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Newton Polyhedra and Igusa's Local Zeta Function

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Introduction

0.1 Definition of Igusa's local zeta function, Newton polyhedron and non-degeneracy

For p prime, denote the field of p-adic numbers by $\mathbb{Q}_p$, the ring of p-adic integers by $\mathbb{Z}_p$, and the finite field of p elements by $\mathbb{F}_p$. If R is a commutative ring with identity, we denote the set of its units by $R^\times$.

Definition 0.1.1. Let $f(x) = f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ with p prime. For $z \in \mathbb{Q}_p$, $\operatorname{ord} z \in \mathbb{Z} \cup \{\infty\}$ denotes the valuation, $|z| = p^{-\operatorname{ord} z}$ and $\operatorname{ac}(z) = p^{-\operatorname{ord} z} z$ denotes the angular component. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a character of $\mathbb{Z}_p^\times$, i.e., a group homomorphism with finite image. We formally put $\chi(0) = 0$. To the above data we associate the following two Igusa local zeta functions (the global and the local one):

$$\begin{aligned} Z_f(s, \chi) &= \int_{\mathbb{Z}_p^n} \chi(\operatorname{ac} f(x)) |f(x)|^s |dx|, \\ \text{and } Z_{f,0}(s, \chi) &= \int_{(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x)) |f(x)|^s |dx|, \end{aligned}$$

for $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^n$ so normalized that $\mathbb{Z}_p^n$ has measure 1. If χ is the trivial character χ_{triv}, then we will write $Z_f(s)$ (resp. $Z_{f,0}(s)$) instead of $Z_f(s, \chi_{\text{triv}})$ (resp. $Z_{f,0}(s, \chi_{\text{triv}})$). See Appendix A for more information about the Haar measure on $\mathbb{Q}_p^n$.

Igusa's local zeta function $Z_f(s)$ for the trivial character contains all information about the numbers of solutions modulo p^e of the congruences $f(x) \equiv 0 \bmod p^e$, $e = 1, 2, 3, \ldots$ (see Lemma 0.1.3 or [Igu78, pp. 97–98], [Den91, Section 1.2]).

Definition 0.1.2. Let $f(x) = f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ with p a prime. We define the Poincaré series of f as the formal power series

$$P(t) = \sum_{e=0}^{\infty} N_e p^{-ne} t^e,$$

where N_e is the number of elements in the set

$$\{x + p^e\mathbb{Z}_p^n \mid x \in \mathbb{Z}_p^n \text{ and } f(x) \equiv 0 \bmod p^e\}$$

for $e \geq 1$ and $N_0 = 1$.

There exists a relation between the Poincaré series of f and Igusa's local zeta function $Z_f(s)$ associated to f.

Lemma 0.1.3. *Let $f(x) = f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$, p a prime and $t = p^{-s}$ with $s \in \mathbb{C}$, $\mathrm{Re}\,(s) > 0$. Then*

$$Z_f(s) = P(t) - \frac{P(t)-1}{t} \qquad \textit{and} \qquad P(t) = \frac{1 - tZ_f(s)}{1-t}.$$

Proof. We can write Igusa's local zeta function associated to f as

$$\begin{aligned} Z_f(s) &= \int_{\mathbb{Z}_p^n} |f(x)|^s |dx| \\ &= \sum_{e=0}^{\infty} \int_{\{x \in \mathbb{Z}_p^n \mid \mathrm{ord}\, f(x) = e\}} |f(x)|^s |dx| \\ &= \sum_{e=0}^{\infty} p^{-es} \cdot \text{measure of } \{x \in \mathbb{Z}_p^n \mid \mathrm{ord}\, f(x) = e\}. \end{aligned}$$

We have

$$\begin{aligned} &\text{measure of } \{x \in \mathbb{Z}_p^n \mid \mathrm{ord}\, f(x) = e\} \\ &= \text{measure of } \left(\{x \in \mathbb{Z}_p^n \mid f(x) \equiv 0 \bmod p^e\} \setminus \{x \in \mathbb{Z}_p^n \mid f(x) \equiv 0 \bmod p^{e+1}\}\right) \\ &= \text{measure of } \{x \in \mathbb{Z}_p^n \mid f(x) \equiv 0 \bmod p^e\} \\ &\quad - \text{measure of } \{x \in \mathbb{Z}_p^n \mid f(x) \equiv 0 \bmod p^{e+1}\} \\ &= \text{measure of } \bigcup_{\substack{a \in \{0,1,\ldots,p^e-1\}^n \\ f(a) \equiv 0 \bmod p^e}} a + p^e\mathbb{Z}_p^n \\ &\quad - \text{measure of } \bigcup_{\substack{a \in \{0,1,\ldots,p^{e+1}-1\}^n \\ f(a) \equiv 0 \bmod p^{e+1}}} a + p^{e+1}\mathbb{Z}_p^n \\ &= N_e p^{-en} - N_{e+1} p^{-(e+1)n}. \end{aligned}$$

So,

$$\begin{aligned} Z_f(s) &= \sum_{e=0}^{\infty} p^{-es}\left(N_e p^{-en} - N_{e+1}p^{-(e+1)n}\right) \\ &= \sum_{e=0}^{\infty} N_e p^{-ne}(p^{-s})^e - \sum_{e=0}^{\infty} N_{e+1}p^{-n(e+1)}(p^{-s})^e \\ &= P(t) - \frac{P(t)-1}{t}. \end{aligned}$$

□

Using resolution of singularities, Igusa [Igu74a] proved that $Z_f(s,\chi)$ is a rational function of p^{-s} (see also [Igu78]). An entirely different proof was obtained ten years later by Denef [Den84] using p-adic cell decomposition instead of resolution of singularities. We denote the meromorphic continuation of $Z_f(s,\chi)$ again by $Z_f(s,\chi)$.

