Zeros of quasi-paraorthogonal polynomials and positive quadrature

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Dedicated to the memory of Luc Wuytack.

Abstract

In this paper we illustrate that paraorthogonality on the unit circle $\mathbb{T}$ is the counterpart to orthogonality on $\mathbb{R}$ when we are interested in the spectral properties. We characterize quasi-paraorthogonal polynomials on the unit circle as the analogues of the quasi-orthogonal polynomials on $\mathbb{R}$. We analyze the possibilities of preselecting some of its zeros, in order to build positive quadrature formulas with prefixed nodes and maximal domain of validity. These quadrature formulas on the unit circle are illustrated numerically.

Keywords: Paraorthogonality, quasi-orthogonality, Szegő-type quadrature

1. Introduction

It is very well known that the orthogonal polynomials $\{\rho_n\}_{n\geq 0}$ associated with a positive Borel measure μ supported on the real line (OPRL) have many properties that allow us, both to emulate the measure, and describe its support. For example, the zeros of ρ_n are simple, they lie on the open convex hull of $\text{supp}(\mu)$ and there is one zero on the closure of each connected component of the complement of $\text{supp}(\mu)$, at most. Moreover, the zeros of ρ_{n+1} and ρ_n interlace and in general, they end up filling the whole support. These zeros describe μ as $n \to \infty$ because they allow us to construct a sequence of measures $\{\mu_n\}_{n\geq 1}$ that converges weakly to μ.

Orthogonal polynomials on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (OPUC), or Szegő polynomials, were introduced in [47, Chapter 11]. Since then, they have been widely studied, not only because of their own interest [45], but also in many applications such as the trigonometric moment problem [1], complex approximation [49], probability and statistics [32], prediction theory [50], systems theory, networks, circuits and scattering

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[24], signal processing [21], and many more, but also because of their intimate connection with OPRL (see e.g. [8, 18]). However, these polynomials present important differences with respect to OPRL, in particular, concerning the above properties, since their zeros are located outside of the support of the measure. As a consequence, these zeros cannot be used for interpolation processes on the unit circle, as nodes for quadrature formulas on the unit circle, and they lack importance in the resolution of the trigonometric moment problem, among other relevant properties that hold for OPRL. These drawbacks were not solved until the concept of paraorthogonality ([34], see also [28, Theorem III]) was introduced in this setting. These paraorthogonal polynomials (POPUC) have their zeros in the support of the measure when the support is $\mathbb{T}$. In this context, focussing on their spectral properties, quadrature, and associated problems, these polynomials are a natural counterpart on the unit circle of the orthogonal polynomials on the real line. Some of the properties that we have mentioned above and many others formulated for a measure supported on the real line, have been adapted for measures on the unit circle, see e.g. [12, 31, 46, 51].

Since the spectra of the orthogonal and paraorthogonal polynomials are directly connected to their respective type of orthogonality, we shall also look at what can be said about the zeros as we relax some of the orthogonality conditions of these polynomials, leading to quasi-orthogonal polynomials on the real line and the quasi-paraorthogonal counterpart on the unit circle.

In the case of the real line, the real polynomials $Q_{n,d}$ of degree n that are orthogonal to $\mathbb{P}_{n-d-1}$ (the vector space of all polynomials of degree at most $n-d-1$), $0 \leq d < n$ are well studied and they are called quasi-orthogonal polynomials (QOPRL) of degree n and order d or (n,d)-QOPRL. Clearly the orthogonal polynomial ρ_n appears as a special case for $d=0$: $\rho_n = Q_{n,0}$. The QOPRL $Q_{n,d}$ have at least $n-d$ changes of sign on the open convex hull of $\mathrm{supp}(\mu)$. This concept was first introduced in 1923 by M. Riesz ([40]) for $d=1$ in relation to moment problems, and then by L. Fejèr ($d=2$) and by J.A. Shohat ($d \geq 1$) in 1933 and 1937, respectively ([27, 44]), in the context of quadrature formulas (see also [16, 25, 26]).

The unit circle analogue of these QOPRL should naturally be called quasi-paraorthogonal polynomials (QPOPUC). Their properties however, are less developed in the literature. This is one of the main purposes of this paper: to define and analyse the spectral properties of these QPOPUC and how these relate to their orthogonality conditions.

The outline of the paper is as follows. In Section 2 we introduce the concept of quasi-paraorthogonal polynomials on the unit circle, where we shall generalize the classical concept of paraorthogonality to QPOPUC of order $2\ell+1$ by requiring orthogonality to spaces of decreasing dimension. The classical concept of paraorthogonality corresponds to quasi-paraorthgonality of order 1. We prove the main result in Section 3: a link between the number of paraorthogonality conditions and the number of zeros of odd multiplicity in the interior of the support of the measure, when it is supported on an arc of the unit circle. We also investigate how we can use the free parameters to prefix some of the zeros. As an application we consider in Section 4 how this can be used to construct positive Szegő-type quadrature formulas on an arc or on $\mathbb{T}$ that are the analogues of the Gauss-type formulas

on an interval or $\mathbb{R}$. Finally, in Section 5 we shall include some numerical experiments to illustrate these quadrature formulas.

We finish this introductory section by defining some notation and abbreviations that are used throughout this paper. We denote by $\mathbb{P}$ the vector space of all polynomials and by $\mathbb{P}_n = \operatorname{span}\{z^k : k = 0, \ldots, n\}$ the $(n+1)$-dimensional space of polynomials of degree at most n. The vector space of all Laurent polynomials is $\mathbb{L} = \operatorname{span}\left\{z^k : k \in \mathbb{Z}\right\}$, with subspaces $\mathbb{L}_{p,q} = \operatorname{span}\{z^p, \ldots, z^q\}$, $p \leq q$, $p, q \in \mathbb{Z}$. $\mathbb{R}$ is the real line, $\mathbb{T}$ the complex unit circle and μ is a positive Borel measure that is supported on a subset S of $\mathbb{R}$ or $\mathbb{T}$. If $S \subsetneq \mathbb{R}$ or $S \subsetneq \mathbb{T}$ we assume it is compact and connected, i.e. a closed interval on $\mathbb{R}$ or a closed arc on $\mathbb{T}$. We denote by $\langle \cdot, \cdot \rangle$ the inner product induced by μ and we assume that $\mathbb{P}$ belongs to the corresponding Hilbert space L^2_μ. Orthogonality will always refer to this space. The acronyms OPRL and OPUC are used for orthogonal polynomials on the real line and the unit circle respectively. POPUC refers to paraorthogonal polynomials on the unit circle and if the letter Q is added in the beginning, it refers to quasi-versions, thus QOPRL means quasi-orthogonal polynomials on the real line and QPOPUC denotes quasi-paraorthogonal polynomials on the unit circle. The precise definitions follow below. By $\mathbb{D}$ and $\mathbb{E}$ we denote the interior and exterior of the closed unit disk, respectively and by $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$. $P^*(z) = z^n \overline{P\left(1/\overline{z}\right)}$ is the reciprocal (or reversed) polynomial of $P \in \mathbb{P}_n$. Throughout the paper, ρ_n will denote the monic orthogonal polynomial of degree n in L^2_μ. For $S \subseteq \mathbb{T}$, the Schur parameters for the OPUC will be denoted by $\{\delta_k\}_{k=0}^\infty$ where $\delta_0 = 1$ and $\delta_k = \rho_k(0) \in \mathbb{D}$, i.e., $\rho_k(z) = z\rho_{k-1}(z) + \delta_k \rho^*_{k-1}(z)$ for $k \geq 1$. All our spaces $\mathbb{Q}_{n,k}$ will be subspaces of $\mathbb{P}_n$, and when we write $\mathbb{Q}^\perp_{n,k}$, we mean its orthogonal complement in $\mathbb{P}_n$. We denote by $\varphi_a(z) = \frac{z+a}{1+\overline{a}z}$, the Möbius Transform associated with $a \in \mathbb{C}\backslash\mathbb{T}$. We also assume that $\sum_{k=a}^b x_k$ is 0 and $\prod_{k=a}^b x_k$ is 1 if $b < a$.

2. Quasi-paraorthogonal polynomials

As we have already said in the Introduction, quasi-orthogonal polynomials with respect to a measure μ on the real line have real coefficients, and as a consequence, their zeros are real or appear in complex conjugate pairs (symmetric with respect to $\mathbb{R}$). The analogue on the unit circle, is that zeros are on $\mathbb{T}$ or appear in symmetric pairs with respect to $\mathbb{T}$. That is: if α is a zero then also $1/\overline{\alpha}$ is a zero. Note that this implies that if $P \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$, then $\alpha = 0$ cannot be a zero of P. Thus, if the zeros are $\alpha_k \neq 0$, $k = 1, \ldots, n$, then we are looking for polynomials satisfying

$$\begin{aligned} P(z) &= \nu \prod_{k=1}^n (z - \alpha_k) = \nu \prod_{k=1}^n (z - 1/\overline{\alpha_k}) = \nu c' z^n \prod_{k=1}^n (\overline{1/\overline{z} - \alpha_k}), \quad c' = \prod_{k=1}^n (-\alpha_k) \in \mathbb{T} \\ &= \frac{\nu}{\overline{\nu}} c' z^n \overline{P\left(1/\overline{z}\right)} = \tau z^n \overline{P}\left(1/z\right) = \tau P^*(z), \quad \tau = c' \frac{\nu}{\overline{\nu}} \in \mathbb{T}. \end{aligned}$$

The notation $P^*(z) = z^n \overline{P(1/\overline{z})}$ is somewhat standard in the OPUC literature to denote the reciprocal of a polynomial P. We have indeed that if $P(z) = \sum_{k=0}^n p_k z^k$ then $P^*(z) = \sum_{k=0}^n \overline{p_{n-k}} z^k$. This explains the following definition.

Definition 2.1 (invariant). *We say that the polynomial $P \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ is invariant if and only if it satisfies $P = \tau P^*$, $\tau \in \mathbb{T}$. The parameter τ is called the invariance parameter and P is said to be τ-invariant.*

Remark 2.2. *Note that if P is a monic invariant polynomial, then its invariance parameter equals $P(0) \in \mathbb{T}$ and if $\nu \in \mathbb{C}\backslash\{0\}$, and P is τ-invariant, then νP is $\tilde{\tau}$-invariant with $\tilde{\tau} = \tau\nu/\overline{\nu}$.*

Thus, since we are interested in the symmetry of the zeros, we should only consider *invariant* polynomials. Szegő polynomials ρ_n have all their zeros in $\mathbb{D}$ and thus they are not invariant. Quasi versions are obtained by imposing certain orthogonality conditions to subspaces of dimension $n-d$. What should this subspace be like for an invariant polynomial $Q_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$?

Let Q_n be an invariant polynomial on $\mathbb{T}$. Then

$$\langle Q_n, z^k\rangle = \tau\langle Q_n^*, z^k\rangle = \tau\langle z^n\overline{Q_n}, z^k\rangle = \tau\langle z^{n-k}, Q_n\rangle = \tau\overline{\langle Q_n, z^{n-k}\rangle}, \quad \tau \in \mathbb{T}.$$

Thus $\langle Q_n, z^{n-k}\rangle = 0 \Leftrightarrow \langle Q_n, z^k\rangle = 0$, for $0 \le k \le n$. An invariant polynomial on $\mathbb{T}$ should always be orthogonal to a subspace that is spanned by powers of z that are centrosymmetric in the sequence $(0, 1, \ldots, n)$, that is, if it includes z^k, it should include z^{n-k} as well. Note that for QOPRL the number of orthogonality conditions is reduced by 1 as the order increases by 1. However the symmetry in paraorthogonality implies that if we remove orthogonality to $z^{n-\ell}$ then we also remove orthogonality to z^ℓ. The number of orthogonality conditions, and thus also the order of QPOPUC, changes in steps of two. So we define a nested sequence of subspaces of dimension $n - 2\ell - 1$ in $\mathbb{P}_n$:

$$\mathbb{Q}_{n,2\ell+1} = \operatorname{span}\{z^k : k = \ell+1, \ldots, n-\ell-1\}, \quad 0 \le \ell \le \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Invariant polynomials, orthogonal to this subspace for $\ell = 0$ were coined *paraorthogonal* as they were introduced in [34], (see also [28, Theorem III]), and they were later studied by many [22, 23, 12, 13, 15, 14, 31, 51, 36, 38]. We shall refer to them as quasi-paraorthogonal polynomials (QPOPUC) of order 1. More generally, we define the set of $(n, 2\ell+1)$-QPOPUC of order $2\ell+1$ as all invariant polynomials of degree n in $\mathbb{Q}_{n,2\ell+1}^{\perp}$. The prefix quasi refers to the fact that they satisfy less paraorthogonality (i.e., symmetric orthogonality) conditions.

We recover a representation of $(n, 2\ell+1)$-QPOPUC in terms of the orthogonal polynomials obtained by Peherstorfer in [39].

Theorem 2.3. *The monic $(n, 2\ell+1)$-QPOPUC are given by*

$$Q_{n,2\ell+1} = zp_\ell\rho_{n-\ell-1} + \tau p_\ell^*\rho_{n-\ell-1}^*, \quad p_\ell \in \mathbb{P}_\ell\backslash\mathbb{P}_{\ell-1} \text{ monic and } Q_{n,2\ell+1}(0) = \tau \in \mathbb{T}.$$

The set of all monic $(n, 2\ell+1)$-QPOPUC $Q_{n,2\ell+1}$ depends on $2\ell+1$ real free parameters.

Proof. We first note that $\mathbb{Q}_{n,2\ell+1}^{\perp}$ has dimension $n+1-(n-2\ell-1)=2\ell+2$. Furthermore, $0=\langle z\rho_{n-\ell-1},z^k\rangle=\langle z^{t+1}\rho_{n-\ell-1},z^{k+t}\rangle=\langle\rho_{n-\ell-1}^*,z^k\rangle=\langle z^t\rho_{n-\ell-1}^*,z^{k+t}\rangle$, for $k=1,\ldots,n-\ell-1$, $t=0,\ldots,\ell$. Thus the $2\ell+2$ polynomials $\{z^{t+1}\rho_{n-\ell-1},z^t\rho_{n-\ell-1}^*\}_{t=0}^{\ell}$ are in $\mathbb{Q}_{n,2\ell+1}^{\perp}$. Moreover they are independent because $zp_\ell(z)\rho_{n-\ell-1}(z)+q_\ell(z)\rho_{n-\ell-1}^*(z)\equiv 0$ is only possible if $p_\ell=q_\ell\equiv 0$. Indeed, if $p_\ell\not\equiv 0$, then $\frac{z\rho_{n-\ell-1}}{\rho_{n-\ell-1}^*}\equiv-\frac{q_\ell}{p_\ell}$, which is impossible because the rational function $\frac{z\rho_{n-\ell-1}}{\rho_{n-\ell-1}^*}$ is irreducible of degree $n-\ell$ since $z\rho_{n-\ell-1}$ and $\rho_{n-\ell-1}^*$ have no common zeros and the rational function $-\frac{q_\ell}{p_\ell}$ has degree at most $\ell<n-\ell$. We get a similar contradiction if $q_\ell\not\equiv 0$ so that $\mathbb{Q}_{n,2\ell+1}^{\perp}=\{zp_\ell\rho_{n-\ell-1}+q_\ell\rho_{n-\ell-1}^*:p_\ell,q_\ell\in\mathbb{P}_\ell\}$.

Let $Q_n=zp_\ell\rho_{n-\ell-1}+q_\ell\rho_{n-\ell-1}^*$ be in $\mathbb{Q}_{n,2\ell+1}^{\perp}$ with $p_\ell\in\mathbb{P}_\ell\backslash\mathbb{P}_{\ell-1}$ monic, then Q_n is monic of degree n. Because the representation with respect to a basis is unique, Q_n is τ-invariant if and only if $q_\ell=\tau p_\ell^*$. Therefore $Q_{n,2\ell+1}=zp_\ell\rho_{n-\ell-1}+\tau p_\ell^*\rho_{n-\ell-1}^*$. The $2\ell+1$ parameters represented by $\tau=Q_{n,2\ell+1}(0)\in\mathbb{T}$ that depends on a real parameter and by the ℓ complex coefficients p_ℓ, that correspond to 2ℓ real parameters. □

The monic polynomial p_ℓ can be written as

$$p_\ell(z)=\prod_{k=1}^{\ell}(z-\eta_k),\quad \eta_k\in\mathbb{C},\quad k=1,\ldots,\ell.$$

So we shall in the rest of the paper often use the ℓ zeros $\{\eta_k\}_{k=1}^{\ell}$ as parameters instead of the ℓ free coefficients of the monic polynomial p_ℓ.

If $\sigma_{n,\ell}=\overline{p_\ell(0)}-\overline{\tau}\delta_{n-\ell}\neq 0$, then $Q_n=Q_{n,2\ell+1}\notin\mathbb{Q}_{n,2\ell-1}^{\perp}$ (see Lemma A.3 of the Appendix) but there will be some 'extra' orthogonality. Because $\langle Q_n,z^\ell\rangle=\tau\langle Q_n^*,z^\ell\rangle=\tau\overline{\langle Q_n,z^{n-\ell}\rangle}=\tau\overline{a_{n-\ell}}$ with $a_{n-\ell}=\langle Q_n,z^{n-\ell}\rangle\neq 0$. Define $\tilde{\tau}=\tau\frac{\sigma_{n,\ell}}{\overline{\sigma_{n,\ell}}}=\tau\frac{\overline{a_{n-\ell}}}{a_{n-\ell}}\in\mathbb{T}$, then clearly $\langle Q_n,z^{n-\ell}-\tilde{\tau}z^\ell\rangle=0$. Thus every $(n,2\ell+1)$-QPOPUC with $\sigma_{n,\ell}\neq 0$ is orthogonal to $\mathbb{Q}_{n,2\ell+1}\oplus\operatorname{span}\{z^{n-\ell}-\tilde{\tau}z^\ell\}$. We shall refer to $\tilde{\tau}$ as the *orthogonality parameter* of $Q_{n,2\ell+1}$.

If $\sigma_{n,\ell}\neq 0$ then the orthogonality parameter can be made explicit in an alternative expression for $Q_{n,2\ell+1}$ as explained in Lemma A.4 of the Appendix. If $\ell=0$, then $p_\ell\equiv 1$ and thus $\sigma_{n,0}$ is always nonzero.

