Spectrum based stability analysis and stabilization of systems described by delay differential algebraic equations

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1 Introduction

We consider the stability analysis and stabilization of systems described by delay differential algebraic equations (DDAEs), also called descriptor systems, of the form

$$E\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} A_ix(t-\tau_i), \quad x(t) \in \mathbb{R}^n, \tag{1}$$

where E is allowed to be singular. The time-delays $\tau_i, i = 1, \ldots, m$, satisfy

$$0 < \tau_1 < \tau_2 < \ldots < \tau_m$$

and the capital letters are real-valued matrices of appropriate dimensions.

The motivation for the system description (1) in the context of designing controllers lies in its generality in modeling interconnected systems. For instance, the feedback interconnection of the system

$$\left\{ \begin{array}{l} \dot{z}(t) = \sum F_iz(t-r_i) + \sum G_iu(t-r_i) \\ y(t) = \sum H_ix(t-r_i) + \sum L_iu(t-r_i) \end{array} \right. \tag{2}$$

and the controller

$$\left\{ \begin{array}{l} \dot{z}_c(t) = \sum \hat{F}_iz_c(t-s_i) + \sum \hat{G}_iy(t-s_i) \\ u(t) = \sum \hat{H}_iz_c(t-s_i) + \sum \hat{L}_iy(t-s_i) \end{array} \right. \tag{3}$$

can be directly brought in the form (1), where

$$x = [z^T \ z_c^T \ u^T \ y^T], \quad \{\tau_1, \ldots, \tau_m\} = \{r_i\} \cup \{s_i\}.$$

In this way no elimination of inputs and outputs is required, which may even not be possible in the presence of delays [13]. Another favorable property is the linear dependence of the matrices of the closed-loop system on the elements of the matrices of the controller. The increase in the number of equations,

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on the contrary, is a minor problem in most applications because the delay difference equations or algebraic constraints are related to inputs and outputs, as illustrated above, and the number of inputs and outputs is usually much smaller than the number of state variables. Finally, we note that also neutral systems can be dealt with in this framework, by introducing slack variables. As an example the neutral system

$$\frac{d}{dt}(z(t) + Gz(t-\tau)) = Hz(t) \tag{4}$$

can be rewritten as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & H \\ -I & I \end{bmatrix} \begin{bmatrix} v(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} \dot{v}(t-\tau) \\ \dot{z}(t-\tau) \end{bmatrix}, \tag{5}$$

where v is the slack variable.

The stability analysis of the null solution of (1) in this work is based on a spectrum determined growth property of the solutions, which allows us to infer stability information from the location of the characteristic roots. For instance, exponential stability will be related to a strictly negative spectral abscissa (the supremum of the real parts of the characteristic roots). As we shall see, the spectral abscissa of (1) may not be a continuous function of the delays. Moreover, this may lead to a situation where infinitesimal delay perturbations destabilize an exponentially stable system. These properties are very similar to the spectral properties of neutral equations (see, e.g., [23, Section 2]), which are known to be closely related to DDAEs [12]. Since in a practical control design the robustness of stability against infinitesimal changes of parameters is a prerequisite, we will define the concept of strong stability, inspired by the common terminology in the context of neutral equations [16, 17], and we will introduce the notion of the robust spectral abscissa, which explicitly take into account the effect of small parametric perturbations. We will also provide explicit conditions and expressions that eventually lead to numerical algorithms.

In order to compute the rightmost characteristic roots of (1), we combine a spectral discretization of an infinite-dimensional linear system equivalent to (1), inspired by [6, 4], with local corrections of characteristic roots by Newton's method. Such a two-step approach, inferred from the dual interpretation of the eigenvalue problem associated with a linear time-delay system as either infinite-dimensional and linear or finite-dimensional and nonlinear, also lies a the basis of the stability routine for steady state solutions of the package DDE-BIFTOOL [10] and of the method for computing $\mathcal{H}_\infty$ norms presented in [22]. We will analyze the presented method and demonstrate why, for reasons of both accuracy and numerical stability, it is to be preferred over applying a spectral discretization scheme for neutral equations. Finally, by combining the obtained theoretical results on the location and sensitivity of the spectrum with the computation of characteristic roots, we arrive at a computational scheme for the robust spectral abscissa.

The numerical algorithms for the computation of characteristic roots and the robust spectral abscissa are subsequently applied to the design of stabilizing controllers. Similarly to [31], a direct optimization approach towards stabilization is taken, based on minimizing the (robust) spectral abscissa as a function of the parameters of the controller. In the example (2)-(3) these parameters may correspond to elements of the controller matrices. In this way stabilization is achieved on the moment that the objective function becomes strictly negative. This approach allows us to design stabilizing controllers with a prescribed structure or order (dimension). It is also possible to fix elements of the controller matrices, allowing to impose additional structure, e.g., a proportional-integral-derivative (PID)-like structure, or sparsity.

In the context of stability optimization of linear time-invariant (LTI) systems it is well known that the spectral abscissa is in general a nonconvex function of the elements of the system matrices. In addition, it is typically not everywhere differentiable, even not everywhere Lipschitz continuous, although it is differentiable almost everywhere [8, 15]. These properties carry over to the case of the robust spectral abscissa of DDAEs under consideration. Therefore, special optimization methods for nonsmooth, nonconvex problems are required. We will use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method with weak Wolfe line search, whose favorable properties in the context of nonsmooth problems have been reported in [19], combined with the gradient sampling algorithm [9], as implemented in the MATLAB code HANSO[1]. The overall algorithm only requires the evaluation of the objective

[1]Hybrid Algorithm for Nonsmooth Optimization, see [28]

function, as well as its derivatives with respect to the controller parameters, whenever it is differentiable.

The resulting control design methodology extends the recent work on designing stabilizing fixed-order/ fixed structure controllers within an eigenvalue optimization framework, as, e.g., implemented in the package HIFOO [15], in two ways: the system class extends from finite-dimensional LTI systems towards a broad class of infinite-dimensional time-delay systems, and the system description is no longer described by differential equations but differential algebraic equations. Even in the delay-free case the latter has an advantage, since a direct feedthrough term does no longer need to be eliminated as in [26], preserving the linear dependence on the controller parameters. The difference with the approach for retarded time-delay systems in [31] lies, besides the broader system class under consideration and the use of algorithms that exploit the DDAE description, in the fact that the robustness of stability with respect to small delay perturbations is explicitly dealt with. The latter is a major issue for both neutral equations and DDAEs. Finally, the presented approach is frequency domain based and grounded in the eigenvalue based framework developed in [23]. Time-domain methods for the stability analysis and stabilization of DDAEs have been described in [12, 29] and the references therein, based on Lyapunov-Krasovskii functionals.

The structure of the article is as follows. In Section 2 assumptions on the system (1) are made and, subsequently, the existence of solutions and stability notions are addressed. Section 3 is devoted to the description of spectral properties and stability results, emphasizing the effect of small delay perturbations on exponential stability. In Section 4 the computation of characteristic roots and the robust spectral abscissa is presented. Section 5 is devoted to the design of stabilizing controllers. In Section 6 numerical results and experiments with the corresponding software are described. In Section 7 the conclusions are presented.

Notations

The notations are as follows:

j	: the imaginary identity
$\mathbb{C}, \mathbb{R}$	: set of the complex and real numbers
$\mathbb{N}$	: set of natural numbers
$\mathbb{R}^+, \mathbb{R}_0^+$	: set of nonnegative and strictly positive real numbers
A^T, A^*	: transpose, complex conjugate transpose of matrix A
A^{-T}	: transpose of the inverse matrix of A
$A^\perp$	: matrix of full column rank whose columns span the orthogonal complement of the column space of A
$\zeta(a;\ b)$	: function of indeterminate a, depending on parameter b
$\rho(A)$	: spectral radius of matrix A
$\mathrm{vec}(A), A = [a_1 \vert \cdots \vert a_q] \in \mathbb{C}^{p\times q}$	: vectorization of A, $\mathrm{vec}(A) = [a_1^T \cdots a_q^T]^T$
I, I_n	: identity matrix of appropriate dimensions, of dimensions $n \times n$
$\Re(u), \Im(u),\ \bar{u},\ u \in \mathbb{C}$	: real part, imaginary part, complex conjugate of u
$\mathcal{D}(.)$	: domain of an operator
$\mathcal{C}(\mathcal{I},\ X)$	: space of continuous functions from the interval $\mathcal{I}$ to X
$\Vert\phi\Vert_s,\ \phi \in \mathcal{C}(I,\ \mathbb{R}^n)$	: supremum norm, $\Vert\phi\Vert_s = \sup_{s\in\mathcal{I}} \Vert\phi(s)\Vert_2$
$\vec{\tau} \in \mathbb{R}^m,\quad \vec{\theta} \in [0,\ 2\pi]^p, \ldots$	: short notation for $(\tau_1, \ldots, \tau_m)$, $(\theta_1, \ldots, \theta_p)$,...

