A vector equilibrium problem for symmetrically located point charges on a sphere

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Abstract

We study the equilibrium measure on the two dimensional sphere in the presence of an external field generated by $r+1$ equal point charges that are symmetrically located around the north pole. The support of the equilibrium measure is known as the droplet. The droplet has a motherbody which we characterize by means of a vector equilibrium problem (VEP) for r measures in the complex plane.

The model undergoes two transitions which is reflected in the support of the first component of the minimizer of the VEP, namely the support can be a finite interval containing 0, the union of two intervals, or the full half-line. The two interval case corresponds to a droplet with two disjoint components, and it is analyzed by means of a genus one Riemann surface.

1 Introduction

1.1 Equilibrium on the sphere

This paper deals with an electrostatic equilibrium problem for free charges on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with logarithmic interaction under the influence of a finite number of fixed point charges [7, 11, 16, 39]. Suppose there are $r+1$ fixed charges at points $p_0, \ldots, p_r$ on $\mathbb{S}^2$, and each p_j carries a charge $a_j > 0$, leading to a charge distribution

$$\sigma = \sum_{j=0}^{r} a_j \delta_{p_j}. \tag{1.1}$$

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Then there exists an equilibrium measure μ_σ in the presence of the fixed charges that is the unique probability measure on $\mathbb{S}^2$ that satisfies for some constant ℓ,

$$\begin{aligned} U^{\mu_\sigma} + U^\sigma &= \ell, \quad \text{on } D_\sigma = \operatorname{supp}(\mu_\sigma), \\ U^{\mu_\sigma} + U^\sigma &\geq \ell, \quad \text{on } \mathbb{S}^2, \end{aligned} \tag{1.2}$$

where we use

$$U^\mu(x) = \int \log \frac{1}{\|x-y\|} d\mu(y)$$

to denote the logarithmic potential of a measure μ. The domain D_σ is known as the droplet, and it determines the measure μ_σ since

$$\mu_\sigma = (\lambda(D_\sigma))^{-1} \lambda_{D_\sigma} \tag{1.3}$$

where λ_D denotes the restriction to D of the normalized Lebesgue measure λ on the sphere. It is known that

$$\lambda(D_\sigma) = \frac{1}{1+\sigma(\mathbb{S}^2)} = \frac{1}{1+\sum_{j=0}^r a_j}. \tag{1.4}$$

see e.g. [11, Appendix A].

A motherbody (or a potential theoretic skeleton [25]) for D_σ is a probability measure σ^* supported on a one-dimensional subset of $\mathbb{S}^2$ (i.e., a curve, or a system of curves) such that for some constant ℓ^*,

$$\begin{aligned} U^{\sigma^*} &= U^{\mu_\sigma} + \ell^*, \quad \text{on } \mathbb{S}^2 \setminus D_\sigma, \\ U^{\sigma^*} &\geq U^{\mu_\sigma} + \ell^*, \quad \text{on } \mathbb{S}^2. \end{aligned} \tag{1.5}$$

Motherbodies are connected to a variety of topics in applied complex analysis, such as quadrature domains and Schwarz functions [1, 12, 24, 34], partial balayage and Hele-Shaw flows [23], orthogonal polynomials in the complex plane [3, 4, 38] and normal matrix models [5, 48].

The aim of this paper is to construct such a motherbody by means of a vector equilibrium problem in the special situation where the points are in a symmetric position around a distinguished point on the unit sphere, that without loss of generality we can take as the north pole. More precisely, we assume that the distance to the north pole is the same for each point p_j, which means that the points are on a circle of constant latitude. On this circle the points are evenly distributed, like vertices of a regular $r+1$-gon. We also assume

$$a_j = a, \qquad \text{for } j = 0, \ldots, r.$$

In this situation we are able to compute the motherbody, which, because of rotational symmetry, is supported on $r+1$ meridians (lines of constant longitude) that connect the north and south poles. From the motherbody we go on to construct the droplet D_σ.

With fixed points $p_0, \ldots, p_r$, the droplet and the support of the motherbody decrease as we increase a. We find three possible situations and the transitions between them.

- For small $a > 0$, the droplet is big and the complement $\mathbb{S}^2 \setminus D_\sigma$ consists of $r+1$ disjoint spherical caps, one around each of the points p_j. The motherbody is supported on the full meridians with a positive density.

- For a first critical value $a_{1,cr}$, the spherical caps are tangent to each other. The density of the motherbody becomes zero at the points of tangency.

- For $a > a_{1,cr}$ the droplet is no longer the complement of disjoint spherical caps. For a somewhat larger than $a_{1,cr}$ the droplet will have two connected components (provided $r \geq 2$), one containing the north pole and the other one the south pole. The motherbody is not fully supported anymore. On each meridian the support has two parts, one with the north pole and one with the south pole.

- For a second critical value $a_{2,cr}$ one of the components disappers. If the points p_j are in the northern hemisphere, then the component containing the north pole disappears. Also the parts of the motherbody containing the north pole have disappeared at the second critical value.

- For larger $a > a_{2,cr}$ the droplet D_σ is simply connected containing the south pole (assuming again that the points p_j are in the northern hemisphere). The support of the motherbody consists of $r+1$ segments containing the south pole, one segment along each meridian.

- As $a \to \infty$, the droplet and the support of the motherbody further shrink to the south pole.

1.2 The case $r = 1$

For $r = 1$ the two spherical caps are tangent at the north pole at the critical value $a_{1,cr}$. Then there is no second critical a-value since for each $a > a_{1,cr}$ the droplet is simply connected. The support of the motherbody is an

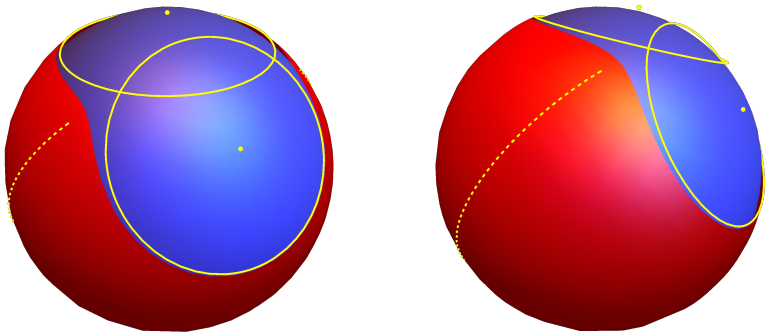


Figure 1: Picture of the droplet (red region) in case $r = 1$ and $a > a_{1,cr}$. The spherical caps centered at p_0 and p_1 with geodesic radii $a/(1 + 2a)$ are also represented, as well as the support of the motherbody (dashed line inside the droplet). The boundary of the droplet is mapped by stereographic projection onto an ellipse in the complex plane [11].

interval along the big circle that separates the two points p_0 and p_1. See Figure 1 that is taken from [11] and compare also with [7, Figure 4]. This situation was analyzed in [11] and it was shown that the boundary of the droplet is mapped by stereographic projection to an ellipse in the complex plane. This fact can also be deduced from earlier work by Gustafsson and Tkachev in [26, Example 3].

The approach of [11] is to first characterize the motherbody by means of an equilibrium problem from logarithmic potential theory [44, 45]. This equilibrium problems asks for the minimizer of

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + 2 \int V(x) d\mu(x) \tag{1.6}$$

among probability measures μ on $\mathbb{R}$, with

$$V(x) = \frac{1+a}{2} \log(x^2 + b^{-2}) - \frac{a}{2} \log(x^2 + b^2), \tag{1.7}$$

where $\pm ib$, $b > 1$, are the images of the two points p_0, p_1 under stereographic projection onto the complex plane. The minimizer is calculated explicitly in [11, Theorem 1.6], see also [43]. The external field is only weakly admissible

[27, 47] and for a fixed $b > 1$ there is a critical value a_{cr} such that the minimizer μ_V is compactly supported if and only if $a > a_{cr}$. Out of the Stieltjes transform of μ_V a meromorphic function S is then constructed that is shown to be the spherical Schwarz function of a certain domain Ω in the sense that its boundary is characterized by

$$\partial\Omega : \quad S(z) = \frac{\bar{z}}{1+|z|^2}.$$

After pulling back to the sphere with inverse stereographic projection, the domain Ω is then proved to give the droplet D_σ and μ_V gives the mother-body.

1.3 Stereographic projection and removal of symmetry

In this paper we extend the approach of [11] to $r+1$ points on the sphere. As in [11] we project onto the complex plane where we do all calculations. Instead of the equilibrium problem (1.6), (1.7) we study a vector equilibrium problem for a vector of r measures. This will be described in section 2.1 below. In this section we first describe what we aim to achieve in the complex plane.

We move from the sphere to the complex plane by stereographic projection, where the south pole is mapped to 0 and the north pole to ∞. The points $p_0, \ldots, p_r$ are projected to $r+1$ points with absolute value $q^{-\frac{1}{r+1}}$ for some number $q > 0$. The projected points will be the solutions of the equation $z^{r+1} + q = 0$, namely

$$p_j \mapsto q^{-\frac{1}{r+1}} e^{i\theta_j}, \qquad \theta_j = \frac{\pi}{r+1} + \frac{2j\pi}{r+1}, \qquad \text{for } j = 0, 1, \ldots, r. \tag{1.8}$$

The case $q < 1$ corresponds to points p_j in the northern hemisphere, and $q > 1$ to points in the southern hemisphere.

The angles θ_j are chosen in such a way that the meridians separating the points $p_0, \ldots, p_r$ at equal distances are mapped to the $r+1$-star

$$\{z \in \mathbb{C} \mid z^{r+1} \in [0, \infty)\}.$$

The droplet D_σ is mapped to a domain $\Omega \subset \mathbb{C} \cup \{\infty\}$, and μ_σ is mapped to its pushforward on Ω which takes the form

$$d\mu_\Omega(z) = \frac{dA(z)}{t\pi(1+|z|^2)^2}\bigg|_\Omega \tag{1.9}$$

where $dA(z)$ is the planar Lebesgue measure on $\mathbb{C}$. and

$$t = \frac{1}{1+\sigma(\mathbb{S}^2)} = \frac{1}{1+(r+1)a} \tag{1.10}$$

The properties (1.2) translate into

$$U^{\mu_\Omega}(z) + a\log\frac{1}{|z^{r+1}+q^{-1}|} + \frac{1+(r+1)a}{2}\log\left(1+|z|^2\right)\begin{cases} = c_1, & z\in\Omega, \\ \geq c_1, & z\in\mathbb{C}. \end{cases} \tag{1.11}$$

for some constant c_1.

The motherbody σ^* (that we are looking for in this paper and whose existence we do not a priori assume) satisfying (1.5) corresponds to a probability measure μ^* on $\{z \mid z^{r+1} \in [0,\infty)\}$ with the property that

$$\begin{aligned} U^{\mu^*} &= U^{\mu_\Omega} + c_2, \quad \text{on } \mathbb{C}\setminus\Omega, \\ U^{\mu^*} &\geq U^{\mu_\Omega} + c_2, \quad \text{on } \mathbb{C}, \end{aligned} \tag{1.12}$$

for some other constant c_2. The aim of the paper is to construct the domain Ω and measures μ^* and μ_Ω satisfying the conditions (1.11) and (1.12).

The probability measures μ^* and μ_Ω will be invariant under rotations around the origin over angle $\frac{2\pi}{r+1}$. For our computations it will be convenient to remove the rotational symmetry, and change variables $z \mapsto z^{r+1}$. Then μ^* will correspond to a probability measure μ_1 on $[0,\infty)$, and μ_Ω to a probability measure μ_U on the set

$$U = \{z^{r+1} \mid z \in \Omega\} \tag{1.13}$$

and μ_U takes the form

$$d\mu_U(z) = \frac{1}{(r+1)t\pi}\frac{dA(z)}{|z|^{\frac{2r}{r+1}}\left(1+|z|^{\frac{2}{r+1}}\right)^2}\bigg|_U. \tag{1.14}$$

which comes from applying the change of variables to (1.9)

Our approach will be to construct μ_1 first as the first component of the minimizer of a vector equilibrium problem (VEP) for r measures. Besides μ_1 there will be further measures $\mu_2,\ldots,\mu_r$ that play auxiliary roles. They do not have a direct interpretation for the problem at hand, though.

In the next section we will state the VEP without trying to motivate the form that it takes. It is actually by no means obvious that this VEP

is relevant for our problem, and it will be our main result that μ_1 after symmetrization gives indeed a measure μ^* that can be identified as the image of the motherbody under stereographic projection. However, for $r = 1$, the VEP is an equilibrium problem for one measure that, after symmetrization, can be identified with (1.7).

The VEP gives rise to an algebraic structure and this will allow us to find a domain U with a measure (1.14). Through (1.13) we find a domain Ω with rotational symmetry and the measure μ_Ω as in (1.9). We prove that it has the properties (1.11) and (1.12).

The VEP depends on two parameters $q > 0$ and $t \in (0, 1)$, that ultimately will play the roles of the parameters appearing in (1.8) and (1.10), as we will show in the end.

2 Statement of results

2.1 Vector equilibrium problem

Let $r \geq 2$ be an integer, and let $q > 0$, $0 < t < 1$ be real parameters. Our starting point is a vector equilibrium problem that asks to minimize the energy functional

$$\mathcal{E}(\mu_1, \mu_2, \ldots, \mu_r) = \sum_{j=1}^{r} I(\mu_j) - \sum_{j=1}^{r-1} I(\mu_j, \mu_{j+1}) + \frac{1-t}{t} I\left(\mu_1, \delta_{-q^{-1}}\right) - \frac{r+t}{t} I\left(\mu_r, \delta_{(-1)^r q}\right), \quad (2.1)$$

depending on r measures. Here $\delta_{-q^{-1}}$ and $\delta_{(-1)^r q}$ denote Dirac point masses. As usual we write

$$I(\mu, \nu) = \int U^\mu d\nu = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y)$$

for the mutual logarithmic energy of μ an ν, and $I(\mu) = I(\mu, \mu)$ for the logarithmic energy of μ.

Our aim is to minimize (2.1) over a vector of measures satisfying certain conditions. We emphasize that a measure (without any adjective) will always refer to a positive measure. We also encounter negative measures or signed measures in this paper, but in such a context the adjective will always be mentioned.

Definition 2.1. The vector equilibrium problem (VEP) asks to minimize the energy functional (2.1) over vectors $(\mu_1, \dots, \mu_r)$ of measures subject to the conditions

(a) $\operatorname{supp}(\mu_j) \subset \Delta_j$ for every j, where

$$\Delta_j = \begin{cases} [0, \infty), & \text{if } j \text{ is odd,} \\ (-\infty, 0], & \text{if } j \text{ is even,} \end{cases} \tag{2.2}$$

(b) the total mass of μ_j is

$$\mu_j(\Delta_j) = 1 + \frac{j-1}{t}, \quad \text{for } j = 1, \dots, r. \tag{2.3}$$

Throughout the paper we will write

$$\mu_0 = \left(1 - \frac{1}{t}\right) \delta_{-q^{-1}}, \qquad \mu_{r+1} = \left(1 + \frac{r}{t}\right) \delta_{(-1)^r q}. \tag{2.4}$$

Then (2.3) is also satisfied for $j \in \{0, r+1\}$, but note that μ_0 is a negative measure (since $0 < t < 1$). Moreover, (2.1) takes the compact form

$$\mathcal{E}(\mu_1, \dots, \mu_r) = \sum_{j=1}^{r} I(\mu_j) - \sum_{j=0}^{r} I(\mu_j, \mu_{j+1}), \tag{2.5}$$

that includes μ_0 and μ_{r+1} as well, but μ_0 and μ_{r+1} remain fixed in the VEP.

Vector equilibrium problems were first introduced by Gonchar and Rakhmanov in their study of Hermite-Padé approximation [21, 22], see also [42]. They also appear in ensembles of random matrices that are related to multiple orthogonal polynomials, see [2, 28] and references cited therein.

The energy functional (2.5) involves an attraction between neighboring measures that is of Nikishin type, and this has appeared in a number of situations before. What is special is that the total masses (2.3) are in an arithmetic progression that is increasing with steps $1/t$. It is more common that the masses are in an arithmetic progression that decreases from 1 to 0 see e.g. [17] and the examples in [2, 28].

The VEP of Definition 2.1 is weakly admissible in the sense of [27] as we show next.

Lemma 2.2. *The vector equilibrium problem is weakly admissible. There is a unique minimizer, denoted* $(\mu_1, \dots, \mu_r)$. *The measures* $\mu_2, \dots, \mu_r$ *have full supports*

$$\operatorname{supp}(\mu_j) = \Delta_j = (-1)^{j-1}[0, \infty), \quad \textit{for } j = 2, \dots, r. \tag{2.6}$$

Proof. To check the conditions in Assumption 2.1 of [27], we write the energy functional (2.5) in the form

$$\sum_{1\le i,j\le r} c_{ij} I(\mu_i,\mu_j) + \sum_{j=1}^{r} \int V_j d\mu_j$$

with

$$c_{ij} = \begin{cases} 1 & \text{if } i=j, \\ -\frac{1}{2} & \text{if } |i-j|=1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

and

$$V_j(x) = \begin{cases} -U^{\mu_0}(x) = (1-\frac{1}{t})\log|x+q^{-1}|, & \text{if } j=1, \\ -U^{\mu_{r+1}}(x) = (1+\frac{r}{t})\log|x-(-1)^r q|, & \text{if } j=r, \\ \equiv 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

The interaction matrix $C = (c_{ij})$ is symmetric and positive definite, and each V_j is continuous on Δ_j, since $-q^{-1} \notin \Delta_1$ and $(-1)^r q \notin \Delta_r$.

The prescribed total masses $m_j = \mu_j(\Delta_j)$ from (2.3) come in an arithmetic progression which implies by (2.7) that

$$\sum_{j=1}^{r} c_{ij} m_j = 0, \qquad \text{for } i = 2,\dots,r-1, \tag{2.9}$$

and also

$$\sum_{j=1}^{r} c_{ij} m_j = \begin{cases} \frac{1}{2}m_0 = \frac{1}{2}(1-\frac{1}{t}), & \text{for } i=1, \\ \frac{1}{2}m_{r+1} = \frac{1}{2}(1+\frac{r}{t}), & \text{for } i=r. \end{cases} \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10) that, for every $i = 1,\dots,r$,

$$V_i(x) - \left(\sum_{j=1}^{r} c_{i,j}\mu_j(\Delta_j)\right)\log(1+|x|^2) \to 0, \qquad \text{as } x \in \Delta_j \to \pm\infty.$$

Thus all conditions of Assumption 2.1 in [27] are satisfied, and the VEP is weakly admissible. Then there is a unique minimizer by [27, Corollary 2.7].

