sigma_{k+p} > \sigma_{k+p+1}$$

$H^{(k)}$ is MDA McMillan deg $= k$

$$H^{(k)}(z) = [T(z)\hat{P}^{-1}(z)]_- = [\hat{Q}^{*-1}(z) W^*(z)]_-$$

$$X(z) := [\eta^{(k+1)}(z) \cdots \eta^{(k+p)}(z)] := \frac{1}{a^*(z)} P(z)$$

$$Y(z) := [\zeta^{(k+1)}(z) \cdots \zeta^{(k+p)}(z)] := \frac{1}{a(z)} Q(z)$$

STEP 1 : $H(z) = \frac{1}{a(z)} N(z)$

cfr. VARGA (Int. J. contr. 1981) – VERHAEGHEN (1985)

$H(z) = C(zI-A)^{-1}B \rightarrow h_{ij} = c_i^T (zI-A)^{-1} b_j$

1. controllable part of (A, b, c^T)

$(A,b,c^T) \rightarrow (QAQ^T, Qb, c^TQ^T) := (\tilde{A}, \tilde{b}, \tilde{c}) \quad (Q^{-1}=Q^T)$

$[\tilde{A}, \tilde{b}]$ = lower Hessenberg $= \begin{bmatrix} \tilde{A}_1 & 0 & : 0 \\ \tilde{A}_3 & \tilde{A}_2 & : \tilde{b}_2 \end{bmatrix}$

$\rightarrow (\tilde{A}_2, \tilde{b}_2, \tilde{c}_2^T)$

2. observable part of (A, b, c^T)

similar : $\begin{bmatrix} A \\ c^T \end{bmatrix} \rightarrow$ Upper Hessenberg $= \begin{bmatrix} \hat{A}_1 & \hat{A}_3 \\ 0 & \hat{A}_2 \\ 0 & \hat{c}_2^T \end{bmatrix}$

$\rightarrow (\hat{A}_2, \hat{b}_2, \hat{c}_2^T)$

3. poles of (A, b, c^T)

$= \lambda(A)$

4. zeros of (A, b, c^T) = finite solutions of

$$\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix} x = \mu \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x$$

↑ upper Hessenberg ↑ upper triang.

5. gain factors : $h(z) = k \prod_{i}(z-\mu_i) / \prod_{j=1}^{q}(z-\lambda_j)$

$k = c^T(z_0 I - A)^{-1} b \prod_{j=1}^{q}(z_0-\lambda_j) / \prod_{i}(z_0-\mu_i) \quad , |z_0| > \max\{|\mu_i|, |\lambda_j|\}$

$H := z_0 I - A = [\eta_{ij}] \quad ; \quad b = [\beta_i] \quad ; \quad c^T = [\gamma_i]$

$QH = \triangledown$ (↳ UpHess.) , $b = Qb \rightarrow k = \frac{\gamma_q \beta_q}{\eta_{qq}} \frac{\prod(z_0-\lambda_j)}{\prod(z_0-\mu_i)}$