Note that the fact that $Z_f(s)$ is a rational function in p^{-s} implies that infinitely many information, namely the numbers N_e for all $e \in \mathbb{N}$, is contained in the finite number of coefficients of the denominator and the numerator of the rational zeta function. The poles of $Z_f(s)$ and their orders determine the behaviour of the numbers of solutions of $f(x) \equiv 0 \bmod p^e$ for e big enough. Moreover, the poles with largest real part give the largest contribution to these numbers (for e big enough). If $f(0) = 0$, all N_e in Definition 0.1.2 are > 0. So, in this case, Definition 0.1.2, the relation between $Z_f(s)$ and $P(t)$ from Lemma 0.1.3 and the rationality of both imply that $Z_f(s)$ has at least one pole.

There are several conjectures about the poles of Igusa's local zeta function, namely a connection with topological invariants of the singularities of the hypersurface $f(x) = 0$. At this moment nobody has a clue to prove these conjectures in general. The aim of this thesis is to study the poles of Igusa's local zeta function for a special but important class of polynomials, namely the polynomials that are non-degenerated with respect to their Newton polyhedron. It is the famous Russian school of V. Arnold that discovered in the 1970's that for such polynomials a lot of aspects can be derived from the geometrical properties of their Newton polyhedron. The study of Igusa's local zeta function restricted to the polynomials that are non-degenerated with respect to their Newton polyhedron was first started by Lichtin and Meuser [LM85] for polynomials in two variables.

We will study polynomials in many variables that are non-degenerated with respect to their Newton polyhedron. For such polynomials we will give an explicit formula for Igusa's local zeta function, which we will use to obtain information about its poles.

For some preliminaries about convex geometry, we refer to [Roc70] or Appendix B. In Appendix B, most of the cited properties and definitions of [Roc70]

are included.

Definition 0.1.4. Let $f(x) = f(x_1, \ldots, x_n) = \sum_{\omega \in \mathbb{N}^n} a_\omega x_1^{\omega_1} \cdots x_n^{\omega_n}$ be a non-zero polynomial over $\mathbb{Z}_p$ with $f(0) = 0$. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\operatorname{supp}(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}$ the support of f. The *Newton polyhedron* $\Gamma(f)$ of f is defined as the convex hull in $(\mathbb{R}^+)^n$ of the set

$$\bigcup_{\omega \in \operatorname{supp}(f)} \omega + (\mathbb{R}^+)^n.$$

The *global Newton polyhedron* $\Gamma_{\mathrm{gl}}(f)$ of f is defined as the convex hull of $\operatorname{supp}(f)$. It is easy to verify that $\Gamma(f) = \Gamma_{\mathrm{gl}}(f) + (\mathbb{R}^+)^n$.

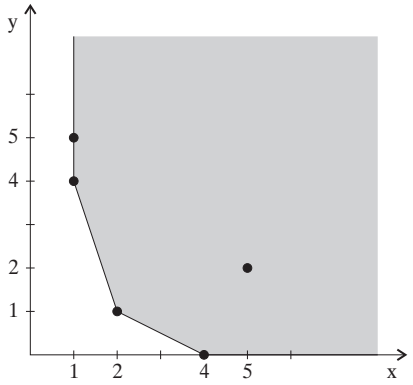


Figure 0.1.1. Newton polyhedron of $xy^4 + xy^5 + x^2y + x^4 + x^5y^2$

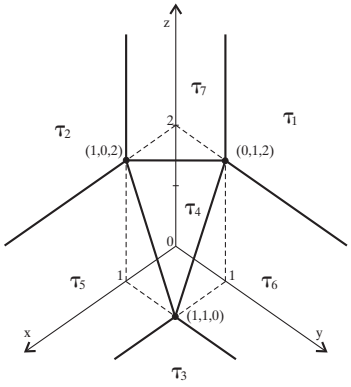


Figure 0.1.2. Newton polyhedron of $xy + xz^2 + yz^2$

Because a Newton polyhedron is a polyhedron, every proper face is an exposed face.[1] So, each proper face of $\Gamma(f)$ is the intersection of $\Gamma(f)$ with a supporting hyperplane [Roc70, pp. 99–100]. **By the faces of $\Gamma(f)$ we mean the proper faces of $\Gamma(f)$ and the Newton polyhedron $\Gamma(f)$ itself.**

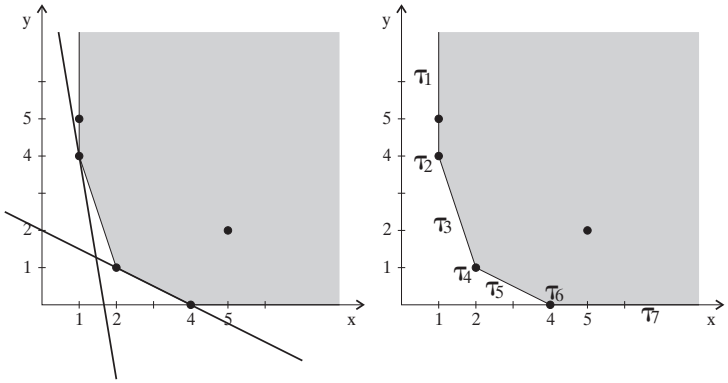


Figure 0.1.3. The proper faces of a Newton polyhedron

Definition 0.1.5. Let p be a prime number. Let f be as in Definition 0.1.4. For every face τ of the Newton polyhedron $\Gamma(f)$ of f, we define

$$f_\tau(x) = \sum_{\omega \in \tau} a_\omega x^\omega,$$

and the polynomial $\bar{f}_\tau(x)$ with coefficients in $\mathbb{F}_p$ by reducing each coefficient a_ω of f_τ modulo $p\mathbb{Z}_p$.