2 Preliminaries and assumptions

Let matrix E in (1) satisfy

$$\mathrm{rank}(E) = n - \nu,$$

with $1 \leq \nu < n$, and let the columns of matrix $U \in \mathbb{R}^{n\times\nu}$, respectively $V \in \mathbb{R}^{n\times\nu}$, be a (minimal) basis for the right, respectively left nullspace of E, which implies

$$U^T E = 0, \quad EV = 0. \tag{6}$$

Throughout the paper we make the following assumption.

Assumption 1 *The matrix $U^T A_0 V$ is nonsingular.*

The equations (1) can be separated into coupled delay differential and delay difference equations. When we define

$$\mathbf{U} = \begin{bmatrix} U^\perp & U \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V^\perp & V \end{bmatrix},$$

a pre-multiplication of (1) with $\mathbf{U}^T$ and the substitution

$$x = \mathbf{V} \; [x_1^T \; x_2^T]^T,$$

with $x_1(t) \in \mathbb{R}^{n-\nu}$ and $x_2(t) \in \mathbb{R}^\nu$, yield the coupled equations

$$\begin{array}{rcl} E^{(11)}\dot{x}_1(t) & = & \sum_{i=0}^m A_i^{(11)} x_1(t-\tau_i) + \sum_{i=0}^m A_i^{(12)} x_2(t-\tau_i), \\ 0 & = & A_0^{(22)} x_2(t) + \sum_{i=1}^m A_i^{(22)} x_2(t-\tau_i) + \sum_{i=0}^m A_i^{(21)} x_1(t-\tau_i), \end{array} \tag{7}$$

where

$$E^{(11)} = {U^\perp}^T E V^\perp \tag{8}$$

and

$$\begin{array}{llll} A_i^{(11)} = {U^\perp}^T A_i V^\perp, & A_i^{(12)} = {U^\perp}^T A_i V, & & \\ A_i^{(21)} = U^T A_i V^\perp, & A_i^{(22)} = U^T A_i V, & i = 0, \ldots, m. & \end{array} \tag{9}$$

Matrix $E^{(11)}$ is invertible, following from

$$\operatorname{rank}(E^{(11)}) = \operatorname{rank}(\mathbf{U}^T E \mathbf{V}) = \operatorname{rank}(E) = n - \nu.$$

In addition, Assumption 1 corresponds to the invertibility of matrix $A_0^{(22)}$.

Equations (7) are semi-explicit delay differential algebraic equations of index 1, because delay differential equations are obtained by differentiating the second equation. This precludes the occurrence of impulsive solutions [12]. Moreover, the invertibility of $A_0^{(22)}$ prevents that the equations are of *advanced* type and, hence, non-causal. This further motivates why Assumption 1 is natural in the delay case considered.

A forward solution of (1) on the interval $[0,\ t_0]$, $t_0 > 0$, with initial condition $x(t) = \phi(t), t \leq 0$, is an absolutely continuous function that satisfies the differential equation (1) almost everywhere on the interval $[0,\ t_0]$. In [11] it is shown that for every initial condition $\phi \in X$, where

$$X := \left\{ \phi \in \mathcal{C}([-\tau_m,\ 0],\ \mathbb{R}^n) :\ U^T A_0 \phi(0) + \sum_{i=1}^m U^T A_i \phi(-\tau_i) = 0 \right\}, \tag{10}$$

and for every $t_0 > 0$, a forward solution $x(\phi)$ exists and is uniquely defined on $[0,\ t_0]$. The constraint in (10) expresses that the initial function must satisfy the second equation of (7) at $t = 0$. It reduces to an algebraic constraint in the delay free case.

3 Spectral properties and stability

In this section the spectral properties of equation (1) are discussed. In the technical derivation connections with the neutral equation

$$\left\{ \begin{array}{rcl} E^{(11)}\dot{x}_1(t) & = & \sum_{i=0}^m A_i^{(11)} x_1(t-\tau_i) + \sum A_i^{(12)} x_2(t-\tau_i), \\ A_0^{(22)}\dot{x}_2(t) & = & -\sum_{i=1}^m A_i^{(22)} \dot{x}_2(t-\tau_i) - \sum_{i=0}^m A_i^{(21)} \dot{x}_1(t-\tau_i), \end{array} \right. \tag{11}$$

obtained by differentiating the second equation in (7), play an important role.

3.1 Exponential stability

Exponential stability of the null solution of (1) is defined as follows.

Definition 2 *The null solution of (1) is exponentially stable if and only if there exist constants $\delta > 0$ and $\gamma > 0$ such that*

$$\forall \phi \in X,\ \forall t \geq 0 :\ \|x(\phi)(t)\| \leq \delta e^{-\gamma t}\|\phi\|_s.$$

Stability conditions can be expressed in terms of the position of the *characteristic roots*, i.e., the roots of the equation

$$\det \Delta(\lambda) = 0, \tag{12}$$

where Δ is the characteristic matrix,

$$\Delta(\lambda) := \lambda E - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda \tau_i}.$$

In particular, we have the following result.

Proposition 3 *The null solution of (1) is exponentially stable if and only if $c < 0$, where c is the* spectral abscissa*,*

$$c := \sup\{\Re(\lambda) :\ \det \Delta(\lambda) = 0\}.$$

The proof can be found in Appendix A.

3.2 Continuity of the spectral abscissa and strong stability

We discuss the dependence of the spectral abscissa of (1) on the delay parameters $\vec{\tau} = (\tau_1, \ldots, \tau_m)$. In general the function

$$\vec{\tau} \in (\mathbb{R}_0^+)^m \mapsto c(\vec{\tau}) \tag{13}$$

is not everywhere continuous, which carries over from the spectral properties of delay difference equations (see, e.g., [2, 21, 25]). In the light of this we first outline properties of the function

$$\vec{\tau} \in (\mathbb{R}_0^+)^m \mapsto c_D(\vec{\tau}) := \sup\{\Re(\lambda) :\ \det \Delta_D(\lambda;\ \vec{\tau}) = 0\}, \tag{14}$$

with

$$\Delta_D(\lambda;\ \vec{\tau}) := U^T A_0 V + \sum_{i=1}^{m} U^T A_i V e^{-\lambda \tau_i}. \tag{15}$$

Note that (15) can be interpreted as the characteristic matrix of the delay difference equation associated with the neutral equation (11).

The property that the function (14) is not continuous led in [24] to the smallest upper bound, which is '*insensitive*' to small delay changes.

Definition 4 *For $\vec{\tau} \in (\mathbb{R}_0^+)^m$, let $C_D(\vec{\tau}) \in \mathbb{R}$ be defined as*

$$C_D(\vec{\tau}) := \lim_{\epsilon \to 0+} c_D^{\epsilon}(\vec{\tau}),$$

where

$$c_D^{\epsilon}(\vec{\tau}) := \sup\{c_D(\vec{\tau} + \delta\vec{\tau}) :\ \delta\vec{\tau} \in \mathbb{R}^m \text{ and } \|\delta\vec{\tau}\| \leq \epsilon\}.$$

Several properties of this upper bound on c_D are listed below (see [23, Chapter 2] for an overview).