3. Zeros of quasi-paraorthogonal polynomials

In the previous section we have described all monic and invariant quasi-paraorthogonal polynomials. We are now in a position to prove the main result that is an analogue on the unit circle of a very well known important result when dealing with measures supported on the real line that connects the number of orthogonality conditions with the number of zeros of odd multiplicity in the interior of the support of the measure.

It has been shown in the literature (see [44, 35, 37, 7, 3]) that an (n,d)-QOPRL associated with a measure whose support is an interval $[a,b]$ has at least $n-d$ zeros of odd multiplicity in (a,b). For $d=1$ we can use the parameter to fix one zero in (a,b), for $d=3$, we can use two of the three parameters to fix a and b as nodes and the third to fix some $\alpha\in(a,b)$, or possibly to fix the orthogonality parameter $\tilde{\tau}$, while in all these cases

it is guaranteed that there are n simple zeros in the support $[a,b] \subset \mathbb{R}$ of the measure. These QOPRL for $d=1$ or $d=2$ are mainly derived in the context of Gauss-Radau and Gauss-Lobatto quadrature formulas respectively.

In this section we shall obtain similar results for QPOPUC assuming that the measure is supported on an arc $S \subset \mathbb{T}$.

3.1. A general statement

The next Lemma gives a property of invariant polynomials that will be of interest for our purposes.

Lemma 3.1. *Let $P \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ be an invariant polynomial. Then,*

1. Factoring structure on $\mathbb{T}$*: if we denote by $\omega_1, \ldots, \omega_m$ the zeros of P of odd multiplicity on $\mathbb{T}$, then*

$$P(z) = K \cdot z^r \prod_{i=1}^{m} (z - \omega_i) \prod_{k=1}^{r} |z - \eta_k|^2, \quad K \neq 0, \quad z \in \mathbb{T}, \tag{1}$$

   where $m + 2r = n$ and η_k are zeros of P that may possibly coincide with ω_i.
2. *If n is odd, then the number of zeros of odd multiplicity on $\mathbb{T}$ is odd. If n is even, then the number of zeros of odd multiplicity on $\mathbb{T}$ is even (possibly zero).*

Proof. The second property is a direct consequence of the first one. If $\alpha \notin \mathbb{T}$ is a zero of P, then $\frac{1}{\overline{\alpha}} = \frac{\alpha}{|\alpha|^2}$ is also a zero of P. So, the factorization of P will have the factors

$$(z - \alpha)\left(z - \frac{\alpha}{|\alpha|^2}\right) = -\frac{\alpha}{|\alpha|^2} z \, |z - \alpha|^2, \ z \in \mathbb{T}. \tag{2}$$

If α is a zero of P on $\mathbb{T}$ and it has multiplicity at least two, then P will have among its factors

$$(z - \alpha)^2 = -\alpha z |z - \alpha|^2, \ z \in \mathbb{T}. \tag{3}$$

Thus, if m is the number of zeros of odd multiplicity on $\mathbb{T}$, then P will have m simple factors and the other factors are of the form (2)-(3), yielding (1). □

We shall denote an oriented arc as an interval $[a,b] \subset \mathbb{T}$ running counterclockwise from a to b. If $[a,b] \subset \mathbb{T}$ denotes a (closed) arc, then the complementary arc with the same orientation is (b,a). The square or rounded brackets are used as in the case of a real interval to indicate that boundary points are included or not. Moreover, an arc can be indicated by three points like $\mathrm{arc}(a,c,b)$, then this defines its orientation: from a over c to b, and we write $a < c < b$.

Theorem 3.2. *For $n \geq 2$, $0 \leq \ell \leq \lfloor \frac{n}{2} \rfloor - 1$, and $\mathrm{supp}(\mu) = S = [a,b] \subset \mathbb{T}$, an $(n, 2\ell+1)$-QPOPUC $Q_{n,2\ell+1}$ has at least $n-2\ell$ zeros of odd multiplicity on the unit circle and $n-2\ell-1$ of them in $\mathring{S} = (a,b)$.*

Proof. Let us first consider the case $n = 2s + 1$. Since a QPOPUC $Q_{n,2\ell+1}$ is invariant, we know from Lemma 3.1 that $Q_{n,2\ell+1}$ has an odd number of zeros of odd multiplicity on $\mathbb{T}$ and that it can be factored as

$$Q_{n,2\ell+1}(z) = Kz^{s-r}\prod_{j=1}^{t}(z-\omega_j)\prod_{j=1}^{u}(z-\nu_j)\prod_{j=1}^{s-r}|z-\eta_j|^2, \qquad K \neq 0, \qquad z = e^{i\theta} \in \mathbb{T}$$

with $t + u = 2r + 1$ odd. We have distinguished between the η_j with even multiplicity and for those of odd multiplicity we have the zeros ν_j that are in $\mathring{S}$ and the distinct ω_i that are not in $\mathring{S}$, that is $\omega_i \in \mathbb{T}\backslash\mathring{S} = [b, a]$. Without loss of generality we order the ω_i such that $b \leq \omega_1 < \omega_2 < \cdots < \omega_t \leq a$. We define the arcs $S_i = [\omega_{2i-1}, \omega_{2i}]$ with midpoints χ_i and set $\psi_i(z) = \frac{z-\omega_{2i-1}}{z-\omega_{2i}}$ for $i = 1, 2, \ldots, \lfloor \frac{t}{2} \rfloor$.

Our aim is to prove that $u \geq n - 2\ell - 1$ which we do by contraposition, thus suppose that $u \leq n - 2\ell - 2$. We first treat the case where t is even. Thus suppose $t = 2k$, $k \in \mathbb{N}$, then

$$\begin{aligned}\prod_{j=1}^{k}(z-\omega_{2j-1})\prod_{j=1}^{k}(z-\omega_{2j}) &= z^k\prod_{j=1}^{k}(-\omega_{2j})\left[\prod_{j=1}^{k}(z-\omega_{2j-1})\prod_{j=1}^{k}(\overline{z}-\overline{\omega_{2j}})\right] \\ &= z^k\prod_{j=1}^{k}(-\omega_{2j})\left[\prod_{j=1}^{k}\psi_j(z)\prod_{j=1}^{k}|z-\omega_{2j}|^2\right], \quad z \in \mathbb{T}.\end{aligned}$$

Furthermore, if

$$\Theta(z) = K\left(\prod_{j=1}^{k}\frac{-\omega_{2j}}{\overline{\psi_j(\chi_j)}}\right)z^{s-r+k}\prod_{j=1}^{u}(z-\nu_j),$$

then

$$Q_{n,2\ell+1}(z)\overline{\Theta(z)} = |K|^2\prod_{j=1}^{s-r}|z-\eta_j|^2\prod_{j=1}^{u}|z-\nu_j|^2\prod_{j=1}^{k}|z-\omega_{2j}|^2\prod_{j=1}^{k}\frac{\psi_j(z)}{\psi_j(\chi_j)}, \quad z \in \mathbb{T}.$$

By Lemma A.1,

$$z \in \mathring{S} \Rightarrow \frac{\psi_j(z)}{\psi_j(\chi_j)} < 0, \quad j = 1, \ldots, k,$$

so that $Q_{n,2\ell+1}(z)\overline{\Theta(z)}$, has a constant sign in $\mathring{S}$.

But $\langle Q_{n,2\ell+1}, \Theta\rangle = 0$ because $\Theta \in \mathbb{Q}_{n,2\ell+1}$. Indeed, since $\Theta \in \operatorname{span}\left\{z^{s-r+k}, \ldots, z^{s-r+k+u}\right\}$, it only remains to prove that $s - r + k \geq \ell + 1$ and that $s - r + k + u \leq n - \ell - 1$. The assumptions are $n = 2s + 1$, $t + u = 2r + 1$, $u \leq n - 2\ell - 2$ and $t = 2k$ (notice that u is odd). So,

$$s - r + k = s - r + \frac{2r + 1 - u}{2} = s + \frac{1}{2} - \frac{u}{2} \geq \ell + 1$$

and

$$s - r + k + u = s + \frac{1}{2} + \frac{u}{2} \leq s + \frac{1}{2} + \frac{n - 2\ell - 2}{2} = n - \ell - 1.$$

This proves that $Q_{n,2\ell+1}$ has at least $n-2\ell-1=2(s-\ell)$ zeros of odd multiplicity in $\mathring{S}=(a,b)$, and since n is odd, $Q_{n,2\ell+1}$ has $n-2\ell$ zeros on the unit circle.

Now, we consider $t=2k+1$, then u is even and $u\leq 2(s-\ell)-2$. We consider

$$\Theta(z)=K\left(\prod_{j=1}^{k}\frac{-\omega_{2j}}{\overline{\psi_j(\chi_j)}}\right)z^{s-r+k}(z-\omega_{2k+1})\prod_{j=1}^{u}(z-\nu_j),$$

then

$$Q_{n,2\ell+1}(z)\overline{\Theta(z)}=|K|^2\prod_{j=1}^{s-r}|z-\eta_j|^2\prod_{j=1}^{u}|z-\nu_j|^2\prod_{j=1}^{k}|z-w_{2j}|^2|z-\omega_{2k+1}|^2\prod_{j=1}^{k}\frac{\psi_j(z)}{\psi_j(\chi_j)},\quad z\in\mathbb{T}.$$

By the same argument, $Q_{n,2\ell+1}(z)\overline{\Theta(z)}$ has a constant sign in $\mathring{S}$, which is a contradiction since $\Theta\in\mathbb{Q}_{n,2\ell+1}$ and thus $\langle Q_{n,2\ell+1},\Theta\rangle=0$. Indeed, $\Theta\in\operatorname{span}\left\{z^{s-r+k},\ldots,z^{s-r+k+u+1}\right\}$ and now, $2k+1+u=2r+1$ (u is even) and we have supposed that $u\leq 2(s-\ell)-1$, hence $u\leq 2(s-\ell-1)$. So, $s-r+k=s-\frac{u}{2}\geq\ell+1$ and $s-r+k+u+1=s+1+\frac{u}{2}\leq s+1+(s-\ell-1)=n-\ell-1$. Again in this case $u\geq 2(s-\ell)$. This proves that $Q_{n,2\ell+1}$ has at least $n-2\ell-1=2(s-\ell)$ zeros of odd multiplicity in $\mathring{S}=(a,b)$, and since n is odd, $Q_{n,2\ell+1}$ has $n-2\ell$ zeros on the unit circle.

When n is even the proof is similar. □

3.2. $Q_{n,1}$ and one prescribed zero

If we can employ the invariance parameter τ (i.e. $Q_{n,1}(0)$ for a monic $Q_{n,1}$) to place an extra zero of odd multiplicity on $\mathring{S}$, then this would imply that the n simple zeros of $Q_{n,1}$ are in $\mathring{S}$. We shall study the situation using Blaschke products. Since the zeros $z_{k,n}$ of the Szegő polynomial ρ_n lie in $\mathbb{D}$, we can define the Blaschke products of degree n (with zeros in $\mathbb{D}$) as follows:

$$F_n(z)=z\frac{\rho_{n-1}(z)}{\rho_{n-1}^*(z)}=\prod_{s=0}^{n-1}\varphi_{z_{s,n}}(z)\ \text{ where }\ z_{0,n}=0\ \text{ and }\ \rho_k(z)=\prod_{l=1}^{k}(z-z_{l,k}),\quad k\in\mathbb{N}.\tag{4}$$

By the Argument Principle, $F_n(z)$ goes around the unit circle exactly n times when z wraps around the origin (a n-to-1 map), and preserves the orientation. If in addition the support of the measure is an arc S, as it is in our case, the following result is a consequence of the previous theorem.

Corollary 3.3. *Let μ be a positive measure supported on an arc $S\subset\mathbb{T}$ with associated orthogonal polynomials $\{\rho_k\}_{k\in\mathbb{N}}$ and let F_n be the Blaschke product as defined above in (4). Then for given $t\in\mathbb{T}$, $F_n(z)=t$ will have at least $n-1$ solutions in $\mathring{S}$.*

Theorem 3.4. *Let μ be a positive measure supported on $S=[a,b]\subsetneq\mathbb{T}$, and $n\geq 2$. Consider the QPOPUC $Q_{n,1}(z)=z\rho_{n-1}(z)+\tau\rho_{n-1}^*(z)$ and define $\tau_a=F_n(a)$ and $\tau_b=F_n(b)$ with F_n as in (4). Then $Q_{n,1}$ has all its simple zeros in $\mathring{S}$ (or in S) if and only if $-\tau\in(\tau_a,\tau_b)$ (or $-\tau\in[\tau_a,\tau_b]$ respectively). Moreover $\alpha\in\mathbb{T}$ is one of these zeros if and only if $-\tau=F_n(\alpha)$.*

Proof. The zeros $\{z_i\}_{i=1}^n$ of $Q_{n,1}$ are the roots of the equation $F_n(z) = -\tau$. We denote the (ordered) roots of $F_n(z) = \tau_a$ by $\{\alpha_i\}_{i=1}^n$ and the (ordered) roots of $F_n(z) = \tau_b$, by $\{\beta_i\}_{i=1}^n$, with $\alpha_1 = a$ and $\beta_n = b$. By the properties of F_n, $z_i \in (\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n$. □

Remark 3.5. *Note also that the only prefixed zero $\alpha \in \mathring{S}$ that is allowed must be in one of the n intervals (α_i, β_i), $i = 1, \ldots, n$. Compare with [7, Theorem 2.9] for QOPRL.*

3.3. $Q_{n,3}$ and 2 or 3 prescribed zeros

A monic QPOPUC

$$Q_{n,3}(z) = z(z-\eta)\rho_{n-2}(z) + \tau(1-\overline{\eta}z)\rho^*_{n-2}(z) \tag{5}$$

of order 3, has at least $n-2$ zeros of odd multiplicity on $\mathbb{T}$, $n-3$ of them are in the interior of the arc where the measure is supported, and it depends on three real parameters that can be used to fix up to three zeros, ensuring that all the zeros of $Q_{n,3}$ are simple and placed on the support of the measure. Let us start by fixing 2 nodes.

If $\eta \in \mathbb{D}$, then $B_n(z) = \frac{z(z-\eta)\rho_{n-2}(z)}{(1-\overline{\eta}z)\rho^*_{n-2}(z)}$ is a Blaschke product of degree n and $B_n(z) = -\tau \in \mathbb{T}$ has n simple solutions on $\mathbb{T}$. The conditions $Q_{n,3}(\alpha_1) = Q_{n,3}(\alpha_2) = 0$ lead to

$$\alpha_i(\alpha_i-\eta)\rho_{n-2}(\alpha_i) + \tau(1-\overline{\eta}\alpha_i)\rho^*_{n-2}(\alpha_i) = 0, \quad i = 1, 2, \tag{6}$$

or equivalently,

$$(\alpha_i - \eta) + \tau f_i(1-\overline{\eta}\alpha_i) = 0, \quad f_i = \overline{F_{n-1}(\alpha_i)}, \quad i = 1, 2. \tag{7}$$

This gives the system

$$\begin{bmatrix} 1 & \tau f_1\alpha_1 \\ 1 & \tau f_2\alpha_2 \end{bmatrix} \begin{bmatrix} \eta \\ \overline{\eta} \end{bmatrix} = \begin{bmatrix} \tau f_1 + \alpha_1 \\ \tau f_2 + \alpha_2 \end{bmatrix}. \tag{8}$$

If $f_1\alpha_1 \neq f_2\alpha_2$, this represents two secants for the circle $\mathbb{T}$

$$\eta + \tau f_1\alpha_1\overline{\eta} = \tau f_1 + \alpha_1 \quad \text{and} \quad \eta + \tau f_2\alpha_2\overline{\eta} = \tau f_2 + \alpha_2, \quad \eta \in \mathbb{C}. \tag{9}$$

The first secant passes through α_1 and τf_1 and the second passes through α_2 and τf_2. If $f_1 = f_2$, then $\eta = \tau f_1 \in \mathbb{T}$, and this should be excluded because η must be in $\mathbb{D}$. These secants intersect in exactly one finite point if and only if $\alpha_1 f_1 \neq \alpha_2 f_2$ and the intersection point is

$$\eta = c_{12} + \tau a_{12}, \quad \overline{c_{12}} = \frac{f_1 - f_2}{f_1\alpha_1 - f_2\alpha_2}, \quad \overline{a_{12}} = \frac{\alpha_1 - \alpha_2}{f_1\alpha_1 - f_2\alpha_2}. \tag{10}$$

By Lemma A.2 of the Appendix, $\eta \in \mathbb{D}$ if and only if the arcs $(\alpha_1, \tau f_1, \alpha_2)$ and $(\alpha_1, \tau f_1, \tau f_2)$ have opposite orientation. (See also Figure 1.) Note that we can only define $(\alpha_1, \tau f_1, \alpha_2)$ and $(\alpha_1, \tau f_1, \tau f_2)$ as arcs if $\tau f_1 \neq \alpha_2$ and $\tau f_2 \neq \alpha_1$ which is implicitly assumed also in the next Theorem.