We say f is *non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron* $\Gamma(f)$ if for every[2] face τ of $\Gamma(f)$ the locus of the polynomial $\bar{f}_\tau$ has no singularities in $(\mathbb{F}_p^\times)^n$ or equivalently the set of congruences

$$\begin{cases} f_\tau(x) & \equiv & 0 \bmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) & \equiv & 0 \bmod p, \quad i = 1, \ldots, n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^n$.

We say f is *non-degenerated over $\mathbb{F}_p$ with respect to the compact faces of its Newton polyhedron*, if we have the same condition, but only for the compact faces of $\Gamma(f)$.

We say f is *non-degenerated over $\mathbb{Q}_p$ with respect to all the faces of its Newton polyhedron*, if for every face τ of $\Gamma(f)$ the locus of f_τ has no singularities in

[1] Face and exposed face in the sense of [Roc70, p. 162].
[2] Thus also for $\Gamma(f)$.

$(\mathbb{Q}_p^\times)^n$. We say f is *non-degenerated over $\mathbb{Q}_p$ with respect to the compact faces of its Newton polyhedron*, if we have the same condition, but only for the compact faces of $\Gamma(f)$.

Remarks.

1. For a compact face τ of $\Gamma(f)$ the following two conditions are equivalent:

   (i) The set of equations $\begin{cases} f_\tau(x) &= 0 \\ \frac{\partial f_\tau}{\partial x_i}(x) &= 0, \quad i = 1, \ldots, n \end{cases}$ has no solution in $(\mathbb{Q}_p^\times)^n$.

   (ii) The set of equations $\frac{\partial f_\tau}{\partial x_i}(x) = 0, \quad i = 1, \ldots, n$ has no solution in $(\mathbb{Q}_p^\times)^n$.

   The same is true if we replace $\mathbb{Q}_p$ by $\mathbb{F}_p$ and if $p \gg 0$ (for a fixed Newton polyhedron). The notation $p \gg 0$ means that p must be big enough.

   Condition (ii) clearly implies condition (i). Conversely, condition (i) implies condition (ii). Indeed, because τ is a compact face, τ is contained in a supporting hyperplane H that does not contain the origin. Thus we can find an equation for H of the form $\sum_{i=1}^n \lambda_i \omega_i = \beta$ with $\lambda_1, \ldots, \lambda_n, \beta \in \mathbb{Z}$ and $\beta \neq 0$. Therefore

   $$\beta f_\tau(x) = \sum_{i=1}^n \lambda_i x_i \frac{\partial f_\tau}{\partial x_i}(x).$$

   So, every solution in $(\mathbb{Q}_p^\times)^n$ of the congruences $(\partial f_\tau / \partial x_i)(x) = 0$ for $i = 1, \ldots, n$ is also a solution of $f_\tau(x) = 0$. The same holds if we replace $\mathbb{Q}_p$ by $\mathbb{F}_p$ and if $p \gg 0$. Indeed, if $p \gg 0$, then $p \nmid \beta$.

   Some authors call *f non-degenerated over $\mathbb{Q}_p$ with respect to its Newton polyhedron* if for all compact faces τ of $\Gamma(f)$ condition (ii) holds.

2. Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{Q}_p$ with respect to all the faces of its Newton polyhedron. Then, if $p \gg 0$, the polynomial f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron by the paper of Greenleaf [Gre65]; see also [Gre74].

3. Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the faces of its Newton polyhedron (i.e., for every face τ of $\Gamma(f)$ the locus of f_τ has no singularities in $(\mathbb{C}^\times)^n$). Then f is non-degenerated over $\mathbb{F}_p$, with respect to all the faces of its Newton polyhedron, for almost every prime p. Indeed,

for a field K, the condition "the set of equations

$$\begin{cases} h_1(x) &= 0 \\ & \vdots \\ h_r(x) &= 0 \end{cases}$$

has no solution in $(K \setminus \{0\})^n$ " is equivalent to the condition "the set of equations

$$\begin{cases} h_1(x) &= 0 \\ & \vdots \\ h_r(x) &= 0 \\ x_i z_i - 1 &= 0, \quad \text{i} = 1,\ldots,\text{n} \end{cases}$$

has no solution in K^{2n}."

By the Weak Nullstellensatz [Ful69, p. 20] the following is well-known: if some polynomials $g_1, \ldots, g_s$ with coefficients in $\mathbb{Z}$ have no common zero in $\mathbb{C}$, then the polynomials $\bar{g}_1, \ldots, \bar{g}_s$ over $\mathbb{F}_p$, obtained by reducing the coefficients of $g_1, \ldots, g_s$ modulo $p\mathbb{Z}_p$, have no common zero in $\mathbb{F}_p$ for almost every prime p.

4. Fix a Newton polyhedron P in $\mathbb{R}^n$. "Almost all" polynomials $f(x_1, \ldots, x_n)$ over $\mathbb{C}$ (resp. $\mathbb{Q}_p$) with Newton polyhedron equal to P are non-degenerated over $\mathbb{C}$ (resp. $\mathbb{Q}_p$) with respect to all the faces of $\Gamma(f)$. To a polynomial f with Newton polyhedron P (and where the support is contained in a given bounded set), we can associate an n_P-tuple consisting of the coefficients of f. By "almost all" we mean that the n_P-tuples of those polynomials that are degenerated, are contained in the zero-set of a non-zero polynomial g over $\mathbb{C}$ (resp. $\mathbb{Q}_p$). The proof is similar to [AVGZ86, p. 157]; see also Appendix C.

