Creating number. How humans developed natural, complex, real and infinite numbers *Claude John N. Crossley.* World Scientific Publishing, 2026 (xx+282 p.) isbn: 978-981-98=2623-0

The author is a British-Australian mathematician and logician who in this book has continued a theme from his previous book *The Emergence of Number* (World Scientific, 1987) where he discussed the natural, complex, and real numbers. In this book he adds a fourth part on infinite numbers. As he is particularly interested in how concepts evolve depending on human understanding and social and technological necessities, this is a natural sequel.

His first part about the natural numbers has been told before. He relies on paleolithic findings and psychological studies to conclude that the ability to count out small quantities was present in primitive civilizations and is possessed by babies and some animals. Out of necessity larger numbers were needed, creating the problem of naming and denoting them in a systematic way. There are great cultural differences (Roman, Greek, Arabic, Hindu,...). Even today, apart from mathematical notation, we have no systematic way of naming arbitrarily large or arbitrarily small numbers. New terminology such as nano-, pica-, femto-, etc. was introduced, but it cannot go on forever. Formal definitions were given by Frege and Dedekind based on the concept of a proof by induction. Frege's cardinal approach started with 0 (there can be none), whereas Dedekind's ordinal approach starting with 1 (you need to have one to define the next one). Later, Peano formalized this using logic, thus resolving the paradoxes in Frege's approach.

The second part deals with complex numbers. For the Greeks, numbers had a geometric meaning such as length or area. They were aware that irrational numbers such as $\sqrt{2}$ and π existed, but there was no need for negative numbers: a square with negative area does not exist. Complex numbers emerged from a practical need. During the 16th century, competitions were organized in Italy to solve quadratic and cubic equations. The algorithmic steps of al-Khwarizmi's *Algebra* were followed, and in some cases this required square roots of negative numbers which gave meaningful results even though the intermediate steps involved "imaginary" numbers. Here we read about the well-known dispute among Tartaglia, dal Ferro, and Cardano about a "secret" formula to solve the equations. The idea was passed on, but the nature of complex numbers only became clear in 1799 when Wessels

interpreted complex numbers as points in a plane. Crossley explains how this insight arose from connections with calculus, e.g. irrational numbers, series expansions, integration, logarithms, and trigonometric functions were intertwined in Euler's formula $e^{ix} = \cos(x) + i\sin(x)$.

In part three on real numbers, we have to travel back in time to the engineering problems of Babylon and Egypt where good approximations were needed (for example, approximations of π or the hypothenuse to construct right angles). The geometry of the Greeks, on the other hand, distinguished between numbers in geometry (which were commensurable) and their magnitudes in arithmetic (which could vary continuously). The Euclidean algorithm of two numbers defines a greatest common divisor so that these numbers were commensurable; their ratio is again a number. But they also knew by the Pythagorean Theorem that $\sqrt{2}$ was not rational. Only in the 19th century were serious attempts made to define real numbers. Cantor introduced the notion of cuts, and Dedekind formalized it. Thus, the set of points on the real line could be identified with the set of real numbers by requiring an additional axiom. Hamilton's quaternions are added as a coda to this part.

With the last part about infinite numbers, we have to dive into the foundations of mathematics, logic and model theory. The definition of the real numbers already required some precise definitions, logical formulations, and set theory. These become only more delicate when considering infinite numbers. One needs to define what the expression "and so on" means within an induction proof or a recursive definition. What if the counting does not end? Cantor required additional axioms such as well-ordering (WO), the axiom of choice (AC) (which is equivalent to WO), the power set axiom (PA) (all subsets of a set form a set) and several others. These allowed him to define transfinite cardinal numbers (indicating quantities, denoted as $\aleph_\alpha$) and ordinal numbers (indicating ordering, such as ω). $\aleph_0$ is the cardinality of $\mathbb{N}$ and can be identified with the ordinal ω. Cantor's diagonal selection technique proved that $\aleph_1$, the cardinality of the reals was larger than $\aleph_0$. Gödel showed that it can not be proved within set theory whether $2^{\aleph_0} = \aleph_1$, i.e. that the cardinality of the power set of $\mathbb{N}$ is $\aleph_1$. This is the subject of the continuum hypothesis (CH): there is no other infinity between $\aleph_0$ and $\aleph_1$.

A separate chapter is devoted to infinitely small and infinitely large numbers. It highlights how Abraham Robinson in the 1960s revived the role of infinitesimals that were previously abandoned due to the critique that they were smaller than anything and yet not zero. Robinson just changed the ax-

ioms and introduced a nonstandard model. Also, the surreal numbers defined by Conway are introduced as a curiosity. Crossley then engages in a discussion of different philosophical views on mathematics. Logicists assume, as Plato did, that mathematics exists and we just discover it. The intuitionists, such as Brouwer, say that mathematics is a creation of our mind, and something "exists" if it can be constructed. The formalists consider mathematics as a game to be played by following the rules. Also, "feasible numbers" are introduced, a term from computer science which relates to the N vs. NP problem. Another group, the finitists, emphasize that all computations are discrete and finite; hence, infinity does not exist in practice. Feasible numbers are the only ones that we can practically handle and hence are the only ones that actually matter.

An epilogue summarizes Crossley's findings: numbers are an invention of the human mind. Mathematics expands when subjective knowledge is transformed into generally accepted knowledge.

This is partly a history book, similar to many others, telling how mathematics evolved from elementary counting to the abstraction of our current mathematical constructs. The theme throughout the story is how number systems were accepted and how they evolved according to necessity. There is a rather strong philosophical aspect because the numbers we work with define in a sense how we conceive mathematics and what we use it for. And yet, the book remains quite readable for an interested reader without advanced mathematical skills. There are formulas (mostly formulations in logic) and there is an occasional theorem and proof. Mathematics is the core of the story, not the mathematicians (no biographies are included). It is well documented (26 pages of references). The history of zero and negative numbers is not explicitly discussed. For this Crossley refers to the literature.

I spotted some minor typos: on page 228, the formula

$$\sum_{n=1}^{\infty} 10^{-n!} = 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + \cdots$$

(the minus signs before 6 and 24 are missing) and page 211: "So long as there us only..."; page 221: "This guarantees at least one...". The index is only 4 pages, and not all occurrences of an item are listed. For example, for Gödel there is only a reference to page 93 (where the name first appears), but Gödel occurs on several other pages too.

To conclude, most of what is being discussed here can be found in other books, but the approach is original: How were humans forced to introduce

new kinds of numbers, and how did these concepts evolve from prehistory to our age of computer science?

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