Theorem 3.6. *Let $n \geq 3$. Given $\tau \in \mathbb{T}$, and distinct $\{\alpha_i\}_{i=1}^2 \subset \mathbb{T}$. Define $\{f_i\}_{i=1}^2 \subset \mathbb{T}$ as in (7). Assume the f_i are distinct and $\alpha_1 f_1 \neq \alpha_2 f_2$. Then the monic $(n,3)$-QPOPUC $Q_{n,3}$ of (5) is uniquely defined and has all its zeros simple and on $\mathbb{T}$, α_1 and α_2 being two of them if τ is such that the arcs $(\alpha_1, \tau f_1, \alpha_2)$ and $(\alpha_1, \tau f_1, \tau f_2)$ have opposite orientation. The parameter η in (5) belongs to $\mathbb{D}$ and depends on τ as given by (10).*

Remark 3.7. *Note that there are infinitely many τ that allow us to fix α_1 and α_2 ensuring that $Q_{n,3}$ has all its zeros simple on $\mathbb{T}$.*

A further geometric interpretation can be obtained as follows (see also [33]). Because $\tau \in \mathbb{T}$, the identity (10) says that $\eta \in \mathbb{T}_{12}$ which is the circle with centre c_{12} and with radius $r_{12} = |a_{12}|$, i.e. (see also Figure 1)

$$\mathbb{T}_{12} = \{\eta(t) = c_{12} + t \cdot r_{12} : t \in \mathbb{T}\}.$$

The η from (10) corresponds to $t = \tau t_0$ where $t_0 = a_{12}/|a_{12}|$. For $t_1 = \alpha_1 \overline{f_2} t_0$ we get $\eta(t_1) = \alpha_1$ and for $t_2 = \alpha_2 \overline{f_1} t_0$ we get $\eta(t_2) = \alpha_2$. Thus $\mathbb{T} \cap \mathbb{T}_{12} = \{\alpha_1, \alpha_2\}$. Since there are two different points of intersection, this means that there are values for $t \in \mathbb{T}$ that will deliver $\eta(t) \in \mathbb{D}$ and others for which $\eta(t) \in \mathbb{E}$. It is clear that the line connecting the origin with c_{12}, i.e., the mediatrix of the chord $[\alpha_1, \alpha_2]$, will intersect $\mathbb{T}$ at the points $\pm\sqrt{\alpha_1 \alpha_2}$ and the circle $\mathbb{T}_{12}$ at two points: one in $\mathbb{D}$ that is closest to the origin and one in $\mathbb{E}$ that is the farthest from the origin. The former is given by $\eta_e = c_{12}(1 - \frac{r_{12}}{|c_{12}|})$, which is $\eta(t_3)$ with $t_3 = -\frac{c_{12}}{|c_{12}|}$. Thus $\eta(t) \in \mathbb{T}_{12} \cap \mathbb{D}$ if and only if t is in the interior of the $\text{arc}(t_1, t_3, t_2) \subset \mathbb{T}$ for which $\eta(t)$ describes the $\text{arc}(\eta(t_1), \eta(t_3), \eta(t_2)) = \text{arc}(\alpha_1, \eta_e, \alpha_2) = \mathbb{T}_{12} \cap \mathbb{D}$. Taking into account the value of t_0, $t \in \text{arc}(t_1, t_3, t_2)$ if and only if $\tau \in \text{arc}\left(\alpha_1 \overline{f_2}, -\frac{\overline{f_1} - \overline{f_2}}{|f_1 - f_2|} \frac{\alpha_1 - \alpha_2}{|\alpha_1 - \alpha_2|}, \alpha_2 \overline{f_1}\right)$.

All this assumes $f_1 \alpha_1 \neq f_2 \alpha_2$, i.e., the two secants have a single finite intersection point. If $f_1 \alpha_1 = f_2 \alpha_2$, then the secants are parallel or coincide.

1. $(\alpha_1 - \alpha_2) + \tau(f_1 - f_2) \neq 0$, then the two equations of (9) are contradictory. The two secants are parallel and intersect at $\eta = \infty$.
2. $(\alpha_1 - \alpha_2) + \tau(f_1 - f_2) = 0$, then the two equations of the system (8) are identical and reduce to $\eta + \overline{\eta} \alpha_1 \alpha_2 = \alpha_1 + \alpha_2$, the secant through $\alpha_1 = \tau f_2$ and $\alpha_2 = \tau f_1$ (See Figure 1). In this case there are infinitely many solutions for η given by $\eta = \alpha_1 + t(\alpha_2 - \alpha_1)$ with $\eta \in \mathbb{D}$ if $0 < t < 1$. This line is in fact a degenerate case of the circle $\mathbb{T}_{12}$ with an infinite radius and a centre at infinity. The mediatrix of the chord $[\alpha_1, \alpha_2]$ intersects $\mathbb{T}_{12}$ at the centre $\eta_e = (\alpha_1 + \alpha_2)/2$ and at ∞. Since η has to be in $\mathbb{D}$, it should be on the degenerate $\text{arc}(\alpha_1, \eta_e, \alpha_2) = \mathbb{T}_{12} \cap \mathbb{D}$ which is the open interval (α_1, α_2).

Remark 3.8. *Note that $Q_{n,3}$ has 3 real parameters incorporated by $\eta \in \mathbb{C}$ and $\tau \in \mathbb{T}$. In the generic case ($\alpha_1 f_1 \neq \alpha_2 f_2$) we can, depending on a chosen τ, fix the zeros α_1 and α_2 by selecting an appropriate η. However, to guarantee that all n zeros are simple and on $\mathbb{T}$, we had to restrict τ to a subarc of $\mathbb{T}$. In the degenerate case (2) above, α_1 and α_2 are chosen such that $\alpha_1 \overline{f_2} = \alpha_2 \overline{f_1} = \tau$, which means that now τ is fixed, but it leaves η still free*

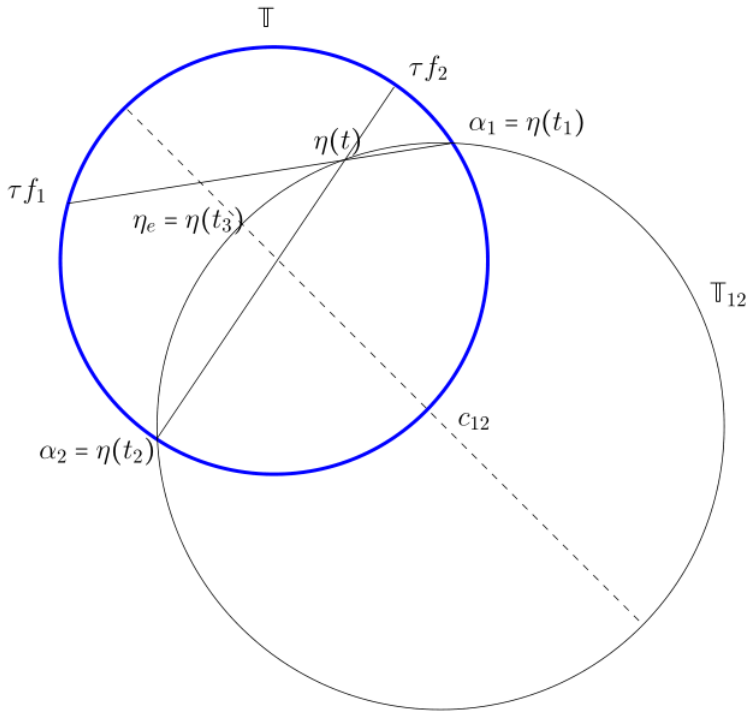

Figure 1: The unit circle $\mathbb{T}$ and the circle $\mathbb{T}_{12}$ with $\mathbb{T} \cap \mathbb{T}_{12} = \{\alpha_1, \alpha_2\}$ and the two secant lines through $(\alpha_1, \tau f_1)$ and $(\alpha_2, \tau f_2)$ intersecting at $\eta(\tau)$.

to vary on the secant line through α_1 and α_2, which corresponds to one real parameter t. To guarantee also in this case that all n zeros are simple and on $\mathbb{T}$, we should restrict η to the subinterval (α_1, α_2), i.e. $0 < t < 1$. The latter can be seen as an arc of the degenerate circle $\mathbb{T}_{12}$ through α_1 and α_2 and centre at ∞.

With this analysis we can turn the existence Theorem 3.6 into an explicit characterization of τ as follows.

Theorem 3.9. *Let $n \geq 3$. Given $\tau \in \mathbb{T}$, and $\{(\alpha_i, f_i)\}_{i=1}^2 \subset \mathbb{T} \times \mathbb{T}$, satisfying the conditions of Theorem 3.6. Then the monic $(n,3)$-QPOPUC $Q_{n,3}$ as in (5) with n simple zeros on $\mathbb{T}$, α_1 and α_2 being two of them, is uniquely defined if $\tau \in \operatorname{arc}\left(\alpha_1\overline{f_2}, -\frac{\overline{f_1}-\overline{f_2}}{|f_1-f_2|}\frac{\alpha_1-\alpha_2}{|\alpha_1-\alpha_2|}, \alpha_2\overline{f_1}\right)$. The parameter η of (5) belongs to $\mathbb{D}$ and depends on τ as given by (10).*

The previous argument shows that if the two prefixed zeros allow a solution that guarantees n simple zeros on $\mathbb{T}$ then we should choose free parameter (τ or t) such that η is on some subarc in $\mathbb{D}$. This remaining freedom will allow us to fix a third zero under certain conditions as we shall currently show.

The conditions $Q_{n,3}(\alpha_1) = Q_{n,3}(\alpha_2) = Q_{n,3}(\alpha_3) = 0$ lead to

$$\alpha_i(\alpha_i - \eta)\rho_{n-2}(\alpha_i) + \tau(1 - \overline{\eta}\alpha_i)\rho^*_{n-2}(\alpha_i) = 0, \quad i = 1, 2, 3$$

or equivalently,

$$\overline{\tau}(\alpha_i - \eta) + f_i(1 - \alpha_i\overline{\eta}) = 0, \quad f_i = \overline{F_{n-1}(\alpha_i)}, \qquad i = 1, 2, 3,$$

where $\eta \in \mathbb{C}$ and $\tau \in \mathbb{T}$. If $1 - \alpha_i\overline{\eta} \neq 0$, $i = 1, 2, 3$, we can reformulate the system as

$$\overline{\tau}\varphi_{-\eta}(\alpha_i) = -f_i, \quad i = 1, 2, 3. \tag{11}$$

Then, $Q_{n,3}$ has n simple zeros if the solution of the above interpolation problem is an automorphism of the unit disk. It is well known that this automorphism exists and is unique if and only if $\operatorname{arc}(\alpha_1, \alpha_2, \alpha_3)$ and $\operatorname{arc}(f_1, f_2, f_3)$ have the same orientation, [42, p. 397], see also [41, p. 280 and p. 296, exercise 30], and that it is a particular case of a more general situation: see [5, Theorem 4.1 and Example 5.1].

Theorem 3.10. *Let $n \geq 3$. Given distinct $\{\alpha_i\}_{i=1}^3 \subset \mathbb{T}$, and assume $f_i = \overline{F_{n-1}(\alpha_i)}$, $i = 1, 2, 3$ with F_{n-1} as in (4), are also distinct. Then there exists a unique monic $(n, 3)$-QPOPUC $Q_{n,3}$ as in (5), having all its zeros simple on $\mathbb{T}$, with α_1, α_2 and α_3 being three of them if $\operatorname{arc}(\alpha_1, \alpha_2, \alpha_3)$ and $\operatorname{arc}(f_1, f_2, f_3)$ have the same orientation. The required η of (5) belongs to $\mathbb{D}$.*

If the support of μ is an arc $S = [a, b] \subsetneq \mathbb{T}$, then Theorem 3.2 guarantees that $Q_{n,3}$ has $n - 2$ zeros of odd multiplicity on $\mathbb{T}$ of which only $n - 3$ are in the interior of the support. We can however prove that *all* the zeros are in the support if we choose $\alpha_1 = a$ and $\alpha_2 = b$ in the following way.

The interpolation conditions (7) become for $\eta \notin \{a, b\}$,

$$\varphi_{-\eta}(a) = -\tau f_a \ \text{ and } \ \varphi_{-\eta}(b) = -\tau f_b \ \text{ where } f_a = \overline{F_{n-1}(a)} \text{ and } f_b = \overline{F_{n-1}(b)}. \tag{12}$$

We know this has a unique solution η for each $\tau \in \mathbb{T}$ if and only if $f_a \neq f_b$ and $af_a \neq bf_b$ and the latter will always be true in this case as a consequence of Corollary 3.3. For this η, $Q_{n,3}$ has zeros at a and b, and thus $Q_{n,3}(z) = (z-a)(z-b)P_{n-2}(z)$ with $P_{n-2} \in \mathbb{P}_{n-2}\backslash\mathbb{P}_{n-3}$ monic. These boundary zeros a and b are in addition to the $(n-3)$ zeros that $Q_{n,3}$ has in (a, b) according to Theorem 3.2. As $Q_{n,3}$ is τ-invariant and orthogonal to $\mathbb{Q}_{n,3}$, P_{n-2} is $\hat{\tau}$-invariant with $\hat{\tau} = \overline{ab}\tau$ and orthogonal to $\mathbb{Q}_{n-2,1}$ with respect to

$$d\hat{\mu}(z) = \sqrt{\overline{ab}}(z - a)(z - b)\overline{z}d\mu(z), \quad z \in S. \tag{13}$$

We assume the square root is chosen such that this measure $\hat{\mu}$ is positive on $\mathring{S}$, which is possible because with $z = e^{i\theta}$, $\sqrt{a} = e^{i\theta_a/2}$, and $\sqrt{b} = e^{i\theta_b/2}$,

$$\sqrt{\overline{ab}}\, e^{-i\theta}(e^{i\theta} - e^{i\theta_a})(e^{i\theta} - e^{i\theta_b}) = 4\sin\Big(\frac{\theta - \theta_a}{2}\Big)\sin\Big(\frac{\theta_b - \theta}{2}\Big) > 0, \quad \forall\theta \in (\theta_a, \theta_b).$$

Thus, P_{n-2} is a monic $\hat{\tau}$-invariant, $(n-2, 1)$-QPOPUC for the positive measure $\hat{\mu}$. Let $\{\hat{\rho}_n\}_{n\in\mathbb{N}}$ be the sequence of monic orthogonal polynomials associated with $\hat{\mu}$, then $P_{n-2} = z\hat{\rho}_{n-3} + \hat{\tau}\hat{\rho}^*_{n-3}$.

Applying Theorem 3.4 thus gives us the following result.

Theorem 3.11. *Let μ be a measure supported on an arc $S = [a, b] \subsetneq \mathbb{T}$ and define the associated positive measure $\hat{\mu}$ as in (13) with monic orthogonal polynomial sequence $\{\hat{\rho}_n\}_{n\in\mathbb{N}}$. For $k \in \mathbb{N}$, let F_k be defined as in (4) and $\hat{F}_k(z) = z\hat{\rho}_{k-1}(z)/\hat{\rho}^*_{k-1}(z)$. Assume $n \geq 3$ and define $f_a = \overline{F_{n-1}(a)}$, $f_b = \overline{F_{n-1}(b)}$ $\hat{\tau}_a = \hat{F}_{n-2}(a)$, $\hat{\tau}_b = \hat{F}_{n-2}(b)$, and $\hat{\tau} = \overline{ab}\tau$. Furthermore set $\eta = c_{12} + ab\hat{\tau}a_{12}$, where $\overline{c_{12}} = \frac{f_a - f_b}{af_a - bf_b}$, $\overline{a_{12}} = \frac{a-b}{af_a - bf_b}$ and $\hat{\tau} \in \mathbb{T}$. Finally, let $Q_{n,3}$ as in (5) be a monic $(n, 3)$-QPOPUC. Then*

1. *$Q_{n,3}$ has all its simple zeros on S where a and b are two of them if and only if $-\hat{\tau} \in (\hat{\tau}_a, \hat{\tau}_b)$.*
2. *$Q_{n,3}$ has all its simple zeros in S where a, b and $\alpha \in (a, b)$ are three of them if and only if $-\hat{\tau} = \hat{F}_{n-2}(\alpha) \in (\hat{\tau}_a, \hat{\tau}_b)$.*

3.4. $Q_{n,2\ell+1}$ with 2ℓ or $2\ell + 1$ prescribed zeros

For the more general case where $\ell > 1$, the analysis becomes very hard when the measure is supported on a subarc of $\mathbb{T}$. So for the rest of this section we shall assume that the measure is supported on the whole circle $\mathbb{T}$.

Let us first fix 2ℓ zeros for the $(n, 2\ell + 1)$-QPOPUC

$$Q_{n,2\ell+1}(z) = zP_\ell(z)\rho_{n-\ell-1}(z) + \tau P^*_\ell(z)\rho^*_{n-\ell-1}(z), \quad P_\ell(z) = \prod_{k=1}^{\ell}(z - \eta_k), \quad \eta_k \in \mathbb{C}. \tag{14}$$

From Theorem 3.2 we know it has at least $(n - 2\ell)$ zeros of odd multiplicity on $\mathbb{T}$. It remains to verify which conditions on $\tau \in \mathbb{T}$ and $\{\eta_i\}_{i=1}^{\ell}$ must be satisfied to allow us to fix 2ℓ nodes, while ensuring that all the zeros of $Q_{n,2\ell+1}$ are simple and on $\mathbb{T}$.

If $\eta_i \in \mathbb{D}$, $i = 1, \ldots, \ell$, then $B_n(z) = \frac{zP_\ell(z)\rho_{n-\ell-1}(z)}{P^*_\ell(z)\rho^*_{n-\ell-1}(z)}$ is a Blaschke product of degree n with all zeros in $\mathbb{D}$ and thus $B_n(z) = \tau \in \mathbb{T}$ has n simple solutions on $\mathbb{T}$. We already know that the conditions $Q_{n,2\ell+1}(\alpha_j) = 0$, $j = 1, \ldots, 2\ell$, lead to

$$\alpha_i P_\ell(\alpha_i)\rho_{n-\ell-1}(\alpha_i) + \tau P^*_\ell(\alpha_i)\rho^*_{n-\ell-1}(\alpha_i) = 0, \quad i = 1, \ldots, 2\ell,$$

or equivalently, with $f_i = \overline{F_{n-\ell}(\alpha_i)}$, $i = 1, \ldots, 2\ell$, ($F_{n-\ell}$ as in (4)):

$$P_\ell(\alpha_i) + \tau f_i P^*_\ell(\alpha_i) = 0, \quad i = 1, \ldots, 2\ell. \tag{15}$$

Obviously, a possible solution will depend on the selection of the nodes $\{\alpha_i\}$ and how the corresponding $\{f_i\}$ behave. This problem has a matrix interpretation, that extends the case $\ell = 1$.