Proposition 5 *The following assertions hold:*

1. *the function*

$$\vec{\tau} \in (\mathbb{R}_0^+)^m \mapsto C_D(\vec{\tau})$$

is continuous;

2. *for every* $\vec{\tau} \in (\mathbb{R}_0^+)^m$, *the quantity* $C_D(\vec{\tau})$ *is equal to the unique zero of the strictly decreasing function*

$$\zeta \in \mathbb{R} \to f(\zeta;\ \vec{\tau}) - 1, \tag{16}$$

where $f:\ \mathbb{R} \to \mathbb{R}^+$ *is defined by*

$$f(\zeta;\ \vec{\tau}) := \max_{\vec{\theta} \in [0,\ 2\pi]^m} \rho\left(\sum_{k=1}^{m} (U^T A_0 V)^{-1} (U^T A_k V) e^{-\zeta \tau_k} e^{j\theta_k} \right); \tag{17}$$

3. $C_D(\vec{\tau}) = c_D(\vec{\tau})$ *for rationally independent*[2] $\vec{\tau}$;

4. *for all* $\vec{\tau}_1, \vec{\tau}_2 \in (\mathbb{R}_0^+)^m$, *we have*

$$\operatorname{sign}\left(C_D(\vec{\tau}_1)\right) = \operatorname{sign}\left(C_D(\vec{\tau}_2)\right) := \Xi; \tag{18}$$

5. $\Xi < 0\ (> 0)$ *holds if and only if* $\gamma_0 < 1\ (> 1)$ *holds, where*

$$\gamma_0 := \max_{\vec{\theta} \in [0,\ 2\pi]^m} \rho\left(\sum_{k=1}^{m} (U^T A_0 V)^{-1} (U^T A_k V) e^{j\theta_k} \right). \tag{19}$$

For the single delay case, some of the expressions can be simplified.

corollary 6 *If* $m = 1$ *then we have*

$$C_D(\vec{\tau}) = \frac{1}{\tau_1} \log\left\{ \rho((U^T A_0 V)^{-1} (U^T A_1 V)) \right\}$$

and

$$\gamma_0 = \rho((U^T A_0 V)^{-1} (U^T A_1 V)).$$

We now come back to the DDAE (1), more precisely, to the properties of the spectral abscissa function (13). The following two technical lemmas make connections between the characteristic roots of (1) and the zeros of (15). Their proofs are similar to the proofs of the corresponding results for neutral equations, in particular [23, Propositions 1.26 and 1.27].

Lemma 7 *There exists a sequence* $\{\lambda_k\}_{k \geq 1}$ *of characteristic roots of (1) satisfying*

$$\lim_{k \to \infty} \Re(\lambda_k) = c_D, \quad \lim_{k \to \infty} \Im(\lambda_k) = \infty.$$

Lemma 8 *For every* $\epsilon > 0$ *there exists a continuous function* $R:\ \left(\mathbb{R}_0^+\right)^m \to \mathbb{R}^+$, *such the characteristic roots of (1) in the half plane*

$$\left\{ \lambda \in \mathbb{C}:\ \Re(\lambda) \geq C_D(\vec{\tau}) + \epsilon \right\}. \tag{20}$$

have modulus smaller than $R(\vec{\tau})$, *for all* $\vec{\tau} \in \left(\mathbb{R}_0^+\right)^m$.

The lack of continuity of the spectral abscissa function (13) leads us again to an upper bound that takes into account the effect of small delay perturbations.

Definition 9 *For* $\vec{\tau} \in (\mathbb{R}_0^+)^m$, *let the* robust spectral abscissa $C(\vec{\tau})$ *be defined as*

$$C(\vec{\tau}) := \lim_{\epsilon \to 0+} c^{\epsilon}(\vec{\tau}), \tag{21}$$

where

$$c^{\epsilon}(\vec{\tau}) := \sup\left\{ c(\vec{\tau} + \delta\vec{\tau}):\ \delta\vec{\tau} \in \mathbb{R}^m \text{ and } \|\delta\vec{\tau}\| \leq \epsilon \right\}.$$

[2]The m components of $\vec{\tau} = (\tau_1, \ldots, \tau_m)$ are rationally independent if and only if $\sum_{k=1}^{m} n_k \tau_k = 0,\ n_k \in \mathbb{Z}$ implies $n_k = 0,\ \forall k = 1, \ldots, m$. For instance, two delays τ_1 and τ_2 are rationally independent if their ratio is an irrational number.

The following characterization of the robust spectral abscissa (21) constitutes the main result of this section.

Proposition 10 *The following assertions hold:*

1. *the function*
$$\vec{\tau} \in (\mathbb{R}_0^+)^m \mapsto C(\vec{\tau}) \tag{22}$$
is continuous;

2. *for every $\vec{\tau} \in (\mathbb{R}_0^+)^m$, we have*
$$C(\vec{\tau}) = \max(C_D(\vec{\tau}), c(\vec{\tau})). \tag{23}$$

Proof. We start by proving the expression (23), where we distinguish between two cases.

Case 1: $c(\vec{\tau}) > C_D(\vec{\tau})$. Take $\epsilon > 0$ such that $\epsilon < c(\vec{\tau}) - C_D(\vec{\tau})$. From Lemma 8 and the fact that the characteristic function is analytic, there are only a finite number of characteristic roots in the half plane (20). Moreover, this number remains invariant under sufficiently small delay perturbations, provided that for the nominal delays $\vec{\tau}$ there are no characteristic roots with real part equal to $C_D + \epsilon$. By the continuity properties of the individual characteristic roots w.r.t. the delay parameters we conclude $C(\vec{\tau}) = c(\vec{\tau})$.

Case 2: $C_D(\vec{\tau}) \geq c(\vec{\tau})$. First, from $c(\vec{\tau}) \geq c_D(\vec{\tau})$ and Lemma 7 we have $C(\vec{\tau}) \geq C_D(\vec{\tau})$. Next, for any $\epsilon > 0$ there are no characteristic roots in the right half plane (20). By Lemma 8 and Rouché's Theorem this remains so for sufficiently small delay perturbations. This fact and the property that $C_D(\vec{\tau})$ is independent of $\vec{\tau}$ lead us to the conclusion $C(\vec{\tau}) = C_D(\vec{\tau})$.

The proof of the continuity of the function (22) follows the same arguments as the proof of [23, Theorem 1.39]. □

In line with the sensitivity of the spectral abscissa with respect infinitesimal delay perturbations, which has been resolved by considering the robust spectral abscissa (21) instead, we define the concept of strong stability[3].

Definition 11 *The null solution of (1) is strongly exponentially stable if there exists a number $\hat{\tau} > 0$ such that the null solution of*

$$E\dot{x}(t) = A_0 + \sum_{k=1}^{m} A_k x(t - (\tau_k + \delta\tau_k))$$

is exponentially stable for all $\delta\vec{\tau} \in (\mathbb{R}^+)^m$ satisfying $\|\delta\vec{\tau}\| < \hat{\tau}$ and $\tau_k + \delta\tau_k \geq 0,\ k = 1, \ldots, m$.

The following result provides necessary and sufficient conditions.

Theorem 12 *The null solution of (1) is strongly exponentially stable if and only if $C(\vec{\tau}) < 0$, or, equivalently, $c(\vec{\tau}) < 0$ and $\gamma_0 < 1$, where γ_0 is defined by (19).*

We illustrate the above results with a numerical example.

Example 13 *Consider the system*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & -\frac{1}{8} \\ -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix} x(t - \tau_1) \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x(t - \tau_2), \tag{24}$$

where a is a parameter. In Figure 1 we plot the rightmost characteristic roots for $a = 1/4$ and two sets of delay values: $\vec{\tau} = (1, 2)$ and $\vec{\tau} = (0.99, 2)$. The dotted line corresponds to the spectral abscissa $c((1, 2))$, the dashed line corresponds to $C_D((1, 2))$. In this case the robust spectral abscissa satisfies

$$C((1, 2)) = c((1, 2)).$$

[3]This terminology is borrowed from the theory of neutral delay differential equations [16, 24].

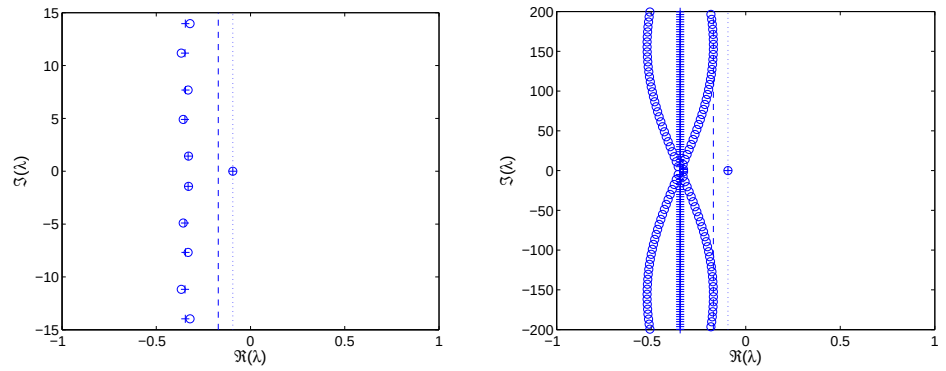


Figure 1: Characteristic roots of (24) for $a = 1/4$ and delays $\vec{\tau} = (1, 2)$ (pluses) and $\vec{\tau} = (0.99, 2)$ (circles). The difference between left and right pane lies in the scaling of the vertical axis.