Given the other measures, the problem for μ_j (for $1 \le j \le r$) is to minimize

$$I(\mu_j) - I(\mu_j, \mu_{j-1} + \mu_{j+1})$$

among measures on Δ_j with total mass (2.3). Since $\mu_{j-1}+\mu_{j+1}$ is a positive measure for $j \geq 2$, it follows that μ_j is a balayage measure (see [45] for the notion of balayage)

$$\mu_j = \frac{1}{2}\,\mathrm{Bal}\,(\mu_{j-1}+\mu_{j+1}, \Delta_j) \qquad \text{for } j = 2, \ldots, r, \tag{2.11}$$

and μ_j has full support for $j \geq 2$, see also (3.2) for the expression of the density of the balayage of a measure on $(-\infty, 0]$ onto $[0, \infty)$. There is a similar formula for the balayage of a measure on $[0, \infty)$ to $(-\infty, 0]$ that shows that it has indeed a full support. □

The balayage property (2.11) means that

$$2U^{\mu_j} = U^{\mu_{j-1}} + U^{\mu_{j+1}} \quad \text{ on } \Delta_j, \qquad \text{for } j = 2, \ldots, r, \tag{2.12}$$

and this will be important for us in what follows.

The measure μ_1 is the main player in the game. The argument in the proof of Lemma 2.2 leading to (2.11) does not work for $j = 1$, since μ_0 is a negative measure. Therefore the balayage of $\mu_0 + \mu_2$ onto $\Delta_1 = [0, \infty)$ is not necessarily positive on the full half-line. However, if it is positive then (2.11) and (2.12) hold for $j = 1$ as well, and then also μ_1 has a full support. It turns out that this happens for t sufficiently large (i.e., sufficiently close to 1).

Our first main result is about the structure of the support $\Sigma_1 = \mathrm{supp}(\mu_1)$ of μ_1. There are four possible cases that will be indicated with acronyms **BIS** = Bounded Interval Support, **UIS** = Unbounded Interval Support, **TIS** = Two Interval Support, and **FIS** = Full Interval Support.

In situations where we want to emphasize the dependence on t of the various notions that we introduced (and of others that are still to come), we append a subscript t. Hence we write for example $\mu_{1,t}$, $\Sigma_{1,t}$, and so on.

Theorem 2.3. *Fix $q > 0$. Let $(\mu_1, \ldots, \mu_r)$ be the minimizer of the vector equilibrium problem depending on the parameter $t \in (0, 1)$.*

(a) *There are four possible cases for $\Sigma_1 = \mathrm{supp}(\mu_1)$, depending on t, namely there exist $0 < x_1 < x_2 < \infty$ such that either **BIS** : $\Sigma_1 = [0, x_1]$, or **UIS** : $\Sigma_1 = [x_2, \infty)$, or **TIS** : $\Sigma_1 = [0, x_1] \cup [x_2, \infty)$, or **FIS** : $\Sigma_1 = [0, \infty)$.*

(b) *For each $j = 1, \ldots, r$ the measure $t\mu_{j,t}$ increases as a function of $t \in (0, 1)$.*

(c) *Suppose* $0 < q < 1$. *Then* 0 *is always in the support of* μ_1 *(and so* ***UIS*** *case does not occur for any* $t \in (0,1)$*).*

(d) *The measure* μ_1 *has a density that is real analytic on the interior of its support with a square-root vanishing at* x_1 *in the* ***BIS*** *and* ***TIS*** *cases, and at* x_2 *in the* ***UIS*** *and* ***TIS*** *cases.*

(e) *There exist constants* $c_0 > 0$ *and* $c_\infty > 0$ *such that*

$$\frac{d\mu_1(x)}{dx} = c_0 x^{-\frac{r}{r+1}} \left(1 + O\left(x^{\frac{1}{r+1}}\right)\right) \qquad \text{as } x \to 0+ \tag{2.13}$$

in ***BIS****,* ***TIS*** *and* ***FIS*** *cases, and*

$$\frac{d\mu_1(x)}{dx} = c_\infty x^{-\frac{r+2}{r+1}} \left(1 + O\left(x^{-\frac{1}{r+1}}\right)\right) \qquad \text{as } x \to \infty \tag{2.14}$$

in ***UIS****,* ***TIS*** *and* ***FIS*** *cases.*

The proof of Theorem 2.3 is in section 3, except for the proof of part (e) which is in section 4.1.2.

It follows from Theorem 2.3 that for $0 < q < 1$, there are two critical values

$$0 \le t_{1,cr} \le t_{2,cr} \le 1, \tag{2.15}$$

depending on q, such that we are in **BIS** case for $0 < t \le t_{1,cr}$, in the **TIS** case for $t_{1,cr} < t < t_{2,cr}$, and in **FIS** case for $t_{2,cr} \le t < 1$. For $r \ge 2$, the inequalities in (2.15) are actually strict inequalities and each of the three possible cases occurs for some values of t.

Ultimately, t will be related by equation (1.10) to the strength a of the fixed charges on the sphere. The critical values $t_{1,cr}$ and $t_{2,cr}$ will correspond to $a_{2,cr}$ and $a_{1,cr}$ (in that order) that are used at the end of section 1.1.

Remark 2.4. There is a symmetry between q and q^{-1} that allows us to restrict attention to $0 < q < 1$.

Let $\vec{\mu} = (\mu_1, \ldots, \mu_r)$ be a vector of measures as in the VEP of Definition 2.1. Let ν_j be the image of μ_j under the inversion $x \mapsto 1/x$, i.e., ν_j is the measure on Δ_j with

$$\int f d\nu_j = \int f\left(\frac{1}{x}\right) d\mu_j(x) \tag{2.16}$$

for a function f on Δ_j. Then it is an easy calculation to show that

$$I(\nu_j, \nu_k) = I(\mu_j, \mu_k) + \int \log|x| d\mu_j(x) \int d\mu_k + \int \log|x| d\mu_k(x) \int d\mu_j. \tag{2.17}$$

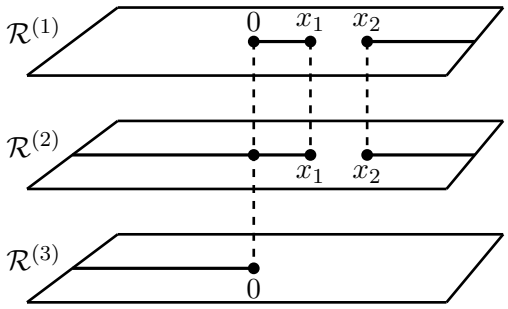


Figure 2: The Riemann surface $\mathcal{R}$ in **TIS** case (for $r = 2$)

Let us use $\mathcal{E}_q$ to denote the energy functional (2.1) corresponding to the parameter $q > 0$. Then using (2.17) and the total masses (2.3) of the measures we find after straightforward calculations that

$$\mathcal{E}_{1/q}(\vec{\nu}) = \mathcal{E}_q(\vec{\mu}) - \frac{r(r+2t-1)}{t^2} \log q.$$

Thus whenever $\vec{\mu} = (\mu_1, \ldots, \mu_r)$ is the minimizer of the VEP with parameter q, then $\vec{\nu} = (\nu_1, \ldots, \nu_r)$ is the minimizer with parameter $1/q$.

Due to this symmetry between q and q^{-1}, the support of μ_1 is always unbounded for $q > 1$, since for $0 < q < 1$ the support contains 0 by part (c) of Theorem 2.3. Instead of the **BIS** case, we then have the **UIS** case for t up to the first critical value. It is then continued with the **TIS** case, and after a second critical value with the **FIS** case.

The above also shows that for $q = 1$, the measure μ_1 is invariant under the inversion $x \mapsto 1/x$. Then we do not have a **BIS** or **UIS** case, but we start with a **TIS** case for t up to a critical value, followed by the **FIS** case.

2.2 A meromorphic function on a Riemann surface

In what follows we restrict to the case $0 < q < 1$. Then $0 \in \mathrm{supp}(\mu_1)$ by Theorem 2.3 (c) and we are in one of the **BIS**, **TIS** or **FIS** cases. We also let $0 < t < 1$. Our further results are based on the consideration of a Riemann surface.

Definition 2.5. The Riemann surface $\mathcal{R}$ (see Figure 2) has $r + 1$ sheets

$\mathcal{R}^{(j)}$, $j = 1, \ldots, r+1$, where

$$\begin{aligned} \mathcal{R}^{(1)} &= \mathbb{C} \setminus \operatorname{supp}(\mu_1), \\ \mathcal{R}^{(j)} &= \mathbb{C} \setminus (\operatorname{supp}(\mu_{j-1}) \cup \operatorname{supp}(\mu_j)), \qquad \text{for } j = 2, \ldots, r+1, \\ \mathcal{R}^{(r+1)} &= \mathbb{C} \setminus \operatorname{supp}(\mu_r) \end{aligned} \tag{2.18}$$

where $(\mu_1, \ldots, \mu_r)$ is the unique minimizer for the VEP of Definition 2.1, and we recall that $\operatorname{supp}(\mu_1) = \Sigma_1 \subset [0, \infty)$ and $\operatorname{supp}(\mu_j) = \Delta_j = (-1)^{j-1}[0, \infty)$ for $j = 2, \ldots, r$. Sheet $\mathcal{R}^{(j)}$ is connected to sheet $\mathcal{R}^{(j+1)}$ along the support of μ_j in the usual crosswise manner for $j = 1, \ldots, r$. We also add two (in **BIS** case) or one (in other cases) points at infinity in order to obtain a compact Riemann surface $\mathcal{R}$.

A count of branch points, together with the Riemann-Hurwitz formula, see e.g. [46], shows that $\mathcal{R}$ has genus zero in the **BIS** and **FIS** cases, while the genus is one in the **TIS** case.

The Stieltjes transform of the measure μ_j is

$$F_j(z) = \int \frac{d\mu_j(x)}{z - x}, \qquad z \in \mathbb{C} \setminus \operatorname{supp}(\mu_j). \tag{2.19}$$

This is also defined for $j = 0$ and $j = r+1$ in which cases we have the simple rational functions

$$F_0(z) = \frac{-1 + t}{t(z + q^{-1})}, \qquad F_{r+1}(z) = \frac{r + t}{t(z - (-1)^r q)} \tag{2.20}$$

We use the Stieltjes transforms to define a function on $\mathcal{R}$.

Definition 2.6. The function Φ is defined on the Riemann surface via its restrictions $\Phi^{(j)}$, $j = 1, \ldots, r+1$, to the various sheets, by

$$\Phi^{(j)}(z) = tF_j(z) - tF_{j-1}(z), \qquad z \in \mathcal{R}^{(j)}, \tag{2.21}$$

for $j = 1, \ldots, r+1$.

Differentiating the identity (2.12) we obtain

$$F_{j,+} - F_{j,-} = F_{j-1} + F_{j+1} \quad \text{on } \Delta_j = \operatorname{supp}(\mu_j), \qquad \text{for } j = 2, \ldots, r, \tag{2.22}$$

which means in view of (2.21) that $\Phi_{\pm}^{(j)} = \Phi_{\mp}^{(j+1)}$ on $\operatorname{supp}(\mu_j)$ for $j = 2, \ldots, r$. Thus Φ is analytic across the cut connecting sheets $\mathcal{R}^{(j)}$ and $\mathcal{R}^{(j+1)}$

for $j \geq 2$. Φ is also analytic across the cut connecting sheets $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, as this follows from the variational condition associated with the VEP

$$2U^{\mu_1} = U^{\mu_0} + U^{\mu_2} + c \quad \text{on } \operatorname{supp}(\mu_1),$$

which upon differentation leads to (2.22) on $\operatorname{supp}(\mu_j)$ for $j = 1$ as well. Thus Φ is meromorphic on $\mathcal{R}$ and it has a number of crucial properties that will be discussed in section 4.1.

2.3 The subset U

With the help of Φ we define a subset U of the complex plane that will lead to the droplet.

Definition 2.7. The set $U \subset \mathbb{C} \cup \{\infty\}$ is defined by

$$U = \overline{\{z \in \mathbb{C} \mid (\operatorname{Im} z) \cdot \operatorname{Im}\left(z\Phi^{(1)}(z)\right) < 0\}}. \tag{2.23}$$

We write U_t if we want to emphasize the dependence of U on the parameter $0 < t < 1$.

Theorem 2.8. *Let $0 < q < 1$ be fixed. Then the following hold.*

(a) *U is a closed set with the properties $\Sigma_1 \subset U$ and $-q^{-1}$.*

(b) *For $0 < t \leq t_{1,cr}$ (the **BIS** case), U is a bounded simply connected set.*

(c) *For $t_{1,cr} < t < t_{2,cr}$ (the **TIS** case), U consists of two disjoint components: a bounded component containing $[0, x_1]$ and an unbounded component containing $[x_2, \infty)$. The complement $\mathbb{C} \setminus U$ is a bounded doubly connected domain.*

(d) *For $t_{2,cr} \leq t < 1$ (the **FIS** case), U is unbounded and connected. The complement $\mathbb{C} \setminus U$ is bounded and simply connected.*

(e) *$t \mapsto U_t$ is increasing with t.*

(f) *$z\Phi^{(1)}(z)$ is real-valued on the boundary ∂U and*

$$z\Phi^{(1)}(z) = \frac{|z|^{\frac{2}{r+1}}}{1 + |z|^{\frac{2}{r+1}}} \qquad \text{for } z \in \partial U. \tag{2.24}$$

The proof of Theorem 2.8 is in section 4. See Figure 3 for plots of U in the three cases.

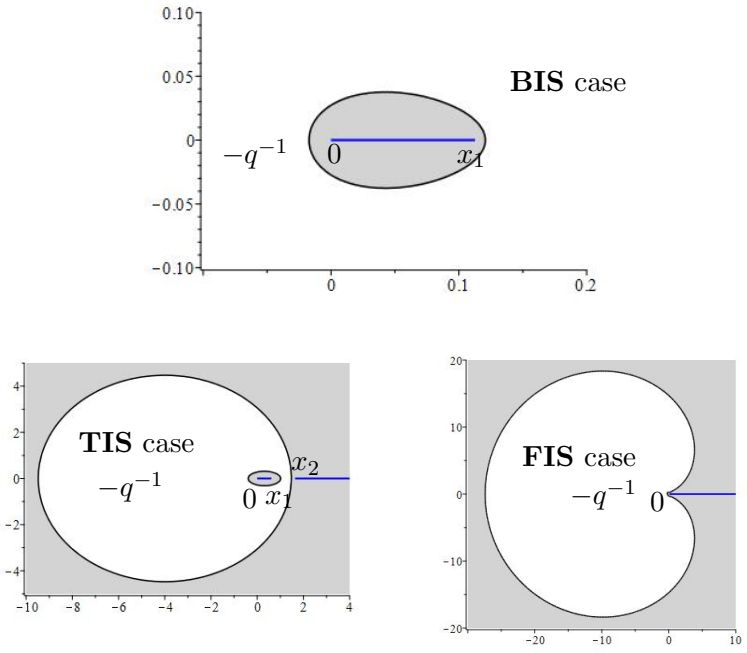


Figure 3: Domain U (shaded region) in the three cases. The blue line denotes the support of μ_1 and it is located within U, while $-q^{-1}$ is located outside U.

2.4 The symmmetric domain Ω with spherical measure

We now introduce $r+1$ fold symmetry.

Definition 2.9. We define a domain Ω (see Figure 4)

$$\Omega = \{z \in \mathbb{C} \mid z^{r+1} \in U\}, \tag{2.25}$$

and a function

$$S(z) = z^r \Phi^{(1)}\left(z^{r+1}\right) \tag{2.26}$$

which we call the spherical Schwarz function of $\partial\Omega$.

We call S the spherical Schwarz function because of the property

$$S(z) = \frac{\bar{z}}{1+|z|^2} \quad \text{for } z \in \partial\Omega, \tag{2.27}$$

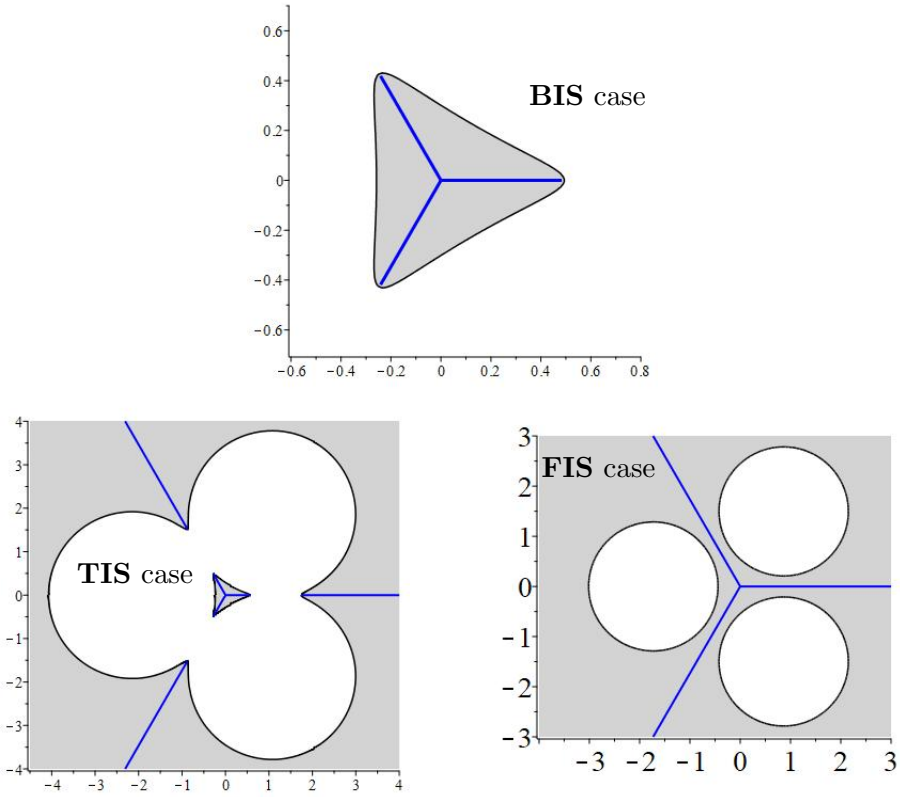


Figure 4: Domain Ω (shaded region) in the three cases for $r = 2$. The blue lines are the support of μ^*.

which follows from (2.24) and the definitions in Definition 2.9. It readily follows from (2.27) that $\frac{S(z)}{1-zS(z)} = \overline{z}$ for $z \in \partial\Omega$, and so $\frac{S(z)}{1-zS(z)}$ is the usual Schwarz function of $\partial\Omega$, and the two notions are very much intertwined, see also the paper [13] on vertex dynamics on the sphere.