Let us assume that there exists a unique solution P_ℓ for the system (15) and that it is given by $P_\ell(z) = p_0 + p_1 z + \cdots + p_{\ell-1}z^{\ell-1} + z^\ell$. Set $\mathcal{V}_1 = [\alpha_i^j]_{i=1,\ldots,\ell}^{j=0,\ldots,\ell-1}$ and $\mathcal{V}_2 = [\alpha_i^j]_{i=\ell+1,\ldots,2\ell}^{j=0,\ldots,\ell-1}$ the Vandermonde matrices associated with $\{\alpha_i\}_{i=1}^{\ell}$ and $\{\alpha_j\}_{j=\ell+1}^{2\ell}$, respectively, and furthermore set

$$\begin{array}{ll}
\mathbf{f}_1 = [f_1, f_2, \ldots, f_\ell]^T, & \mathcal{F}_1 = \operatorname{diag}(\mathbf{f}_1), \\
\mathbf{f}_2 = [f_{\ell+1}, f_{\ell+2}, \ldots, f_{2\ell}]^T, & \mathcal{F}_2 = \operatorname{diag}(\mathbf{f}_2), \\
\mathbf{d}_1 = [\alpha_1^\ell, \alpha_2^\ell, \ldots, \alpha_\ell^\ell]^T, & \mathcal{D}_1 = \operatorname{diag}(\mathbf{d}_1), \\
\mathbf{d}_2 = [\alpha_{\ell+1}^\ell, \alpha_{\ell+2}^\ell, \ldots, \alpha_{2\ell}^\ell]^T, & \mathcal{D}_2 = \operatorname{diag}(\mathbf{d}_2),
\end{array}$$

then, from (15)

$$p_0 + p_1\alpha_i + \cdots + p_{\ell-1}\alpha_i^{\ell-1} + \tau f_i \alpha_i^{\ell} \left(\overline{p}_{\ell-1}\overline{\alpha_i^{\ell-1}} + \cdots + \overline{p}_0 \right) = -\tau f_i - \alpha_i^{\ell}, \quad i = 1, \ldots, 2\ell.$$

Setting $\mathbf{p} = [p_0,\ p_1,\ \cdots,\ p_{\ell-1}]^T$, these equations are equivalent to

$$\mathcal{V}_k \mathbf{p} + \tau \mathcal{F}_k \mathcal{D}_k \overline{\mathcal{V}_k} \overline{\mathbf{p}} = -\tau \mathbf{f}_k - \mathbf{d}_k, \quad k = 1, 2,$$

or in matrix form, the vector $[\mathbf{p}^T, \overline{\mathbf{p}}^T]^T$ is the unique solution of

$$\begin{bmatrix} \mathcal{V}_1 & \tau \mathcal{F}_1 \mathcal{D}_1 \overline{\mathcal{V}_1} \\ \mathcal{V}_2 & \tau \mathcal{F}_2 \mathcal{D}_2 \overline{\mathcal{V}_2} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \overline{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} -\tau \mathbf{f}_1 - \mathbf{d}_1 \\ -\tau \mathbf{f}_2 - \mathbf{d}_2 \end{bmatrix}. \tag{16}$$

The uniqueness of the solution implies that the coefficient matrix is nonsingular.

Theorem 3.12. *Let $n \geq 2\ell + 1$. Given $\tau \in \mathbb{T}$ and distinct $\{\alpha_i\}_{i=1}^{2\ell} \subset \mathbb{T}$, and assume $f_i = \overline{F_{n-\ell}(\alpha_i)}$, $i = 1, \ldots, 2\ell$ with $F_{n-\ell}$ defined in (4) are distinct and satisfy*

$$\prod_{1 \leq i < j \leq \ell} \frac{(\alpha_{\ell+i} - \alpha_{\ell+j})\,(\overline{\alpha_i} - \overline{\alpha_j})}{(\overline{\alpha_{\ell+i}} - \overline{\alpha_{\ell+j}})\,(\alpha_i - \alpha_j)} \neq \prod_{i=1}^{\ell} \frac{f_{\ell+i}\alpha_{\ell+i}^{\ell}}{f_i \alpha_i^{\ell}}. \tag{17}$$

Then the monic τ-invariant $(n, 2\ell{+}1)$-QPOPUC $Q_{n,2\ell+1}(z) = zP_\ell(z)\rho_{n-\ell-1}(z) + \tau P_\ell^*(z)\rho_{n-\ell-1}^*(z)$ with n simple zeros on $\mathbb{T}$, and $\alpha_1, \alpha_2, \ldots, \alpha_{2\ell}$ being 2ℓ of them is uniquely defined if all $s_k(0) \in \mathbb{D}$, $k = 1, \ldots, \ell$, where

$$s_\ell(z) = \frac{P_\ell(z)}{P_\ell^*(z)} \quad \textit{and} \quad s_{k-1}(z) = \frac{1}{z} \frac{s_k(z) - s_k(0)}{1 - \overline{s_k(0)} s_k(z)}, \quad k = \ell, \ell - 1, \ldots, 2, \tag{18}$$

and $P_\ell(z) = p_0 + p_1 z + \cdots + p_{\ell-1} z^{\ell-1} + z^\ell$ where $\mathbf{p} = [p_0\ p_1\ \cdots\ p_{\ell-1}]^T$ is the unique solution of the system (16). The vector $\mathbf{p}$ is given by

$$\mathbf{p} = -\left[\left(\overline{\mathcal{V}_1}\right)^{-1} \overline{\mathcal{D}_1 \mathcal{F}_1} \mathcal{V}_1 - \left(\overline{\mathcal{V}_2}\right)^{-1} \overline{\mathcal{D}_2 \mathcal{F}_2} \mathcal{V}_2 \right]^{-1} \cdot \left[\left(\overline{\mathcal{V}_1}\right)^{-1} \left(\tau \overline{\mathbf{d}_1} + \overline{\mathbf{f}_1}\right) - \left(\overline{\mathcal{V}_2}\right)^{-1} \left(\tau \overline{\mathbf{d}_2} + \overline{\mathbf{f}_2}\right) \right], \tag{19}$$

where we use the notation that was introduced above.

Proof. We consider the following auxiliary problem. Let $P_\ell(z) = p_0 + p_1 z + \cdots + p_{\ell-1} z^{\ell-1} + z^\ell$ and $Q_\ell(z) = 1 + q_1 z + \cdots + q_{\ell-1} z^{\ell-1} + q_\ell z^\ell$ verify

$$p_0 + p_1\alpha_i + \cdots + p_{\ell-1}\alpha_i^{\ell-1} + \tau f_i \alpha_i^{\ell} \left(q_\ell + q_{\ell-1}\overline{\alpha_i} + \cdots + q_1 \overline{\alpha_i}^{\ell-1} \right) = -\tau f_i - \alpha_i^{\ell}, \quad i = 1, \ldots, 2\ell.$$

Setting $\mathbf{p} = [p_0,\ p_1,\ \cdots,\ p_{\ell-1}]^T$ and $\mathbf{q} = [q_1,\ q_2,\ \cdots,\ q_\ell]^T$, these are equivalent to

$$\mathcal{V}_k \mathbf{p} + \tau \mathcal{F}_k \mathcal{D}_k \overline{\mathcal{V}_k} \mathcal{S}_\ell \mathbf{q} = -\tau \mathbf{f}_k - \mathbf{d}_k, \quad k = 1, 2,$$

where $\mathcal{S}_\ell = [\delta_{i,\ell+1-j}]_{i,j=1}^{\ell}$ is the $\ell \times \ell$ flipping matrix which maps a vector upside down. Thus

$$\begin{bmatrix} \mathcal{V}_1 & \tau\mathcal{F}_1\mathcal{D}_1\overline{\mathcal{V}_1} \\ \mathcal{V}_2 & \tau\mathcal{F}_2\mathcal{D}_2\overline{\mathcal{V}_2} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathcal{S}_\ell\mathbf{q} \end{bmatrix} = \begin{bmatrix} -\tau\mathbf{f}_1 - \mathbf{d}_1 \\ -\tau\mathbf{f}_2 - \mathbf{d}_2 \end{bmatrix}. \tag{20}$$

The condition (17) ensures that the associated matrix is not singular, hence the system has a unique solution.

That $P_\ell = Q_\ell^*$ can be seen as follows. Multiply by $\overline{\tau}\mathrm{diag}(\overline{\mathcal{F}_1\mathcal{D}_1}, \overline{\mathcal{F}_2\mathcal{D}_2})$ from the left in (20) to obtain

$$\begin{bmatrix} \overline{\tau\mathcal{F}_1\mathcal{D}_1}\mathcal{V}_1 & \overline{\mathcal{V}_1} \\ \overline{\tau\mathcal{F}_2\mathcal{D}_2}\mathcal{V}_2 & \overline{\mathcal{V}_2} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathcal{S}_\ell\mathbf{q} \end{bmatrix} = \begin{bmatrix} -\overline{\tau\mathbf{d}_1} - \overline{\mathbf{f}_1} \\ -\overline{\tau\mathbf{d}_2} - \overline{\mathbf{f}_2} \end{bmatrix}.$$

By taking the complex conjugate and interchanging the block columns we get

$$\begin{bmatrix} \mathcal{V}_1 & \tau\mathcal{F}_1\mathcal{D}_1\overline{\mathcal{V}_1} \\ \mathcal{V}_2 & \tau\mathcal{F}_2\mathcal{D}_2\overline{\mathcal{V}_2} \end{bmatrix} \begin{bmatrix} \mathcal{S}_\ell\overline{\mathbf{q}} \\ \overline{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} -\tau\mathbf{f}_1 - \mathbf{d}_1 \\ -\tau\mathbf{f}_2 - \mathbf{d}_2 \end{bmatrix}.$$

Hence, from the unicity of the solution, $\mathbf{p} = \mathcal{S}_\ell\overline{\mathbf{q}}$, and so, $P_\ell = Q_\ell^*$.

For the second part, observe that (19) follows directly from (20) by block Gaussian elimination.

The third part described in (18) is the classical Schur-Cohn test (see [43, 17]) for the polynomial P_ℓ. □

Note that the yeast of this proof is essentially the same as in the Theorem 3.6. We require that the system expressing the interpolation conditions has a unique solution, and require that the solution P_ℓ has its zeros in $\mathbb{D}$, which was obtained in Theorem 3.6 and by restricting τ to an arc, which is here replaced by a Schur-Cohn test.

Remark 3.13. *We point out that (17) does not depend on τ, so neither does the existence of $Q_{n,2\ell+1}$. Whether or not $Q_{n,2\ell+1}$ in Theorem 3.12 will have the desired properties depends on the value of $\tau \in \mathbb{T}$. If (17) is satisfied, there will exist a polynomial P_ℓ such that the interpolation conditions are satisfied. However, whether or not the other conditions of the previous theorem can be satisfied, in particular the conditions that the polynomial P_ℓ will have zeros in $\mathbb{D}$, depends continuously on the parameter $\tau \in \mathbb{T}$. All the $s_k(0)$ will be in $\mathbb{D}$ for τ on certain arcs of $\mathbb{T}$. If τ leaves such an arc, this means that some Schur parameter $s_{\ell_0}(0)$, and thus also a zero η of P_ℓ, leaves $\mathbb{D}$ by crossing $\mathbb{T}$. At that moment η becomes one of the zeros of $Q_{n,2\ell+1}$. For τ outside these arcs, $Q_{n,2\ell+1}$ can have zeros of multiplicity larger than 1 and/or pairs of zeros $(z_i, 1/\overline{z_i})$ with $z_i \in \mathbb{D}$. To find out what exactly are the pathological cases requires further analysis of the location and multiplicity of the zeros, which can be obtained by a deeper exploration of the behaviour of the Schur parameters. See for example [4, 48] and similar discussions in the literature. However, keeping in mind Theorem 3.2, we can always assure that there are at least $n - 2\ell$ zeros on $\mathbb{T}$ with odd multiplicity.*

The freedom to choose τ in the indicated arcs of $\mathbb{T}$ may allow us to fix another additional node. More precisely, if we want to fix $2\ell+1$ zeros $\{\alpha_i\}_{j=1}^{2\ell+1} \subset \mathbb{T}$ of $Q_{n,2\ell+1}$, it should be verified that

$$\alpha_i P_\ell(\alpha_i)\rho_{n-\ell-1}(\alpha_i) + \tau P_\ell^*(\alpha_i)\rho_{n-\ell-1}^*(\alpha_i) = 0, \quad j = 1, 2, \ldots, 2\ell+1.$$

Equivalently, if $f_i = \overline{F_{n-\ell}(\alpha_i)} \in \mathbb{T}$ and $\lambda = \sqrt{\tau}$,

$$\lambda P_\ell(\alpha_i) + f_i\overline{\lambda} P_\ell^*(\alpha_i) = 0, \quad i = 1, 2, \ldots, 2\ell+1. \tag{21}$$

Also this problem has a matrix interpretation where the main difference with the previous case is that now $\tau = Q_{n,2\ell+1}(0) = \overline{\lambda^2}$ is a parameter that we have to determine by solving the system.

More precisely, if we denote $P_\ell(z) = \prod_{j=1}^{\ell}(z-\eta_j) = p_0 + p_1 z + \cdots + p_{\ell-1}z^{\ell-1} + z^\ell$, then from (21)

$$\lambda\left(p_0 + p_1\alpha_i + \cdots + p_{\ell-1}\alpha_i^{\ell-1} + \alpha_i^\ell\right) + f_i\overline{\lambda}\left(1 + \overline{p}_{\ell-1}\alpha_i + \cdots + \overline{p}_0\alpha_i^\ell\right) = 0, \quad i = 1, \ldots, 2\ell+1.$$

Setting $\mathbf{p} = \lambda[p_0, p_1, \ldots, p_{\ell-1}, 1]^T$, and $\mathcal{V} = [\alpha_i^j]_{i=1,\ldots,2\ell+1}^{j=0,\ldots,\ell}$, $\mathbf{f} = [f_1, f_2, \ldots, f_{2\ell+1}]^T$ and $\mathcal{F} = \operatorname{diag}(\mathbf{f})$, this is equivalent to the homogeneous system

$$\mathcal{V}\mathbf{p} + \mathcal{F}\mathcal{V}\mathbf{p}^* = 0, \quad \text{or} \quad [\mathcal{V} \;\; \mathcal{F}\mathcal{V}]\begin{bmatrix} \mathbf{p} \\ \mathbf{p}^* \end{bmatrix} = \mathbf{0},$$

and the vector $[\mathbf{p}^T, \mathbf{p}^{*T}]^T$ is the unique solution, up to a multiplicative constant if the matrix $[\mathcal{V} \;\; \mathcal{F}\mathcal{V}] \in \mathbb{C}^{(2\ell+1)\times(2\ell+2)}$ has full rank $2\ell+1$. Since P_ℓ has degree ℓ deleting column $\ell+1$ in this matrix makes it square and nonsingular.

Theorem 3.14. *Let $n \geq 2\ell+1$. Given distinct numbers $\{\alpha_i\}_{i=1}^{2\ell+1} \subset \mathbb{T}$, and let $f_i = \overline{F_{n-\ell}(\alpha_i)}$, $i = 1, \ldots, 2\ell+1$ with $F_{n-\ell}$ as defined in (4) be also distinct. Denote by $\mathcal{V}_1 = [\alpha_i^j]_{i=1,\ldots,\ell}^{j=0,\ldots,\ell-1}$ and $\mathcal{V}_2 = [\alpha_i^j]_{i=\ell+1,\ldots,2\ell+1}^{j=0,\ldots,\ell}$ the square Vandermonde matrices associated with $\{\alpha_i\}_{i=1}^{\ell}$ and $\{\alpha_i\}_{i=\ell+1}^{2\ell+1}$, respectively, and the rectangular Vandermonde matrices $\mathcal{V}_1' = [\alpha_i^j]_{i=1,\ldots,\ell}^{j=0,\ldots,\ell} \in \mathbb{C}^{\ell\times(\ell+1)}$ and $\mathcal{V}_2' = [\alpha_i^j]_{i=\ell+1,\ldots,2\ell+1}^{j=0,\ldots,\ell-1} \in \mathbb{C}^{(\ell+1)\times\ell}$, and furthermore set*

$$\begin{array}{ll} \mathbf{f}_1 = [f_1, f_2, \ldots, f_\ell]^T, & \mathcal{F}_1 = \operatorname{diag}(\mathbf{f}_1), \\ \mathbf{f}_2 = [f_{\ell+1}, f_{\ell+2}, \ldots, f_{2\ell+1}]^T, & \mathcal{F}_2 = \operatorname{diag}(\mathbf{f}_2), \\ \mathbf{d}_1 = [\alpha_1^\ell, \alpha_2^\ell, \ldots, \alpha_\ell^\ell]^T, & \mathbf{d}_2 = [\alpha_{\ell+1}^\ell, \alpha_{\ell+2}^\ell, \ldots, \alpha_{2\ell+1}^\ell]^T, \end{array}$$

and assume

$$\Delta = \det\begin{bmatrix} \mathcal{V}_1 & \mathcal{F}_1\mathcal{V}_1' \\ \mathcal{V}_2' & \mathcal{F}_2\mathcal{V}_2 \end{bmatrix} \neq 0. \tag{22}$$

Then there exists a unique monic invariant $(n, 2\ell+1)$-quasi-paraorthogonal polynomial $Q_{n,2\ell+1}(z) = zP_\ell(z)\rho_{n-\ell-1}(z) + \tau P_\ell^*(z)\rho_{n-\ell-1}^*(z)$ with all its zeros simple on $\mathbb{T}$, with α_1, $\alpha_2, \ldots, \alpha_{2\ell+1}$ being $2\ell+1$ of them if all $s_k(0) \in \mathbb{D}$, $k = 1, \ldots, \ell$, where

$$s_\ell(z) = \frac{P_\ell(z)}{P_\ell^*(z)} \quad \text{and} \quad s_{k-1}(z) = \frac{1}{z}\frac{s_k(z) - s_k(0)}{1 - \overline{s_k(0)}s_k(z)}, \quad k = \ell, \ell-1, \ldots, 2, \tag{23}$$

where $P_\ell(z)=\sum_{k=1}^{\ell-1}p_kz^k+z^\ell$ *is given by* $\mathbf{p}=[p_0,p_1,\dots,p_{\ell-1}]^T$ *such that*

$$\mathbf{p}=\left[\mathcal{V}_1-\mathcal{F}_1\mathcal{V}_1'\mathcal{V}_2^{-1}\overline{\mathcal{F}_2}\mathcal{V}_2'\right]^{-1}\cdot\left[\mathbf{d}_1-\mathcal{F}_1\mathcal{V}_1'\mathcal{V}_2^{-1}\overline{\mathcal{F}_2}\mathbf{d}_2\right]\ \textit{while}\ \tau=(-1)^{\ell+1}\frac{\overline{\Delta}}{\Delta}\prod_{j=1}^{2\ell+1}f_j\alpha_j^\ell. \quad (24)$$