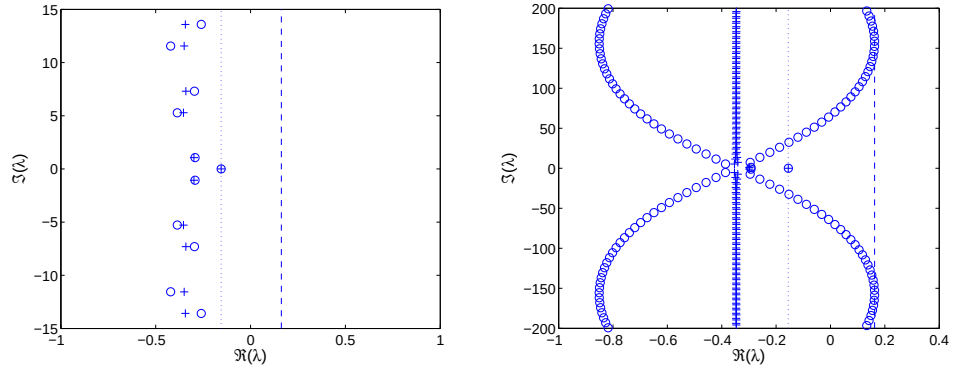


Figure 2: Characteristic roots of (24) for $a = 3/4$ and delays $\vec{\tau} = (1, 2)$ (indicated with '+') and $\vec{\tau} = (0.99, 2)$ (indicated with 'o').

Moreover, since the robust spectral abscissa is strictly negative the null solution of (24) is strongly exponentially stable.

For parameter $a = 3/4$ the analysis has been repeated and the results are displayed in Figure 2. In this case we have

$$C((1, 2)) = C_D((1, 2)) > 0 > c((1, 2)),$$

thus, although the zero solution is asymptotically stable for $\vec{\tau} = (1, 2)$, the stability is not robust with respect to small delay perturbations. This property is consistent with the characteristic roots shown for the perturbed delay values $\vec{\tau} = (0.99, 2)$.

Remark 14 *The spectral abscissa c is continuous with respect to changes of the system matrices A_k, $k = 0, \ldots, m$. We refer once again to [23] for the corresponding result on neutral equations.*

4 Computation of characteristic roots and stability assessment

In §4.1 an approach for the computation of the characteristic roots is presented, which extends the spectral discretization approach of [5] to a system of DDAEs. Connections with spectral discretization schemes for neutral equations are briefly discussed in §4.2 Finally, the computation of the robust spectral abscissa (21) is addressed in §4.3, motivated by the application to stability assessment.

4.1 A spectral discretization scheme

Consider the linear operator $\mathcal{A}$ on the space X, defined by

$$\begin{array}{l} \mathcal{D}(\mathcal{A}) = \{\phi \in X :\ \phi' \in \mathcal{C}([-\tau_m,\ 0],\ \mathbb{R}^n),\ E\phi'(0) = A_0\phi(0) + \sum_{k=1}^m A_k\phi(-\tau_k)\}, \\ \mathcal{A}\ \phi = \phi'. \end{array} \tag{25}$$

The eigenvalue problem for $\mathcal{A}$ is defined as

$$(\lambda I - \mathcal{A})z = 0,\ z \in \mathcal{D}(\mathcal{A}),\ z \neq 0. \tag{26}$$

It can be verified that the characteristic roots of (1) are eigenvalues of the operator $\mathcal{A}$. Moreover, if λ is an eigenvalue of $\mathcal{A}$, then the corresponding eigenfunction takes the form

$$z(\theta) = ve^{\lambda\theta},\ \theta \in [-\tau_m,\ 0], \tag{27}$$

where $v \in \mathbb{C}^n \setminus \{0\}$ satisfies

$$\Delta(\lambda)v = 0. \tag{28}$$

Conversely, if a pair (v, λ) satisfies (28) and $v \neq 0$, then (27) is an eigenfunction of $\mathcal{A}$ corresponding to the eigenvalue λ.

The eigenvalue problem (26) can be discretized using a *spectral method* (see, e.g. [30, 4]). Given a positive integer N, we consider a mesh Ω_N of $N+1$ distinct points in the interval $[-\tau_m,\ 0]$:

$$\Omega_N = \{\theta_{N,i},\ i = 1, \ldots, N+1\},$$

where

$$-\tau_m \leq \theta_{N,1} < \ldots < \theta_{N,N} < \theta_{N,N+1} = 0.$$

This allows to replace X with the space X_N of discrete functions defined over the mesh Ω_N, i.e., any function $\phi \in X$ is discretized into a block vector $\mathbf{x} = [\mathbf{x}_1^T \ \cdots \ \mathbf{x}_{N+1}^T]^T \in X_N$, with components

$$\mathbf{x}_i = \phi(\theta_{N,i}) \in \mathbb{C}^n,\quad i = 1, \ldots, N+1.$$

Let $\mathcal{P}_N\mathbf{x}$, be the unique $\mathbb{C}^n$ valued interpolating polynomial of degree smaller or equal than N, satisfying

$$\left(\mathcal{P}_N\mathbf{x}\right)(\theta_{N,i}) = \mathbf{x}_i,\quad i = 1, \ldots, N+1.$$

In this way we can approximate the eigenvalue problem (26) by the pencil

$$(\lambda E_N - \mathcal{A}_N)\mathbf{x} = 0, \tag{29}$$

where

$$E_N = \begin{bmatrix} I_{nN} & 0 \\ 0 & E \end{bmatrix}$$

and matrix $\mathcal{A}_N :\ X_N \to X_N$ is defined by

$$\begin{array}{lll} (\mathcal{A}_N\ \mathbf{x})_i = & \left(\mathcal{P}_N\mathbf{x}\right)'(\theta_{N,i}), & i = 1, \ldots, N, \\ (\mathcal{A}_N\ \mathbf{x})_{N+1} = & A_0\left(\mathcal{P}_N\mathbf{x}\right)(0) + \sum_{i=1}^m A_i(\mathcal{P}_N\mathbf{x})(-\tau_i). & \end{array} \tag{30}$$

Using the Lagrange representation

$$\mathcal{P}_N\mathbf{x} = \sum_{k=1}^{N+1} l_{N,k}\ \mathbf{x}_k,$$

where the Lagrange polynomials $l_{N,k}$ are real valued polynomials of degree N satisfying

$$l_{N,k}(\theta_{N,i}) = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases}$$

we get an explicit form for the matrix $\mathcal{A}_N$,

$$\mathcal{A}_N = \begin{pmatrix} d_{1,1} & \dots & d_{1,N+1} \\ \vdots & & \vdots \\ d_{N,1} & \dots & d_{N,N+1} \\ a_1 & \dots & a_{N+1} \end{pmatrix} \in \mathbb{R}^{(N+1)n \times (N+1)n}, \tag{31}$$

where

$$\begin{array}{rcll} d_{i,k} & = & l'_{N,k}(\theta_{N,i}) I_n, & i = 1, \dots, N,\ k = 1, \dots, N+1, \\ a_k & = & A_0 l_{N,k}(0) + \sum_{i=1}^m A_i l_{N,k}(-\tau_i), & k = 1, \dots, N+1. \end{array}$$

The following proposition provides an interpretation of the approximation of (26) by (29) in terms of an approximation of the characteristic matrix.

Proposition 15 *For $\lambda \in \mathbb{C}$, let $p_N(\cdot;\ \lambda)$ be the polynomial of degree N satisfying*

$$\begin{cases} p_N(0;\ \lambda) = 1, \\ p'_N(\theta_{N,i};\ \lambda) = \lambda p_N(\theta_{N,i};\ \lambda), \quad 1 \le i \le N. \end{cases} \tag{32}$$

Moreover, let

$$\Delta_N(\lambda) := \lambda E - A_0 - \sum_{i=1}^m A_i p_N(-\tau_i;\ \lambda).$$

Then the following statements are equivalent:

1. $\exists \mathbf{x} \in \mathbb{C}^{(N+1)n},\ \mathbf{x} \neq 0 :\ (\lambda E_N - \mathcal{A}_N)\,\mathbf{x} = 0;$ (33)
2. $\exists v \in \mathbb{C}^n,\ v \neq 0 :\ \Delta_N(\lambda) v = 0.$ (34)

For the proof we refer to Appendix B. In [14] it is shown that the functions

$$\lambda \mapsto p_N(-\tau_i;\ \lambda),\ i = 1, \dots, m, \tag{35}$$

are proper rational functions of order N with common poles, and with the property of uniform convergence on compact sets to the functions

$$\lambda \mapsto e^{-\lambda \tau_i},\ i = 1, \dots, m.$$

Summing up the above results, the characteristic roots of (1) appear either as the solutions of the finite-dimensional nonlinear eigenvalue problem

$$\Delta(\lambda) v = 0,\ v \in \mathbb{C}^n,\ v \neq 0, \tag{36}$$

or as the solutions of the infinite-dimensional linear eigenvalue problem (26), which can be discretized into (29). Both viewpoints can be combined in a computational scheme. The discretization of the linear infinite-dimensional problem allows to obtain estimates for *all* characteristic roots in a region where the approximation is accurate. These estimates can subsequently be corrected by a local method acting on the nonlinear equations (36). This brings us to the following algorithm.