Then S is defined and meromorphic on $\{z \in \mathbb{C} \mid z^{r+1} \notin \operatorname{supp}(\mu_1)\}$ with poles at the solutions of $z^{r+1} = -q^{-1}$, with the behavior $zS(z) \to 1$ as $z \to \infty$. Also S has an analytic continuation to a meromorphic function on a compact $r+1$ sheeted Riemann surface where $S(z) = z^r \Phi^{(j)}(z^{r+1})$ on the jth sheet. This analytic continuation has poles on the $(r+1)$st sheet given by the solutions of $z^{r+1} = (-1)^r q$.

We next define the two measures μ_Ω and μ^*.

Definition 2.10. We define a measure μ_Ω on Ω by

$$d\mu_\Omega(z) = \frac{1}{\pi t}\,\frac{dA(z)}{(1+|z|^2)^2}\bigg|_\Omega, \tag{2.28}$$

and μ^* as the unique measure on the $r+1$ star

$$\{z \mid z^{r+1} \in \mathbb{R}^+\} \tag{2.29}$$

that is invariant under rotation $z \mapsto e^{\frac{2\pi i}{r+1}} z$ and whose pushforward under $z \mapsto z^{r+1}$ is equal to μ_1 (the first component of the minimizer of the VEP).

This leads to the final main result of the paper.

Theorem 2.11. (a) *μ_Ω is a probability measure on Ω, and μ^* is a probability measure on* (2.29).

(b) *$t \mapsto \Omega_t$, $t \mapsto t\mu_{\Omega,t}$ and $t \mapsto t\mu_t^*$ are increasing for $t \in (0,1)$.*

(c) *There is a constant $c_1 = c_{1,t}$ such that*

$$U^{\mu_\Omega}(z) - \frac{1-t}{(r+1)t}\log|z^{r+1}+q^{-1}| + \frac{1}{2t}\log\left(1+|z|^2\right)\begin{cases} = c_1, & z \in \Omega, \\ \geq c_1, & z \in \mathbb{C}. \end{cases} \tag{2.30}$$

If Ω is unbounded, then $c_1 = 0$.

(d) *There is a constant $c_2 = c_{2,t}$ such that*

$$U^{\mu_\Omega}(z) - U^{\mu^*}(z)\begin{cases} = c_2, & z \in \mathbb{C}\setminus\Omega, \\ \leq c_2, & z \in \mathbb{C}. \end{cases} \tag{2.31}$$

If Ω is bounded, then $c_2 = 0$.

The proof of Theorem 2.11 is in section 5.

Parts (c) and (d) of Theorem 2.11 tell us that the equations (1.11) and (1.12) are satisfied provided

$$a = \frac{1-t}{(r+1)t} \tag{2.32}$$

which agrees with (1.10). Thus, as already explained, Theorem 2.11 shows that the image of Ω under inverse stereographic projection is the droplet D_σ on the unit sphere, and the pullback of μ^* is the motherbody σ^*.

Remark 2.12. The domain Ω_t increases with t according to part (b) of Theorem 2.11. It is an instance of Laplacian growth (or Hele-Shaw flow) in the spherical metric. We refer to [25] and the references therein for more on the interesting topic of Laplacian growth and its many connections.

Figure 4 contains a plot of Ω in the various cases for the value $r = 2$. It is interesting to note that in the BIS case Ω coincides with the droplet in the normal matrix model with a cubic potential, see e.g. [4, 18, 48]. The eigenvalues in this random matrix model tend to the droplet with a uniform density (in contrast to (2.28) which is uniform in the spherical metric), and the zeros of related orthogonal polynomials tend to the motherbody. The limiting zero counting measure is characterized by a vector equilibrium problem in [4] that is however different from the VEP of Definition 2.1, see also [31] for the case $r \geq 3$. Our VEP can be seen as a spherical analogue from the VEPs in [4, 31].

In the normal matrix model with a cubic potential the droplet grows up to a critical time and then cusps appear on the boundary of the droplet that cause a breakdown of the model, (see however [33, 35, 36, 37] for continuations beyond breakdown). In our model there is no breakdown since the transition to the **TIS** takes place before we reach the cusp situation.

3 Proof of Theorem 2.3

3.1 Proof of part (a)

3.1.1 A more general result

Given the second component μ_2 of the solution of the VEP of Definition 2.1, μ_1 is the probability measure μ on $[0, \infty)$ that minimizes $I(\mu) - I(\mu, \sigma)$ where $\sigma = \mu_0 + \mu_2$ is a signed measure with integral $\int \sigma = 2$. Part (a) of Theorem 2.3 will follow from the following more general result, where it is important that the negative part of σ is a Dirac point mass. Note that σ used in this section is not related to σ from (1.1).

Proposition 3.1. *Suppose $\sigma = -A\delta_{-q^{-1}} + \sigma^+$ where $A > 0$ and σ^+ is a measure on $(-\infty, 0]$ with $A < \int d\sigma^+ < \infty$. Then there is a unique μ on $[0, \infty)$ that minimizes*

$$I(\mu) - I(\mu, \sigma)$$

among all measures on $[0, \infty)$ with $\int d\mu = \frac{1}{2} \int \sigma$. The support $\Sigma = \operatorname{supp}(\mu)$ takes one of the forms described in Theorem 2.3 (a), namely Σ is a bounded interval $[0, x_1]$ containing 0, an unbounded interval $[x_2, \infty)$ not containing

0, *the disjoint union of two intervals* $[0, x_1] \cup [x_2, \infty)$, *or the full half-line* $[0, \infty)$.

The minimization problem in Proposition 3.1 is again weakly admissible, and there is a unique minimizer μ. If we relax the condition that μ is a measure and also allow signed measures, then the minimizer is the balayage

$$\nu = \frac{1}{2}\,\mathrm{Bal}(\sigma, [0, \infty)) \tag{3.1}$$

which is known to have the density

$$\frac{d\nu}{dx} = \frac{1}{2\pi\sqrt{x}} \int_{-\infty}^{0} \frac{\sqrt{|s|}}{x - s} d\sigma(s), \qquad 0 < x < \infty. \tag{3.2}$$

If the density (3.2) happens to be non-negative on $[0, \infty)$, then $\mu = \nu$ solves the minimization problem and $\operatorname{supp} \mu = [0, \infty)$.

If ν is not a positive measure, then we use the idea of iterated balayage [15, 30]. This method is based on the fact that $\mu \leq \nu^+$ where ν^+ is the positive part of ν in its Jordan decomposition

$$\nu = \nu^+ - \nu^-.$$

In particular $\operatorname{supp}(\mu) \subset \operatorname{supp}(\nu^+)$, see [30, Lemma 3]. With this information we can restrict the minimization problem to measures supported on $\operatorname{supp}(\nu^+)$, and if we also allow signed measures then the minimum is attained by

$$\nu^+ - \mathrm{Bal}(\nu^-, \operatorname{supp}(\nu^+)).$$

If this happens to be a positive measure then it is equal to μ, and we can stop. Otherwise we repeat the above step, which leads to the following iterative procedure.

We put $\nu_1 = \nu$, and iteratively for $k = 1, 2, \ldots$, we write $\nu_k = \nu_k^+ - \nu_k^-$ where ν_k^+ and ν_k^- are the positive and negative parts of ν_k, and we define

$$\nu_{k+1} = \nu_k^+ - \mathrm{Bal}(\nu_k^-, \operatorname{supp}(\nu_k^+)). \tag{3.3}$$

The convergence properties of the sequence $(\nu_k)_k$ are not fully understood, but in cases where we can control the supports of the measures we will have that $\nu_k^- \to 0$ and $\nu_k^+ \to \mu$ as $k \to \infty$.

Under the conditions of Proposition 3.1 we can indeed control the supports, and we will show that for each k the support of ν_k^+ takes one of the forms stated in the proposition, namely $\operatorname{supp}(\nu_k^+)$ is either a bounded

interval $[0, x_{1,k}]$, an unbounded interval $[x_{2,k}, \infty)$, a union of two intervals $[0, x_{1,k}] \cup [x_{2,k}, \infty)$, or the full half-line $[0, \infty)$. Since the supports are decreasing if k increases, the two sequences $(x_{1,k})$ and $(x_{2,k})$ are either finite (maybe even empty), or else they monotonically converge to limits x_1 and/or x_2. In this way we will be able to show that $\mathrm{supp}(\mu)$ has one of the forms in the proposition.

In the first step we show that the support of $\nu^+ = \nu_1^+$ has the required form.

3.1.2 First step: The support of ν^+

For σ as in the statement of Proposition 3.1, the density (3.2) of ν takes the form

$$\frac{v(x)}{2\pi\sqrt{x}}, \quad \text{with} \quad v(x) = \int_0^\infty \frac{d\rho(s)}{x+s} - \frac{A_1}{x+q^{-1}}, \qquad 0 < x < \infty, \tag{3.4}$$

where we put $A_1 = \dfrac{A}{\sqrt{q}} > 0$ and $d\rho(s) = \sqrt{s}\, d\sigma^+(-s)$.

Lemma 3.2. *In the above setting the following hold.*

(a) *v has at most two zeros in $(0, \infty)$.*

(b) *There exist $0 \le x_1 \le x_2 \le \infty$ such that*

$$\mathrm{supp}(\nu^-) = [x_1, x_2] \quad \textit{and} \quad \mathrm{supp}(\nu^+) = \overline{(0, x_1) \cup (x_2, \infty)}.$$

Proof. (a) Suppose, to get a contradiction, that $0 < x_0 < x_1 < x_2 < \infty$ are three zeros of v. Let $s_1 = q^{-1}$ and write $v = f_0 - f_1 + f_2$ with

$$f_0(x) = \int_0^{s_1} \frac{d\rho(s)}{x+s}, \quad f_1(x) = \frac{A}{x+s_1}, \quad f_2(x) = \int_{s_1}^\infty \frac{d\rho(s)}{x+s}. \tag{3.5}$$

Then by (3.5) and the multilinearity of the determinant

$$\det\left[f_k(x_j)\right]_{j,k=0}^2 = A_1 \int_0^{s_1} d\rho(s_0) \int_{s_1}^\infty d\rho(s_2) \det\left[\frac{1}{x_j + s_k}\right]_{j,k=0}^2. \tag{3.6}$$

There is an explicit formula for the determinant (Cauchy determinant)

$$\det\left[\frac{1}{x_j + s_k}\right]_{j,k=0}^2 = \frac{\prod\limits_{0 \le j < k \le 2} (x_k - x_j) \prod\limits_{0 \le j < k \le 2} (s_k - s_j)}{\prod\limits_{j=0}^2 \prod\limits_{k=0}^2 (x_j + s_k)}.$$

In the integral in (3.6) we have $0 < s_0 < s_1 < s_2$, and since also $0 < x_0 < x_1 < x_2$, we see that the Cauchy determinant is > 0. Since $A_1 > 0$ it follows from (3.6) that $\det[f_k(x_j)] > 0$ and the matrix $[f_k(x_j)]_{j,k=0}^2$ is invertible.

However, since $v = f_0 - f_1 + f_2$ by (3.4) and (3.5), and since $v(x_j) = 0$ for $j = 0, 1, 2$, it follows that

$$\begin{pmatrix} f_0(x_0) & f_1(x_0) & f_2(x_0) \\ f_0(x_1) & f_1(x_1) & f_2(x_1) \\ f_0(x_2) & f_1(x_2) & f_2(x_2) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} v(x_0) \\ v(x_1) \\ v(x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and this is a contradiction, since the matrix is invertible.

(b) From part (a) we know that v has at most two zeros in $(0, \infty)$. By continuity, v can also have at most two sign changes in $(0, \infty)$.

If v has no sign changes then $v \geq 0$ on $(0, \infty)$, since due to the fact that σ has the density (3.4) and

$$\int d\sigma = \int_0^\infty \frac{v(x)}{2\pi\sqrt{x}} dx > 0 \tag{3.7}$$

it cannot be fully ≤ 0. Then we take $x_2 = x_1$ in the lemma.

If v has one sign change, say at $x^* > 0$, and if there is no other zero of v, then v is either > 0 on $(0, x^*)$ and < 0 on (x^*, ∞), or vice versa. In the former case we take $x_1 = x^*$ and $x_2 = \infty$ and in the latter case we take $x_1 = 0$ and $x_2 = x^*$. If there is another zero in $(0, \infty)$, then the inequality is not strict at this one zero, but we still take x_1 and x_2 as above, and the conclusion of part (b) holds true if v has one sign change. [It is actually not possible that there is another zero, but we do not need this fact.]

If v has two sign changes, say at $0 < x_1 < x_2 < \infty$, then v is either positive on $(0, x_1)$, negative on (x_1, x_2), and positive again on (x_2, ∞), or the other way around negative on $(0, x_1)$, positive on (x_1, x_2) and negative on (x_2, ∞). [Now we can be sure that the inequalities are strict since there are no more than two zeros by part (a).] The latter possibility cannot happen, which we can see by adding $\epsilon\delta_{-\epsilon^2}$ to ν_0 for some small $\epsilon > 0$. Part (a) continues to apply and it follows that

$$v_\epsilon(x) = v(x) + \frac{\epsilon}{x + \epsilon^2} \tag{3.8}$$

has at most two sign changes on $(0, \infty)$. Since $v_\epsilon(0) > 0$ for sufficiently small $\epsilon > 0$, the set where $v_\epsilon < 0$ is then at most a single interval. Letting $\epsilon \to 0+$, we then arrive at a contradiction in case v is negative on $(0, x_1) \cup (x_2, \infty)$.

This proves part (b) of the lemma in all cases. □

3.1.3 Second step: Monotonicity of v on $\mathrm{supp}(\nu^+)$

In order to make the induction step in the iterated balayage argument that follows we need the following behavior of v on the parts where it is positive.

Lemma 3.3. *Under the same conditions as in Lemma 3.2 where we let x_1, x_2 be as in part (b) of Lemma 3.2, the following hold.*

(a) *v' and $(xv)'$ have at most two zeros in $(0,\infty)$.*

(b) *If $0 < x_1 < x_2$ then $x \mapsto v(x)$ is strictly decreasing for $x \in (0, x_1)$.*

(c) *If $x_1 < x_2 < \infty$ then $x \mapsto xv(x)$ is strictly increasing for $x \in (x_2, \infty)$.*

Proof. (a) The proof is similar to the proof of part (a) of Lemma 3.2. Note that from (3.4)

$$v'(x) = -\int_0^\infty \frac{d\rho(s)}{(x+s)^2} + \frac{A_1}{(x+q^{-1})^2}. \tag{3.9}$$

We let $s_1 = q^{-1}$ and write $v' = f_0' - f_1' + f_2'$ with f_0, f_1, f_2 as in (3.5) and then

$$\det\left[f_j'(x_k)\right]_{j,k=0}^2 = -A_1 \int_0^{s_1} d\rho(s_0) \int_{s_1}^\infty d\rho(s_2) \det\left[\frac{1}{(x_k+s_j)^2}\right]_{j,k=0}^2. \tag{3.10}$$

A Maple calculation shows that

$$\det\left[\frac{1}{(x_k+s_j)^2}\right]_{j,k=0}^2 = \frac{\prod_{0\le j<k\le 2}(x_k - x_j) \prod_{0\le j<k\le 2}(s_j - s_i)}{\prod_{j=0}^2 \prod_{k=0}^2 (x_k+s_j)^2} \times P(x_0, x_1, x_2; s_0, s_1, s_2)$$

where P is a homogeneous degree six polynomial in the six variables whose coefficients (with respect to the monomial basis) are all positive. Thus $P > 0$ when all its arguments are > 0, and it follows that (3.10) is negative, and in particular non-zero, whenever $0 < x_0 < x_1 < x_2$. Thus v' cannot have more than two positive zeros.

The proof for $(xv)'$ is similar, since

$$(xv)' = \int_0^\infty \frac{s d\rho(s)}{(x+s)^2} - \frac{A_1}{q(x+q^{-1})^2}$$

which has a similar form as (3.9), and the same argument applies.

(b) Suppose $0 < x_1 < x_2$. Then $v(x_1) = 0$ and $v(x) \geq 0$ for $x \in (0, x_1)$ by Lemma 3.2 (b). Also $v(x) \leq 0$ for $x \in (x_1, x_2)$.

Since $v(x) \to 0$ as $x \to \infty$, there is a global negative minimum, say at $x^* > x_1$, where the derivative vanishes and changes sign. According to part (a), there is at most one other sign change of the derivative. If this were in the interval $(0, x_1)$ then it would correspond to a local maximum of v on the interval $(0, x_1)$. Then we modify v to v_ϵ as in (3.8) in the proof of Lemma 3.2. Part (a) applies to v_ϵ and it follows that v'_ϵ has at most two sign changes. For $\epsilon > 0$ small enough one sign change is close to x^*, say at $x^*(\epsilon)$, and v_ϵ has its global minimum there. Since $v'_\epsilon(0) < 0$ for $\epsilon > 0$ sufficiently small, there can be no sign change of v'_ϵ in $(0, x^*(\epsilon))$, and letting $\epsilon \to 0+$ we find that v' has no sign change in $(0, x^*)$. Thus v is strictly decreasing in $(0, x_1)$ as claimed in part (b).

(c) The proof for part (c) is similar. Suppose $x_1 < x_2 < \infty$, so that $v(x_2) = 0$ and $v(x) \geq 0$ for $x \in (x_2, \infty)$ by Lemma 3.2 (b).

Since $xv(x) \to 0$ as $x \to 0$, there is a global minimum of $x \mapsto xv(x)$, at $x^* \in (x_1, x_2)$ say, where the derivative is zero and changes sign. There is at most one more zero by part (a). If there were a sign change of $(xv)'$ in (x_2, ∞), then that would give us a maximum of $x \mapsto xv(x)$ on (x_2, ∞). We again modify v to v_ϵ as in (3.8). Part (a), applied to v_ϵ, tells us that $(xv_\epsilon)'$ has at most two sign changes on $(0, \infty)$. For small $\epsilon > 0$ one sign change is close to x^*, say at $x^*(\epsilon)$ where $xv_\epsilon(x)$ has its global minimum. Since

$$\lim_{x \to \infty} (xv_\epsilon(x))' > 0$$

for small enough $\epsilon > 0$, this derivative then has no sign change in $(x^*(\epsilon), \infty)$ and therefore $xv_\epsilon(x)$ is strictly increasing in $(x^*(\epsilon), \infty)$. Letting $\epsilon \to 0+$ it follows that $xv(x)$ increases strictly in (x^*, ∞), and a fortiori in (x_2, ∞). $\square$

3.1.4 Third step: Iterated balayage

In the final step we use the iterated balayage to complete the proof of Proposition 3.1. We take $\nu_1 = \nu$ where ν is the signed measure on $[0, \infty)$ with density (3.2). If $\nu_1 \geq 0$ then $\nu_1 = \mu$ and we are in the full interval support (**FIS**) case $\Sigma = [0, \infty)$.