Proof. The proof is analogous to the previous case. We consider the following auxiliary problem. Let $\tilde{P}_\ell(z)=\tilde{p}_0+\tilde{p}_1z+\cdots+\tilde{p}_{\ell-1}z^{\ell-1}+\tilde{p}_\ell z^\ell$ with $\tilde{p}_\ell\neq 0$ and $Q_\ell(z)=q_0+q_1z+\cdots+q_{\ell-1}z^{\ell-1}+q_\ell z^\ell$ satisfy

$$\tilde{p}_0+\tilde{p}_1\alpha_i+\cdots+\tilde{p}_\ell\alpha_i^\ell+f_i\left(q_0+q_1\alpha_i+\cdots+q_\ell\alpha_i^\ell\right)=0,\quad i=1,\dots,2\ell+1.$$

It is easy to verify that $Q_\ell=\tilde{P}_\ell^*$, then setting $\tilde{\mathbf{p}}=[\tilde{p}_0,\tilde{p}_1,\dots,\tilde{p}_{\ell-1}]^T$ and $\mathbf{q}=\left[\overline{\tilde{p}_\ell},\overline{\tilde{p}_{\ell-1}},\dots,\overline{\tilde{p}_0}\right]^T$, the above equations are equivalent to the linear system

$$\begin{bmatrix}\mathcal{V}_1 & \mathcal{F}_1\mathcal{V}_1'\\ \mathcal{V}_2' & \mathcal{F}_2\mathcal{V}_2\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{p}}\\ \mathbf{q}\end{bmatrix}=-\tilde{p}_\ell\begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{bmatrix},\ \ \tilde{p}_\ell\neq 0$$

which has a unique solution by (22). Then, (24) follows directly from block Gaussian elimination taking into account that $\mathcal{V}_1$ and $\mathcal{F}_2\mathcal{V}_2$ are nonsingular and

$$\frac{\overline{\tilde{p}_\ell}}{\tilde{p}_\ell}=-\det\begin{bmatrix}\mathcal{V}_1' & \mathcal{D}_1\mathcal{F}_1\mathcal{V}_1\\ \mathcal{V}_2 & \mathcal{D}_2\mathcal{F}_2\mathcal{V}_2'\end{bmatrix}/\det\begin{bmatrix}\mathcal{V}_1 & \mathcal{F}_1\mathcal{V}_1'\\ \mathcal{V}_2' & \mathcal{F}_2\mathcal{V}_2\end{bmatrix}=-\frac{1}{\Delta}\det\begin{bmatrix}\mathcal{V}_1' & \mathcal{D}_1\mathcal{F}_1\mathcal{V}_1\\ \mathcal{V}_2 & \mathcal{D}_2\mathcal{F}_2\mathcal{V}_2'\end{bmatrix},$$

where $\mathcal{D}_1=\operatorname{diag}(\alpha_1,\dots,\alpha_\ell)$, $\mathcal{D}_2=\operatorname{diag}(\alpha_{\ell+1},\dots,\alpha_{2\ell+1})$. Since all the entries in the matrices are on $\mathbb{T}$, we can rewrite the determinant of the numerator as

$$\det\begin{bmatrix}\mathcal{D}_1^\ell\mathcal{F}_1 & \\ & \mathcal{D}_2^\ell\mathcal{F}_2\end{bmatrix}\det\begin{bmatrix}\overline{\mathcal{F}_1\mathcal{V}_1} & \overline{\mathcal{V}_1'}\\ \overline{\mathcal{F}_2\mathcal{V}_2'} & \overline{\mathcal{V}_2}\end{bmatrix}=(-1)^\ell\,\overline{\Delta}\prod_{j=1}^{2\ell+1}f_j\alpha_j^\ell\ \text{thus}\ \frac{\overline{\tilde{p}_\ell}}{\tilde{p}_\ell}=(-1)^{\ell+1}\frac{\overline{\Delta}}{\Delta}\prod_{j=1}^{2\ell+1}f_j\alpha_j^\ell.$$

Moreover, if $s_\ell(z)=P_\ell(z)/P_\ell^*(z)$, with $P_\ell(z)=\tilde{P}_\ell(z)/\tilde{p}_\ell=\sum_{k=1}^{\ell-1}(\tilde{p}_k/\tilde{p}_\ell)z^k+z^\ell$, then P_ℓ will have all its zeros in $\mathbb{D}$ if and only if all the $s_k(0)\in\mathbb{D}$ for $k=\ell,\dots,1$ by the Schur-Cohn test. Setting $P_\ell(z)=\prod_{i=1}^{\ell}(z-\eta_i)$ with $\eta_i\in\mathbb{D}$, $i=1,\dots,\ell$, it is clear that

$$Q_{n,2\ell+1}(z)=z\prod_{i=1}^{\ell}(z-\eta_i)\rho_{n-\ell-1}(z)+\frac{\overline{\tilde{p}_\ell}}{\tilde{p}_\ell}\prod_{i=1}^{\ell}(1-\overline{\eta_i}z)\rho_{n-\ell-1}^*(z)$$

is the monic invariant $(n,2\ell+1)$-QPOPUC with invariance parameter $\tau=Q_{n,2\ell+1}(0)=\frac{\overline{\tilde{p}_\ell}}{\tilde{p}_\ell}$ and all its zeros simple on $\mathbb{T}$, with $\alpha_1,\alpha_2,\dots,\alpha_{2\ell+1}$ being $2\ell+1$ of them. □

Remark 3.15. *Note that prefixing* $2\ell+1$ *zeros will in general define* $Q_{n,2\ell+1}$ *uniquely, which includes its invariance parameter* τ*. Thus there is no 'varying'* τ*. Generically, it is impossible to fix more than* $2\ell+1$ *prefixed zeros, unless we select a zero* $\alpha_{2\ell+2}$ *that happens to be one of the zeros of* $Q_{n,2\ell+1}$ *that is already uniquely defined by the first* $2\ell+1$ *zeros. A discussion somewhat similar to Remark 3.13 could be made here if we choose* $\{\alpha_i\}_{i=1}^{2\ell}$ *fixed and consider* $\alpha_{2\ell+1}$ *as a varying parameter defining* τ*, but we shall not elaborate on that.*

We note also that, with a different normalization for P_ℓ, a similar technique was used in [29, Theorem 2.2, Corollary 2.3] to find Blaschke product interpolants of degree ℓ for $2\ell+1$ points on $\mathbb{T}$.

4. Applications to quadrature formulas on the unit circle

In this section we shall be concerned with the numerical estimation of integrals of the form

$$I(f) = \int f(z) d\mu(z), \tag{25}$$

f being a Riemann-Stieltjes integrable function with respect to a measure μ supported on an arc $S \subseteq \mathbb{T}$, by means of quadrature formulas (q.f.) of the form

$$I_n(f) = \sum_{s=1}^{n} \lambda_s f(z_s), \tag{26}$$

with distinct nodes $\{z_s\}_{s=1}^n \subset \text{supp}(\mu)$ and weights defined by $\lambda_s = I(L_{n,s})$ with $\{L_{n,s}\}_{s=1}^n$ the Lagrange Laurent polynomial interpolation basis for the nodes $\{z_s\}_{s=1}^n$ in an n dimensional subspace of the form $\mathbb{L}_{-p,q}$, $q+p = n-1$, with $p, \ell \in \mathbb{N}$ such that $0 < 2\ell + 1 < n$ and $\ell \leq p \leq n - \ell - 1$. This q.f. is obviously exact in $\mathbb{L}_{-p,q}$. If all weights in (26) are positive we call $I_n(f)$ a positive quadrature formula (positive q.f.). Let us denote the nodal polynomial by $W_n(z) = \prod_{s=1}^n (z - z_s)$. Note that it is $W_n(0)$-invariant.

It is well known that the (n, d)-QOPRL with d prefixed zeros can be used as the nodal polynomials of Gauss-type formulas, i.e., positive quadrature formulas with maximal domain of exactness like Gauss-Radau ($d = 1$) or Gauss-Lobatto ($d = 2$) or more general quadrature formulas with prescribed nodes ([7]).

The purpose of this section is to apply $(n, 2\ell + 1)$-QPOPUC as nodal polynomials to obtain similar results for Szegő-type q.f. namely to characterize positive quadrature formulas with maximal domain of validity when some of the nodes are prefixed. This will correspond to q.f. whose nodal polynomial is a $(n, 2\ell + 1)$-QPOPUC with 2ℓ or $2\ell + 1$ prefixed nodes.

4.1. Szegő-type quadrature

As a consequence of the density of Laurent polynomials in the space of continuous functions defined on an arc $S \subseteq \mathbb{T}$ with respect to the uniform norm, q.f. like (26) are usually constructed by imposing $I(L) = I_n(L)$ for all $L \in \text{span}\{z^{-p}, \ldots, z^q\} = \mathbb{L}_{-p,q}$, $p, q \in \mathbb{N}$ with $p + q$ as large as possible (in this case, we say that $I_n(f)$ is exact in $\mathbb{L}_{-p,q}$ and it has at least degree of exactness $p+q$). The *Szegő q.f.* introduced in [34] are positive q.f. whose degree of exactness is $2n - 2$. The nodal polynomial is $W_n = Q_{n,1}$, a monic invariant $(n, 1)$-QPOPUC $Q_{n,1} = z\rho_{n-1} + \tau\rho_{n-1}^* = \nu_{n,1}^{-1}[\rho_n + \tilde{\tau}\rho_n^*]$ in which the invariance parameter $\tau = Q_{n,1}(0)$ or, equivalently, the orthogonality parameter $\tilde{\tau}$, is free. It is exact in $\mathbb{L}_{-(n-1),n-1}$, that is a subspace of Laurent polynomials of dimension $2n-1$, despite the fact that the q.f. depends on $2n$ parameters (n nodes and n weights, all varying with τ). Recall that Gaussian q.f. also depend on $2n$ parameters but they are exact in $\mathbb{P}_{2n-1}$, a subspace of dimension $2n$. So we may expect that there must also be some subspace of dimension $2n$ in which the Szegő quadrature is exact which corresponds to the orthogonality represented by the orthogonality parameter $\tilde{\tau}$. This was first observed in [38] and [9]. The authors in

[38], found that the Szegő q.f. also integrated exactly $z^n - \frac{\tilde{\tau}\tau}{z^n}$ thus achieving a subspace of dimension $2n$. So we shall consider subspaces of Laurent polynomials that are invariant under the involution $f(z) \mapsto f_*(z) = \overline{f(1/\overline{z})}$ and we shall say that a q.f. on $\mathbb{T}$ of the form (26) has degree of exactness e if it is exact for such an invariant subspace of dimension e. This and other issues in the construction of q.f. on the unit circle, were studied in a more general context by some of the authors in [19]. In that paper, it was proved that the resulting q.f. always have real weights $\{\lambda_k\}_{k=1}^n$, at least $\lfloor \frac{e}{2} \rfloor + 1$ of them are positive where $e \geq n-1$ is the degree of exactness of the q.f. [19, Theorem 2.3] and there always exist two positive q.f. that are exact in a maximal domain of dimension $2n$ [19, Corollary 2.9] when $\mathrm{supp}(\mu) = \mathbb{T}$. Also an analogue of a Jacobi-type theorem (see Theorem 4.1 below) was obtained [19, Theorem 2.6].

This idea of constructing a q.f. whose nodal polynomial is a QPOPUC is briefly summarized as follows. As in the real line situation, we need to define a nested sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of subspaces of Laurent polynomials, such that $\mathcal{L}_0 = \mathrm{span}\,\{1\}$, $\mathcal{L}_{n-1} \subset \mathcal{L}_n$, $\dim(\mathcal{L}_n) = n+1$ and $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n = \mathbb{L}$. This can be done by starting from a sequence $\{\omega_k\}_{k\geq 1} \subset \mathbb{T}$ and defining

$$\mathcal{L}_{2k} := \mathbb{L}_{-k,k}, \qquad \mathcal{L}_{2k+1} = \mathcal{L}_{2k+1}(\omega_{k+1}) := \mathcal{L}_{2k} \oplus \mathrm{span}\left\{ z^{k+1} - \frac{\omega_{k+1}}{z^{k+1}} \right\}, \quad k \geq 0. \tag{27}$$

Note that $\mathcal{L}_n$ is invariant under the involution $L \mapsto L_*$, that is $L(z) \in \mathcal{L}_n \Leftrightarrow \overline{L(1/\overline{z})} \in \mathcal{L}_n$. The next result gives the paraorthogonality conditions that must be satisfied by the nodal polynomial of a q.f. $I_n(f)$ of the form (26) to be exact in subspaces $\mathcal{L}_{n-1+k}$, $1 \leq k \leq n$ of the form (27) and not in $\mathcal{L}_{n+k}$. It was proved in [38] for $k = n-1$ and in the context of Laurent polynomials in [19, Theorem 2.6] for any k. For completeness we include a proof using the notation and the definitions introduced in Section 2 inspired by the results obtained in [38] for $k = n-1$. The result can be stated as follows:

Theorem 4.1. *Let $n, \ell, p \in \mathbb{N}$ such that $0 < 2\ell + 1 < n$, $\ell \leq p \leq n - \ell - 1$. Let $I_n(f)$ be a q.f. of the form (26) exact in $\mathbb{L}_{-p,n-1-p}$. Let W_n be its nodal polynomial and $a_{n-\ell} = \langle W_n, z^{n-\ell} \rangle \neq 0$. Then with $\tau = W_n(0)$*

1. *$I_n(f)$ is exact in $\mathcal{L}_{2(n-\ell-1)}$ if and only if W_n is a monic $(n, 2\ell+1)$-QPOPUC.*
2. *$I_n(f)$ is exact in $\mathcal{L}_{2(n-\ell)-1}(\omega)$ with $\omega = \tilde{\tau}\tau$ if and only if W_n is an $(n, 2\ell+1)$-QPOPUC and $\tilde{\tau} = \tau \frac{\overline{a_{n-\ell}}}{a_{n-\ell}}$.*

Proof. Let $I_n(f)$ be exact in $\mathbb{L}_{-p,q}$, and denote by $L_{n-1}(f,z)$ the Laurent polynomial interpolating f in $\mathbb{L}_{-p,n-1-p}$ in the nodes $\{x_i\}_{i=1}^n$ of the q.f. Let $L \in \mathcal{L}_{2(n-\ell-1)}$, then

$$L(z) - L_{n-1}(L,z) = \frac{W_n(z)P(z)}{z^{n-\ell-1}}, \quad P \in \mathbb{P}_{n-2\ell-2},$$

integrating the equality,

$$I(L) = I_n(L),\ \forall\, L \in \mathcal{L}_{2(n-\ell-1)} \quad \Leftrightarrow \quad 0 = I\left(\frac{W_n P}{z^{n-\ell-1}}\right) = \langle W_n, Q \rangle,\ \forall Q \in \mathbb{Q}_{n,2\ell+1}.$$

It only remains to prove that additionally $z^{n-\ell}-\omega z^{-(n-\ell)}$ will be integrated exactly if and only if $\omega=\tilde{\tau}\tau$.

Therefore we proceed as follows. Assume the nodal polynomial is $Q_n := Q_{n,2\ell+1}$, a monic $(n,2\ell+1)$-QPOPUC with $\langle Q_n, z^{n-\ell}\rangle \neq 0$ then its orthogonality parameter $\tilde{\tau}$ is well defined. If $\tau = Q_n(0)$, then

$$z^{-\ell}[z^n - Q_n] \in z^{-\ell}\mathbb{L}_{0,n-1} = \mathbb{L}_{-\ell,n-\ell-1} \quad \text{and} \quad z^{-(n-\ell)}[1-\overline{\tau}Q_n] \in z^{-(n-\ell)}\mathbb{L}_{1,n} = \mathbb{L}_{-(n-\ell-1),\ell} \tag{28}$$

and because both $\mathbb{L}_{-\ell,n-\ell-1}$ and $\mathbb{L}_{-(n-\ell-1),\ell}$ are subspaces of $\mathbb{L}_{-(n-\ell-1),n-\ell-1}$, (recall that $2\ell+1<n$) the quadrature will be exact for these Laurent polynomials:

$$[I-I_n]\Big(z^{n-\ell}-z^{-\ell}Q_n\Big)=0 \quad \text{and} \quad [I-I_n]\Big(z^{-(n-\ell)}-\overline{\tau}z^{-(n-\ell)}Q_n\Big)=0$$

but because Q_n is the nodal polynomial, $I_n(z^{-\ell}Q_n)=0=I_n(z^{-(n-\ell)}Q_n)$, so that

$$I(z^{n-\ell}) = I_n(z^{n-\ell}) + I(Q_n z^{-\ell}) \quad \text{and} \quad I(z^{-(n-\ell)}) = I_n(z^{-(n-\ell)}) + \overline{\tau} I(Q_n z^{-(n-\ell)}).$$

Thus for any $\omega\in\mathbb{T}$

$$I\left(z^{n-\ell}-\frac{\omega}{z^{n-\ell}}\right) = I_n\left(z^{n-\ell}-\frac{\omega}{z^{n-\ell}}\right) + \langle Q_n, z^{\ell}\rangle - \omega\overline{\tau}\langle Q_n, z^{n-\ell}\rangle.$$

We know that $I(Q_n z^{-(n-\ell)}) = \langle Q_n, z^{n-\ell}\rangle \neq 0$ and because Q_n is $\tilde{\tau}$-orthogonal, $\langle Q_n, z^{\ell}\rangle = \tilde{\tau}\langle Q_n, z^{n-\ell}\rangle$ so that

$$\langle Q_n, z^{\ell}\rangle - \omega\overline{\tau}\langle Q_n, z^{n-\ell}\rangle = (\tilde{\tau}-\omega\overline{\tau})\langle Q_n, z^{n-\ell}\rangle = 0 \Leftrightarrow \omega = \tau\tilde{\tau}.$$

□

This theorem states a one-to-one correspondence between all QPOPUC that have all their zeros simple and in the support of the measure and all q.f. exact in a certain subspace $\mathcal{L}_m$. We have seen in previous sections that if $\mathrm{supp}(\mu)$ is an arc $S\subsetneq\mathbb{T}$, then $Q_{n,2\ell+1}$ has at least $n-2\ell$ zeros of odd multiplicity on $\mathbb{T}$ and $n-2\ell-1$ of them in $\mathring{S}$. It depends on $2\ell+1$ free real parameters that can be used to preselect up to $2\ell+1$ zeros simple and on S. It remains to analyse when this will deliver positive q.f. with maximal domain of exactness.