Algorithm 16 *1. Fix N and compute the eigenvalues of the pencil (E_N, A_N).*

2. Correct these approximate characteristic roots by applying Newton's method to the equations (36).

In our implementation, the grid points in the discretization of (26) are specified as

$$\theta_{N,i} = \frac{\tau_m}{2}(\alpha_{N,i} - 1), \quad \alpha_{N,i} = -\cos \frac{\pi i}{N+1},\ i = 1, \dots, N+1, \tag{37}$$

i.e., the nonzero grid points are the (scaled and shifted) zeros of the Chebyshev polynomial of the second kind and order N. With this choice of grid points the convergence of the individual eigenvalues of the pencil $(E_N, \mathcal{A}_N)$ to corresponding characteristic roots of (1) is fast. More precisely, spectral accuracy (approximation error $O(N^{-N})$) is obtained because asymptotic distribution is equal to the distribution of (scaled and shifted) zeros of Chebyshev polynomials. For the proof we refer to the corresponding result for retarded time-delay systems in [4]. It makes use of the interpretation in Proposition 15.

Remark 17 *The interpretation of the discretized eigenvalue problem in Proposition 15, along with the fact that the functions (35) are independent on the system data (excepting the dependence on τ_m, which can be removed by a preliminary scaling of the problem) allow to choose the value of N necessary to capture all characteristic roots in a prescribed compact set or a prescribed half plane $\{\lambda \in \mathbb{C} : \Re(\lambda) > \mu\}$, with $\mu > C_D$. We refer to [22, Section 4.3] where a similar idea is elaborated in the context of $\mathcal{H}_\infty$ norm computations. The software corresponding to this article contains an automatic selection of N.*

4.2 Connection with the spectral discretization of neutral equations

We show that the discretization approach presented in §4.1 is advantageous compared to a (standard) spectral discretization of the neutral equations (11), whose spectrum consists of the spectrum of (1), possibly extended with a characteristic root at zero.

The approximate characteristic roots of the neutral equation (11), obtained by the spectral method described in [7], can be interpreted as the roots of the equation

$$\det\left\{\lambda\begin{bmatrix} E^{(11)} & 0 \\ 0 & -A_0^{(22)} \end{bmatrix} - \sum_{i=0}^{m}\begin{bmatrix} A_i^{(11)} & A_i^{(12)} \\ 0 & 0 \end{bmatrix} p_N(-\tau_i;\ \lambda) - \begin{bmatrix} 0 & 0 \\ A_0^{(21)} & 0 \end{bmatrix} p_N{}'(0;\ \lambda) - \sum_{i=1}^{m}\begin{bmatrix} 0 & 0 \\ A_i^{(21)} & A_i^{(22)} \end{bmatrix} p_N{}'(-\tau_i;\ \lambda)\right\}, \quad (38)$$

where the functions p_N are defined in Proposition 15. A comparison of (38) with the characteristic equation of the neutral equation (11) learns that the effect of the discretization can be interpreted as the effect of the substitutions

$$e^{-\lambda\tau_i} \leftarrow p_N(-\tau_i;\ \lambda), \quad (39)$$

$$\left(\lambda e^{-\lambda\tau_i}\right) \leftarrow p_N{}'(-\tau_i;\ \lambda),\ i = 1, \ldots, m, \quad (40)$$

in the characteristic matrix.

Similarly, the approximations obtained in the first step of Algorithm 16 correspond, by Proposition 15, to the roots of the equation

$$\det\left\{\lambda\begin{bmatrix} E^{(11)} & 0 \\ 0 & 0 \end{bmatrix} - \sum_{i=0}^{m}\begin{bmatrix} A_i^{(11)} & A_i^{(12)} \\ A_i^{(21)} & A_i^{(22)} \end{bmatrix} p_N(-\tau_i;\ \lambda)\right\} = 0, \quad (41)$$

which is obtained from the characteristic equation of (7) solely by the substitution

$$e^{-\lambda\tau_i} \leftarrow p_N(-\tau_i;\ \lambda),$$

implying

$$\left(\lambda e^{-\lambda\tau_i}\right) \leftarrow \lambda p_N(-\tau_i;\ \lambda),\ i = 1, \ldots, m. \quad (42)$$

Clearly, the substitutions (42) are preferred on (40) because of the adverse effect of differentiating on the accuracy of a polynomial approximation. This is illustrated in Figure 3, which shows the results of the computation of the relative errors

$$e_1 := \frac{\left|j\omega e^{-j\omega} - j\omega p_N(-1;\ j\omega)\right|}{|j\omega e^{-j\omega}|} \quad (43)$$

and

$$e_2 := \frac{\left|j\omega e^{-j\omega} - p_N'(-1;\ j\omega)\right|}{|j\omega e^{-j\omega}|}, \quad (44)$$

as a function of ω, for $N = 20$. Here, the functions p_N are defined on the grid (37), with $\tau_m = 1$.

The effect on the numerical stability of the type of equation discretized (DDAE or neutral equation) is illustrated with the following example.

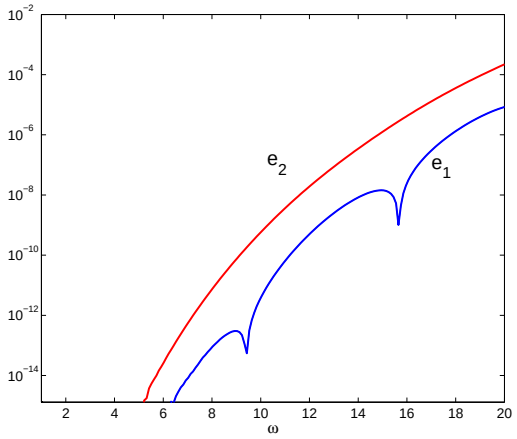


Figure 3: Comparison of (43) and (44) as a function of ω.

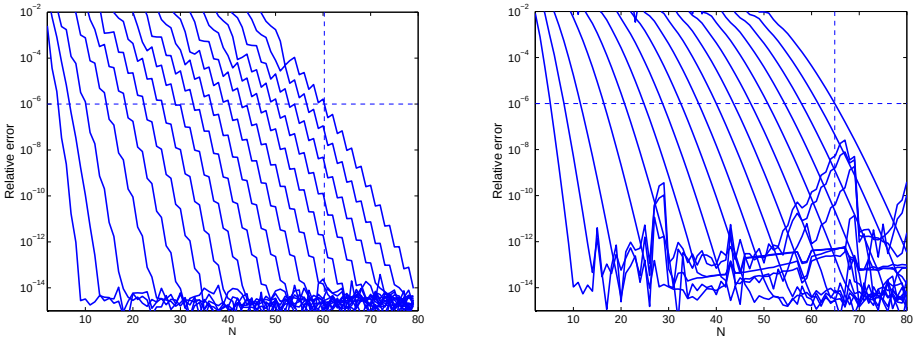


Figure 4: Relative error on the 32 smallest characteristic roots obtained by a spectral discretization of (45) (left pane) and (46) (right pane).

Example 18 *Figure 4 (left) shows the relative error on the 32 smallest characteristic roots of the DDAE*

$$\begin{cases} \dot{x}_1(t) &= x_2(t-1) \\ 0 &= -x_2(t) + x_1(t) + \frac{1}{2}x_2(t-1), \end{cases} \tag{45}$$

approximated as in the first step of Algorithm 16, in function of the number of discretization points, N. Differentiating the second equation in (45) leads to the neutral equation

$$\begin{cases} \dot{x}_1(t) &= x_2(t-1) \\ \dot{x}_2(t) &= x_2(t-1) + \frac{1}{2}\dot{x}_2(t-1). \end{cases} \tag{46}$$

A spectral discretization of (46) results in the approximation error displayed in Figure (4) (right). A comparison between the left and right pane of Figure 4 indicates that a method based on that discretizing (46) is much more prone to rounding errors. This can again be attributed to the approximation of derivatives in the righthand side of (46).