0.2 Main results of this thesis

To calculate $Z_f(s)$ explicitly for a given polynomial f, and to determine its poles, is often a difficult problem. This has been done for polynomials in two variables by Strauss [Str83], Meuser [Meu83], Igusa [Igu85], Lichtin and Meuser [LM85], Goldman [Gol86], Loeser [Loe88] and Veys [Vey90]. Many results have been obtained by Igusa when f is a relative invariant of a prehomogeneous vector space; see [Igu96] and the references in it. The monodromy conjecture of Igusa (see, e.g., [Den91, Section 2.3]) predicts a remarkable relation between the poles of $Z_f(s, \chi)$ and topological invariants of the singularities of the complex locus of $f = 0$, for any polynomial f. Many results towards this conjecture have been obtained by Veys [Vey93] for polynomials in three variables. From Igusa's

work in [Igu74a], it follows that one can determine Igusa's local zeta function associated to $f(x)+g(y)$ for the trivial character, if one knows the zeta functions for the trivial and the non-trivial characters of $f(x)+a$ and $g(y)+b$ for suitable a, b (see also [Den91, Section 5.1] and [DV95]). Some other cases are treated in [Gol83] and [Zn01].

An important class of polynomials are the ones that are non-degenerated over $\mathbb{Q}_p$ with respect to all the faces of their Newton polyhedron (see Definition 0.1.5 above). For such polynomials one has a very explicit resolution of singularities of the (p-adic) locus of $f = 0$, which is called a toric resolution. The analogue over $\mathbb{R}$ (instead of $\mathbb{Q}_p$) was first treated in the classical paper [Var76a] of Varchenko. The material in Varchenko's paper can be adapted to the p-adic case in a straightforward way. But Varchenko's method yields a list of candidate poles for $Z_f(s)$ that is much too large. This was illustrated for polynomials in two variables by Lichtin and Meuser [LM85]. Using a different method (non-birational change of variables) Denef [Den95] obtained a much smaller list of candidate poles. (See [DS89] for the analogue of this method over $\mathbb{R}$.) Both methods yield formulas for $Z_f(s)$ which are not very explicit. But under the slightly stronger condition that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron, Varchenko's method (based on toric resolution) can be further enhanced to obtain a very explicit formula for $Z_f(s)$. This is the first main result, namely Theorem 2.3.2. Actually, in this thesis, we preferred to obtain this explicit formula by using a different method, which is entirely elementary. We have written down the proof in a very detailed and self-contained way to make this material available to a wider audience. Chapter 1 contains preliminaries about Newton polyhedra and Chapter 2 contains our explicit formula.

In fact, the formula from Theorem 2.3.2 holds even for $Z_f(s, \chi)$ with χ a character of $\mathbb{Z}_p^\times$ with conductor $c_\chi = 1$ (and if f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron). The conductor c_χ is the smallest $c \in \mathbb{N} \setminus \{0\}$ such that χ is trivial on $1 + p^c\mathbb{Z}_p$. So, $c_{\chi_{\text{triv}}}$ is also equal to 1. It is well-known (see [Den91, Theorem 3.3]) that $Z_f(s, \chi)$ is constant as a function of s for $p \gg 0$, if the conductor $c_\chi > 1$ and f is a polynomial over $\mathbb{Z}$. For the exact condition of a similar result for f a polynomial over $\mathbb{Z}_p$ and p a fixed prime number, we refer to the same reference. So, Theorem 2.3.2 gives a formula for $Z_f(s, \chi)$ for the relevant cases. The actors in this formula are the cones Δ_τ associated to the faces τ of $\Gamma(f)$ (see Definition 1.1.4) and, the numbers of elements N_τ of the sets $\{x \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(x) = 0\}$, if $\chi = \chi_{\text{triv}}$, or the sums $\sum_{a \in (\mathbb{F}_p^\times)^n} \chi(f_\tau(a))$, if $\chi \neq \chi_{\text{triv}}$. See also [DH01] and [Hoo01]. Independently, Zúñiga-Galindo [Zn99] obtained the same formula, as a special case of a more general result.

In collaboration with Davy Loots we developed software, based on this formula, to calculate $Z_f(s)$ for a concrete polynomial f. This program, Polygusa [HL00] (see also Appendix D), is written in Maple and gives also other relevant informa-

tion: for instance, a picture of $\Gamma(f)$, the cones and the numbers N_τ associated to the faces τ of $\Gamma(f)$, the topological zeta function of f. Polygusa is a useful tool to generate examples and is already used by several mathematicians who study Igusa zeta functions.

In Chapter 3, we study the poles of Igusa's local zeta function using the formula from Theorem 2.3.2. We obtain a set of candidate poles for $Z_f(s,\chi)$ together with their expected orders, i.e., upperbounds for the orders if these candidate poles are actual poles (see Proposition 3.1.1, Definition 3.1.2, Proposition 3.2.5 and Proposition 3.3.3). This can all be expressed in terms of the faces of $\Gamma(f)$ and their dimension. The formula from Theorem 2.3.2 also enables us to prove Theorem 3.4.1, which is the second main result. It states that $Z_f(s)$ has always at least one real pole, assuming f is non-degenerated in the above sense. In particular, this theorem gives more information about the largest real pole of $Z_f(s)$: it is either -1 or $-1/t_0$, where $(t_0, t_0, \ldots, t_0)$ is the intersection point of the diagonal $D = \{(t, t, \ldots, t) \in \mathbb{R}^n\}$ with the boundary of $\Gamma(f)$.