4.2. Szegő and Szegő-Radau quadrature

Since $\nu_{n,1}=1+\tilde{\tau}\overline{\delta_n}\neq 0$, it follows from Lemma A.4 in the Appendix, that $Q_{n,1}(z) = z\rho_{n-1}+\tau\rho^*_{n-1}(z) = \nu_{n,1}^{-1}[\rho_n(z)+\tilde{\tau}\rho^*_{n-1}(z)]$ and

$$\tau = Q_{n,1}(0) = \tilde{\tau}\frac{\overline{\nu_{n,1}}}{\nu_{n,1}} = \varphi_{\delta_n}(\tilde{\tau}) \Leftrightarrow \tilde{\tau} = \varphi_{-\delta_n}(\tau).$$

We know that $Q_{n,1}$ has always n simple zeros on $\mathbb{T}$. It was shown in [34] that Szegő's q.f. has $Q_{n,1}$ as its nodal polynomial, it is exact in $\mathcal{L}_{2n-2}=\mathbb{L}_{-(n-1),n-1}$ and its weights

are positive. By [38] or Theorem 4.1, we know that it is also exact in $\mathcal{L}_{2n-1}(\omega)$ with $\omega = \tilde{\tau}\tau = \tilde{\tau}\varphi_{\delta_n}(\tilde{\tau}) = \tau\varphi_{-\delta_n}(\tau)$. The freedom to pick an arbitrary $\tau \in \mathbb{T}$ allows us to choose

$$\tau = -F_n(\alpha) = -\frac{\alpha\rho_{n-1}(\alpha)}{\rho_{n-1}^*(\alpha)} \Leftrightarrow \tilde{\tau} = -\frac{\rho_n(\alpha)}{\rho_n^*(\alpha)}$$

which fixes a node at $\alpha \in \mathbb{T}$.

All this is well known but with our results of the previous section this can be generalized to the case where $S = \operatorname{supp}(\mu)$ is an arc on $\mathbb{T}$.

Theorem 4.2 (Szegő and Szegő-Radau q.f.). *Let μ be a positive measure supported on an arc $S = [a, b] \subset \mathbb{T}$ and let $I_n(f)$ be a q.f. of the form (26).*

1. *$I_n(f)$ is a Szegő quadrature formula exact in $\mathcal{L}_{2n-1}(\omega)$ if and only if its nodal polynomial is the τ-invariant $(n, 1)$-QPOPUC satisfying Theorem 3.4 and $\omega = \tau\varphi_{-\delta_n}(\tau)$.*
2. *$I_n(f)$ is the unique Szegő-Radau quadrature formula exact in $\mathcal{L}_{2n-2}$ and prefixed node $\alpha \in \mathring{S} \subseteq \mathbb{T}$ if and only if its nodal polynomial is the τ-invariant $(n, 1)$-QPOPUC with $-\tau = \alpha\rho_{n-1}(\alpha)/\rho_{n-1}^*(\alpha)$ satisfying Theorem 3.4. If $\omega = \tau\varphi_{-\delta_n}(\tau)$, then the q.f. is exact in $\mathcal{L}_{2n-1}(\omega)$.*

Proof. The original proof of exactness in $\mathcal{L}_{2n-2}$ and the positivity of the weights that is given in [34] for a measure supported on $\mathbb{T}$ still holds if the support is an arc $[a, b] \subset \mathbb{T}$ and all the nodes are in the open arc (a, b). We note that the positivity of the weights also follows directly from the exactness in $\mathcal{L}_{2n-2}$. Indeed, as we mentioned above, by [19, Theorem 2.3], the exactness in $\mathcal{L}_{2n-2}$ implies the positivity of $\lfloor\frac{2(n-1)}{2}\rfloor + 1 = n$ weights. The statement about exactness in $\mathcal{L}_{2n-1}(\omega)$ is a consequence of Theorem 4.1 and Lemma A.4. □

Remark 4.3. *The q.f. using $Q_{n,1}$ as nodal polynomial where we preselect τ and hence select some particular value of ω that defines the domain of exactness is the analogue of the Gaussian rule for a measure supported on $[a, b] \subset \mathbb{R}$. If $\operatorname{supp}(\mu) = \mathbb{T}$ this was considered in [19] where it was proved that there are two positive Szegő q.f. exact in $\mathcal{L}_{2n-1}(\omega)$ for any $\omega \in \mathbb{T}$.*

The q.f. using $Q_{n,1}$ as nodal polynomial where τ is fixed by the preselected node α is the analogue of the Gauss-Radau rule for a measure supported on $[a, b] \subset \mathbb{R}$. Also this was considered in the case that $\operatorname{supp}(\mu) = \mathbb{T}$ in e.g. [9].

To the best of our knowledge our more general results are new when $\operatorname{supp}(\mu)$ is an arc $S \subsetneq \mathbb{T}$.

4.3. Positive Szegő-type q.f. with 2 or 3 prefixed nodes

The classical Gauss-Lobatto quadrature formulas fix zeros at the boundaries of $\operatorname{supp}(\mu) = [a, b] \subset \mathbb{R}$. For Szegő-Lobatto q.f. when $\operatorname{supp}(\mu) = \mathbb{T}$, there are no boundary points and therefore two distinct points α_1 and α_2 are preselected on $\mathbb{T}$ and a positive q.f. having these nodes are constructed like in [33]. For more details about Szegő-Lobatto q.f. see also [19, 18]. In [10] Szegő-Lobatto quadrature is discussed in a more general context of orthogonal rational functions, but it contains the polynomials as a special case.

We summarize the results as follows.

Theorem 4.4 (**Szegő-Lobatto quadrature formula**). *Let $n \geq 3$. Given $\tau \in \mathbb{T}$ and distinct $\{\alpha_i\}_{i=1}^2 \subset \mathbb{T}$, we define $\{f_i\}_{i=1}^2 \subset \mathbb{T}$ as in (7). Assume the f_i are distinct and $\alpha_1 f_1 \neq \alpha_2 f_2$. Let $I_{n,3}(f)$ be a q.f. of the form (26) with nodal polynomial W_n. Then $I_{n,3}(f)$ is a Szegő-Lobatto q.f exact in $\mathcal{L}_{2n-4}$ whose nodes include α_1 and α_2 if and only if $W_n = Q_{n,3}$ a QPOPUC as in (5) that satisfies the conditions of Theorem 3.6 or equivalently Theorem 3.9. If moreover τ satisfies $\tau\overline{\eta}+\delta_{n-1} \neq 0$ with η given by (10), then $\omega = \frac{\tau\overline{\eta}+\delta_{n-1}}{\overline{\tau}\eta+\overline{\delta}_{n-1}} \in \mathbb{T}$ is well defined and then $I_{n,3}(f)$ is exact in $\mathcal{L}_{2n-1}(\omega)$.*

Proof. It has been proved by Theorems 3.6 and 3.9 under what conditions $Q_{n,3}$ has simple nodes on $\mathbb{T}$ and by Theorem 4.1(1) we know when $I_{n,3}(f)$ is exact in $\mathcal{L}_{2n-4}$. If $\tau\overline{\eta}+\delta_{n-1} \neq 0$, then the orthogonality parameter $\tilde{\tau}$ and ω will be well defined by Lemma A.4 in the Appendix and the quadrature will be exact in $\mathcal{L}_{2n-3}(\omega)$ with $\omega = \tau\tilde{\tau}$ by Theorem 4.1(2).

It only remains to prove that the weights are positive. Therefore note that

$$W_n(z) = z\hat{\rho}_{n-1}(z) - \tau\hat{\rho}_{n-1}^*(z), \quad \text{with} \quad \hat{\rho}_{n-1}(z) = z\rho_{n-2}(z) - \tau\overline{\eta}\rho_{n-2}^*(z), \quad \tau\overline{\eta} \in \mathbb{D}.$$

By Favard's theorem on the unit circle (see [34]), we conclude that $\hat{\rho}_{n-1}$ is a monic Szegő polynomial and W_n is a monic $(n,1)$-QPOPUC associated with a positive modified measure $\hat{\mu}$, then our $I_{n,3}(f)$ is also a Szegő-Radau q.f. associated to $\hat{\mu}$ that coincides with our q.f. in $\mathcal{L}_{2n-3}(\omega)$, and hence it is a positive q.f.

So far we proved that the conditions of Theorems 3.6 or 3.9 imply the existence of the prescribed positive q.f. The inverse implication is a consequence of [39, Theorem 3.1]. □

Since the previous Theorem 4.4 holds for τ belonging to an arc, we may use it to fix a third node.

Theorem 4.5 ($(n,3)$-**Szegő-Peherstorfer quadrature**). *Let $n \geq 3$ and let $I_{n,3}(f)$ be the q.f. of the from (26) with nodal polynomial W_n. Then $I_{n,3}(f)$ is the unique Szegő-Peherstorfer q.f. exact in $\mathcal{L}_{2n-4}$ with prescribed nodes α_1, α_2, and α_3 if and only if $W_n = Q_{n,3}$ as in (5) satisfying the conditions of Theorem 3.10. If moreover τ verifying (11) satisfies $\tau\overline{\eta} + \delta_{n-1} \neq 0$ with η given by (10), then $I_{n,3}(f)$ is exact in $\mathcal{L}_{2n-3}(\omega)$ with $\omega = \frac{\tau\overline{\eta}+\delta_{n-1}}{\overline{\tau}\eta+\overline{\delta}_{n-1}}$.*

Proof. By Theorem 3.10, we know that if $\operatorname{arc}(\alpha_1, \alpha_2, \alpha_3)$ and $\operatorname{arc}(f_1, f_2, f_3)$ have the same orientation, then there exists a unique $Q_{n,3}$ with simple zeros on $\mathbb{T}$ and α_1, α_2 and α_3 are three of them. Thus $I_{n,3}(f)$ will be exact in $\mathcal{L}_{2n-4}$ by Theorem 4.1(1). The weights are positive by the same argument as used in the previous theorem 4.4. The reciprocal is again a consequence of [39, Theorem 3.1]. The expression for ω follows from Theorem 4.1 and Lemma A.4. □

Theorems 4.4 and 4.5 hold again for integrals where $\operatorname{supp}(\mu) = \mathbb{T}$. If $\operatorname{supp}(\mu)$ is an arc $[a,b] \subsetneq \mathbb{T}$, these theorems only guarantee $n-3$ zeros of odd multiplicity on the $\operatorname{arc}(a,b)$. The 'classical' versions of these Theorems do hold when two of the prefixed points are the boundary points a and b.

Theorem 4.6 (**Classical Szegő-Lobatto and Szegő-Peherstorfer q.f.**). *Let μ be a measure supported on an arc $S = [a, b] \subsetneq \mathbb{T}$. Let $n \geq 3$, $\tau \in \mathbb{T}$, $f_a = F_{n-1}(a)$ and $f_b = F_{n-1}(b)$ as in (7) and let $I_{n,3}(f)$ be the q.f. of the form (26) with nodal polynomial W_n.*

1. *$I_{n,3}(f)$ is a Szegő-Lobatto quadrature formula (fixing nodes in a and b) exact in $\mathcal{L}_{2n-3}(\omega)$ if and only if $W_n = Q_{n,3}$ as in (5) satisfying the first condition of Theorem 3.11, with $\eta \in \mathbb{D}$, $\tau\overline{\eta} + \delta_{n-1} \neq 0$ and $\omega = \frac{\tau\overline{\eta}+\delta_{n-1}}{\overline{\tau}\eta+\overline{\delta_{n-1}}}$. Note that η given by (10) is in $\mathbb{D}$ if $\operatorname{arc}(a, \tau f_a, b)$ and $\operatorname{arc}(a, \tau f_a, \tau f_b)$ have opposite orientation.*
2. *$I_{n,3}(f)$ is the unique Szegő-Peherstorfer quadrature formula exact in $\mathcal{L}_{2n-4}$ and prefixed nodes a, b and $\alpha \in (a, b)$ if and only if $W_n = Q_{n,3}$ as in (5) satisfying the second condition of Theorem 3.11 with $\eta \in \mathbb{D}$, i.e. $\operatorname{arc}(a, \alpha, b)$ and $\operatorname{arc}(f_a, f_\alpha, f_b)$ have the same orientation, where $f_\alpha = \overline{F_{n-1}(\alpha)}$. If moreover τ is such that $\tau\overline{\eta} + \delta_{n-1} \neq 0$ then $I_{n,3}(f)$ is exact in $\mathcal{L}_{2n-3}(\omega)$ with $\omega = \frac{\tau\overline{\eta}+\delta_{n-\ell}}{\overline{\tau}\eta+\overline{\delta_{n-\ell}}}$.*

Proof. By Theorems 3.11, it is sufficient that the nodal polynomial $Q_{n,3}$ satisfies the conditions mentioned, for both q.f. to have n simple nodes in the arc $[a, b] \subset \mathbb{T}$. The positivity of the weights is deduced by the same argument used in the previous theorem. The necessity of the conditions is a consequence of [39, Theorem 3.1]. □

4.4. Positive Szegő-type q.f. with an arbitrary number of prefixed nodes

For quadrature formulas with more than 3 prefixed nodes, we shall only consider integrals where $\operatorname{supp}(\mu) = \mathbb{T}$ as we did in the previous section for the location of the zeros of QPOPUC of order $2\ell + 1 > 3$.

Theorem 4.7 (**General Szegő-Peherstorfer quadrature formula**). *Let $n > 2\ell + 1$, and let $I_{n,2\ell+1}(f)$ be a q.f. of the form (26) with nodal polynomial W_n.*

1. *$I_{n,2\ell+1}(f)$ is a Szegő-Peherstorfer q.f. exact in $\mathcal{L}_{2(n-\ell)-1}(\omega)$ with prefixed nodes $\{\alpha_i\}_{i=1}^{2\ell} \subset \mathbb{T}$ if and only if $W_n = Q_{n,2\ell+1}$ as in (14) satisfying the hypothesis of Theorem 3.12 where τ is such that $\tau\overline{P_\ell(0)} - \delta_{n-\ell} \neq 0$ and $\omega = \frac{\tau\overline{P_\ell(0)}-\delta_{n-\ell}}{\overline{\tau}P_\ell(0)-\overline{\delta_{n-\ell}}}$.*
2. *$I_{n,2\ell+1}(f)$ is the unique Szegő-Peherstorfer q.f. of the form (26) exact in $\mathcal{L}_{2(n-\ell-1)}$ and prefixed nodes $\{\alpha_i\}_{i=1}^{2\ell+1} \subset \mathbb{T}$ if and only if $W_n = Q_{n,2\ell+1}$ as in (14) satisfying the hypothesis of Theorem 3.14. If moreover τ is such that $\tau\overline{P_\ell(0)} - \delta_{n-\ell} \neq 0$ then $I_{n,2\ell+1}(f)$ is exact in $\mathcal{L}_{2(n-\ell-1)}(\omega)$ with $\omega = \frac{\tau\overline{P_\ell(0)}-\delta_{n-\ell}}{\overline{\tau}P_\ell(0)-\overline{\delta_{n-\ell}}}$.*

Proof. The existence and uniqueness of $W_n = Q_{n,2\ell+1}$ is the result of Theorem 3.12 and Theorem 3.14, and the exactness in $\mathcal{L}_{2(n-\ell)-1}(\omega)$ is an immediate consequence of Theorem 4.1. As in the previous case, we are going to show that $Q_{n,2\ell+1}$ is a $(n, 1)$-QPOPUC of a positive auxiliary measure. Therefore, the associated q.f. is actually a Szegő-Radau quadrature for this auxiliary measure that coincides with our q.f. $I_{n,2\ell}(f)$ in $\mathcal{L}_{2(n-\ell)-1}(\omega)$, hence it is a positive q.f. We know that the Schur-Cohn test for P_ℓ is equivalent with

$$\begin{bmatrix} P_\ell(z) \\ P_\ell^*(z) \end{bmatrix} = \hat{\theta}_\ell\hat{\theta}_{\ell-1} \cdots \hat{\theta}_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{with } \hat{\theta}_j = \begin{bmatrix} 1 & \kappa_j \\ \overline{\kappa_j} & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, \quad \kappa_j = P_j(0) \in \mathbb{D},$$

or, after multiplying the second row with $\tilde{\tau}$ and taking the transpose

$$[z \;\; \tilde{\tau}]\theta_1' \;\cdots\; \theta_\ell' \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}^{-1} = [P_\ell(z) \quad \tilde{\tau} P_\ell^*(z)], \;\; \theta_j' = \begin{bmatrix} 1 & \tilde{\tau}\overline{\kappa_j} \\ \overline{\tilde{\tau}}\kappa_j & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix},$$

while

$$\begin{bmatrix} \rho_{n-\ell-1}(z) \\ \rho_{n-\ell-1}^*(z) \end{bmatrix} = \theta_{n-\ell-1}\theta_{n-\ell-2} \;\cdots\; \theta_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \;\; \theta_j = \begin{bmatrix} 1 & \delta_j \\ \overline{\delta_j} & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$$

where the δ_j is the jth Schur parameter.