In the rest of the proof we assume that ν_1 is not a positive measure. Then iteratively we construct the sequence $(\nu_k)_k$ as in (3.3). Inductively we then have $\int d\nu_k = \int d\nu_1 = \frac{1}{2} \int \sigma$ and $\mu \leq \nu_k^+$ for every k, and in particular

$$\Sigma \subset \operatorname{supp}(\nu_k^+).$$

The sequence $(\int d\nu_k^+)_k$ decreases and if $\int d\nu_k^-$ tends to 0 as $k \to \infty$, then $\nu_k^+ \to \mu$ in the sense of weak* convergence of measures on $[0,\infty]$. In the present situation (with the help of Lemmas 3.2 and 3.3) we can prove that this is indeed the case.

Lemma 3.4. *For every k we have*

(a) $\operatorname{supp}(\nu_k^+) = \overline{(0,x_{1,k})} \cup \overline{(x_{2,k},\infty)}$ *for some* $0 \le x_{1,k} < x_{2,k} \le \infty$, *while* $\operatorname{supp}(\nu_k^-) \subset [x_{1,k}, x_{2,k}]$, *and*

(b) ν_k *has a density* $\frac{v_k(x)}{2\pi\sqrt{x}}$ *where* $x \mapsto v_k(x)$ *strictly decreases on* $(0, x_{1,k})$ *and* $x \mapsto x v_k(x)$ *strictly increases on* $(x_{2,k}, \infty)$.

Assuming that Lemma 3.4 holds, we complete the proof of Proposition 3.1 as follows. The measures (ν_k^+) converges to μ, and $\operatorname{supp}(\mu) = \overline{(0,x_1)} \cup \overline{(x_2,\infty)}$ where $x_1 = \lim_k x_{1,k}$ and $x_2 = \lim_k x_{2,k}$. This establishes Proposition 3.1.

Proof of Lemma 3.4. For $k = 1$, the statements (a) and (b) are contained in Lemmas 3.2 and 3.3.

Suppose the lemma holds for a certain $k \ge 1$. Let us assume that $0 < x_{1,k} < x_{2,k} < \infty$. We use the fact that the balayage of a delta mass δ_t at $t \in (x_{1,k}, x_{2,k})$ onto $[0, x_{1,k}] \cup [x_{2,k}, \infty)$ has the density

$$\frac{c(t)}{2\pi\sqrt{x}} \frac{x + b(t)}{|x-t|} \frac{1}{\sqrt{(x-x_{1,k})(x-x_{2,k})}}$$

with positive constants $b(t) > 0$ and $c(t) > 0$. Then $\operatorname{Bal}(\nu_k^-, \operatorname{supp}(\nu_k^+))$ has the density

$$\frac{1}{\pi\sqrt{x}} \frac{1}{\sqrt{(x-x_{1,k})(x-x_{2,k})}} \int c(t) \frac{x+b(t)}{|x-t|} d\nu_1^-(t), \qquad x \in (0,x_{1,k}) \cup (x_{2,k},\infty).$$

In view of (3.3) and the induction hypothesis we then obtain that ν_{k+1} has the density $\frac{v_{k+1}(x)}{2\pi\sqrt{x}}$ with

$$v_{k+1}(x) = v_k(x) - \frac{1}{\sqrt{(x-x_{1,k})(x-x_{2,k})}} \int_{x_{1,k}}^{x_{2,k}} c(t) \frac{x+b(t)}{|x-t|} d\nu_k^-(t), \quad (3.11)$$

for $x \in (0,x_{1,k}) \cup (x_{2,k},\infty)$. Also by the induction hypothesis $v_k(x)$ is strictly decreasing on $(0,x_{1,k})$. The other term in the right-hand side of (3.11) (including the minus-sign) is also decreasing on $(0,x_{1,k})$, since each

of the factors $\frac{1}{\sqrt{x_{1,k}-x}}$, $\frac{1}{\sqrt{x_{2,k}-x}}$, $x+b(t)$, and $\frac{1}{t-x}$ is positive and strictly increasing for $x \in (0, x_{1,k})$ as $0 < x_{1,k} < t < x_{2,k}$ and $b(t) > 0$. Also note that $c(t)d\nu_k^-(t)$ is a positive measure on $[x_{1,k}, x_{2,k}]$.

Thus v_{k+1} is strictly decreasing on $(0, x_{1,k})$. Then it is either fully negative on $(0, x_{1,k})$, in which case we take $x_{1,k+1} = 0$, or v_{k+1} is positive on some interval $(0, x_{1,k+1})$ with $0 < x_{1,k+1} < x_{1,k}$ and v_{k+1} is negative on $(x_{1,k+1}, x_{1,k})$.

Similar arguments show that $xv_{k+1}(x)$ is strictly increasing on $(x_{2,k}, \infty)$. Here we need to observe that each of the factors $\frac{\sqrt{x}}{\sqrt{x-x_{1,k}}}$, $\frac{\sqrt{x}}{\sqrt{x_{2,k}-x}}$, and $\frac{x+b(t)}{t-x}$ decreases on $(x_{2,k}, \infty)$. Thus v_{k+1} is either fully negative there, in which case we put $x_{2,k+1} = \infty$, or v_{k+1} is positive on some interval $(x_{2,k+1}, \infty)$ with $x_{2,k} < x_{2,k+1} < \infty$ and v_{k+1} is negative on $(x_{2,k}, x_{2,k+1})$.

Parts (a) and (b) of the lemma are thus proved for $k+1$ in case $0 < x_{1,k} < x_{2,k} < \infty$. If $x_{1,k} = 0$ or $x_{2,k} = \infty$, then there is an analogous reasoning (which is simpler). The lemma follows by induction. □

3.2 Proof of part (b)

3.2.1 Definitions

We first define maps M, $\widetilde{M}$ and M_j between signed measures and vectors of signed measures that will be used in the proof of part (b) of Theorem 2.3.

Definition 3.5. (a) For a signed measure σ on $(-\infty, 0]$ with $0 < \int d\sigma < \infty$ we define $M(\sigma) = \mu$ as the measure on $[0, \infty)$ that minimizes

$$I(\mu) - I(\mu, \sigma) \tag{3.12}$$

among $\mu \geq 0$ with $\int d\mu = \frac{1}{2}\int d\sigma$.

(b) Similarly, for a signed measure σ on $[0, \infty)$ with $0 < \int d\sigma < \infty$ we define $\widetilde{M}(\sigma) = \mu$ as the measure on $(0, -\infty]$ that minimizes (3.12) among $\mu \geq 0$ with $\int d\mu = \frac{1}{2}d\sigma$.

(c) Consider vectors $\vec{\nu} = (\nu_0, \ldots, \nu_{r+1})$ of signed measures of length $r+2$, such that ν_j is supported on $(-1)^j[0, \infty)$ for $j = 0, 1, \ldots, r+1$ and $0 < \int d\nu_{j-1} + \int d\nu_{j+1} < \infty$ for $j = 1, \ldots, r$. For such $\vec{\nu}$ we define

$$M_j\vec{\nu} = (\nu_0, \ldots, \nu_{j-1}, \widehat{\nu}_j, \nu_{j+1}, \ldots, \nu_{r+1}), \qquad j = 1, \ldots, r, \tag{3.13}$$

where

$$\widehat{\nu}_j = \begin{cases} M(\nu_{j-1} + \nu_{j+1}), & \text{if } j \text{ is odd,} \\ \widetilde{M}(\nu_{j-1} + \nu_{j+1}), & \text{if } j \text{ is even,} \end{cases}$$

with M and $\widetilde{M}$ as defined in parts (a) and (b).

Some remarks are in order.

Remark 3.6. (a) The measures $M(\sigma)$ and $\widetilde{M}(\sigma)$ in parts (a) and (b) of Definition 3.5 are minimizers of weakly admissible equilibrium problems. The minimizers uniquely exist [27].

(b) If $\sigma \geq 0$ then $\mu = M(\sigma)$ is the balayage measure $\mu = \frac{1}{2}\,\mathrm{Bal}(\sigma, [0,\infty))$. In this case we have a monotonicity result

$$0 \leq \sigma \leq \widetilde{\sigma} \implies M(\sigma) \leq M(\widetilde{\sigma}) \tag{3.14}$$

for measures σ and $\widetilde{\sigma}$ on $(-\infty, 0]$.

(c) Similarly

$$0 \leq \sigma \leq \widetilde{\sigma} \implies \widehat{M}(\sigma) \leq \widehat{M}(\widetilde{\sigma}) \tag{3.15}$$

for measures σ and $\widetilde{\sigma}$ on $[0,\infty)$.

(d) The maps are positive homogeneous in the sense that $M(c\sigma) = cM(\sigma)$, $\widehat{M}(c\sigma) = c\widehat{M}(\sigma)$ and $M_j(c\vec{\nu}) = cM_j(\vec{\nu})$ if $c > 0$.

(e) If $(\mu_{1,t}, \ldots, \mu_{r,t})$ is the solution of the VEP of Definition 2.1 for some $q > 0$ and $t \in (0,1)$, and $\vec{\mu}_t = (\mu_{0,t}, \mu_{1,t}, \ldots, \mu_{r,t}, \mu_{r+1,t})$, then

$$M_j(\vec{\mu}_t) = \vec{\mu}_t, \qquad j = 1, \ldots, r. \tag{3.16}$$

That is, $\vec{\mu}_t$ is a common fixed point for the mappings M_j. It is the only common fixed point among vectors $\vec{\mu}$ with μ_0 and μ_{r+1} given by (2.4).

3.2.2 Monotonicity of M

We are going to apply M only to positive measures and to signed measures whose negative part is a single point mass at $-q^{-1}$ (as in Proposition 3.1). We need the extension of the monotonicity result (3.14) to such signed measure. It could be that the monotonicity result is valid more generally, but we do not consider it here since this is all we need for our present purposes.

For such signed measures σ we have the information about the supports of $M(\sigma)$ from Proposition 3.1, and we also rely on the iterated balayage that was used in the proof of Proposition 3.1.

Lemma 3.7. *Let $\sigma \le \widetilde{\sigma}$ be signed measure on $(-\infty, 0]$ with $0 < \int d\sigma < \int d\widetilde{\sigma} < \infty$ whose negative parts are single point masses at $-q^{-1}$ only. Then $M(\sigma) \le M(\widetilde{\sigma})$.*

Proof. Under the assumptions of the lemma, the signed measures take the form $\sigma = -A\delta_{-q^{-1}} + \sigma^+$ and $\widetilde{\sigma} = -\widetilde{A}\delta_{-q^{-1}} + \widetilde{\sigma}^+$ with $A \ge \widetilde{A} \ge 0$, and $0 \le \sigma^+ \le \widetilde{\sigma}^+$. We write $\mu = M(\sigma)$ and $\widetilde{\mu} = M(\widetilde{\sigma})$.

We recall the iterated balayage algorithm from the proof of Proposition 3.1, see in particular Lemma 3.4, and we apply it to the signed measure $\widetilde{\sigma}$. That is, we start with $\nu_1 = \frac{1}{2}\operatorname{Bal}(\widetilde{\sigma}, [0, \infty))$, and from there we construct the sequence $(\nu_k)_k$ inductively by

$$\nu_{k+1} = \nu_k^+ - \operatorname{Bal}(\nu_k^-, \operatorname{supp}(\nu_k^+)), \qquad k = 1, 2, \dots.$$

Then (ν_k) converges to $\widetilde{\mu} = M(\widetilde{\sigma})$ as was shown in the proof of Lemma 3.4.

Next we define a second sequence $(\rho_k)_k$ by $\rho_1 = \frac{1}{2}\operatorname{Bal}(\sigma, [0, \infty))$, and

$$\rho_{k+1} = \rho_k^+ - \operatorname{Bal}(\rho_k^-, \operatorname{supp}(\nu_k^+)), \qquad k = 1, 2, \dots. \tag{3.17}$$

Since $\sigma \le \widetilde{\sigma}$ we have $\rho_1 \le \nu_1$, and then by induction it easily follows that $\rho_k \le \nu_k$ for every k. Note that we deviate from the earlier construction by taking in (3.17) the balayage of ρ_k^+ onto $\operatorname{supp}(\nu_k^+)$ and not onto $\operatorname{supp}(\rho_k^+)$. Since $\operatorname{supp}(\nu_k^+) \supset \operatorname{supp}(\rho_k^+)$, we however still find (by induction) that $\mu = M(\sigma) \le \rho_k^+$ for every k. Then $\rho_\infty = \lim_{k\to\infty} \rho_k$ is a signed measure with $\rho_\infty \le \widetilde{\mu}$ and $\mu \le \rho_\infty^+$. Thus $\mu \le \widetilde{\mu}$ as claimed in the lemma. □

3.2.3 M-convexity

We need two more definitions. Note that M-convexity is not a standard terminology, but it is introduced here to help the exposition.

Definition 3.8. Let $\vec{\nu}$ be as in Definition 3.5 (c), and let M_j be as in (3.13). Then we say that $\vec{\nu}$ is M-convex if

$$\vec{\nu} \le M_j(\vec{\nu}) \qquad \text{for every } j = 1, \dots, r.$$

Definition 3.9. The set $\mathcal{M}_q$ contains those vectors $\vec{\nu} = (\nu_0, \nu_1, \dots, \nu_{r+1})$ satisfying

- ν_j is a positive measure on $(-1)^{j-1}[0, \infty)$ for $j = 1, \dots, r+1$,
- $\nu_0 = -A\delta_{-q^{-1}}$ for some $A < \int d\nu_2$.

Then we have the following properties.

Lemma 3.10. *Suppose $\vec{\nu} \in \mathcal{M}_q$.*

(a) *Then $M_j(\vec{\nu}) \in \mathcal{M}_q$ for every $j = 1, \dots, r$.*

(b) *If $\vec{\nu} \le \vec{\rho} \in \mathcal{M}_q$ then $M_j(\vec{\nu}) \le M_j(\vec{\rho})$ for every $j = 1, \dots, r$.*

(c) *If $\vec{\nu}$ is M-convex then so is $M_j\vec{\nu}$ for every $j = 1, \dots, r$.*

(d) *If $\vec{\nu}$ is M-convex and $c > 0$ then $\vec{\nu} + (c\delta_{-q^{-1}}, 0, 0, \dots, 0)$ is M-convex.*

(e) *If $\vec{\nu}$ is M-convex and $\rho \ge 0$ is a measure on $(-1)^r[0, \infty)$ then $\vec{\nu} + (0, 0, \dots, 0, \rho)$ is M-convex.*

Proof. (a) Obvious.

(b) This follows from the monotonicity of M and $\widetilde{M}$ on positive measures, see (3.14) and (3.15), and the monotonicity of M on signed measures whose negative part only contains a point mass at $-q^{-1}$, see Lemma 3.7.

(c) Since $\vec{\nu}$ is M-convex we have $\vec{\nu} \le M_k\vec{\nu}$ for every k. The maps M_k and M_j commute if $|j - k| \ne 1$. Thus it follows from part (b) that

$$M_j\vec{\nu} \le M_jM_k\vec{\nu} = M_kM_j\vec{\nu}, \qquad k \ne \{j-1, j+1\}.$$

For $k \in \{j-1, j+1\}$ we can verify by direct inspection that the inequality between $M_j\vec{\nu}$ and $M_kM_j\vec{\nu}$ also holds. The two vectors only differ at positions $k = j \pm 1$, which for $M_j\vec{\nu}$ is equal to $\nu_{j\pm1}$, and for $M_{j\pm1}M_j\vec{\nu}$ it is $M(\nu_{j\pm2}+\widehat{\nu}_j)$ or $\widetilde{M}(\nu_{j\pm2}+\widehat{\nu}_j)$ (depending on the parity of j) with $\widehat{\nu}_j$ as in (3.13). By M-convexity of $\vec{\nu}$ we have $\nu_j \le \widehat{\nu}_j$ and by the monotonicity properties of M and $\widetilde{M}$ and M-convexity once more, we have

$$\nu_{j-1} \le M(\nu_{j\pm2} + \nu_j) \le M(\nu_{j\pm2} + \widehat{\nu}_j) \quad \text{if } j \text{ is odd}$$

with M replaced by $\widetilde{M}$ if j is even. This proves $M_j\vec{\nu} \le M_kM_j\vec{\nu}$ also in case $|j - k| = 1$ and part (c) follows.

(d) and (e) are straightfoward verifications. □

3.2.4 Proof of Theorem 2.3 (b)

Proof. Let us take $0 < s < t < 1$. Then we have to show that $s\mu_{j,s} \le t\mu_{j,t}$ for every $j = 0, 1, \dots, r+1$. This is clear for $j = 0$ and $j = r+1$ due to the definitions (2.4).

Write $\vec{\mu}_s = (\mu_{0,s}, \mu_{1,s}, \dots, \mu_{r+1,s})$ and similarly for $\vec{\mu}_t$. Then by the definition of the operators M_j, we have

$$M_j(s\vec{\mu}_s) = s\vec{\mu}_s, \quad M_j(t\vec{\mu}_t) = t\vec{\mu}_t.$$

see also Remark 3.6 (d) and (e).

Now we put

$$\begin{aligned}\vec{\nu}_1 &= s\vec{\mu}_s + ((t-s)\delta_{-q^{-1}}, 0, \dots, 0, (t-s)\delta_{(-1)^r q}) \\ &= (t\mu_{0,t}, s\mu_{1,s}, s\mu_{2,s}, \dots, s\mu_{r,s}, t\mu_{r+1,t}). \end{aligned} \tag{3.18}$$

This is the vector $s\vec{\mu}_s$ with the 0th and $r+1$st components replaced by those of $t\vec{\mu}_t$. Then $s\vec{\mu}_s \le \vec{\nu}_1$ and $\vec{\nu}_1$ is M-convex by Lemma 3.10 (d) and (e) and the fact that $s\vec{\mu}_s$ is M-convex.