We have special interest for the largest real candidate pole of $Z_f(s,\chi)$ different from -1. It is well-known (see Proposition 3.2.5) that this candidate pole equals $-1/t_0$, if $t_0 \neq 1$, and that its expected order equals κ, where κ is the codimension of the smallest face τ_0 of $\Gamma(f)$ that contains $(t_0, t_0, \ldots, t_0)$. It is well-known, see Theorem 3.4.1(i), that the actual largest real pole of Igusa's local zeta function $Z_f(s)$ for the trivial character equals $-1/t_0$ with actual order equal to κ, if $t_0 > 1$, (assuming f is non-degenerated in the above sense). Our third main result, Theorem 4.1.11, states that this remains true, for almost all p, when $t_0 < 1$, for polynomials f with coefficients in $\mathbb{Z}$ that are sufficiently non-degenerated, if no vertex of τ_0 belongs to $\{0,1,2\}^n$. The proof is contained in Chapter 4 (see also [DH01]). It is based on our explicit formula for $Z_f(s)$ and a very deep result of Adolphson and Sperber on estimates for exponential sums. With similar techniques, we also prove Theorem 4.1.13, which is an analogous result for the case when $t_0 = 1$. Under the same condition as in Theorem 4.1.11 (or even a less stronger condition), $-1/t_0$ will then be the largest real pole of $Z_f(s)$ with order equal to $\kappa + 1$.

The main purpose of Chapter 5 (see also [Hoo02]) is to prove one of the conjectures of Denef and Sargos [DS92, Section 6.2] for non-degenerated polynomials in the p-adic case:

> *if τ_0 is an unstable face of $\Gamma(f)$ over $\mathbb{F}_p$ with respect to f, and if $-1/t_0 \neq -1$ is a pole of $Z_f(s,\chi)$, then its order will be smaller than expected, i.e., $< \kappa$.*

The notion "unstable" (see Definition 5.3.1) was first established by Denef and Sargos in [DS92]. The analogue of the above conjecture for the real case has been proved by Denef, Laeremans and Sargos [DLS97].

Inspired by the proof for the real case, we tried to reduce our question about the polynomial f to the same question about the polynomial f_{τ_0}. Namely Theorem 5.1.3, our fourth main result, states that $-1/t_0$ is an actual pole of $Z_f(s,\chi)$ of the expected order if and only if the same holds for $Z_{f_{\tau_0}}(s,\chi)$. In particular, if $-1/t_0$ is not a pole of $Z_{f_{\tau_0}}(s,\chi)$ of the expected order, the same holds for $Z_f(s,\chi)$. Theorem 5.2.1 extends this result to arbitrary poles. In Theorem 5.3.3, we will give a formula for $Z_f(s,\chi)$ that holds if $f = f_\sigma$, where $\sigma \subset \Gamma(f)$ is an unstable convex set over $\mathbb{F}_p$ with respect to f (see Definition 5.3.1), and if f is non-degenerated over $\mathbb{F}_p$ with respect to a particular set of faces of $\Gamma(f)$. The formula in this theorem, which is our fifth main result, is similar to the one in Theorem 2.3.2, but can provide a smaller set of candidate poles for $Z_f(s,\chi)$ or smaller upperbounds for their orders (see Example 5.4.1). Finally, Theorem 5.3.6 uses this formula together with Theorem 5.1.3 to obtain a proof of the mentioned conjecture of Denef and Sargos. Theorem 5.3.8 gives an analogous result for arbitrary poles.

In Chapter 6, we study the real poles of maximal order of $Z_f(s,\chi)$ and $Z_{f,0}(s,\chi)$ for a character χ with conductor equal to 1. We say that a pole has maximal order if its order is equal to the number of variables of f. If the polynomial f is sufficiently non-degenerated, we can prove the following results, using our formula from Theorem 2.3.2: Igusa's local zeta functions $Z_f(s,\chi)$ and $Z_{f,0}(s,\chi)$ for $\chi \neq \chi_{\text{triv}}$ and $Z_{f,0}(s)$ have at most one real pole of maximal order. On the other hand, Igusa's local zeta function $Z_f(s)$ has at most two real poles of maximal order. The method we follow is similar to the method and the proofs of [LV99] or [Lae97, Chapter 6] in which similar results are derived for the topological zeta function.

0.3 Open problems for further investigation

In the context of this thesis, we would like to propose the following projects for eventual further investigation.

Suppose that the polynomial $f \in \mathbb{Z}[x_1,\ldots,x_n]$ is non-degenerated over $\mathbb{C}$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and $p \gg 0$. Denef and Sargos conjectured that there exists a character χ such that $-1/t_0$ is a pole of $Z_f(s,\chi)$ of order κ if and only if τ_0 is not unstable over $\mathbb{C}$ with respect to f [DS92, Section 6.2]. We proved one part of this conjecture; see Theorem 5.3.6, Remark 3 after Definition 0.1.5 and [Den91, Theorem 3.3] (or p. viii). The following other part remains to be checked.

Conjecture 0.3.1. *Let $f(x_1,\ldots,x_n)$ be a polynomial over $\mathbb{Z}_p$ with $f(0) = 0$ and p a prime number. Suppose that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and that the origin is a singular*

point of f. Suppose that τ_0 is not unstable over $\mathbb{F}_p$ with respect to f. If $t_0 < 1$, then $-1/t_0$ is a pole of $Z_f(s,\chi)$ of order κ for at least one character χ.

This conjecture might even be true for $\chi = \chi_{\text{triv}}$. We have not found a counter example yet.

Conjecture 0.3.2. *Let $f(x_1,\ldots,x_n)$ be a polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and that the origin is a singular point of f.*

(i) If $t_0 < 1$, then $-1/t_0$ is a pole of $Z_f(s)$ for $p \gg 0$.