4.3 Computation of the robust spectral abscissa and stability assessment

Assessing the growth rate of solutions and stability (which both require checking the presence of characteristic roots in a right half plane), cannot always be reduced to computing the characteristic roots in a compact set, since system (1) may have infinite series of characteristic roots whose imaginary parts tend to infinity, yet whose real parts have a finite limit (see, e.g., Lemma 7). Moreover, the presence of such root chains may lead to a discontinuity of the spectral abscissa function with respect to the delays. The latter can be resolved by considering the robust spectral abscissa (21).

Based on the characterization (23) and Lemma 8 the robust spectral abscissa can be calculated by complementing the computation of characteristic roots, as presented in §4.1, with the evaluation of C_D. The latter is outlined in what follows.

We take a predictor-corrector approach to compute C_D, based on the second assertion of Proposition 5. In the prediction step we use Dekker-Brent's method [1] to find a zero of the function (16), where the function evaluations of f are approximated by restricting $\vec{\theta}$ in (17) to a grid. Note that for $m \geq 2$ we can write

$$f(\zeta;\ \vec{\tau}) = \max_{\vec{\theta} \in [0,\ 2\pi]^{m-1}} \rho\left(H_1 e^{-\zeta\tau_1} + \sum_{k=2}^{m} H_k e^{-\zeta\tau_k} e^{j\theta_k} \right), \tag{47}$$

with

$$H_k = (U^T A_0 V)^{-1}(U^T A_k V),\ k = 1, \ldots, m,$$

hence, a grid on the space $[0,\ 2\pi]^{m-1}$ is sufficient. If a high accuracy of C_D is required, then one may want to use a local corrector, based on the equations

$$\begin{cases} \left(H_1 e^{-\zeta\tau_1} + \sum_{k=2}^{m} H_k e^{-\zeta\tau_k} e^{j\theta_k}\right) v = \lambda v \\ u^* \left(H_1 e^{-\zeta\tau_1} + \sum_{k=2}^{m} H_k e^{-\zeta\tau_k} e^{j\theta_k}\right) = \lambda u^* \\ n(u) = 1 \\ u^* v = 1 \\ \lambda^* \lambda = 1 \\ \Im\left(e^{-\zeta\tau_k} e^{j\theta_k} (u^* H_k v)\bar{\lambda}\right) = 0,\ k = 2, \ldots, m \end{cases}, \tag{48}$$

where $n(u) = 1$ is a normalization constraint. These equations express that for the desired value of ζ, the matrix

$$H_1 e^{-\zeta\tau_1} + \sum_{k=2}^{m} H_k e^{-\zeta\tau_k} e^{j\theta_k}$$

has an eigenvalue on the unit circle and that the derivatives of the modulus of this eigenvalue with respect to $\theta_2, \ldots, \theta_m$ are equal to zero. Since the (overdetermined) equations have an exact solution, the Gauss-Newton method exhibits quadratic convergence whenever the solution is isolated, see Section 10.2 of [27].

Remark 19 *To assess strong stability of the null solution of (1), it is not necessary to compute C_D, in addition to computing rightmost characteristic roots. By Theorem 12 it is only needed to check whether $C_D < 0$, which is equivalent to $\gamma_0 < 1$. Because*

$$\gamma_0 = f(0;\ \vec{\tau}),$$

this amounts to evaluating the function f in one point, instead of finding the zero of (16). The value of γ_0 can be extracted from the solutions of (47) and (48), where ζ is set to zero and the equation $\lambda^*\lambda = 1$ in (48) is dropped.

Remark 20 *The computational cost of C_D is dominated by the evaluation of the right hand side of* (47)*, where gridding leads to an exponential growth in the number of terms and restricts the approach to a small number of delays. However, in most practical problems, the number of delays to be considered in (47) is much smaller than the number of system delays, m, because most of the terms in (47) are zero. Note that in the context of feedback control a nonzero term corresponds to a high frequency feedthrough over the whole control loop.*

An alternative approach, based on a characterization in terms of positive polynomials matrices, is presented in [18].

5 Robust stabilization by eigenvalue optimization

We now consider the equations

$$E\dot{x}(t) = A_0(\vec{p})x(t) + \sum_{i=1}^{m} A_i(\vec{p})x(t-\tau_i), \tag{49}$$

where the system matrices linearly depend on parameters $\vec{p} \in \mathbb{R}^{n_p}$. In control applications these parameter usually correspond to controller parameters. For example, in the feedback interconnection (2)-(3) they may arise from a parameterization of the matrices $(\hat{F}_i, \hat{G}_i, \hat{H}_i, \hat{L}_i)$. In §5.1 the stabilization problem for (49) is related to two optimization problems and in §5.2 the corresponding optimization algorithms are briefly discussed.

5.1 An optimization point of view

To impose exponential stability of the null solution of (49) it is necessary to find values of $\vec{p}$ for which the spectral abscissa is strictly negative. If the achieved stability is required to be robust against small delay perturbations, this requirement must be strengthened to the negativeness of the robust spectral abscissa. This brings us to the optimization problem

$$\inf_{\vec{p}} C(\vec{\tau};\ \vec{p}). \tag{50}$$

Strongly stabilizing values of $\vec{p}$ exist if the objective function can be made strictly negative. By Theorem 12 the latter can be evaluated as

$$C_D(\vec{\tau};\ \vec{p}) = \max(c(\vec{\tau};\ \vec{p}), C_D(\vec{\tau};\ \vec{p})). \tag{51}$$

An alternative approach consists of solving the constrained optimization problem

$$\inf_{\vec{p}} c(\vec{\tau};\ \vec{p}),\ \text{subject to}\ \ \gamma_0(\vec{p}) < \gamma, \tag{52}$$

where $\gamma < 1$. If the objective function is strictly negative, then the satisfaction of the constraint implies strong stability.

The advantage of solving problem (50) is that for all values of $\vec{p}$ a bound on the exponential growth rate of the solutions is assured, which takes into account delay perturbations. The advantage of solving (52) instead is that evaluating $\gamma_0(\vec{p})$ is less computationally demanding than evaluating $C_D(\vec{\tau};\ \vec{p})$, see Remark 19.

5.2 Algorithms

We start with problem (50). Lemma 8 illustrates that the spectrum of the DDAE (49) behaves similarly as the spectrum of a delay differential equation of retarded type whenever the parameters are restricted to the set

$$\{\vec{p} \in \mathbb{R}^{n_p} : \ c(\vec{\tau};\ \vec{p}) > C_D(\vec{\tau};\ \vec{p})\}\,. \tag{53}$$

Also the spectral properties for retarded systems derived in [31] carry over whenever (53) is satisfied. In particular, the spectral abscissa function $\vec{p} \mapsto c(\vec{\tau};\ \vec{p})$ may be not everywhere differentiable, even not everywhere Lipschitz continuous. A lack of differentiability may occur when there is more than one *active* characteristic root, i.e., a characteristic root whose real part equals the spectral abscissa. A lack of Lipschitz continuity may occur when an active characteristic roots is multiple and non-semisimple. On the contrary, the spectral abscissa function is differentiable at points where there is only one active characteristic root with multiplicity one. Since this is the case with probability one when randomly sampling parameters from the set (53), the spectral abscissa is smooth almost everywhere. The function f in Proposition 5 is a maximum eigenvalue function, similarly to the spectral abscissa function. Therefore, the above properties also hold for the function

$$\vec{p} \mapsto C_D(\vec{\tau};\ \vec{p})$$

and they are preserved by the maximum operator in (51). We conclude that the objective function in (50), the robust spectral abscissa, is not everywhere differentiable, not everywhere Lipschitz continuous, but it is differentiable almost everywhere.

The properties of the problem (50) preclude the use of standard optimization methods, developed for smooth problems. Instead we use a combination of BFGS with weak Wolfe line search and gradient sampling, as implemented in the MATLAB code HANSO [28]. The overall algorithm only requires the evaluation of the objective function. as well as its derivatives with respect to the controller parameters, *whenever* it is differentiable.