We choose an infinite sequence $(j_k)_k$ in $\{1, \dots, r\}$ where we make sure that every j in $\{1, \dots, r\}$ appears an infinite number of times in the sequence. Then we define a sequence $(\vec{\nu}_k)_k$ by

$$\vec{\nu}_{k+1} = M_{j_k}(\vec{\nu}_k) \quad \text{for } k = 1, 2, \dots.$$

Inductively we find that each $\vec{\nu}_k$ is M-convex by Lemma 3.10(c) and the fact that $\vec{\nu}_1$ is M-convex. Then $\vec{\nu}_k \le \vec{\nu}_{k+1}$ for every k.

Also by induction it is easy to show that

$$\int d(\vec{\nu}_k)_j \le t + j - 1$$

for every k and for every j. Thus the sequence $(\vec{\nu}_k)$ is increasing with a componentwise limit $\vec{\nu}_k \to \vec{\nu}_\infty$, as $k \to \infty$ (with convergence in weak*-sense).

If $j = j_k$ is even, then

$$(\vec{\nu}_{k+1})_j = M\left((\vec{\nu}_k)_{j-1} + (\vec{\nu}_k)_{j+1}\right).$$

while for j is odd we have to replace M by $\widetilde{M}$.

If we take the limit $k \to \infty$ along the subsequence for which $j_k = j$, then it follows from this that $M_j(\vec{\nu}_\infty) = \vec{\nu}_\infty$ for every j. Since the 0th and $r+1$st components are those of $t\vec{\mu}_t$, we we conclude that $\vec{\nu}_\infty = t\vec{\mu}_t$, see also Remark 3.6 (e).

We combine the inequalities to find

$$s\vec{\mu}_s \le \vec{\nu}_1 \le \cdots \le \vec{\nu}_k \le \cdots \le \vec{\nu}_\infty = t\vec{\mu}_t$$

which indeed shows that $s\mu_{j,s} \le t\mu_{j,t}$ for every j. □

3.3 Proof of part (c)

3.3.1 An equivalent equilibrium problem

For $\theta > 0$ and a measure μ on $[0, \infty)$ we write

$$I_\theta(\mu) = \iint \log \frac{1}{|x^\theta - y^\theta|} d\mu(x) d\mu(y) \tag{3.19}$$

which we may call the θ-energy of μ. For $\theta = 1$ it reduces to the usual logarithmic energy $I(\mu)$ of μ.

Proposition 3.11. *Let $(\mu_1, \dots, \mu_r)$ be the solution of the VEP of Definition 2.1 with parameters $q > 0$ and $0 < t < 1$. Then the first component μ_1 minimizes*

$$\frac{1}{2} I(\nu) + \frac{1}{2} I_\theta(\nu) + \int V(x) d\nu(x) \tag{3.20}$$

with $\theta = \frac{1}{r}$ and

$$V(x) = -\frac{1-t}{t} \log\left(x + q^{-1}\right) + \frac{r+t}{t} \log\left(x^{1/r} + q^{1/r}\right) \tag{3.21}$$

among all probability measures ν on $[0, \infty)$.

Energy functionals of the form (3.20) appeared before in the context of Muttalib-Borodin ensembles [6, 41]. These are joint probability densities for n particles on the positive real line of the form

$$\frac{1}{Z_n} \prod_{1 \le i < j \le n} (x_i - x_j)(x_i^\theta - x_j^\theta) \prod_{j=1}^n e^{-nV(x_j)}, \qquad \text{all } x_j > 0 \tag{3.22}$$

where $\theta > 0$ is a positive constant and Z_n is a normalization factor. In the large n limit the particles are distributed according to the minimizer of (3.20), see [10, 19] and see [9, 20, 32, 40] for some recent contributions on Muttalib-Borodin ensembles (3.22). We use the characterization of μ_1 via the equilibrium problem from Proposition 3.11 in the proofs of parts (c) and (d) of Theorem 2.3.

The proof of Proposition 3.11 follows along the lines of the proof of Theorem 1.1 in [29].

Proof of Proposition 3.11. For a probability measure ν on $[0, \infty)$ we define

$$J(\nu) = \min_{\nu_2, \dots, \nu_r} \left(\sum_{j=2}^r I(\nu_j) - \sum_{j=1}^r I(\nu_j, \nu_{j+1}) \right) \tag{3.23}$$

where $\nu_1 = \nu$ and $\nu_{r+1} = \mu_{r+1} = \frac{r+t}{t}\delta_{(-1)^r q}$. The minimization is over all $\nu_2, \ldots, \nu_r$ satisfying the support condition (2.2) and the total mass condition (2.3), i.e., $\operatorname{supp}(\nu_j) \subset \Delta_j$ and $\nu_j(\Delta_j) = 1 + \frac{j-1}{t}$ for $j = 2, \ldots, r$.

This is again a weakly admissible vector equilibrium problem, similar to the VEP from Definition 2.1, and it has a unique solution. It is simpler to solve, since only positive measures are involved and we can be sure that the minimizers $\nu_2, \ldots, \nu_r$ have full supports, with the property

$$2U^{\nu_j} = U^{\nu_{j-1}} + U^{\nu_{j+1}} \quad \text{on } \Delta_j, \qquad \text{for } j = 2, \ldots, r, \tag{3.24}$$

see also (2.12).

From (3.24) we obtain

$$\begin{aligned} I(\nu_j) &= \int U^{\nu_j} d\nu_j = \frac{1}{2} \int \left(U^{\nu_{j-1}} - U^{\nu_{j+1}}\right) d\nu_j \\ &= \frac{1}{2} I(\nu_{j-1}, \nu_j) + \frac{1}{2} I(\nu_j, \nu_{j+1}), \qquad j = 2, \ldots, r. \end{aligned}$$

Hence from (3.23)

$$\begin{aligned} J(\nu) &= -\frac{1}{2} I(\nu, \nu_2) - \frac{1}{2} I(\nu_r, \nu_{r+1}) \\ &= -\frac{1}{2} \int U^{\nu_2} d\nu - \frac{r+t}{2t} U^{\nu_r}((-1)^r q), \end{aligned} \tag{3.25}$$

where ν_2 and ν_r are from the minimizer $(\nu_2, \ldots, \nu_r)$ associated with ν. We are going to calculate U^{ν_2} and U^{ν_r}.

We first do this for a point mass $\nu = \delta_p$ with $p > 0$, and the general case is obtained by averaging over p. So let $(\nu_2, \ldots, \nu_r)$ be the minimizer associated with $\nu = \nu_1 = \delta_p$. We use the Riemann surface $\mathcal{S}$ with r sheets $\mathcal{S}^{(j)}$, $j = 1, \ldots, r$, given by

$$\begin{aligned} &\mathcal{S}^{(1)} = \mathbb{C} \setminus (-\infty, 0], \\ &\mathcal{S}^{(j)} = \mathbb{C} \setminus \mathbb{R}, \qquad\qquad \text{for } j = 2, \ldots, r-1, \\ &\mathcal{S}^{(r)} = \mathbb{C} \setminus \left((-1)^r [0, \infty)\right). \end{aligned} \tag{3.26}$$

Sheet $S^{(j)}$ is connected to sheet $S^{(j-1)}$ along the cut $\Delta_j = (-1)^j [0, \infty)$ for $j = 2, \ldots, r$. We add a point at infinity to obtain a compact Riemann surface.

We define a function Ψ on $\mathcal{S}$ by its restriction $\Psi^{(j)}$ to the jth sheet as follows.

$$\begin{aligned}
\Psi^{(1)}(z) &= \frac{z}{z-p} - z\int \frac{d\nu_2(s)}{z-s},\\
\Psi^{(j)}(z) &= z\int \frac{d\nu_j(s)}{z-s} - z\int \frac{d\nu_{j+1}(s)}{z-s}, \quad \text{for } j=2,\ldots,r-1,\\
\Psi^{(r)}(z) &= z\int \frac{d\nu_r(s)}{z-s} - \frac{r+t}{t}\frac{z}{z-(-1)^r q}.
\end{aligned} \tag{3.27}$$

The conditions (3.24) imply that Ψ is meromorphic on $\mathcal{S}$ with poles at $z=p$ on the first sheet and at $z=(-1)^r q$ on the rth sheet, see also the discussion after Definition 2.6 that shows why Φ is meromorphic on $\mathcal{R}$. [The construction of Ψ is similar to that of Φ.] Due to the total masses of the measures we have $\Psi(z)\to -\frac{1}{t}$ as $z\to\infty$.

The Riemann surface (3.26) has a simple parametrization $z=w^r$, and in the w variable the poles are at $w=p^{1/r}$ and $w=-q^{1/r}$. Taking into account the residues at the poles and the behavior at infinity, we find that

$$\Psi(z) = \frac{1}{r}\frac{p^{1/r}}{w-p^{1/r}} + \frac{r+t}{rt}\frac{q^{1/r}}{w+q^{1/r}} - \frac{1}{t}, \qquad z=w^r. \tag{3.28}$$

Observe also that $\Psi(z)=0$ for $z=w=0$.

Specifying (3.28) to the first sheet, and recalling (3.27) we find

$$\begin{aligned}
\int \frac{d\nu_2(s)}{z-s} &= \frac{1}{z-p} - \frac{1}{z}\Psi^{(1)}(z)\\
&= \frac{1}{z-p} - \frac{1}{rz}\frac{p^{1/r}}{z^{1/r}-p^{1/r}} - \frac{r+t}{rtz}\frac{q^{1/r}}{z^{1/r}+q^{1/r}} - \frac{1}{tz}
\end{aligned} \tag{3.29}$$

with principal branch of the fractional powers. We integrate with respect to z and find after straightforward calculation

$$\int \log(z-s)d\nu_2(s) = \log(z-p) - \log\left(z^{1/r}-p^{1/r}\right) + \frac{r+t}{t}\log\left(z^{1/r}+q^{1/r}\right) \tag{3.30}$$

There is no constant of integration since both sides behave as $(1+t^{-1})\log z + o(1)$ as $z\to\infty$. The real part of (3.30) gives us the logarithmic potential

$$U^{\nu_2}(z) = \log\left|\frac{z^{1/r}-p^{1/r}}{z-p}\right| - \frac{r+t}{t}\log\left|z^{1/r}+q^{1/r}\right|. \tag{3.31}$$

An analogous calculation, based on (3.28) and the expression (3.27) of Ψ on the rth sheet, leads to the logarithmic potential of ν_r,

$$U^{\nu_r}(z) = \frac{r+t}{r} \log \left| \frac{z^{1/r} + q^{1/r}}{z - (-1)^r q} \right| - \log \left| z^{1/r} - p^{1/r} \right| \tag{3.32}$$

with the branch of the rth root that is analytic on $\mathbb{C} \setminus \Delta_r$ and that is real and negative for real $z \in \mathbb{C} \setminus \Delta_r$. However, we emphasize that $p^{1/r}$ and $q^{1/r}$ always denote the positive rth roots. Similarly when we write $x^{1/r}$ with $x > 0$ as for example in (3.20) and in (3.34) below.

Thus (3.31) and (3.32) give the logarithmic potentials of ν_2 and ν_r associated with δ_p. Associated with a general probability measure $\nu = \nu_1$ on $[0, \infty)$, we then have measures ν_2 and ν_r whose logarithmic potentials are obtained from averaging (3.31) and (3.32) over p, that is

$$U^{\nu_2}(z) = \int \log \left| \frac{z^{1/r} - x^{1/r}}{z - x} \right| d\nu(x) - \frac{r+t}{t} \log \left| z^{1/r} + q^{1/r} \right| \tag{3.33}$$

and

$$U^{\nu_r}(z) = \frac{r+t}{r} \log \left| \frac{z^{1/r} + q^{1/r}}{z - (-1)^r q} \right| - \int \log \left| z^{1/r} - x^{1/r} \right| d\nu(x). \tag{3.34}$$

From (3.33) we obtain

$$I(\nu, \nu_2) = I(\nu) - I_{1/r}(\nu) - \frac{r+t}{t} \int \log |x^{1/r} + q^{1/r}| d\nu(x), \tag{3.35}$$

and from (3.34) we obtain, noting that $z^{1/r}$ in (3.34) is negative for $z \in \mathbb{R} \setminus \Delta_r$, and thus in particular for $z = (-1)^r q$,

$$\begin{aligned} U^{\nu_r}((-1)^r q) &= \frac{r+t}{t} \log \lim_{x \to q} \left| \frac{x^{1/r} - q^{1/r}}{x - q} \right| - \int \log \left| -q^{1/r} - x^{1/r} \right| d\nu(x) \\ &= \frac{r+t}{t} \log \left(\frac{1}{r} q^{1/r - 1} \right) - \int \log \left(x^{1/r} + q^{1/r} \right) d\nu(x). \end{aligned} \tag{3.36}$$

Using (3.35) and (3.36) in (3.25) we obtain

$$\begin{aligned} J(\nu) = -\frac{1}{2} I(\nu) + \frac{1}{2} I_{1/r}(\nu) + \frac{r+t}{t} \int \log \left(x^{1/r} + q^{1/r} \right) d\nu(x) \\ - \frac{1}{2} \left(\frac{r+t}{t} \right)^2 \log \left(\frac{1}{r} q^{1/r - 1} \right). \end{aligned} \tag{3.37}$$

Finally, comparing (3.23) with the energy functional (2.5) we obtain for a given ν on $[0,\infty)$ that

$$\min_{\nu_2,\ldots,\nu_r} \mathcal{E}(\nu,\nu_2,\ldots,\nu_r) = I(\nu) - I(\nu,\mu_0) + J(\nu)$$

so that in view of (3.37) and noting that μ_0 is given by (2.4)

$$\begin{aligned}\min_{\nu_2,\ldots,\nu_r} \mathcal{E}(\nu,\nu_2,\ldots,\nu_r) &= \frac{1}{2}I(\nu) + \frac{1}{2}I_{1/r}(\nu) \\ &\quad - \frac{1-t}{t}\int \log\left(x+q^{-1}\right) d\nu(x) + \frac{r+t}{t}\int \log\left(x^{1/r}+q^{1/r}\right) d\nu(x) \\ &\quad - \frac{1}{2}\left(\frac{r+t}{t}\right)^2 \log\left(\frac{1}{r}q^{1/r-1}\right). \quad (3.38)\end{aligned}$$

The left-hand side of (3.38) as a functional on probability measures ν on $[0,\infty)$ attains its minimum at $\nu = \mu_1$. Since the last term on the right in (3.38) is only a constant, independent of ν, the proposition follows. □

3.3.2 Minimum of V is in the support

In the next step we discuss a general fact about the minimizer for a Muttalib-Borodin type energy functional (3.20) with $\theta > 0$, and where $V : [0,\infty) \to \mathbb{R}$ is continuous with

$$\liminf_{x\to\infty} \left(V(x) - (1+\theta)\log x\right) \geq -\infty.$$

Under this condition there is a unique probability measure μ on $[0,\infty)$ that minimizes (3.20).

The following is well-known for the case $\theta = 1$, but apparently has not been observed for general θ.

Lemma 3.12. *If $x_0 \geq 0$ is such that $V(x_0) = \min_{x\geq 0} V(x)$ then $x_0 \in \operatorname{supp}(\mu)$ where μ is the probability measure that minimizes* (3.20).

Proof. In this proof we use the notation

$$U_\theta^\mu(x) = \int \log \frac{1}{|x^\theta - s^\theta|} d\mu(s).$$

The minimizer μ satisfies, for some constant ℓ,

$$U^\mu(x) + U_\theta^\mu(x) + V(x) \begin{cases} = \ell, & \text{on } \operatorname{supp}(\mu), \\ \geq \ell & \text{on } [0,\infty). \end{cases} \quad (3.39)$$

Now,

$$h(x) = U^\mu(x) + U^\mu_\theta(x) = \int \log \frac{1}{|x-s|} d\mu(s) + \int \log \frac{1}{|x^\theta - s^\theta|} d\mu(s),$$

extends into the complex plane where we use the principal branch of x^θ, i.e., with a branch cut along $(-\infty, 0]$. Then h is harmonic in $\mathbb{C} \setminus ((-\infty, 0] \cup \text{supp}(\mu))$ and it tends to $-\infty$ as $|x| \to \infty$. By the maximum principle for harmonic functions, the maximum of h is attained on $\text{supp}(\mu) \cup (-\infty, 0]$ only.

For $x > 0$ and $s > 0$, it is easy to see that $|-x-s| > |x-s|$ and $|(-x)^\theta - s^\theta| = |x^\theta e^{\pi i\theta} - s^\theta| > |x^\theta - s^\theta|$. Therefore $U^\mu(-x) < U^\mu(x)$ and $U^\mu_\theta(-x) < U^\mu_\theta(x)$ for $x > 0$ which means that $h(-x) < h(x)$ for $x > 0$, and therefore the maximum of h is not attained on $(-\infty, 0]$.

Thus the maximum of h is attained on $\text{supp}(\mu)$, say at $x_1 \in \text{supp}(\mu)$.

If $x_0 \notin \text{supp}(\mu)$, then $h(x_0) < h(x_1)$, and since $V(x_0) \leq V(x_1)$ by the assumption in the lemma, we have a strict inequality

$$h(x_0) + V(x_0) < h(x_1) + V(x_1). \tag{3.40}$$

Since $x_1 \in \text{supp}(\mu)$ the right-hand side of (3.40) is equal to ℓ, by the equality in (3.39) It follows that

$$U^\mu(x_0) + U^\mu_\theta(x_0) + V(x_0) < \ell$$

which contradicts the inequality in (3.39). Therefore $x_0 \in \text{supp}(\mu)$. □

3.3.3 Proof of part (c) of Theorem 2.3.

In view of Proposition 3.11 and Lemma 3.12 it is enough to show that the external field (3.21) attains its minimum at $x = 0$ in case $0 < q < 1$. This is what we do in the next lemma, and then part (c) follows.