In Theorem 4.1.11, this conjecture is proved in the special case that no vertex of τ_0 belongs to $\{0,1,2\}^n$. In this case, $-1/t_0$ is even a pole of $Z_f(s)$ of order κ. In Example 4.2.1, this extra condition is not satisfied and $-1/t_0$ is not a pole of $Z_f(s)$ of order κ. Nevertheless, $-1/t_0$ still remains a pole of $Z_f(s)$ in this example. In Example 4.2.2, there is also a vertex of τ_0 contained in $\{0,1,2\}^n$. For $p \equiv 3 \bmod 4$, the number $-1/t_0$ is not a pole of $Z_f(s)$ of order κ but still remains a pole of $Z_f(s)$.

If one finds a counter example for Conjecture 0.3.2, one might replace Assertion (i) by one of the following two weaker assertions:

(ii) If $t_0 < 1$, then $-1/t_0$ is a pole of $Z_f(s)$ for infinitely many p.

(ii) If $t_0 < 1$, then there exists at least one character χ such that $-1/t_0$ is a pole of $Z_f(s,\chi)$ for infinitely many p.

Further, we want to examine if the following conjecture is true.

Conjecture 0.3.3. *Let $f(x_1,\ldots,x_n)$ be a polynomial over $\mathbb{Z}_p$ with $f(0) = 0$ and p a prime number. Suppose that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and suppose that s_1 is a candidate pole of $Z_f(s)$. Suppose that all the facets of $\Gamma(f)$ of pure contribution to the candidate pole s_1 are unstable with respect to the same variable. Suppose that the origin is a singular point of f. If $s_1 \neq -1$, then s_1 is not an actual pole of $Z_f(s)$.*

See Example 3 and Example 6 of Appendix D for an illustration of this conjecture. The analogue of the conjecture above for real analytical functions has been proved by Denef and Sargos [DS92, Théorème 1] in the special case that $s_1 = -1/t_0$. First, one could investigate if the ideas of this proof may be adapted for Igusa's local zeta function. Related to the above conjecture is the following result of Denef.

Theorem 0.3.4. *Let f be a polynomial over $\mathbb{Z}_p$ in n variables. Suppose that f is non-degenerated over $\mathbb{Q}_p$ with respect to the compact faces of its Newton polyhedron. Suppose that $s_1 \neq 1$ is a real candidate pole of $Z_f(s)$ such that there is only one facet μ of pure contribution to s_1. Suppose that μ is a simplex with $n-1$ vertices contained in a coordinate hyperplane and one vertex at distance 1 from that coordinate hyperplane. Then there is no pole of*

$$\int_{p^k \mathbb{Z}_p^n} |f(x)|^s |dx|$$

with real part equal to s_1 for $k \gg 0$.

Finally, the proof of Theorem 5.2.1 shows that the condition "μ is the only face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1" in Theorem 5.3.8 can be replaced by a weaker condition. Examples show that even this condition might be omitted; see Example 4 in Appendix D. We would like to search for a proof or a counterexample.

Chapter 1

More about Newton polyhedra

In Definition 1.1.4, we will give a partition of $(\mathbb{R}^+)^n$ in sets that are closely related to the Newton polyhedron of a polynomial f as in Definition 0.1.4. In order to define these sets Δ_τ and to know more about them, we will present a selection of some well-known definitions and properties in Section 1.1. In Section 1.2, we show that it is always possible to give a partition of such a cone Δ_τ into rational simplicial cones or simple cones. Although all results in this chapter are well-known, we provided most of them with a proof to make this material more easily accessible. For some preliminaries about convex geometry, we refer to [Roc70] or Appendix B. In Appendix B, most of the cited properties and definitions of [Roc70] are included.

1.1 Partition of $(\mathbb{R}^+)^n$ into cones Δ_τ associated to the faces of a Newton polyhedron

Definition 1.1.1. Let f be as in Definition 0.1.4. For $a \in (\mathbb{R}^+)^n$, we define

$$m(a) = \inf_{x \in \Gamma(f)} \{a \cdot x\}$$

and we define the *first meet locus of* a as

$$F(a) = \{x \in \Gamma(f) \mid a \cdot x = m(a)\},$$

where $a \cdot x$ denotes the scalar product $\sum_i^n a_i x_i$ of $a = (a_1, \ldots, a_n)$ and $x = (x_1, \ldots, x_n)$.

Note that the infimum in the definition of $m(a)$ is attained and so $m(a) = \min_{x\in\Gamma(f)} \{a \cdot x\}$. This is clear, because we can take the infimum over the elements in the closed and bounded set $\Gamma_{\mathrm{gl}}(f)$ instead of $\Gamma(f) = \Gamma_{\mathrm{gl}}(f) + (\mathbb{R}^+)^n$. Moreover, we can take the minimum over the elements in $\mathrm{supp}\,(f)$ instead of $\Gamma(f)$ or $\Gamma_{\mathrm{gl}}(f)$, because $\Gamma_{\mathrm{gl}}(f)$ is the convex hull of the (finite) set $\mathrm{supp}\,(f)$.