The evaluation of the robust spectral abscissa can be done as described in §4.3. When this function is differentiable we can express

$$\frac{\partial C}{\partial p_k}(\vec{\tau};\ \vec{p}) = \begin{cases} \frac{\partial c}{\partial p_k}(\vec{\tau};\ \vec{p}), & c(\vec{\tau};\ p) > C_D(\vec{\tau};\ \vec{p}), \\ \\ \frac{\partial C_D}{\partial p_k}(\vec{\tau};\ \vec{p}), & c(\vec{\tau};\ p) < C_D(\vec{\tau};\ \vec{p}), \end{cases}$$

for $k = 1, \ldots, n_p$. In case $c > C_D$, the derivative of the robust spectral abscissa is inferred from the sensitivity of an individual characteristic roots. More precisely, we can express

$$\frac{\partial C}{\partial p_k}(\vec{\tau};\ \vec{p}) = \Re\left(\frac{w^*\left(\frac{\partial A_0}{\partial p_k}(\vec{p}) + \sum_{i=1}^m \frac{\partial A_i}{\partial p_k}(\vec{p}) e^{-\lambda\tau_i}\right) z}{w^*\left(E + \sum_{i=1}^m A_i \tau_i e^{-\lambda\tau_i}\right) z}\right), \tag{54}$$

where the tuple (λ, w, z) satisfies

$$w^*\Delta(\lambda) = 0,\ \Delta(\lambda)z = 0,\ z \neq 0,\ w \neq 0$$

and λ corresponds to the rightmost characteristic roots. If $c < C_D$ then the robust spectral abscissa is differentiable in the generic case where for $\zeta = C_D(\vec{\tau};\ \vec{p})$ the maximum in (47) is isolated. We can then express

$$\frac{\partial C}{\partial p_k}(\vec{\tau};\ \vec{p}) = \frac{\Re\left(\bar{\lambda} u^*\left(\frac{\partial H_1}{\partial p_k}(\vec{p}) e^{-\zeta\tau_1} + \sum_{i=2}^m \frac{\partial H_i}{\partial p_k}(\vec{p}) e^{-\zeta\tau_i} e^{j\theta_i}\right) v\right)}{\Re\left(\bar{\lambda} u^*\left(H_1(\vec{p})\tau_1 e^{-\zeta\tau_1} + \sum_{i=2}^m H_i(\vec{p})\tau_i e^{-\zeta\tau_i} e^{j\theta_i}\right) v\right)}, \tag{55}$$

where $(\zeta, \lambda, u, v, \vec{\theta})$ refers to the corresponding solution of (48). An alternative to the analytic formulae (54) and (55) consists of computing the derivatives by finite differences.

Finally, we come back to the constrained problem (52). It can be solved using the barrier method proposed in [32], which is on its turn inspired by interior point methods for solving convex optimization

problems, see, e.g., [3]. The first step consists of finding a feasible point, i.e., a set of values for $\vec{p}$ satisfying the constraint. If the feasible set is nonempty such a point can be found by solving

$$\min_{\vec{p}} \gamma_0(\vec{p}). \tag{56}$$

Once a feasible point $\vec{p} = \vec{p}_0$ has been obtained one can solve in the next step the unconstrained optimization problem

$$\min_{\vec{p}} \{c(\vec{p}) - r \log(\gamma - \gamma_0(\vec{p}))\} \tag{57}$$

where $r > 0$ is a small number, and $\gamma \in (\gamma_0(\vec{p}_0),\ 1]$. The second term (the barrier) assures that the feasible set cannot be left when the objective function is decreased in a quasi-continuous way (because the objective function will go to infinity when $\gamma_0 \to \gamma$). If (57), with $\gamma = 1$, is repeatedly solved for decreasing values of r and with the previous solution as starting value, a solution of (52) is obtained. Strong *exponential* stability can be imposed by setting γ strictly smaller than one.

In our implementation we use HANSO for solving both (56) and (57). Derivatives of γ_0 exist for almost all values of $\vec{p}$ and can be computed using the formula

$$\frac{\partial \gamma_0(\vec{p})}{\partial p_k} = \frac{1}{|\lambda|} \Re \left(\bar{\lambda} u^* \left(\frac{\partial H_1}{\partial p_k}(\vec{p}) + \sum_{i=2}^{m} \frac{\partial H_i}{\partial p_k}(\vec{p}) e^{j\theta_i} \right) v \right), k = 1, \ldots, n_p,$$

with the quadruple $\left(\lambda, u, v, \vec{\theta}\right)$ solving (48) (where ζ is set to zero and one equation is dropped, see Remark 19). Note that the objective function in (57) still contains an implicit constraint in the sense that it is defined only on the feasible set of (52). This problem can be solved by explicitly setting the objective function to infinity outside the feasible set [32].

6 Numerical Examples

We illustrate the approach with several case-studies. A user-friendly MATLAB implementation of the control design algorithms, as well as all data corresponding to the presented examples, are publicly available [20].

6.1 A stabilization problem with input delay

As a first example we take the system with input delay from [31]:

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \quad y(t) = x(t), \tag{58}$$

where

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \quad \tau = 5. \tag{59}$$

The uncontrolled system is unstable, characterized by the spectral abscissa $c = 0.108$. We design a dynamic controller of the form

$$\begin{cases} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t), \\ u(t) &= C_c x_c(t) + D_c y(t), \quad x_c(t) \in \mathbb{R}^{n_c}, \end{cases} \tag{60}$$

using the approach of Section 5, where we set

$$p = \mathrm{vec} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}.$$

Since the transfer function from u to y is strictly proper, the robust spectral abscissa equals the spectral abscissa, and the optimization problems (50) and (52) reduce to the (unconstrained) minimization of

controller	minimum c
output feedback, $n_c = 0$	-0.1489
output feedback, $n_c = 1$	-0.2293
output feedback, $n_c = 2$	-0.2682
output feedback, $n_c = 3$	-0.4575

Table 1: Results of minimizing the spectral abscissa of (58) and (60).

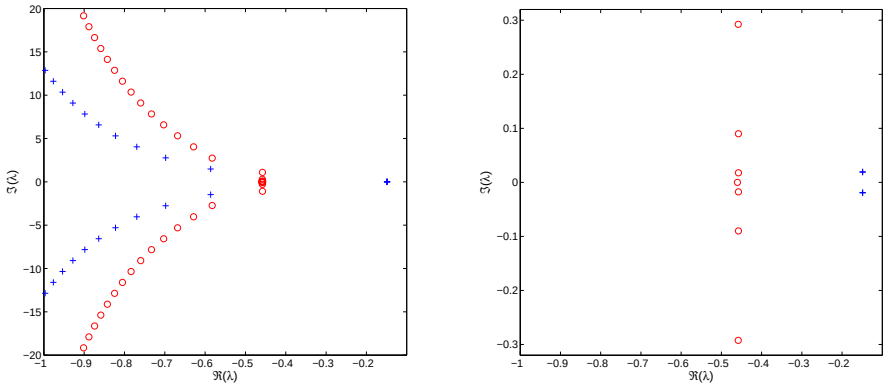


Figure 5: Characteristic roots of (58) and (60), corresponding to a minimum of the spectral abscissa, for $n_c = 0$ (pluses) and $n_c = 3$ (circles). The left and right pane correspond to a different scaling of the imaginary axis.

the spectral abscissa. The results for different controller orders are displayed in Table 1. The case $n_c = 0$ refers to static feedback, also considered in [31]. In Figure 5 the rightmost characteristic roots of the closed-loop system are shown for $n_c = 0$ and $n_c = 3$.

For the second example we assume that the measured output of the system (58) is instead given by

$$\tilde{y}(t) = x(t) + \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} u(t - 2.5) + \begin{bmatrix} 2/5 \\ -2/5 \\ -2/5 \end{bmatrix} u(t - 5) \tag{61}$$

and we design a static controller,

$$u(t) = D_c \tilde{y}(t). \tag{62}$$

In this case there is a high-frequency feedthrough term in the control loop. Solving the optimization problem (50) leads to

$$C = -0.0309, \quad D_c = [0.0409 \;\; 0.0612 \;\; 0.3837]. \tag{63}$$

In Figure 6 we show the rightmost characteristic roots corresponding to the minimum of the robust spectral abscissa (63). The dotted line corresponds to $\Re(\lambda) = c_D$, the dashed line to $\Re(\lambda) = C_D$. The minimum of C is characterized by an equality between C_D and the spectral abscissa c, the latter induced by a rightmost characteristic root with multiplicity three. This is compatible with the three degrees of freedom in the controller. In order to illustrate that we indeed have $c = C_D$ we depict in Figure 7 the rightmost characteristic roots after perturbing the delay value 2.5 in (61) to 2.51.

Solving the optimization problem (57) with $r = 10^{-3}$ and $\gamma = 10^{-3}$ yields

$$c = -0.0345, C_D = -0.00602, \; D_c = [0.0249 \; 0.1076 \; 0.3173].$$

Compared to (63), where we had $C = c = C_D$, a further reduction of the spectral abscissa has been achieved, at the price of an increased value of C_D. This is expected because the constraint $\gamma_0 < 1$ imposes robustness of stability, yet no bound on the exponential decay rate of the solutions.