STEP 2. SOLVE $V(z)\,Z(z) = 0$

$$V(z) = \sum_0^N V_i z^i \quad , \quad V_i \in \mathbb{C}^{2p \times 4p}$$

$$Z(z) = \sum_0^N Z_i z^i \quad , \quad Z_i \in \mathbb{C}^{4p \times 2p}$$

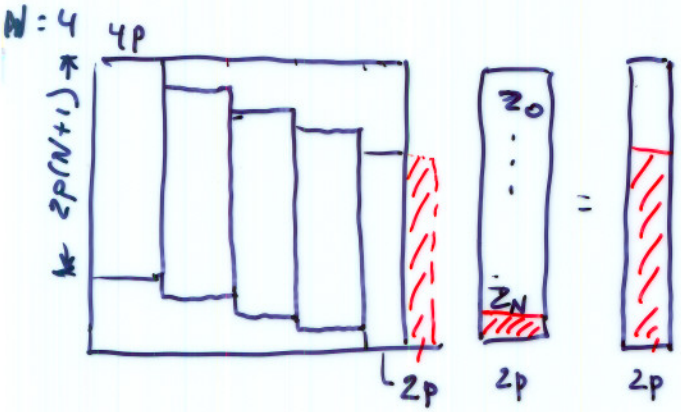
$V(z)\,Z(z) = 0 \iff$

$$\begin{bmatrix} V_0 & & 0 \\ V_1 & V_0 & \\ \vdots & & \ddots \\ V_N & \cdots & V_0 \\ & \ddots & \vdots \\ 0 & & V_N \end{bmatrix} \begin{bmatrix} Z_0 \\ \vdots \\ Z_N \end{bmatrix} = 0$$

$p \notin \sigma\left(\Gamma[H(z)]\right)$

$\Rightarrow \exists$ solution s.t.

$$Z_N = \begin{bmatrix} X \\ \hline I_{2p} \end{bmatrix} \in \mathbb{C}^{4p \times 2p}$$



size :
$2p(2N+1)$

ex : $N = 15$, $p = 3$
$\rightarrow 186$

use projection method (Kung, Kailath Automatica 1980)

$O(N^2 p^3)$ flops
$O(N p^2)$ memory

STEP 3 RICCATI EQ.

cfr Laub (IEEE AC 1979)

STEP 4 ISOLATE H_{MDA}

K & L → construct $[\hat{Q}^{*-1}(z)\, S^{*}(z)]$
+ partial fraction expans. ⇒ $[\ ]_-$

Here: $Q(z), S(z) \rightarrow (A, B, C, D)$
+ isolate stable part of (A, B, C, D)

(note: <u>left</u> MFD because $\underline{Q}_N = I_p$, $\hat{S}_N = -H_0^* = H_0$)

1: $D^{-1}(z) N(z) \longrightarrow (A, B, C, D)$

ctr: Williams (1985) structure thm of Wolovich

2. $(\bar{A}, \bar{B}, \bar{C}) \rightarrow (A, B, C)$ (original dimensions)

3. stable part
ctr. Glover (1984, Int. J. contr)

1. $V_1^T A V_1 = \tilde{A}$ real upper Schur
2. $V_2^T \tilde{A} V_2 = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$ $|\lambda(\tilde{A}_{11})| < 1$ $|\lambda(\tilde{A}_{22})| > 1$ (k)
3. Solve $\tilde{A}_{11} X - X \tilde{A}_{22} + \tilde{A}_{12} = 0 \rightarrow X$

3 $T = V_1 V_2 \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \rightarrow A = T^{-1} \tilde{A} T = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$

$T^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

$C T = [C_1 \ C_2]$

(A_M, B_1, C_1) is stable part!

PERFORMANCE

1. RANK DETERMINATION

- degree of tf. → ok.
- degree of $a(z), N(z)$ in $H(z) = \frac{N(z)}{a(z)}$ → ok.
- stable part (step 4) → rank = McMillan deg. H_{MDA} → OK.
  ! except 1 case where $\rho \approx \sigma_k(\Gamma[H(z)])$ ↔ ρ
  " 0.0115 " 0.01159 " 0.0116 ok.

choice of tolerance = critical !!!

2. SYSTEMS TO BE SOLVED

- $V(z)\,Z(z) = 0$: size $2p(2N+1)$ (*)
- in Riccati eq : " p
- MFD → (A, B, C, D) : " p

condition numbers ok.
EXCEPT: when $\rho \approx \sigma_k(\Gamma[H(z)])$ in (*)

= MOST CRITICAL STEP !
Projection method doesn't work if $\rho \approx \sigma_k$

COMPARISON WITH GLOVER (BETTER)

- cnt
- MNA → take k s.t. $\sigma_k > \rho \geq \sigma_{k+1}$
- state space