Lemma 3.13. *Let V be given by (3.21) with $0 < q < 1$. Then for every $t \in (0, 1)$ it is true that*

$$V(0) = \min_{x \geq 0} V(x).$$

Proof. Note that $\left(x^{1/r} + q^{1/r}\right)^r \geq x + q$ for $x \geq 0$ by the binomial theorem, so that by (3.21)

$$\begin{aligned} &tV(x) - tV(0) \\ &\quad \geq -(1-t)\log\left(x + q^{-1}\right) + \frac{r+t}{r}\log(x+q) - \left(2 - t + \frac{t}{r}\right)\log q. \end{aligned} \tag{3.41}$$

Since $0 < q < 1$, we have $x + q \geq q^2 x + q$ for $x \geq 0$ and therefore we can estimate (3.41) further to obtain

$$\begin{aligned} &tV(x) - tV(0) \\ &\quad \geq -(1-t)\log\left(x + q^{-1}\right) + \frac{r+t}{r}\log\left(q^2 x + q\right) - \left(2 - t + \frac{t}{r}\right)\log q. \end{aligned}$$

This means $V(x) - V(0) \geq \left(1 + \frac{1}{r}\right)\log\left(qx + 1\right)$ and the lemma follows. □

3.4 Proof of part (d)

The measure μ_1 minimizes $I(\mu) - I(\mu, \mu_0 + \mu_2)$ among all probability measures on $[0, \infty)$. Thus it is the equilibrium measure in the external field $-U^{\mu_0+\mu_2}$ which is real analytic on $[0, \infty)$. Then it follows from [14] that μ_1 is absolutely continuous with respect to Lebesgue measure with a density that is real analytic in the interior of its support.

From [14] it also follows that the density at the endpoint x_1 (in **BIS** and **TIS** cases) behaves as $\approx c(x_1 - x)^{\frac{1}{2}+2N}$ as $x \to x_1-$, for a certain non-negative integer N. From the iterated balayage it can be seen that $N = 0$. Indeed, the algorithm in section 3.1.4 gives us the sequence of signed measures $(\nu_k)_k$ with densities that are such that $\sqrt{x}\frac{d\nu_k}{dx}$ is positive and strictly decreasing on $[0, x_1]$ by Lemma 3.4 (b). The proof of that lemma (see (3.11))) actually shows that the decrease gets stronger as k increases. Since $\nu_k \to \mu_1$ as $k \to \infty$, it then follows that the density of μ_1 cannot have a zero derivative at x_1, and thus it vanishes as a square root at x_1 in **BIS** and **TIS** cases.

Similarly, the density vanishes as a square root at x_2 in **UIS** and **TIS** cases.

4 Proof of Theorem 2.8

4.1 Properties of Φ

We start by listing a number of properties of the meromorphic function Φ from Definition 2.6.

4.1.1 Zeros and poles

Lemma 4.1. *Let $q > 0$. The function $z\Phi$ is a degree 2 meromorphic function on the Riemann surface $\mathcal{R}$ with the following properties.*

(a) $z\Phi \to 1$ *as* $z \to \infty$ *on any of the sheets.*

(b) *It has simple poles at* $z = -q^{-1}$ *on the first sheet and at* $z = (-1)^r q$ *on the last sheet, and no other poles.*

(c) *Suppose one of the* ***BIS***, ***TIS***, *or* ***FIS*** *cases, so that* $z \in \operatorname{supp}(\mu_1)$. *Then* $z\Phi$ *has simple zeros at* $z = 0$ *and at a point* x_0 *on the first sheet with*

$$-q^{-1} < x_0 < 0, \tag{4.1}$$

and no other zeros. In ***UIS*** *case, there are two points on the Riemann surface with* $z = 0$ *and* $z\Phi$ *has a simple zero at each of these.*

Proof. We already noted that Φ is meromorphic on $\mathcal{R}$, see the discussion after Definition 2.6. Thus also $z\Phi$ is meromorphic on $\mathcal{R}$. In part (b) we show that it has two simple poles, and no other poles, and therefore its degree is two.

(a) From (2.19) we have $zF_j(z) \to \mu_j(\Delta_j)$ as $z \to \infty$. In view of the total massses (2.3) of the measures and the definition (2.21), part (a) follows.

(b) From (2.20) and (2.21) we see that

$$\begin{aligned} \Phi^{(1)}(z) &= tF_1(z) + \frac{1-t}{z+q^{-1}} \\ &= t\int \frac{d\mu_1(s)}{z-s} + \frac{1-t}{z+q^{-1}}, \end{aligned} \tag{4.2}$$

$$\Phi^{(r+1)}(z) = -tF_r(z) + \frac{r+t}{z-(-1)^r q}, \tag{4.3}$$

and so $z\Phi$ has simple poles at $-q^{-1}$ on the first sheet and at $(-1)^r q$ on the $r+1$-st sheet.

There is no pole at $z = \infty$ because of part (a). There is no pole at $z = 0$ either, since the form (2.19) of F_j as a Stieltjes transform, easily implies that $zF_j(z) \to 0$ as $z \to 0$. Thus also $z\Phi \to 0$ as $z \to 0$. There are no other candidates for poles, and therefore the degree is two.

(c) We already remarked in part (b) that $z\Phi$ vanishes when $z = 0$. In **UIS** case there are two points on the Riemann surface with $z = 0$. In that case both of these are simple zeros, and there are no other zeros, since the degree is two.

In **BIS**, **TIS**, and **FIS** cases, there is only one point $z = 0$, and it is at most a double zero of $z\Phi$. Then $z^{\frac{1}{r+1}}$ is the local coordinate, and z (as a

function on the Riemann surface) has a zero of order $r+1$ at $z=0$. Hence Φ has a pole at $z=0$ of order $\geq r-1$, and so Φ is unbounded at $z=0$ (we may assume $r \geq 2$). Looking on the first sheet, we conclude from (4.2) that F_1 is unbounded at $z=0$, and it dominates the behavior of $\Phi^{(1)}$ as $z \to 0$. Since $F_1(z) < 0$ for negative real z, it then follows that $\Phi^{(1)}(z)$ is negative for negative z close to 0.

From (4.2) we also see that $\Phi^{(1)}(z) \to +\infty$ as $z \to -q^{-1}+$, as the residue at the pole is positive. Thus $\Phi^{(1)}$ changes sign on the interval $(-q^{-1}, 0)$ and hence there is a zero, say at $x_0 \in (-q^{-1}, 0)$. Then x_0 is also a zero of $z\Phi$, and we conclude that both $z = x_0$ and $z = 0$ are simple zeros, and these are the only zeros, as the degree of $z\Phi$ is two. □

4.1.2 Proof of Theorem 2.3 (e)

Proof. Suppose we are in one of the **BIS**, **TIS**, or **FIS** cases, so that $0 \in \operatorname{supp}(\mu_1)$. Then $z=0$ is a simple zero of $z\Phi$, by part (c) of Lemma 4.1. Since $z^{\frac{1}{r+1}}$ is a local coordinate, we find that Φ has a pole of order r at $z=0$. From (4.2) we then get for some non-zero constant C,

$$F_1(z) = Cz^{-\frac{r}{r+1}} \left(1 + O\left(z^{\frac{1}{r+1}}\right)\right) \qquad \text{as } z \to 0$$

and the fractional powers have their branch cut along $[0, \infty)$. By the Stieltjes inversion formula

$$\frac{d\mu_1}{dx} = -\frac{1}{\pi} \lim_{\delta \to 0+} \operatorname{Im} F_1(x + i\delta), \qquad x > 0,$$

and (2.13) follows.

Finally, (2.14) follows from (2.13) and the symmetry between q and $1/q$, see Remark 2.4. □

Remark 4.2. The behavior (2.13) is characteristic for the density of minimizers of Muttalib-Borodin type energy functionals as in (3.20). This was proved by Claeys and Romano [10, Remark 1.9] under general conditions on the external field, which however do not cover the case (3.21).

4.1.3 Critical points

We need to know about the critical points, by which we mean the ramification points of $z\Phi$. There is no ramification at $z=0$ or $z=\infty$, and so we may alternatively characterize the critical points as those points where the derivative of $z\Phi^{(j)}(z)$ vanishes for some $j = 1, \ldots, r+1$.

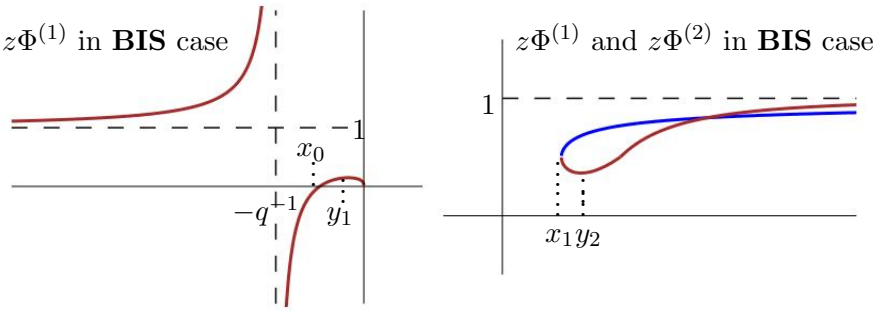


Figure 5: Sketches of the graph of $z\Phi$ in **BIS** case. The left panel shows the graph of $z\Phi^{(1)}(z)$ on the negative real line where it has a pole at $-q^{-1}$, a zero at x_0 and a local maximum at y_1. The right panel shows the graphs of $z\Phi^{(1)}(z)$ (in brown) and $z\Phi^{(2)}$ (in blue) on the interval $[x_1, \infty)$. The graph of $z\Phi^{(1)}(z)$ has a local minimum at y_2, while the graph of $z\Phi^{(2)}(z)$ is strictly increasing. Both graphs tend to 1 at infinity. The graph of $z\Phi^{(r+1)}(z)$ is as in the right panel of Figure 7 below.

Since $z\Phi$ has degree 2 the Riemann-Hurwitz formula [46] tells us that there are two critical points in the **BIS** and **FIS** (genus zero) cases, and four critical points in the **TIS** (genus one) case. The following lemma says that they are all real and on the first sheet.

We fix $0 < q < 1$ and we continue to use x_1, x_2 as in Theorem 2.3 (a) depending on the various cases, and x_0 for the zero of Φ on the first sheet as in Lemma 4.1 (c).

Lemma 4.3. *Let $0 < q < 1$. The critical points of $z\Phi$ are on the real part of the first sheet of the Riemann surface.*

(a) *In all cases there is a critical point y_1 with $y_1 \in (x_0, 0)$.*

(b) *In **BIS** case there is one more critical point $y_2 \in (x_1, \infty)$.*

(c) *In **TIS** case there are three more critical points. A critical point $y_0 \in (-\infty, -q^{-1})$ and two critical points $y_2, y_3 \in (x_1, x_2)$ with $y_2 < y_3$.*

(d) *In **FIS** case there is one more critical point $y_0 \in (-\infty, -q^{-1})$.*

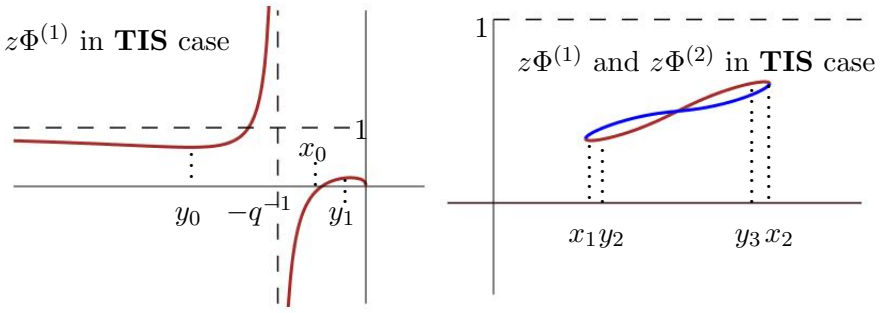


Figure 6: Sketches of the graph of $z\Phi$ in **TIS** case. The left panel shows the graph of $z\Phi^{(1)}(z)$ on the negative real line where it has a pole at $-q^{-1}$, zeros at x_0 and 0, a local minimum at y_0, and a local maximum at y_1. The right panel shows the graphs of $z\Phi^{(1)}(z)$ (in brown) and $z\Phi^{(2)}$ (in blue) on the interval $[x_1, x_2]$. The graph of $z\Phi^{(1)}(z)$ has a local minimum at y_2 and a local maximum at y_3, while the graph of $z\Phi^{(2)}(z)$ is strictly increasing. The graph of $z\Phi^{(r+1)}(z)$ is as in the right panel of Figure 7 below.

Proof. (a) By Lemma 4.1 (c) $z\Phi^{(1)}(z)$ has zeros at $z = x_0$ and at $z = 0$, and in between it is real and positive. So there is a local maximum, and the point $y_1 \in (x_0, 0)$ where it is attained is a critical point in all cases.

The proofs of parts (b)-(d) rely on an inspection of the graph of $z\Phi$ on the real part of the Riemann surface (that is, on the part where both z and Φ are real), see Figures 5, 6, and 7 for sketches of the graphs in the various cases. We infer the following about $z\Phi$ from the behavior at the poles and at infinity,

- every value in $(-\infty, 0)$ is attained once in $(-q^{-1}, x_0)$ on the first sheet and once between 0 and $(-1)^r q$ on the last sheet,
- every value in $(1, \infty)$ is attained once in $(-\infty, -q^{-1})$ on the first sheet and once between $(-1)^r q$ and infinity on the last sheet.

Since $z\Phi$ has degree two, the values in $(-\infty, 0)$ and $(1, \infty)$ are attained nowhere else on the Riemann surface. In particular

$$\begin{array}{ll} 0 < z\Phi^{(1)}(z) < 1, \text{ and} & \begin{cases} \text{for } z \in [x_1, \infty) & \text{in } \textbf{BIS} \text{ case,} \\ \text{for } z \in [x_1, x_2] & \text{in } \textbf{TIS} \text{ case,} \end{cases} \\ 0 < z\Phi^{(2)}(z) < 1, & \end{array} \tag{4.4}$$

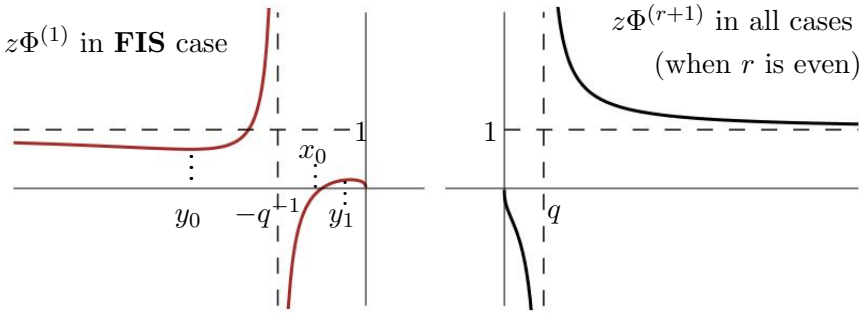


Figure 7: Sketches of the graph of $z\Phi$ in **FIS** case. The left panel shows the graph of $z\Phi^{(1)}(z)$ on $(-\infty, 0]$ where it has a pole at $-q^{-1}$, zeros at x_0 and 0, a local minimum at y_0, and a local maximum at y_1. The right panel shows the graph of $z\Phi^{(r+1)}(z)$ on $(-1)^r[0,\infty)$ which has a pole at $(-1)^r q$. The figure is for $r = 2$ and it has the same features for all cases.

see the right panels of Figures 5 and 6.

From the fact that the density of μ_1 vanishes as a square root at the endpoints x_1, x_2 see Theorem 2.3 (d), it follows that

$$F_1'(z) = -\int \frac{1}{(z-x)^2} d\mu_1(x) \to -\infty \tag{4.5}$$

as $z \to x_1+$ or $z \to x_2-$. Hence by (2.21) we also have

$$\Big(z\Phi^{(1)}(z)\Big)' \to -\infty \quad \text{and} \quad \Big(z\Phi^{(2)}(z)\Big)' \to +\infty \tag{4.6}$$

as $z \to x_1+$ (in **BIS** and **TIS** cases) or $z \to x_2-$ (in **TIS** case).

(b) In **BIS** case we noted in (4.4) that $z\Phi^{(1)}(z)$ takes the value $x_1\Phi^{(1)}(x_1) \in (0,1)$ at x_1, and by (4.6) it starts to decrease if $z \in (x_1, \infty)$ increases. Since it tends to the value 1 at infinity, there will be a local minimum, say at $y_2 \in (x_1, \infty)$. This is a critical point, and part (b) follows.

It also follows that $z\Phi^{(2)}(z)$ strictly increases for $z \in [x_1, \infty)$ since in **BIS** case there are no further critical points.

(c) In **TIS** case we first observe that

$$\frac{d}{dz}\left(zF_1(z)\right) = -\int \frac{x}{(z-x)^2} d\mu_1(x) < 0, \qquad z \in (x_1, x_2), \tag{4.7}$$

$$\frac{d}{dz}\left(zF_2(z)\right) = -\int \frac{x}{(z-x)^2} d\mu_2(x) > 0, \qquad z \in (x_1, x_2). \tag{4.8}$$

The difference in sign is due to the fact that μ_1 is supported on $[0, \infty)$, while μ_2 is supported on $(-\infty, 0]$. Thus by (2.21)

$$\left(z\Phi^{(2)}(z)\right)' > 0 \quad \text{for } z \in (x_1, x_2),$$

and therefore $z\Phi^{(2)}(z)$ strictly increases on (x_1, x_2) and there are no critical points in (x_1, x_2) on the second sheet. We also conclude

$$x_1\Phi^{(1)}(x_1) < x_2\Phi^{(1)}(x_2)$$

but $z\Phi^{(1)}(z)$ will not be monotonic on $[x_1, x_2]$ due to (4.6). Instead it will start to decrease at x_1 to a local minimum, say at y_2, and then increases to a local maximum, say at y_3, and then again decreases. This gives us the critical points $y_2 < y_3$ in (x_1, x_2). We already know $y_1 \in (-x_0, 0)$.

The final critical point is in $(-\infty, -q^{-1})$, and this follows from the observation that

$$zF_1(z) = z\int \frac{d\mu_1(x)}{z-x} = 1 + \int \frac{x d\mu_1(x)}{z-x}$$

with $\int \frac{x d\mu_1(x)}{z-x} < 0$ for $z \in (-\infty, 0]$ and

$$\int \frac{x d\mu_1(x)}{z-x} = -Cz^{-\frac{1}{r+1}}(1 + O(z^{-\frac{1}{r+1}}) \text{ as } z \to -\infty, \tag{4.9}$$

with a positive constant $C > 0$. Also $zF_0(z) = (1 - t^{-1}) + O(z^{-1})$ as $z \to \infty$, so that by (2.21).

$$z\Phi^{(1)}(z) = 1 + \int \frac{x d\mu_1(x)}{z-x} + O(z^{-1}) \tag{4.10}$$

as $z \to -\infty$, where the second term is negative for $z < 0$ and it dominates the $O(z^{-1})$ term as $z \to -\infty$. Therefore $z\Phi^{(1)}(z)$ decreases on an interval $(-\infty, y_0)$ for some $y_0 \in (-\infty, -q^{-1})$, it reaches a local minimum at y_0 and then increases to $+\infty$ as $z \to -q^{-1}-$. See also Figure 6.