Now define

$$\begin{bmatrix} \tilde{\rho}_{n-\ell}(z) \\ \tilde{\rho}_{n-\ell}^*(z) \end{bmatrix} = \tilde{\theta}_{n-\ell} \begin{bmatrix} \rho_{n-\ell-1}(z) \\ \rho_{n-\ell-1}^*(z) \end{bmatrix}, \quad \text{with} \;\; \tilde{\theta}_{n-\ell} = \theta_\ell',$$

and recursively

$$\begin{bmatrix} \tilde{\rho}_{n-\ell+j}(z) \\ \tilde{\rho}_{n-\ell+j}^*(z) \end{bmatrix} = \tilde{\theta}_{n-\ell+j} \begin{bmatrix} \tilde{\rho}_{n-\ell+j-1}(z) \\ \tilde{\rho}_{n-\ell+j-1}^*(z) \end{bmatrix}, \quad j = 1, \ldots, \ell-1$$

where

$$\tilde{\theta}_{n-\ell+j} = \begin{bmatrix} 1 & \tilde{\delta}_{n-\ell+j} \\ \overline{\tilde{\delta}_{n-\ell+j}} & 1 \end{bmatrix} = \theta_{\ell-j}' = \begin{bmatrix} 1 & \tilde{\tau}\overline{\kappa_{\ell-j}} \\ \overline{\tilde{\tau}}\kappa_{\ell-j} & 1 \end{bmatrix}.$$

Thus, we end up with

$$\begin{aligned} Q_{n,2\ell+1}(z) &= [zP_\ell(z) \;\; \tilde{\tau}P_\ell^*(z)] \begin{bmatrix} \rho_{n-\ell-1}(z) \\ \rho_{n-\ell-1}^*(z) \end{bmatrix} \\ &= [z \quad \tilde{\tau}]\tilde{\theta}_{n-1} \;\cdots\; \tilde{\theta}_{n-\ell}\theta_{n-\ell-1} \;\cdots\; \theta_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= z\tilde{\rho}_{n-1}(z) + \tilde{\tau}\tilde{\rho}_{n-1}^*(z). \end{aligned}$$

In other words, the $\tilde{\rho}_{n-1}$ are monic Szegő polynomials for a positive measure on the unit circle by Favard's theorem since it has parameters $\{\delta_1, \delta_2, \ldots, \delta_{n-\ell-1}\tilde{\delta}_{n-\ell}, \ldots, \tilde{\delta}_{n-1}\}$ that are all in $\mathbb{D}$. Therefore all the weights are positive. This shows the existence of a positive q.f. if all the conditions of Theorem 3.12 or Theorem 3.14 are satisfied. As in previous theorems, the reciprocal is a consequence of [39, Theorem 3.1]. □

Note that F. Peherstorfer in Theorem 3.1 of his paper [39] characterized positive quadratures exact in the spaces $\mathcal{L}_{2(n-\ell-1)}$ in terms of quasi-paraorthogonal polynomials of odd order but without observing the orthogonality conditions that these verify. However, he did not provide strategies for the construction of these positive q.f., and this has been precisely one of our purposes, taking advantage of the use of QPOPUC. This is why we used the term Szegő-Peherstorfer quadrature.

Our theorems recover these results but they are important because we also show the influence of the invariance parameter on the nodes and on the maximal domain of exactness. It opens the road to an analysis of the dynamics of the nodes as they vary in terms of this

invariance parameter. But more importantly, our theorems are constructive and actually describe a computational procedure to obtain the Schur coefficients needed to generate the nodal polynomial, and from these, it is possible to compute the nodes and weights of the q.f. (see below in the numerical section).

If $2\ell+1$ nodes are fixed then there can be at most one solution for the $Q_{n,2\ell+1}$ and thus also the positive q.f. with these prefixed nodes and exact in $\mathcal{L}_{2(n-\ell-1)}$ will be unique if it exists. It will also be exact in $\mathcal{L}_{2(n-\ell)-1}(\omega)$ with $\omega = \tau\tilde{\tau}$ where τ and $\tilde{\tau}$ are defined by the fixed nodes. If however we only fix 2ℓ nodes, there may be an infinite set of $Q_{n,2\ell+1}$, depending on its invariance parameter $\tau = Q_{n,2\ell+1}(0)$ that may vary in certain arcs on $\mathbb{T}$. The corresponding q.f., if it exists, is exact in a space $\mathcal{L}_{2(n-\ell)-1}(\omega)$, where now ω depends on τ. The freedom to choose τ in certain arcs has been used before to fix an extra node. We could use the same freedom to fix ω instead. Given $2\ell+1$ nodes, find the corresponding solution $Q_{n,2\ell+1}$, i.e., find P_ℓ and τ in (14), that matches these nodes results in the linear problem that we discussed above. In essence P_ℓ is used to match 2ℓ nodes and the parameter τ is used to match the last node. Given 2ℓ nodes and ω, find the corresponding $Q_{n,2\ell+1}$ is the same when using P_ℓ to match the 2ℓ nodes, but finding a τ that gives a prescribed ω is a difficult nonlinear problem that does not allow an explicit solution. However, generically there will be at most two values for τ as we prove in Theorem A.5 of the Appendix. In both cases, if a solution $Q_{n,2\ell+1}$ of these matching problems exists, then it remains to check that this results in a positive q.f. In practice, this is verified by the Schur-Cohn test for the polynomial P_ℓ.

5. Numerical experiments

The aim of this section is to present, some illustrative numerical examples in the construction of positive q.f. on the unit circle with a number of prescribed nodes greater than two and maximal domain of exactness. We consider the approximation of the integral (25) where μ is one of the most important measures on the unit circle: the Rogers-Szegő (normalized) weight function (also known as "wrapped" Gaussian measure), given by

$$\mu(\theta,q) = \mu(\theta) = \frac{1}{\sqrt{2\pi\log(1/q)}} \sum_{j=-\infty}^{\infty} \exp\left(-\frac{(\theta-2\pi j)^2}{2\log(1/q)}\right), \quad 0<q<1. \tag{29}$$

Properties and applications of Rogers-Szegő polynomials, the family of orthogonal polynomials on $\mathbb{T}$ with respect to μ given by (29), have been extensively studied in recent years. In particular, quadrature rules associated with this measure were considered in [20].

It is very well known that for this weight function, the trigonometric moments and monic Szegő polynomials are explicitly given by $\mu_k = q^{\frac{k^2}{2}}$, $k\in\mathbb{Z}$, and by

$$\rho_n(z) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{n-j}{2}} z^j,$$

where for all $0<q<1$, the usual q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(n)_q}{(j)_q (n-j)_q}, \quad \text{where} \quad (n)_q = (1-q)(1-q^2)\cdots(1-q^n) \quad \text{and} \quad (0)_q = \begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q \equiv 1,$$

respectively. So, $\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(1-q^n)\ \cdots\ (1-q^{n-j+1})}{(1-q)\ \cdots\ (1-q^j)}$, $\delta_n = (-1)^n q^{\frac{n}{2}}$ and $\rho_n^*(z) = (-1)^n q^{-\frac{n}{2}} \rho_n(qz)$.

The numerical computations were implemented in Matlab software with double precision.

Suppose first we want to construct a 16-point rule with 6 prescribed nodes ($\ell = 3$): $\alpha_1 = e^{-\frac{3\pi}{4}i}$, $\alpha_2 = -i$, $\alpha_3 = 1$, $\alpha_4 = e^{\frac{\pi}{4}i}$, $\alpha_5 = \overline{\alpha_2}$ and $\alpha_6 = \overline{\alpha_1}$ and we want to guarantee positive weights and a maximal degree of exactness, which is a space $\mathcal{L}_{25}(\omega)$ for some ω.

We know $Q_{16,7}$ is given by

$$Q_{16,7}(z) = zP_3(z)\rho_{12}(z) + \tau_3 P_3^*(z)\rho_{12}^*(z), \quad P_3(z) = (z-\eta_1)(z-\eta_2)(z=\eta_3),$$

and the maximal domain of exactness of the positive q.f. $I_{16,7}(f)$ is

$$\mathcal{L}_{25}(\omega_3) = \mathbb{L}_{-12,12} \oplus \text{span}\left\{ z^{13} - \frac{\omega_3}{z^{13}} \right\}, \quad \omega_3 = \tau_3 \tilde{\tau}_3, \quad \tilde{\tau}_3 = \frac{\overline{\eta_1\eta_2\eta_3} + \overline{\tau_3}\delta_{13}}{\overline{\tau_3}\eta_1\eta_2\eta_3 + \overline{\delta_{13}}}.$$

Observe that the nodes of the q.f. that were not prefixed, the $\{\eta_i\}_{i=1}^3$, the orthogonality parameter $\tilde{\tau}_3$ and thus also ω_3 all depend on the choice for the invariance parameter τ_3. We have chosen $\tau_3 = e^{0.9\pi i}$, resulting in $\tilde{\tau} \approx e^{-0.87834\pi i}$ and thus $\omega_3 \approx e^{0.02166\pi i}$ for the example in Table 1 below. The domain of exactness is maximized for this ω_3.

Using the procedure developed in Theorem 4.7 we can compute the Schur parameters $\tilde{\delta}_{13}, \tilde{\delta}_{14}, \tilde{\delta}_{15}$ defined by the preselected nodes. These parameters, together with the Schur parameters δ_j, $j = 1, \ldots, 12$, for the Rogers-Szegő polynomials allow us to compute the nodes and weights from the CMV matrix associated with them. See [12, 13, 11, 6]. The results for $\tau_3 = e^{0.9\pi i}$ are displayed in Table 1 and the nodes are also graphically represented in Figure 2(A).

As we have mentioned before, positive quadrature formulas will be guaranteed for τ_3 in certain arcs of $\mathbb{T}$. In Figure 2(D) we have indicated these as green arcs. In our example, these are the arcs given approximately by $\{e^{\theta\pi i} : \theta \in (0.251, 0.499) \cup (0.765, 1.229) \cup (1.505, 1.995)\}$. For each τ in the three green arcs, the conditions of Theorems 3.12 and 4.7 are satisfied and we can obtain a positive q.f. with the 6 nodes prefixed and we obtain exactness in $\mathcal{L}_{23}(\omega_3)$.

Note that the boundaries of the arcs seem to coincide with the fixed nodes, but that is not true. We know that at such a boundary point, one of the η's crosses $\mathbb{T}$ and at the moment it is on $\mathbb{T}$, it must coincide with (at least one) zero of $Q_{16,7}$. Experimentally we observed that this η leaves $\mathbb{T}$ by passing through one of the prefixed α's. This creates some 'degenerate situation' where one of the 'moving zeros' of $Q_{16,7}$, (i.e. the zeros varying with τ) collides with that α (as in Figure 2(C) where the collision happens in $\alpha = \exp(0.75\pi i)$ and only 15 distinct nodes are shown). Since that α is then a double zero, it will also be a zero of the derivative $Q'_{16,7}$, while all the other zeros of $Q'_{16,7}$ stay in $\mathbb{D}$. It is intuitively clear that the 'moving zeros' closest to τ will be most sensitive to varying τ. That explains why the collision of a moving zero and a prefixed α will happen near the value of τ, and thus why the boundary points for the arcs of τ will be near the prefixed zeros.

The complementary red arcs correspond to situations where at least one of the η's, and hence one of the Schur parameters for P_3 is not in $\mathbb{D}$. When τ enters a red arc, no positive

Nodes	Weights
$-0.942694568084625 - 0.333656936543722i$	$0.8839144135446476e-03$
$\alpha_1 = \frac{\sqrt{2}}{2}(-1-i)$	$0.3573449079562991e-02$
$-0.379700471214962 - 0.925109481174597i$	$0.1167016324025051e-01$
$\alpha_2 = -i$	$0.3157192607103394e-01$
$0.382004048856982 - 0.924160649809800i$	$0.6909738483841478e-01$
$0.706940179145082 - 0.707273343984008i$	$0.1207223162006649e-00$
$0.923913888542018 - 0.382600479036773i$	$0.1684088258652674e-00$
$\alpha_3 = 1$	$0.1880771416745326e-00$
$0.923939153768060 + 0.382539462192281i$	$0.1683601716568688e-00$
$\alpha_4 = \frac{\sqrt{2}}{2}(1+i)$	$0.1206961198605809e-00$
$0.382340759272763 + 0.924021397911716i$	$0.6914095657372876e-01$
$\alpha_5 = i$	$0.3164220053779544e-01$
$-0.381050525980387 + 0.924554215095074i$	$0.1167384545514241e-01$
$\alpha_6 = \frac{\sqrt{2}}{2}(-1+i)$	$0.3486160898429015e-02$
$-0.908886480405499 + 0.417043601720617i$	$0.5376356190583758e-03$
$-0.974806799005855 + 0.223050901392394i$	$0.4577880151194835e-03$

Table 1: A 16-point q.f. for the Rogers-Szegő weight function with prescribed nodes $\{\alpha_i\}_{i=1}^{6}$ and maximal domain of exactness $\mathcal{L}_{25}(\omega)$. We have taken $q = 0.5$ and $\tilde{\tau}_3 = e^{0.9\pi i}$.

quadrature exist for several reasons. What happens there depends on the location of the zeros of the derivative $Q'_{16,7}$. If all the zeros of $Q'_{16,7}$ are in $\mathbb{D}$, then the zeros of $Q_{16,7}$ are simple and on $\mathbb{T}$. If $Q'_{16,7}$ has a zero in $\mathbb{E}$, then there is at least one pair of zeros of $Q_{16,7}$ not on $\mathbb{T}$ as in Figure 2(B). Since a zero of $Q'_{16,7}$ can only leave $\mathbb{D}$ by crossing $\mathbb{T}$, it can only pass $\mathbb{T}$ through a multiple zero of $Q_{16,7}$. We observe that this happens in a red arc when two moving zeros collide.

The remaining Figure 2(D) shows the green and red arcs for τ. The nodes are plotted for the situation $\tau = 1$ where there are 16 simple nodes on $\mathbb{T}$ but not all the weights are positive. It seems that the figure shows only 15 nodes but a moving zero almost perfectly coincides with the prefixed $\alpha_3 = 1$ so that they are not seen as distinct on the figure. As we have pointed out in Remark 3.13, for a deeper exploration it is necessary to make a detailed study of the behaviour of the corresponding Schur parameters.

The freedom to choose the τ in these green arcs allows us to fix a seventh node. This is described by Theorems 3.14 and 4.7. As an example we choose again the Rogers-Szegő

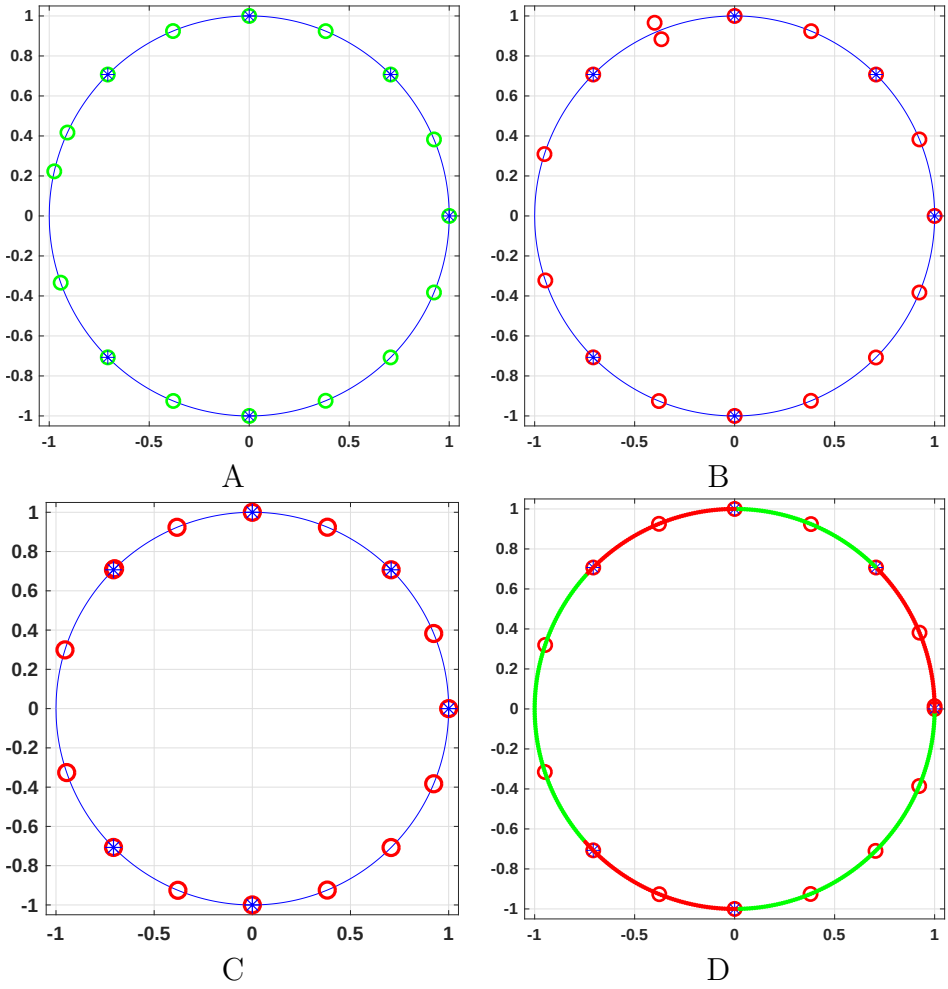

Figure 2: The plot shows the distribution of nodes in the Rogers-Szegő example. The circles show the locations of the computed nodes of the q.f. with the stars indicating the preselected nodes. The figure corresponds to the nodal polynomial $Q_{16,7}$ with 6 preselected nodes. The q.f. is exact in $\mathcal{L}_{25}(\omega)$. (A): $\tau = \exp(0.9\pi i)$ corresponding to $\omega \approx \exp(0.02166\pi i)$. (B): $\tau = \exp(0.63\pi i)$ there is one pair of nodes not on $\mathbb{T}$. (C): $\tau \approx \exp(0.7575\pi i)$ one moving zero coincides with one of the prefixed (in this case with $\exp(0.75\pi i)$). (D): Nodes for $\tau = 1$. There are 16 simple nodes, but not all the weights are positive. Seemingly only 15 nodes are plotted, but $\alpha_3 = 1$ is a nearly double node. For τ in a green arc, the weights are positive. For τ in a red arc the weights are not positive.

polynomials but now select seven nodes that are less symmetric on the circle. The results are shown for the nodal polynomial $Q_{16,7}$ in Table 2 and Figure 3.