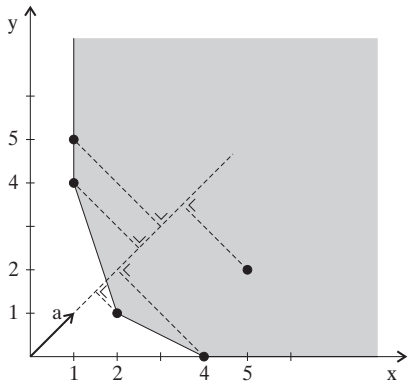


Figure 1.1.1. Calculation of $m(a)$

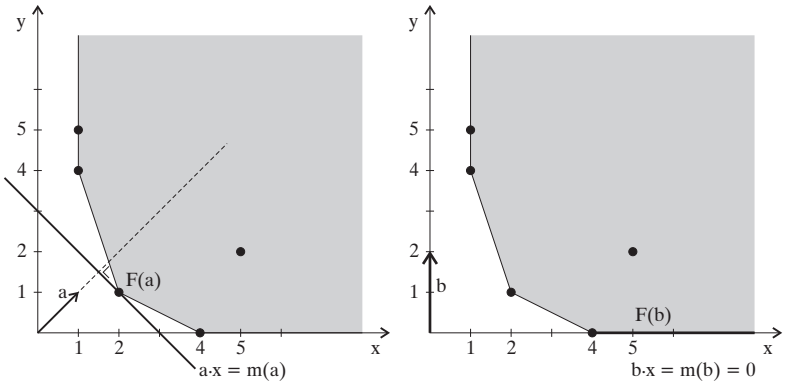


Figure 1.1.2. The first meet loci of $a = (1, 1)$ and $b = (0, 2)$

The following two lemmas are easy to prove.

Lemma 1.1.2. *Let f be as in Definition 0.1.4 and $a \in (\mathbb{R}^+)^n$. Then $F(a)$, the first meet locus of a, is a face of $\Gamma(f)$. In particular $F(0) = \Gamma(f)$ and $F(a)$ is a proper face of $\Gamma(f)$, if $a \neq 0$. Moreover, $F(a)$ is a compact face if and only if $a \in (\mathbb{R}^+ \setminus \{0\})^n$.*

Lemma 1.1.3. *Let $a_1, \ldots, a_r \in (\mathbb{R}^+)^n \setminus \{0\}$. Suppose that $F(a_1) \cap F(a_2) \cap \cdots \cap F(a_r) \neq \emptyset$. Let $\lambda_i \in \mathbb{R}$ and $\lambda_i > 0$ for $i = 1, \ldots, r$. Then*

$$F(\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_r a_r) = F(a_1) \cap F(a_2) \cap \cdots \cap F(a_r).$$

Remark. Figure 1.1.3 shows that the condition $F(a_1) \cap F(a_2) \cap \cdots \cap F(a_r) \neq \emptyset$ is necessary.

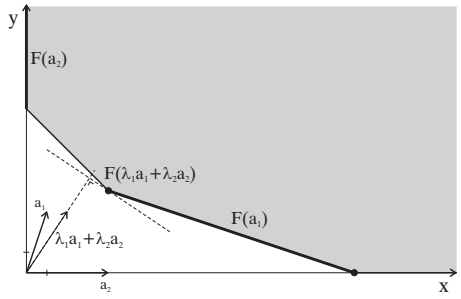


Figure 1.1.3. The condition of Lemma 1.1.3 is necessary.

Definition 1.1.4. We define an equivalence relation on $(\mathbb{R}^+)^n$ by

$$a \sim a' \qquad \text{if and only if} \qquad F(a) = F(a').$$

If τ is a face of $\Gamma(f)$, we define the *cone associated to* τ as

$$\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}.$$

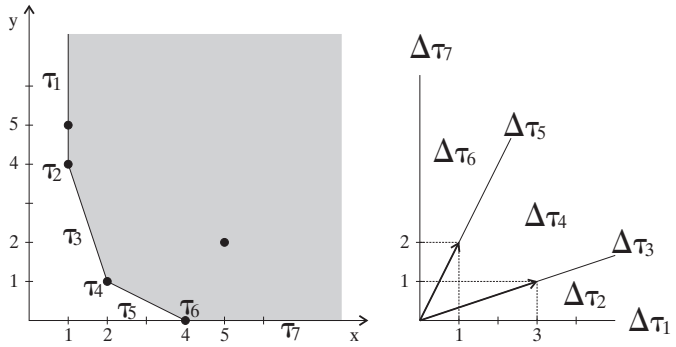


Figure 1.1.4. The cones associated to the faces of a Newton polyhedron

Note that $\Delta_{\Gamma(f)} = \{0\}$. We will now study the other equivalence classes Δ_τ in order to give an interesting description of them in Proposition 1.1.10.

Lemma 1.1.5. *Let f be as in Definition 0.1.4 and let τ be a proper face of $\Gamma(f)$. Then*

(i) Δ_τ is a relatively open subset of $(\mathbb{R}^+)^n$;

(ii) $\bar{\Delta}_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$ and is a polyhedral cone;

(iii) the function m of Definition 1.1.1 is linear on $\bar{\Delta}_\tau$.

Proof. Suppose that $\Gamma(f)$ is the convex hull of the points $P_1, \ldots, P_s$ and the directions of recession[1] $e_1, \ldots, e_n$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. By [Roc70, Theorem 18.3], we may suppose that τ is the convex hull of the points $\{P_1, \ldots, P_r\} = \tau \cap \{P_1, \ldots, P_r, P_{r+1}, \ldots, P_s\}$ and the directions of recession $e_1, \ldots, e_k$ with $r \leq s$ and $k \leq n$. Now one easily proves that $\Delta_\tau = \{a = (a_1, \ldots, a_n) \in \mathbb{R}^n \mid a \cdot P_1 = a \cdot P_2 = \cdots = a \cdot P_r, a \cdot P_1 < a \cdot P_{r+1}, \ldots, a \cdot P_1 < a \cdot P_s, a_i = 0 \text{ for } i = 1, \ldots, k, a_j > 0 \text{ for } j = k+1, \ldots, n\}$, which is clearly a relatively open set. It follows that $\bar{\Delta}_\tau = \{a \in \mathbb{R}^n \mid a \cdot P_1 = a \cdot P_2 = \cdots = a \cdot P_r, a \cdot P_1 \leq a \cdot P_{r+1}, \ldots, a \cdot P_1 \leq a \cdot P_s, a_i = 0 \text{ for } i = 1, \ldots, k, a_j \geq 0 \text{ for } j = k+1, \ldots, n\} = \{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$. Finally, (ii) implies that $m(a) = a \cdot x_\tau$ for every $a \in \bar{\Delta}_\tau$, where x_τ is a fixed element of τ. This shows that m is linear on $\bar{\Delta}_\tau$. □