(d) In **FIS** case the expansions (4.9) and (4.10) remain valid, as does the conclusion that $z\Phi^{(1)}(z)$ has a local minimum at some $y_0 \in (-\infty, -q^{-1})$, and y_0 is a critical point. □

After these preparations we turn to the proof of Theorem 2.8.

4.2 Proof of part (a)

Proof. For $x \in \mathrm{supp}(\mu_1)$, we have by (2.21) and the Stieltjes inversion formula

$$x \operatorname{Im} \Phi_{\pm}^{(1)}(x) = xt \operatorname{Im} (F_1)_{\pm}(x) = \mp \frac{xt}{\pi} \frac{d\mu_1(x)}{dx}. \tag{4.11}$$

Here the subscript $\pm$ denotes the limiting value from the upper $(+)$ or lower $(-)$ half plane. Then by (2.23) we find $\mathrm{supp}(\mu_1) \subset U$.

from the Cauchy-Riemann equations and the definition (2.23) of U, we obtain that the parts of the real line where $z\Phi^{(1)}(z)$ is real and decreasing belong to U, while those parts where $z\Phi^{(1)}(z)$ is real and increasing do not belong to U. Then in view of the behavior of $z\Phi^{(1)}(z)$ on the realline that we see in Figures 5, 6, 7, and the fact that $\mathrm{supp}(\mu_1) \subset U$, we conclude that

$$U \cap (\mathbb{R} \cup \{\infty\}) = \begin{cases} [y_1, y_2] & \text{in } \textbf{BIS} \text{ case}, \\ [-\infty, y_0] \cup [y_1, y_2] \cup [y_3, \infty] & \text{in } \textbf{TIS} \text{ case}, \\ [-\infty, y_0] \cup [y_1, \infty] & \text{in } \textbf{FIS} \text{ case}. \end{cases} \tag{4.12}$$

In particular $-q^{-1} \notin U$. This proves part (a). □

4.3 Proof of part (b)

Proof. Since the y_j's are critical points, we have that $z\Phi^{(1)}(z)$ is also real on certain contours that emanate from each y_j into the complex plane. These contours are going to be the boundary ∂U of U.

The labelling of the critical points in Lemma 4.3 is such that y_j is a local minimum of $z\Phi^{(1)}(z)$ if j is even, and a local maximum if j is odd, when we restrict to the real line. It means that $z\Phi^{(1)}(z)$ is real and increasing on ∂U when we move away from y_j with j odd, and decreasing from y_j with j even.

Noting that

$$y_1\Phi^{(1)}(y_1) < y_2\Phi^{(1)}(y_2) \quad \text{in } \textbf{BIS} \text{ case}, \tag{4.13}$$

we conclude that the part of ∂U that emanates from y_1 will end at y_2 in **BIS** case. Since $\Sigma_1 = \mathrm{supp}(\mu_1) \subset U$, we also see that ∂U consists of a simple closed contour surrounding Σ_1 and U is a bounded simply connected domain in the **BIS** case. □

4.4 Proof of part (c)

Proof. In **TIS** case, we have four critical points and instead of (4.13) we have

$$y_1\Phi^{(1)}(y_1) < y_2\Phi^{(1)}(y_2) < y_3\Phi^{(1)}(y_3) < y_0\Phi^{(1)}(y_0) \quad \text{in } \mathbf{TIS} \text{ case.} \tag{4.14}$$

Then ∂U consists of two closed contours, one containing y_1 and y_2, and one containing y_0 and y_3. Both closed contours go around Σ_1. It follows that U has two components, namely the bounded domain that is enclosed by the inner component of ∂U, and the unbounded domain that is outside of the outer component of ∂U. This proves part (c). □

4.5 Proof of part (d)

Proof. In **FIS** case we have two critical points $y_0 < y_1 < 0$ with

$$y_1\Phi^{(1)}(y_1) < y_0\Phi^{(1)}(y_0), \quad \text{in } \mathbf{FIS} \text{ case.} \tag{4.15}$$

Then ∂U is a closed contour containing y_0 and y_1, and U is the domain that is exterior to this contour. □

4.6 Proof of part (e)

Proof. We know from Theorem 2.3 (b) that $t \mapsto t\mu_{j,t}$ increases with t for every j, and $t\mu_{j,t}$ has total mass $t + j - 1$. Then

$$\rho_j = \rho_{j,t} = \frac{\partial(t\mu_{j,t})}{\partial t}, \quad j = 0, \dots, r+1 \tag{4.16}$$

is a probability measure on $\mathrm{supp}(\mu_j)$ for every j. In particular $\rho_0 = \delta_{-q^{-1}}$ by (2.3).

Thus by differentiating (2.21) for $j = 1$ with respect to t,

$$\frac{\partial\Phi^{(1)}(z)}{\partial t} = \int \frac{d\rho_1(x)}{z-x} - \frac{1}{z+q^{-1}}.$$

Since ρ_0 and ρ_1 are both probability measures we obtain from this

$$\frac{\partial(z\Phi^{(1)}(z))}{\partial t} = \int \frac{x d\rho_1(x)}{z-x} + \frac{q^{-1}}{z+q^{-1}}. \tag{4.17}$$

For $\operatorname{Im} z > 0$ both terms in the right-hand side of (4.17) have negative imaginary parts (since $xd\rho_1(x)$ is a positive measure), while for $\operatorname{Im} z < 0$ the two terms have positive imaginary parts. In other words

$$\frac{\partial}{\partial t} \operatorname{Im}\left(z\Phi^{(1)}(z)\right) \begin{cases} < 0 & \text{for } \operatorname{Im} z > 0, \\ > 0 & \text{for } \operatorname{Im} z < 0, \end{cases}$$

Therefore the part in the upper half plane where $\operatorname{Im}\left(z\Phi^{(1)}(z)\right) < 0$ increases with t. Similarly, the part in the lower half plane where $\operatorname{Im}\left(z\Phi^{(1)}(z)\right) > 0$ increass with t, which proves part (e) in view of the definition (2.23) of U. □

4.7 Proof of part (f)

Proof. It is clear from (2.23) that $z\Phi^{(1)}(z)$ is real-valued for $z \in \partial U$. The point of part (f) is that ∂U is characterized by (2.24).

Being a meromorphic function on a compact Riemann surface, Φ satisfies the algebraic equation

$$\prod_{j=1}^{r+1}\left(\Phi - \Phi^{(j)}(z)\right) = \Phi^{r+1} + \sum_{k=1}^{r+1}(-1)^k e_k(z)\Phi^{r+1-k} = 0 \tag{4.18}$$

where $e_k(z)$ is the kth elementary symmetric function in $\Phi^{(1)}, \ldots, \Phi^{(r+1)}$, i.e.,

$$e_k(z) = \sum_{1 \le j_1 < \cdots < j_k \le r+1} \prod_{l=1}^{k} \Phi^{(j_l)}(z). \tag{4.19}$$

Each e_k is a rational function of $z \in \mathbb{C}$ with real coefficients and simple poles at $-q^{-1}$ and at $(-1)^r q$, due to the simple poles of $\Phi^{(1)}$ and $\Phi^{(r+1)}$ at these respective values, see (4.2) and (4.3). Since $z\Phi^{(j)}(z) \to 1$ as $z \to \infty$ for every j by Lemma 4.1 (a), and there are $\binom{r+1}{k}$ terms in (4.19) we have

$$z^k e_k(z) \to \binom{r+1}{k} \quad \text{as } z \to \infty.$$

By Lemma 4.1 (c) $z = 0$ is a simple zero of $z\Phi$ on the Riemann surface, and from (4.19) we get that $z^k e_k(z)$ becomes zero for $z = 0$. Thus

$$z^k e_k(z) = \binom{r+1}{k} \frac{z(z + A_k)}{(z + q^{-1})(z - (-1)^r q)}$$

for some real value A_k.

Using this in (4.18) and clearing denominators by multiplying with $z^r(z+q^{-1})(z-(-1)^r q)$ we obtain

$$z^r\left(z+q^{-1}\right)\left(z-(-1)^r q\right)\Phi^{r+1} + \sum_{k=1}^{r+1}(-1)^k\binom{r+1}{k}(z+A_k)\left(z\Phi\right)^{r+1-k}=0. \quad (4.20)$$

We separate terms that are polynomial in $z\Phi$ to rewrite (4.20) as (with $A_0 = q^{-1}-(-1)^r q$)

$$\begin{aligned}\sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k}A_k(z\Phi)^{r+1-k} &= (-1)^r z^{-1}\left(z\Phi\right)^{r+1} - z\sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k}\left(z\Phi\right)^{r+1-k}\\ &= (-1)^r z^{-1}\left(z\Phi\right)^{r+1} + z\left(z\Phi-1\right)^{r+1}. \quad (4.21)\end{aligned}$$

In the last step we used the binomial theorem.

Let $z\in\partial U$. Then $z\Phi^{(1)}(z)$ is real and it satisfies the equation (4.21), which means that the left-hand side is real since each A_k is real. Thus the right-hand side is real as well, and taking imaginary parts we obtain since $z\Phi^{(1)}(z)$ is real,

$$0=(-1)^{r+1}\frac{\operatorname{Im} z}{|z|^2}\left(z\Phi^{(1)}(z)\right)^{r+1}+\operatorname{Im} z\left(z\Phi^{(1)}(z)-1\right)^{r+1}, \qquad \text{for } z\in\partial U.$$

This equation leads to (2.24) whenever $\operatorname{Im} z\neq 0$. Thus (2.24) holds for $z\in\partial U\setminus\mathbb{R}$ and by continuity it also holds for $z\in\partial U\cap\mathbb{R}$. □

5 Proof of Theorem 2.11

5.1 Proof of part (a)

Proof. Since μ^* is the symmetric pullback of the probability measure μ_1, it is also a probability measure. That μ_Ω is a probability measure as well can be seen from the formulas in Lemma 5.2 below, by letting $z\to\infty$ in either (5.3) (in case Ω is bounded), or (5.4) (in case Ω is unbounded). □

5.2 Proof of part (b)

Proof. Since $t\mu_{1,t}$ is increasing by Theorem 2.3 (b), also $t\mu_t^*$ increases with t.

The domains U_t increase with t by Theorem 2.8. Then also Ω_t increases with t and then $t\mu_{\Omega,t}$ also increases with t, since by (2.28) this is just the spherical area measure $\frac{1}{\pi}\frac{dA(z)}{(1+|z|^2)^2}$ restricted to Ω_t. □

5.3 Stieltjes transform of μ_Ω

The proofs of parts (c) and (d) are modelled after the proofs in the paper [11] that deals with the case $r = 1$. See in particular the proof of Proposition 4.1 in [11]. As a preparation we need the following formula for the spherical Schwarz function from (2.26), which is the analogue of [11, (5.1)].

Lemma 5.1. *We have*

$$S(z) = \frac{(1-t)z^r}{z^{r+1}+q^{-1}} + t\int \frac{d\mu^*(x)}{z-x} \tag{5.1}$$

Proof. Since μ^* is the symmetric pullback of μ_1 under the mapping $z \mapsto z^{r+1}$, we can easily verify that their logarithmic potentials are related via

$$U^{\mu^*}(z) = \frac{1}{r+1}U^{\mu_1}(z^{r+1}).$$

and also, with appropriate branches of the logarithm,

$$\int \log(z-x)d\mu^*(x) = \frac{1}{r+1}\int \log\left(z^{r+1}-x\right)d\mu_1(x).$$

Taking the z-derivative we find

$$\int \frac{d\mu^*(x)}{z-x} = z^r F_1(z^{r+1}). \tag{5.2}$$

Then combining (2.26), (4.2), and (5.2), we find (5.1). □

The following lemma is the analogue of [11, Lemma 5.2] and its proof is also very similar.

Lemma 5.2. *The Stieltjes transform of μ_Ω satisfies*

$$\int \frac{d\mu_\Omega(x)}{z-x} = \int \frac{d\mu^*(x)}{z-x}, \qquad z \in \mathbb{C}\setminus\Omega, \tag{5.3}$$

$$\int \frac{d\mu_\Omega(x)}{z-x} = -\frac{(1-t)z^r}{t(z^{r+1}+q^{-1})} + \frac{\bar{z}}{t(1+|z|^2)}, \quad z \in \Omega. \tag{5.4}$$

Proof. Take $z \in \mathbb{C} \setminus \Omega$ first. Then by (1.9), (2.28), the complex Green's formula, and the property (2.27) of the spherical Schwarz function, we find if Ω is bounded,

$$\begin{aligned} t \int \frac{d\mu_\Omega(s)}{z-s} &= \frac{1}{\pi} \int_\Omega \frac{dA(s)}{(z-s)(1+|s|^2)^2} \\ &= \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\bar{s}}{(z-s)(1+|s|^2)} ds \\ &= \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{S(s)}{z-s} ds \\ &= -\frac{1}{2\pi i} \oint_{\partial(\mathbb{C}\setminus\Omega)} \frac{S(s)}{z-s} ds. \end{aligned} \tag{5.5}$$

Since the complex Green's formula applies to bounded domains, one has to modify the calculation in case Ω is unbounded. Then one first makes a cut-off to $\{z \in \Omega \mid |z| \leq R\}$ with a large $R > 0$. The Green's formula then produces an additional integral over $|z| = R$, which however tends to zero as $R \to \infty$, due to the fact that $\frac{S(s)}{z-s} = O(s^{-2})$ as $s \to \infty$. Thus (5.5) also holds in the unbounded case.

The remaining integral in (5.5) is evaluated with the residue theorem for $\mathbb{C} \setminus \Omega$. The spherical Schwarz function S has $r+1$ simple poles at the solutions of $s^{r+1} = -q^{-1}$, the poles are all in $\mathbb{C} \setminus \Omega$, and from (5.1) it can be checked that S has the same residue $\frac{1-t}{r+1}$ at each of the poles. Together they give the contribution

$$-\frac{1-t}{r+1} \sum_{s^{r+1}=-q^{-1}} \frac{1}{z-s} = -\frac{(1-t)z^r}{z^{r+1}+q^{-1}} \tag{5.6}$$

to the integral (5.5). There is an additional pole in (5.5) at $s = z$ with the contribution $S(z)$. Finally, note that there is no contribution from infinity in case $\mathbb{C} \setminus \Omega$ is unbounded, since the integrand in (5.5) is $O(s^{-2})$ as $s \to \infty$. In total we get

$$t \int_\Omega \frac{d\mu_\Omega(s)}{z-s} = -\frac{(1-t)z^r}{z^{r+1}+q^{-1}} + S(z), \qquad z \in \mathbb{C} \setminus \Omega,$$

and (5.4) follows because of (5.1).

Let $z \in \Omega \setminus \partial\Omega$. Take $\varepsilon > 0$ such that the disk $D(z, \varepsilon)$ of radius ε around z is contained in Ω. Then by a calculation similar to (5.5), with complex

Green's theorem and the spherical Schwarz function

$$t\int_{\Omega\setminus D(z,\varepsilon)}\frac{d\mu_\Omega(s)}{z-s}=-\frac{1}{2\pi i}\oint_{\partial(\mathbb{C}\setminus\Omega)}\frac{S(s)}{z-s}ds \\ -\frac{1}{2\pi i}\oint_{\partial D(z,\varepsilon)}\frac{\bar{s}}{(z-s)(1+|s|^2)}ds. \qquad (5.7)$$

The integral over $\partial(\mathbb{C}\setminus\Omega)$ is again evaluated using the residue theorem, but in the present situation there is no contribution from $s=z$, but only the combined contribution (5.6) from the poles of S. The integral over the circle $\partial D(z,\varepsilon)$ (including the prefactor $-\frac{1}{2\pi i}$) tends to $\frac{\bar{z}}{1+|z|^2}$ as $\varepsilon\to 0+$. Thus letting $\varepsilon\to 0+$ in (5.7) we obtain (5.4). □

5.4 The log integral of μ_Ω

We will need the following result in the **TIS** case for the proof of Lemma 5.5 below.

Lemma 5.3. *In **FIS** and **TIS** cases we have*

$$U^{\mu_\Omega}(0)+\frac{1-t}{(r+1)t}\log q=0. \qquad (5.8)$$

In **BIS** case we have

$$U^{\mu_\Omega}(0)+\frac{1-t}{(r+1)t}\log q=\frac{1}{2t(r+1)}\int_{x_1}^{\infty}\left(\Phi^{(1)}(x)-\Phi^{(2)}(x)\right)dx\le 0$$

but we will not prove this as we do not need it for the proof of parts (c) and (d) of Theorem 2.11.

Proof of Lemma 5.3. In the proof we use $\log x=\log|x|+i\arg x$ with $0<\arg x<2\pi$, and we are going to show that

$$t\int\log(x)d\mu_\Omega(x)=\frac{1-t}{r+1}\log q+t\pi i \quad \text{ in } \textbf{FIS} \text{ and } \textbf{TIS} \text{ cases} \qquad (5.9)$$

and then (5.8) will follow by taking the real parts on both sides. The evaluation of (5.9) follows along the lines of the proof of Lemma 5.2 but there is a non-trivial extra step required in the **TIS** case.