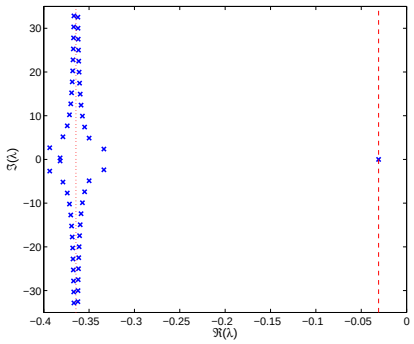


Figure 6: Characteristic roots corresponding to the minimum of the robust spectral abscissa of (58) and (61), for the control law (62). The rightmost characteristic roots, $\lambda \approx -0.0309$, has multiplicity three.

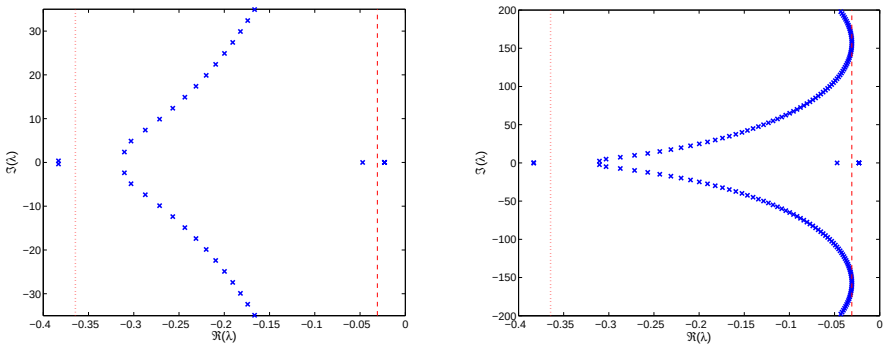


Figure 7: Effect on the characteristic roots of a perturbation of the delays $(2.5, 5)$ in (61) to $(2.51, 5)$. The difference between left and right pane lies in the scaling of the imaginary axis.

6.2 Heating system

In [33] a linear model of an experimental heat transfer set-up at the CTU in Prague is proposed, consisting of 10 delay differential equations. The inclusion of an integrator to achieve a zero steady state error to a set-point of one of the state variables eventually leads to equations of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{5} A_i x(t-\tau_i) + Bu(t-\tau_6), \quad x(t) \in \mathbb{R}^{11\times 11},\ u(t) \in \mathbb{R}, \tag{64}$$

see [33] for the corresponding matrices and delay values. We consider two outputs

$$y(t) = \begin{bmatrix} x_5(t) + x_6(t) \\ -x_{10}(t) + x_{11}(t) \end{bmatrix}. \tag{65}$$

The spectral abscissa of the uncontrolled system is equal to zero. In Table 2 we show the result of optimizing the spectral abscissa using static state feedback, $u(t) = Kx(t),\ K \in \mathbb{R}^{1\times 11}$, and dynamic output feedback (60).

controller	minimum c
static state feedback,	-0.0577
output feedback, $n_c = 0$	-0.0187
output feedback, $n_c = 1$	-0.0218
output feedback, $n_c = 2$	-0.0237

Table 2: Results of minimizing the spectral abscissa of (64)-(65) for static state feedback and for dynamic output feedback (60) .

7 Concluding remarks

We presented an eigenvalue based solution to the stabilization problem for linear interconnected systems with time-delays, which can be modeled in a natural way by DDAEs. The approach has several advantages. First, the system description (1) allows to model a broad class of interconnected systems, as we have illustrated. Moreover, in the application of the method there is no restriction on the number of delays, even not on the number of delays in the delay-difference equation associated with the neutral equation (11), and the robustness of stability against small delay perturbations is explicitly taken into account. Second, the structure and dimension of the controller can be chosen by the user. Third, since the stabilization method is directly based on the location of the spectrum, it is non-conservative in the sense that a strongly stabilizing controller can be computed whenever it exists, in contrast to approaches inferred from sufficient (but not necessary) stability conditions. The approach has been implemented in generic software which is publicly available [20].

As a by-product, we demonstrated in §4.2 the advantage of applying a spectral discretization directly to the DDAE description of the problem, instead of transforming the equations to neutral delay differential equations and applying standard techniques. Vice versa, for characteristic roots computations for neutral equations, it may be beneficial to transform them to DDAEs first (like (4) is transformed to (5)) and, next, apply the discretization scheme presented in §4.1. This may double the dimension, but the accuracy and stability of the method will be significantly improved. The latter can be explained by the fact that derivatives no longer need to be approximated in the spectral discretization.

Finally, we note that in a practical control design the stabilization phase is usually only a first step in the overall design procedure. The (subsequent) fixed-order $\mathcal{H}_\infty$ control design is addressed in the accompanying paper [13].

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A Proof of Proposition 3

The proof of the implication '$\Rightarrow$' is by contradiction. If $c \geq 0$ then for all $\delta > 0$, there exists a characteristic root λ satisfying $\Re(\lambda) > -\delta$. The presence of the corresponding exponential solution contradicts the exponentially stability property.

We now prove the implication '$\Leftarrow$'. Following from $c < 0$, there is no characteristic root at zero. Hence, the differentiation of the delay difference equations that leads to the neutral equation (11) introduces a characteristic root at zero with multiplicity m. Since the corresponding null vectors of the characteristic matrix have the form $[0\ \beta^T]^T$, with $\beta \in \mathbb{C}^\nu$, the introduced characteristic root at zero is semi-simple.

Denote by $\mathcal{T}(t),\ t \geq 0$, the time-integration operator associated with the solutions of the neutral equation (11), i.e.

$$\mathcal{T}(t)\phi = x_t(\phi),$$

where $x_t(\phi) \in \mathcal{C}([-\tau_m,\ 0], \mathbb{C}^n)$ is defined by $x_t(\phi)(\theta) = x(\phi)(t+\theta),\ \theta \in [-\tau_m,\ 0]$. Note that $\mathcal{T}(t)$ is a strongly continuous semi-group. The characteristic roots of (11), which are the eigenvalues of its infinitesimal generator $\mathcal{A}$, are infinite in number but countable. Denote these eigenvalues by $\lambda_i,\ i \geq 0$, with $\lambda_0 = 0$, and let P_{λ_i} be the spectral projection onto the corresponding generalized eigenspace $\mathcal{M}_{\lambda_i}$. We can decompose a solution $x(\phi)(t)$ on an interval $[t_1,\ t_2]$, where $n\tau_m \leq t_1 < t_2$, in the following way:

$$x_t(\phi) = \mathcal{T}(t)\ \phi \quad = \textstyle\sum_{i=0}^{\infty} \mathcal{T}(t) P_{\lambda_i}\ \phi. \tag{66}$$

Because $\mathcal{M}_0 = \left\{\theta \in [-\tau_m,\ 0] \mapsto [0\ \beta^T]^T :\ \beta \in \mathbb{C}^m\right\}$ we get

$$x_t(\phi) = \begin{bmatrix} 0 \\ \beta_0 \end{bmatrix} + \sum_{i=1}^{\infty} \mathcal{T}(t) P_{\lambda_i} \phi, \tag{67}$$

hence, we can write

$$x_t(\phi)(\theta) = \begin{bmatrix} 0 \\ \beta_0 \end{bmatrix} + \sum_{i=1}^{\infty} \beta_i e^{\lambda_i (t+\theta)}, \qquad \theta \in [-\tau_m,\ 0], \tag{68}$$

where the constants $\beta_0 \in \mathbb{C}^m$ and $\beta_i \in \mathbb{C}^n,\ i \geq 1$, depend on the initial condition ϕ.

When the initial condition ϕ is an element of the space X, the corresponding solutions of (1) and (11) coincide since both are uniquely defined. Since (1) does not have an equilibrium different from the null solution we conclude that $\beta_0 = 0$. Since $c < 0$ there exists a number $\gamma > 0$ such that $\Re(\lambda_i) \leq \gamma,\ \forall i \geq 1$. From this and the expansion (68) the assertion follows.

B Proof of Proposition 15

For the implication 1. $\Rightarrow$ 2. we observe that the N first (block) equations of (33) express that

$$\left(\mathcal{P}_N \mathbf{x}\right)(t) = p_N(t;\ \lambda)\mathbf{x}_{N+1},$$

hence, $\mathbf{x}_{N+1} \neq 0$. Substituting this in the last equation of (33) yields the second statement, with $v = \mathbf{x}_{N+1}$. For the implication 2. $\Rightarrow$ 1., we expand

$$p_N(t;\ \lambda) = \sum_{i=1}^{N} l_{N,i}(t) c_i + l_{N,N+1}(t).$$

Substituting this expression in (32) and (34) we obtain (33), with $\mathbf{x}_{N+1} = v$ and $\mathbf{x}_i = c_i v,\ i = 1, \ldots, N$.

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