Nodes	Weights
$-0.826939158283098 - 0.562291408878033i$	$0.2376579706882289e-02$
$-0.467939713417261 - 0.883760388684045i$	$0.1037662006497247e-01$
$-0.043436557033519 - 0.999056187365392i$	$0.3149875854919127e-01$
$0.369113471779490 - 0.929384336510408i$	$0.7101663025883392e-01$
$* \ 0.707106781186549 - 0.707106781186549i$	$0.1233651358289617e-00$
$0.925520203512843 - 0.378698234600514i$	$0.1687187874278502e-00$
$* \ 1.000000000000000 - 0.000000000000001i$	$0.1840313040017806e-00$
$* \ 0.929776485888252 + 0.368124552684676i$	$0.1575356797392286e-00$
$* \ 0.809016994374943 + 0.587785252292476i$	$0.2978440722236509e-01$
$* \ 0.707106781186542 + 0.707106781186551i$	$0.8780562736213628e-01$
$* \ 0.637423989748693 + 0.770513242775784i$	$0.1721291309058466e-01$
$0.384703682295919 + 0.923040127420233i$	$0.6764455894241819e-01$
$* \ 0.000000000000000 + 1.000000000000000i$	$0.3284367751303363e-01$
$-0.408105016619360 + 0.912934989695385i$	$0.1198741905917201e-01$
$-0.772470256507957 + 0.635050945051283i$	$0.3103537707867246e-02$
$-0.996767928597351 + 0.080334902251429i$	$0.6983635247225544e-03$

Table 2: A 16-point q.f. for the Rogers-Szegő weight function with prescribed nodes $\{\alpha_i\}_{i=1}^{7}$ (in blue and marked by *) and maximal domain of exactness. The required τ_3 was $\tau_3 = -0.363884303133021 - 0.931444155026694i \approx e^{-0.6\pi i}$ and the formula is exact in $\mathcal{L}_{25}(\omega_3)$ with $\omega_3 \approx e^{-0.8\pi i}$.

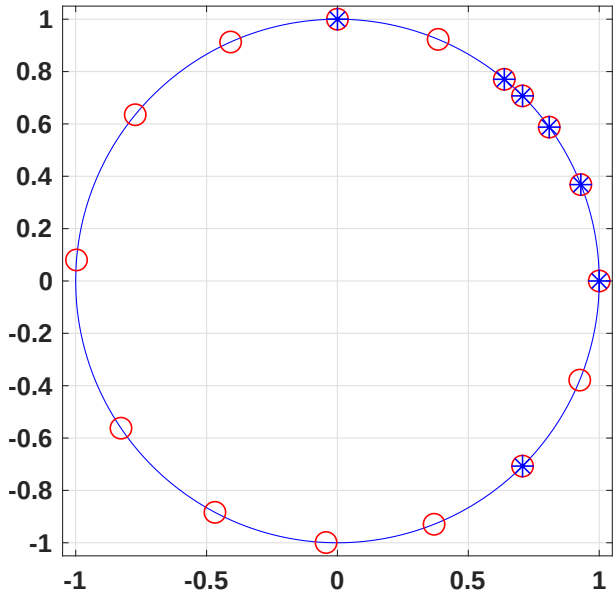

Figure 3: The plot shows the distribution of nodes in the Rogers-Szegő example. Seven nodes are prefixed based on the nodal polynomial $Q_{16,7}$ with an appropriate choice of τ_3 The red circles show the locations of the computed nodes of the q.f. with the stars indicating the preselected nodes. The quadrature is exact in $\mathcal{L}_{25}(\omega_3)$ with $\omega_3 \approx e^{0.8\pi i}$.

Appendix

In this addendum we collect some auxiliary properties that are used to prove some results in the body of this paper.

The first Lemma relates to points on $\mathbb{T}$ and it is quoted from [2, Chapter 3].

Lemma A.1. *Let $[z_1, z_3]$ be an arc on $\mathbb{T}$ and $z_2 \in (z_1, z_3)$. Then*

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \in \mathbb{R} \Leftrightarrow z \in \mathbb{T}.$$

Moreover,

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \in \mathbb{R}^+ \Leftrightarrow z \in (z_1, z_3),$$

The next lemma is a simple geometric observation.

Lemma A.2. *Let $\alpha, \beta, \gamma, \delta$ be four distinct points on $\mathbb{T}$ with $\alpha\delta - \beta\gamma \neq 0$ then the secants $\alpha\beta$ and $\gamma\delta$ intersect in $\mathbb{D}$ if and only if* $\operatorname{arc}(\alpha, \beta, \gamma)$ *and* $\operatorname{arc}(\alpha, \beta, \delta)$ *have opposite orientation.*

Proof. The condition $\alpha\delta - \beta\gamma \neq 0$ implies that the two secants are not parallel. The secant $\alpha\beta$ divides $\mathbb{T}$ in two arcs A_1 and A_2. It is obvious that the secant $\gamma\delta$ will intersect the secant $\alpha\beta$ in $\mathbb{D}$ if and only if both γ and δ belong to the interiors of different arcs. If γ belongs to $\mathring{A}_1$, then δ must belong to $\mathring{A}_2$ and vice versa. This is equivalent with saying that $\operatorname{arc}(\alpha, \beta, \gamma)$ and $\operatorname{arc}(\alpha, \beta, \delta)$ have opposite orientation. Similarly, it holds that the secants intersect in $\mathbb{E}$ if and only if $\operatorname{arc}(\alpha, \beta, \gamma)$ and $\operatorname{arc}(\alpha, \beta, \delta)$ have the same orientation. □

The following relations are easy to verify.

Lemma A.3. *Let $n > 2\ell + 1$. Let $Q_{n,2\ell+1}(z) = zp_\ell(z)\rho_{n-\ell-1}(z) + \tau p^*_\ell(z)\rho^*_{n-\ell-1}(z)$ with $p_\ell \in \mathbb{P}_\ell \backslash \mathbb{P}_{\ell-1}$ monic and $\tau \in \mathbb{T}$.*
Define $a_k = \langle Q_{n,2\ell+1}, z^k \rangle$, $k = 0, \ldots, n$, then $a_k = \tau\overline{a_{n-k}}$ and $a_k = 0$, $k = \ell+1, \ldots, n-\ell-1$ while

$$a_{n-\ell} = (p_\ell(0) - \tau\overline{\delta_{n-\ell}})\|\rho_{n-\ell-1}\|^2 \quad \text{and} \quad \langle Q_{n,2\ell+1}, z^{n-\ell} - \tilde{\tau} z^\ell \rangle = a_{n-\ell} - \overline{\tilde{\tau}}\tau\overline{a_{n-\ell}}.$$

If $a_{n-\ell} = 0$ or equivalently $\sigma_{n,\ell} = \overline{p_\ell(0)} - \overline{\tau}\delta_{n-\ell} = 0$ then there exists a monic $p_{\ell-1} \in \mathbb{P}_{\ell-1} \backslash \mathbb{P}_{\ell-2}$ such that

$$Q_{n,2\ell+1}(z) = zp_{\ell-1}\rho_{n-\ell}(z) + \tau p^*_{\ell-1}(z)\rho^*_{n-\ell}(z) = Q_{n,2\ell-1}(z).$$

If $a_{n-\ell} \neq 0$ then

$$\langle Q_{n,2\ell+1}, z^{n-\ell} - \tilde{\tau} z^\ell \rangle = 0 \Leftrightarrow \tilde{\tau} = \tau\frac{\overline{a_{n-\ell}}}{a_{n-\ell}} = \frac{\tau\overline{p_\ell(0)} - \delta_{n-\ell}}{p_\ell(0) - \tau\overline{\delta_{n-\ell}}}.$$

Let $\tilde{Q}_{n,2\ell+1}(z) = q^*_\ell(z)\rho_{n-\ell}(z) + \tilde{\tau} q_\ell(z)\rho^*_{n-\ell}(z)$, with $q_\ell \in \mathbb{P}_\ell \backslash \mathbb{P}_{\ell-1}$ monic and $\tilde{\tau} \in \mathbb{T}$.
Define $\tilde{a}_k = \langle \tilde{Q}_{n,2\ell+1}(z), z^k \rangle$ then $\tilde{a}_k = \tilde{\tau}\overline{\tilde{a}_{n-k}}$, $k = 0, \ldots, n$, and $\tilde{a}_k = 0$ for $k = \ell +$

$1,\dots,n-\ell-1$ *while* $\tilde{a}_{n-\ell}=\|\rho_{n-\ell}\|^2\neq 0$, *thus* $\tilde{Q}_{n,2\ell+1}\in\mathbb{Q}^{\perp}_{n,2\ell+1}\backslash\mathbb{Q}^{\perp}_{n,2\ell-1}$.
If $\nu_{n,\ell}=\overline{q_\ell(0)}+\tilde{\tau}\overline{\delta_{n-\ell}}=0$ *then* $\tilde{Q}_{n,2\ell+1}\in\mathbb{P}_{n-1}$ *and* $\tilde{Q}_{n,2\ell+1}(0)=0$ *and thus* $\tilde{Q}_{n,2\ell+1}(z)=zQ_{n-2,2\ell-1}(z)$ *with* $\langle Q_{n-2,2\ell-1},z^{n-\ell-1}\rangle\neq 0$.
If $\nu_{n,\ell}\neq 0$ *then* $\tilde{Q}_{n,2\ell+1}\in\mathbb{Q}^{\perp}_{n,2\ell+1}\backslash\mathbb{Q}^{\perp}_{n,2\ell-1}$ *is in* $\mathbb{P}_n\backslash\mathbb{P}_{n-1}$, *and if* $\sigma_{n,\ell}\neq 0$ *then the monic polynomial* $\nu_{n,\ell}^{-1}\tilde{Q}_{n,2\ell+1}\in\mathbb{Q}^{\perp}_{n,2\ell+1}\backslash\mathbb{Q}^{\perp}_{n,2\ell-1}$.
Thus if $\nu_{n,\ell}\sigma_{n,\ell}\neq 0$ *then* $\nu_{n,\ell}^{-1}\tilde{Q}_{n,2\ell+1}(z)=Q_{n,2\ell+1}$ *are equivalent representations of all monic polynomials of degree* n *in* $\mathbb{Q}^{\perp}_{n,2\ell+1}\backslash\mathbb{Q}^{\perp}_{n,2\ell-1}$.

Note that $\tilde{a}_{n-\ell}=\tilde{\tau}\overline{\tilde{a}_\ell}>0$ so that $\langle\tilde{Q}_{n,2\ell+1},z^{n-\ell}-\tilde{\tau}z^\ell\rangle=0$.

The next result gives the relations between $\tau,\tilde{\tau},p_\ell$ and q_ℓ for monic polynomials in $\mathbb{Q}^{\perp}_{n,2\ell+1}\backslash\mathbb{Q}^{\perp}_{n,2\ell-1}$.

Lemma A.4. *Let* p_ℓ *and* q_ℓ *be monic polynomials of degree* n *and assume* $\nu_{n,\ell}=\overline{q_\ell(0)}+\tilde{\tau}\overline{\delta_{n-\ell}}\neq 0$, *and* $\sigma_{n,\ell}=\overline{p_\ell(0)}-\overline{\tau}\delta_{n-\ell}\neq 0$ *then*

$$\begin{aligned}Q_{n,2\ell+1}(z) &= \nu_{n,\ell}^{-1}[q_\ell^*(z)\rho_{n-\ell}(z)+\tilde{\tau}q_\ell(z)\rho_{n-\ell}^*(z)],\\ &= zp_\ell(z)\rho_{n-\ell-1}(z)+\tau p_\ell^*(z)\rho_{n-\ell-1}^*(z),\end{aligned}$$

represent all monic $(n,2\ell+1)$*-QPOPUC not in* $\mathbb{Q}^{\perp}_{n,2\ell-1}$ *and*

$$\begin{aligned}&p_\ell=\nu_{n,\ell}^{-1}[q_\ell^*+\tilde{\tau}\overline{\delta_{n-1}}q_\ell],\quad \nu_{n,\ell}=\overline{q_\ell(0)}+\tilde{\tau}\overline{\delta_{n-\ell}}\\ &q_\ell=\sigma_{n,\ell}^{-1}[p_\ell^*-\overline{\tau}\delta_{n-\ell}p_\ell],\quad \sigma_{n,\ell}=\overline{p_\ell(0)}-\overline{\tau}\delta_{n-\ell}\\ &\nu_{n,\ell}\overline{\sigma_{n,\ell}}=\overline{\nu_{n,\ell}}\sigma_{n,\ell}=1-|\delta_{n-\ell}|^2>0\\ &\tau=\tilde{\tau}\overline{\nu_{n,\ell}}/\nu_{n,\ell}=\tilde{\tau}\overline{\sigma_{n,\ell}}/\sigma_{n,\ell}\\ &\omega=\tau\tilde{\tau}=\frac{\tilde{\tau}q_\ell(0)+\delta_{n-\ell}}{\overline{\tilde{\tau}}\overline{q_\ell(0)}+\overline{\delta_{n-\ell}}}=\frac{\tau\overline{p_\ell(0)}-\delta_{n-\ell}}{\overline{\tau}p_\ell(0)-\overline{\delta_{n-\ell}}}.\end{aligned}$$

Proof. Using the Szegő recursion we get

$$\begin{aligned}Q_{n,2\ell+1} &= \nu_{n,\ell}^{-1}[z\rho_{n-\ell-1}+\delta_{n-\ell}\rho_{n-\ell-1}^*]q_\ell^*+\tilde{\tau}\nu_{n,\ell}^{-1}[\rho_{n-\ell-1}^*+\overline{\delta_{n-\ell}}z\rho_{n-\ell-1}]q_\ell\\ &= \nu_{n,\ell}^{-1}[q_\ell^*+\tilde{\tau}\overline{\delta_{n-\ell}}q_\ell]z\rho_{n-\ell-1}+\nu_{n,\ell}^{-1}[q_\ell^*\delta_{n-\ell}+\tilde{\tau}q_\ell]\rho_{n-\ell-1}^*\\ &= zp_\ell\rho_{n-\ell-1}+\tau p_\ell^*\rho_{n-\ell-1}^*,\quad p_\ell=\nu_{n,\ell}^{-1}[q_\ell^*+\tilde{\tau}\overline{\delta_{n-\ell}}q_\ell]\end{aligned}$$

with $\nu_{n,\ell}=\overline{q_\ell(0)}+\tilde{\tau}\overline{\delta_{n-\ell}}$ because q_ℓ is monic and $\tau=\tilde{\tau}\overline{\nu_{n,\ell}}/\nu_{n,\ell}$. For the inverse relation we use

$$\begin{aligned}p_\ell^*-\overline{\tau}\delta_{n-\ell}p_\ell &= \overline{\nu_{n,\ell}^{-1}}(q_\ell+\overline{\tilde{\tau}}\delta_{n-\ell}q_\ell^*)-\overline{\tau}\delta_{n-\ell}\nu_{n,\ell}^{-1}(q_\ell^*+\tilde{\tau}\overline{\delta_{n-\ell}}q_\ell)\\ &= (\overline{\nu_{n,\ell}^{-1}}-\overline{\tau}\delta_{n-\ell}\nu_{n,\ell}^{-1}\tilde{\tau}\overline{\delta_{n-\ell}})q_\ell+(\overline{\nu_{n,\ell}^{-1}}\overline{\tilde{\tau}}\delta_{n-\ell}-\overline{\tau}\delta_{n-\ell}\nu_{n,\ell}^{-1})q_\ell^*\\ &= \overline{\nu_{n,\ell}^{-1}}\left(1-\overline{\tau}\tilde{\tau}\frac{\nu_{n,\ell}^{-1}}{\overline{\nu_{n,\ell}^{-1}}}|\delta_{n-\ell}|^2\right)q_\ell+\delta_{n-\ell}(\overline{\nu_{n,\ell}^{-1}}\overline{\tilde{\tau}}-\overline{\tau}\nu_{n,\ell}^{-1})q_\ell^*\\ &= \overline{\nu_{n,\ell}^{-1}}(1-|\delta_{n-\ell}|^2)q_\ell,\end{aligned}$$

where the last line follows from $\tau = \tilde{\tau}\frac{\nu_{n,\ell}^{-1}}{\overline{\nu_{n,\ell}^{-1}}}$. Because q_ℓ is monic, $\sigma_{n,\ell} = \overline{\nu_{n,\ell}^{-1}}(1 - |\delta_{n-\ell}|^2) = \overline{p_\ell(0)} - \overline{\tau}\delta_{n-\ell}$. The remaining equalities are immediately verified. □

Let $I_{n,2\ell+1}(f)$ be a positive Szegő-Peherstorfer quadrature formula as described in Theorem 3.14 with 2ℓ prefixed nodes and exact in $\mathcal{L}_{2(n-\ell)+1}(\omega)$ with ω depending on τ as described in the theorem. Given τ, we can compute P_ℓ and ω. The converse of finding τ, given ω is a highly nonlinear problem. However, generically, there will be two solutions as follows by the next theorem.

Theorem A.5. *Let P_ℓ be defined by the system (16) parametrized in τ then for $\omega \in \mathbb{T}$, the nonlinear equation $\omega = \frac{\tau\overline{P_\ell(0)} - \delta_{n-\ell}}{\overline{\tau}P_\ell(0) - \overline{\delta_{n-\ell}}}$, will in general have at most two different solutions for $\tau \in \mathbb{T}$.*

Proof. The system (16) can be written as

$$\mathcal{M}\begin{bmatrix} I & \\ & \tau I \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ \overline{\mathbf{p}} \end{bmatrix} = -\tau\mathbf{f} - \mathbf{d}.$$

If $\det \mathcal{M} \neq 0$, then

$$\begin{bmatrix} \mathbf{p} \\ \overline{\mathbf{p}} \end{bmatrix} = -\begin{bmatrix} \tau I & \\ & I \end{bmatrix}\mathcal{M}^{-1}\mathbf{f} - \begin{bmatrix} I & \\ & \overline{\tau} I \end{bmatrix}\mathcal{M}^{-1}\mathbf{d}.$$

The element on the first row is $P_\ell(0) = \prod_{i=1}^{\ell}(-\eta_i)$ which has the form $A\tau + B$ with $[A, B] = -[1\ 0\ \cdots\ 0]\mathcal{M}^{-1}[\mathbf{f}, \mathbf{d}]$. Substituting this into the equation for ω and writing D for $\overline{\delta_{n-\ell}}$ gives $\omega[\overline{\tau}(A\tau + B) - D] = \tau(\overline{A}\overline{\tau} + \overline{B}) - \overline{D}$. Since we are looking for $\tau \in \mathbb{T}$, this results in a constrained quadratic equation

$$\overline{B}\tau^2 + \tau[(\overline{A} - \overline{D}) - \omega(A - D)] - \omega B = 0, \quad \tau \in \mathbb{T}.$$

Thus except when $B = 0$, which means that $\omega = \frac{\overline{A}-\overline{D}}{A-D}$ does not depend on τ, and thus any τ will do, there will be at most two solutions for $\tau \in \mathbb{T}$. □

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