We start with the **FIS** case. Consider the cut-off domain

$$\Omega_{R,\delta}=\{z\in\Omega\mid |z|\le R,\operatorname{dist}(z,[0,\infty))>\delta\}$$

with large $R > 0$ and small $\delta > 0$. Due to our definition of the logarithm with the branch cut along $[0, \infty)$ we can apply the complex Green's theorem to the integral over $\Omega_{R,\delta}$ and we find in **FIS** case

$$\begin{aligned}\frac{1}{\pi}\int_{\Omega_{R,\delta}} \log x \frac{dA(x)}{(1+|x|^2)^2} &= \frac{1}{2\pi i}\oint_{\partial\Omega_{R,\delta}} \log s \frac{\overline{s}}{1+|s|^2} ds \\ &\to \frac{1}{2\pi i}\oint_{\Omega} \log s \frac{\overline{s}}{1+|s|^2} ds + \frac{1}{2\pi i}\oint_{|s|=R} \log s \frac{\overline{s}}{1+|s|^2} ds \\ &\qquad - \int_0^R \frac{x}{1+x^2} dx \end{aligned} \tag{5.10}$$

as $\delta \to 0+$. The last term in (5.10) is the combined contribution of the upper and lower sides of the branch cut of the logarithm. It yields

$$\int_0^R \frac{x}{1+x^2} dx = \frac{1}{2}\log\left(1+R^2\right) = \log R + o(1) \quad \text{as } R \to \infty. \tag{5.11}$$

The second integral in the right-hand side of (5.10) is evaluated with parametrization $s = Re^{i\theta}$, $0 < \theta < 2\pi$, to give

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|s|=R} \log(s) \frac{\overline{s}}{1+|s|^2} ds &= \frac{R^2 \log R}{1+R^2} + \frac{R^2}{1+R^2}\pi i \\ &= \log R + \pi i + o(1) \quad \text{as } R \to \infty. \end{aligned} \tag{5.12}$$

In the first term we use (2.27) and then evaluate the integral by a residue calculation over $\mathbb{C} \setminus \Omega$. The complement of Ω consists of $r+1$ disjoint disks in **FIS** case, see Figure 4, and $S(s)$ is meromorphic with one simple pole at the solution of $s^{r+1} + q^{-1}$ in each of the disks with residue $\frac{1-t}{r+1}$. Therefore

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\Omega} \log s \frac{\overline{s}}{1+|s|^2} ds &= -\frac{1}{2\pi i}\oint_{\partial(C\setminus\Omega)} (\log s) S(s) ds \\ &= -\frac{1-t}{r+1} \sum_{s: s^{r+1} = -q^{-1}} \log s \\ &= \frac{1-t}{r+1}\log q - (1-t)\pi i \end{aligned} \tag{5.13}$$

Letting $R \to \infty$ in (5.10) we find from (2.28), (5.11), (5.12) and (5.13) that

$$\begin{aligned} t\int \log(x) d\mu_\Omega(x) &= \lim_{R\to\infty}\lim_{\delta\to 0+} \frac{1}{\pi}\int_{\Omega_{R,\delta}} \log x \frac{dA(x)}{(1+|x|^2)^2} \\ &= \frac{1-t}{r+1}\log q + t\pi i \end{aligned}$$

as claimed in (5.9) in **FIS** case.

In **TIS** case we have to adjust the above calculation in two ways. First, since $\Omega \cap [0,\infty) = [0, y_2^*] \cup [y_3^*, \infty)$, with $y_j^* = y_j^{1/(r+1)}$, the integral over $[0, R]$ in (5.10) is replaced by

$$\left(\int_0^{y_2^*} + \int_{y_3^*}^R \right) \frac{x}{1+x^2} dx = - \int_{y_2^*}^{y_3^*} \frac{x}{1+x^2} dx + \log R + o(1) \quad \text{as } R \to \infty.$$

Second, in the evaluation (5.13) of the integral over Ω, there is a contribution from the intersection $[y_2^*, y_3^*]$ of $\mathbb{C} \setminus \Omega$ with the positive real line, due to the discontinuity of the logarithm. Instead of (5.13) we get

$$\frac{1}{2\pi i} \oint_\Omega \log s \frac{\overline{s}}{1+|s|^2} ds = \frac{1-t}{r+1} \log q - (1-t)\pi i - \int_{y_2^*}^{y_3^*} S(x) dx,$$

and the result is the formula

$$t \int \log(x) d\mu_\Omega(x) = \frac{1-t}{r+1} \log q + t\pi i - \int_{y_2^*}^{y_3^*} \left(S(x) - \frac{x}{1+x^2} \right) dx$$

for the **TIS** case.

To obtain (5.9) it remains to prove that the integral in the right-hand side vanishes, and we do this by showing the identity (5.14) in Lemma 5.4 below. The right-hand side of (5.14) is zero and to see this we recall (2.21) from which we get

$$\begin{aligned} \Phi^{(1)}(x) - \Phi^{(2)}(x) &= t(2F_1(x) - F_0(x) - F_2(x)) \\ &= -t \frac{d}{dx} \left(2U^{\mu_1}(x) - U^{\mu_0}(x) - U^{\mu_2}(x) \right), \quad \text{for } x_1 < x < x_2. \end{aligned}$$

We also recall that $2U^{\mu_1} - U^{\mu_0} - U^{\mu_2}$ vanishes on the support of μ_1, and hence in particular at both x_1 and x_2. Then the right-hand side is indeed 0 by the fundamental theorem of calculus.

Lemma 5.3 is thus proved, pending the proof of the remarkable identity (5.14). Since the proof of this identity uses new ideas that were not in [11], we decided to give it in a separate lemma. □

Lemma 5.4. *In the **TIS** case we have*

$$\int_{y_2^*}^{y_3^*} \left(S(x) - \frac{x}{1+x^2} \right) dx = \frac{1}{2(r+1)} \int_{x_1}^{x_2} \left(\Phi^{(1)}(x) - \Phi^{(2)}(x) \right) dx, \quad (5.14)$$

where $y_j^* = y_j^{\frac{1}{r+1}}$ for $j = 2, 3$.

Proof. Consider $\omega = \Phi dz$ as a meromorphic differential on the Riemann surface. Then

$$\oint_a \omega = \int_{x_1}^{x_2} \left(\Phi^{(1)}(x) - \Phi^{(2)}(x) \right) dx \tag{5.15}$$

for the cycle a that goes from x_1 to x_2 on the first sheet, and back from x_2 to x_1 on the second sheet, cf. Figure 2. The meromorphic differential has simple poles at $-q^{-1}$ on first sheet, at $(-1)^r q$ on last sheet, and at ∞ with respective residues $1-t$, $r+t$, and $-1-r$.

Since $z\Phi$ is a degree two meromorphic function we can represent the Riemann surface $\mathcal{R}$ by the equations

$$\eta^2 = \prod_{j=0}^{3} (\zeta - \zeta_j), \qquad \zeta = z\Phi \tag{5.16}$$

with $\zeta_j = y_j \Phi^{(1)}(y_j)$, for $j = 0, 1, 2, 3$, being the four branch points of ζ with $y_0 < y_1 < y_2 < y_3$ by Lemma 4.3 and

$$0 < \zeta_1 < \zeta_2 < \zeta_3 < \zeta_0 < 1,$$

see also Figure 6. In the new coordinates $\mathcal{R}$ is a two sheeted cover of the ζ-plane, with branch cuts $[\zeta_1, \zeta_2]$ and $[\zeta_3, \zeta_0]$. We label the sheets so that $z = -q^{-1}$ corresponds to $\zeta = \infty$ on the first sheet and $z = (-1)^r q$ to $\zeta = \infty$ on the second sheet. Then η is positive for real $\zeta > \zeta_0$ on the first sheet. The point $z = \infty$ corresponds to the point $\zeta = 1$ on the second sheet. The a-cycle goes from ζ_2 to ζ_3 on the first sheet and back from ζ_3 to ζ_2 on the second sheet.

The meromorphic differential ω has simple poles at the two points at $\zeta = \infty$ with residues $1-t$ and $r+t$, and at $\zeta = 1$ on the second sheet with residue $-1-r$. Then

$$2\omega + (r+1)\frac{d\zeta}{\zeta - 1} \tag{5.17}$$

has residues $\pm(1-2t-r)$ at the two points at infinity, and residues $\pm(r+1)$ at the two points with $\zeta = 1$. Thus (5.17) has an anti-symmetry with respect to the involution $(\zeta, \eta) \mapsto (\zeta, -\eta)$ of $\mathcal{R}$. It follows that

$$2\omega + (r+1)\frac{d\zeta}{\zeta - 1} = \frac{A\zeta^2 + B\zeta + C}{\zeta - 1}\frac{d\zeta}{\eta}$$

for certain constants A, B, and C. From this form we conclude that

$$\oint_a \left(2\omega + (r+1)\frac{d\zeta}{\zeta - 1} \right) d\zeta = 2 \int_{\zeta_2}^{\zeta_3} \left(2\omega + (r+1)\frac{d\zeta}{\zeta - 1} \right) d\zeta$$

with integration on the first sheet. Clearly $\oint_a \frac{d\zeta}{\zeta-1} = 0$ and therefore

$$\begin{aligned}\oint_a \omega &= \int_{\zeta_2}^{\zeta_3} \left(2\omega + (r+1)\frac{d\zeta}{\zeta-1}\right) \\ &= 2\int_{y_2}^{y_3} \Phi^{(1)}(z)dz + (r+1)\left(\log(1-\zeta_3) - \log(1-\zeta_2)\right) \end{aligned} \tag{5.18}$$

since $[\zeta_2, \zeta_3]$ on the first sheet corresponds to $[y_2, y_3]$ on the first sheet in the original z-variable where $\omega = \Phi^{(1)}(z)dz$.

Changing variable $z = x^{r+1}$ and using (2.26) we have

$$\int_{y_2}^{y_3} \Phi^{(1)}(z)dz = (r+1)\int_{y_2^*}^{y_3^*} S(x)dx \tag{5.19}$$

since $y_j^* = y_j^{\frac{1}{r+1}}$. For the last term on the right of (5.18) we recall that $\zeta_j = y_j\Phi^{(1)}(y_j)$ and y_j belongs to ∂U. Therefore it satisfies the equation (2.24), that is,

$$\zeta_j = \frac{y_j^{\frac{2}{r+1}}}{1+y_j^{\frac{2}{r+1}}} = \frac{(y_j^*)^2}{1+(y_j^*)^2}, \qquad \text{for } j = 2, 3,$$

which we rewrite as

$$\log(1-\zeta_j) = -\log\left(1+(y_j^*)^2\right), \qquad \text{for } j = 2, 3. \tag{5.20}$$

Hence

$$\begin{aligned}\log(1-\zeta_2) - \log(1-\zeta_3) &= \log\left(1+(y_3^*)^2\right) - \log\left(1+(y_2^*)^2\right) \\ &= 2\int_{y_2^*}^{y_3^*} \frac{x}{1+x^2}dx. \end{aligned} \tag{5.21}$$

Combining (5.15), (5.18), (5.19), (5.21) we obtain the equality of the two integrals in (5.14). □

5.5 Measures ν_t and ρ_t

For the proofs of parts (c) and (d), we also need to consider the dynamical picture where we vary t, see [11, section 6]. To emphasize the t-dependence we attach a subscript t to the notions that vary with t.

We already observed in part (b) that $t\mu_t^*$ and $t\mu_{\Omega,t}$ increase with t. The derivatives

$$\rho_t = \frac{\partial}{\partial t}\left(t\mu_t^*\right), \qquad \nu_t = \frac{\partial}{\partial t}\left(t\mu_{\Omega,t}\right) \tag{5.22}$$

therefore exist for almost every t, as can be proved as in [8, Theorem 2], but in our case the derivatives actually exist for every $t \in (0,1)$.

Both ρ_t and ν_t are probability measures, with $\operatorname{supp}(\rho_t) = \operatorname{supp}(\mu_t^*)$ and $\operatorname{supp}(\nu_t) = \partial\Omega_t$, see (2.28). Indeed ν_t measures how the domain Ω_t grows in the spherical metric as t increases.

Applying $\frac{\partial}{\partial t}t$ to the identities (5.3) and (5.4) for the Stieltjes transforms, and using (5.22), we get

$$\int \frac{d\nu_t(s)}{z-s} = \begin{cases} \displaystyle\int \frac{d\rho_t(s)}{z-s}, & z \in \mathbb{C}\setminus\Omega_t, \\ \displaystyle\frac{z^r}{z^{r+1}+q^{-1}}, & z \in \Omega_t. \end{cases} \tag{5.23}$$

Lemma 5.5. *There are constants $C_{1,t}$ and $C_{2,t}$ such that the following hold.*

(a) *We have*

$$U^{\nu_t}(z) \le U^{\rho_t}(z) + C_{2,t}, \qquad z \in \mathbb{C}, \tag{5.24}$$

with equality for $z \in \mathbb{C}\setminus\Omega_t$.

(b) *We have*

$$U^{\nu_t}(z) \le -\frac{1}{r+1}\log|z^{r+1}+q^{-1}| + C_{1,t}, \quad z \in \mathbb{C}, \tag{5.25}$$

with equality for $z \in \Omega_t$.

Proof. (a) The first identity in (5.23) implies that $U^{\nu_t} - U^{\rho_t}$ is constant on each connected component of $\mathbb{C}\setminus\Omega_t$. Thus for some constant $C_{2,t}$,

$$U^{\nu_t}(z) = U^{\rho_t}(z) + C_{2,t}, \quad z \in \mathbb{C}\setminus\Omega_t, \tag{5.26}$$

since $\mathbb{C}\setminus\Omega_t$ is either connected (in **BIS** and **TIS** cases), or consists of $r+1$ disjoint components (in **FIS** case) where due to $r+1$-fold rotational symmetry the constant is the same on each component. If Ω_t is bounded (the **BIS** case) then $C_{2,t} = 0$, since both potentials in (5.26) behave as $-\log|z| + o(1)$ as $z \to \infty$.

Since ν_t is supported on $\partial\Omega_t$, the function $U^{\rho_t} - U^{\nu_t}$ is superharmonic on the interior of Ω_t, including at ∞ if Ω_t is unbounded. By the minimum

principle for superharmonic functions we find the corresponding inequality (5.24) on Ω_t, and part (a) follows.

(b) For part (b) we argue similarly, but there is an additional twist when Ω_t is not connected (the **TIS** case). Using the second identity of (5.23), we apply similar reasoning to U^{ν_t} and $-\frac{1}{r+1}\log|z^{r+1}+q^{-1}|$, which is the logarithmic potential of the discrete measure with mass $\frac{1}{r+1}$ at each solution of $z^{r+1}+q^{-1}=0$. We find that

$$U^{\nu_t}(z)+\frac{1}{r+1}\log|z^{r+1}+q^{-1}|$$

is constant on each connected component of Ω_t.

In **BIS** and **FIS** cases we have that Ω_t is connected and therefore for some constant $C_{1,t}$,

$$U^{\nu_t}(z)=-\frac{1}{r+1}\log|z^{r+1}+q^{-1}|+C_{1,t},\qquad z\in\Omega_t, \tag{5.27}$$

in **BIS** and **FIS** cases. If Ω_t is unbounded then we let $z\to\infty$ in (5.27) and we find that $C_{1,t}=0$ in **FIS** case.

In **TIS** case we have that Ω_t has two connected components. We find that (5.27) holds with $C_{1,t}=0$ in the unbounded component for the same reason that $C_{1,t}=0$ in **FIS** case. The bounded component could potentially have a different constant. However we are able to compute $U^{\nu_t}(z)$ at $z=0$ because of Lemma 5.3 which says that

$$tU^{\mu_{\Omega,t}}(0)=-\frac{1-t}{r+1}\log q$$

in the **TIS** case. Then taking the t-derivative and using the definition (5.22) of ν_t, we obtain

$$U^{\nu_t}(0)=\frac{1}{r+1}\log q,$$

which implies that (5.27) with $C_{1,t}=0$ holds for $z=0$ and thus throughout the bounded component as well in the **TIS** case.

From (5.27) and the fact that ν_t is supported on $\partial\Omega_t$, we obtain the inequality (5.25) in all cases (by the minimum principle, as in the proof of part (a)) and part (b) follows. □

Remark 5.6. The identity (5.26) and the fact that $\mathrm{supp}(\nu_t)=\partial\Omega_t$ show in fact that

$$\nu_t=\mathrm{Bal}(\rho_t,\partial\Omega_t).$$

Similarly (5.27) gives that

$$\nu_t = \mathrm{Bal}\left(\frac{1}{r+1}\sum_{z^{r+1}=-q^{-1}} \delta_z, \partial\Omega_t\right).$$

Thus ν_t is a balayage measure onto $\partial\Omega_t$ from two sides. It is the balayage of ρ_t which is supported inside Ω_t, and it is also the balayage of a discrete measure supported in the complement on Ω_t.

5.6 Proof of part (c)

Proof. Integrating the identities (5.22) we obtain the identities

$$t\mu_t^* = \int_0^t \rho_s ds, \quad \text{and} \quad t\mu_{\Omega,t} = \int_0^t \nu_s ds, \tag{5.28}$$

which are analogous to the formulas of Buyarov and Rakhmanov [8] for varying families of measures on the real line. We also have

$$\begin{aligned} t\mu_{\Omega,t} &= \lim_{\tau\to 1-} \tau\mu_{\Omega,\tau} - \int_t^1 \nu_s ds \\ &= \frac{dA(z)}{\pi(1+|z|^2)^2} - \int_t^1 \nu_s ds. \end{aligned} \tag{5.29}$$

We can calculate the logarithmic potential

$$-\int_{\mathbb{C}} \log|z-s| \frac{dA(s)}{\pi(1+|s|^2)} = -\frac{1}{2}\log\left(1+|z|^2\right), \qquad z\in\mathbb{C}.$$

Therefore by (5.29) and (5.25), we have for every $z\in\mathbb{C}$,

$$\begin{aligned} tU^{\mu_{\Omega,t}}(z) &= -\frac{1}{2}\log\left(1+|z|^2\right) - \int_t^1 U^{\nu_s}(z) ds \\ &\geq -\frac{1}{2}\log\left(1+|z|^2\right) + \int_t^1 \left(\frac{1}{r+1}\log\left|z^{r+1}+q^{-1}\right| - C_{1,s}\right) ds \\ &= -\frac{1}{2}\log\left(1+|z|^2\right) + \frac{1-t}{r+1}\log\left|z^{r+1}+q^{-1}\right| + tc_{1,t}, \end{aligned} \tag{5.30}$$

with $c_{1,t} = -\frac{1}{t}\int_t^1 C_{1,s} ds$.

Equality holds in (5.25) for $z\in\Omega_t$ which implies that equality holds in (5.30) for

$$z \in \bigcap_{t\leq s<1} \Omega_s = \Omega_t,$$

since $\Omega_t \subset \Omega_s$ whenever $t<s$. The proof of part (c) is complete. □

5.7 Proof of part (d)

Proof. We obtain for every $z \in \mathbb{C}$, using (5.28) and (5.24),

$$\begin{aligned} tU^{\mu_{\Omega,t}}(z) &= \int_0^t U^{\nu_s}(z) ds \\ &\leq \int_0^t \left(U^{\rho_s}(z) + C_{2,s}\right) ds \\ &= tU^{\mu_t^*}(z) + tc_{2,t} \quad \text{with } c_{2,t} = \frac{1}{t}\int_0^t C_{2,s} ds. \end{aligned}$$

Equality holds, by Lemma 5.5 (a), for

$$z \in \bigcap_{0<s\leq t} (\mathbb{C} \setminus \Omega_s) = \mathbb{C} \setminus \Omega_t,$$

since $\Omega_s \subset \Omega_t$ whenever $s < t$. □

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