Definition 1.1.6. A *fan* $\mathcal{F}$ is a finite set of rational polyhedral cones such that

1. every (non-empty) face of a cone of $\mathcal{F}$ is contained in $\mathcal{F}$ and
2. the intersection of two arbitrary cones C_i and C_j in $\mathcal{F}$ is a face of both C_i and C_j.

Lemma 1.1.7. *Let f be as in Definition 0.1.4. The closures $\bar{\Delta}_\tau$ of the cones associated to the faces of $\Gamma(f)$ form a fan in $(\mathbb{R}^+)^n$. Moreover, we have the following:*

(i) Let τ be a proper face of $\Gamma(f)$. Then the order-reversing map

$$\{\textit{faces of } \Gamma(f) \textit{ that contain } \tau\} \to \{\textit{(non-empty) faces of } \bar{\Delta}_\tau\} : \sigma \mapsto \bar{\Delta}_\sigma$$

is one-to-one and onto.

(ii) Let τ_1, τ_2 be faces of $\Gamma(f)$. Suppose that τ_1 is a facet[2] of τ_2. Then $\bar{\Delta}_{\tau_2}$ is a facet of $\bar{\Delta}_{\tau_1}$.

[1] See Definition B.1.8.

[2] Recall that a facet is an $(n-1)$-dimensional face.

Proof. We first prove (i) and that the cones $\bar{\Delta}_\tau$ form a fan. In Proposition 1.1.10, we will prove that the cones $\bar{\Delta}_\tau$ are *rational* polyhedral cones. Fix a face $\sigma \neq \tau$ of $\Gamma(f)$ that contains the face τ. Suppose that $\Gamma(f)$ is the convex hull of the points

$$P_1, \ldots, P_k, P_{k+1}, \ldots, P_{k+t}, \ldots, P_l \tag{1.1.1}$$

and the directions of recession

$$e_1, \ldots, e_d, e_{d+1}, \ldots, e_{d+s}, \ldots, e_n, \tag{1.1.2}$$

with $P_1, \ldots, P_k$ (resp. $P_1, \ldots, P_{k+t}$) the only points from (1.1.1) that belong to τ (resp. σ) and with $e_1, \ldots, e_d$ (resp. $e_1, \ldots, e_{d+s}$) the only vectors from (1.1.2) that are directions of recession of τ (resp. σ). Now one proves that $\bar{\Delta}_\sigma$ is a face of $\bar{\Delta}_\tau$: the equation of the supporting hyperplane is

$$x \cdot ((P_{k+1} - P_1) + \cdots + (P_{k+t} - P_1) + e_{d+1} + \cdots + e_{d+s}) = 0.$$

It easily follows from Lemma 1.1.5(ii) that the mentioned map in the assertion is one-to-one and order-reversing. We now prove that it is also onto. Fix an arbitrary face γ of $\bar{\Delta}_\tau$ and an x in the relative interior of γ. Put $\sigma = F(x)$. Then σ is a face of $\Gamma(f)$ that contains τ, because $x \in \gamma \subset \bar{\Delta}_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$. Now, the faces γ and $\bar{\Delta}_\sigma$ of $\bar{\Delta}_\tau$ must be equal, because their relative interiors have a common element, x (see [Roc70, Corollary 18.1.2]).

Finally, suppose that τ_1, τ_2 are faces of $\Gamma(f)$. Then $\bar{\Delta}_{\tau_1} \cap \bar{\Delta}_{\tau_2} = \bar{\Delta}_{\tau_3}$ with τ_3 the smallest face of $\Gamma(f)$ that contains $\tau_1 \cup \tau_2$, i.e., the intersection of all the faces of $\Gamma(f)$ that contain $\tau_1 \cup \tau_2$.

Assertion (ii) follows from (i). The cone $\bar{\Delta}_{\tau_2}$ is clearly a (proper) face of $\bar{\Delta}_{\tau_1}$. So, if it is not a facet, there exists a facet F of $\bar{\Delta}_{\tau_1}$ such that $\bar{\Delta}_{\tau_2} \subsetneq F \subsetneq \bar{\Delta}_{\tau_1}$. Then there exists a face γ of $\Gamma(f)$ that contains τ_1 and satisfies $F = \bar{\Delta}_\gamma$. But then $\tau_2 \supsetneq \gamma \supsetneq \tau_1$, which is not possible. □

Definition 1.1.8. A vector $a \in \mathbb{R}^n$ is called *primitive* if the components of a are integers whose greatest common divisor is 1.

Lemma 1.1.9. *Let f be as in Definition 0.1.4.*

(i) If H is a supporting hyperplane of the Newton polyhedron $\Gamma(f)$ of f and $a \in \mathbb{R}^n \setminus \{0\}$ is perpendicular to H, then a (or $-a$) $\in (\mathbb{R}^+)^n$.

(ii) If τ is a facet, there exists an $a \in \mathbb{N}^n \setminus \{0\}$ such that a is perpendicular to τ.

Proof.

(i) Because $a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$ is perpendicular to H, there exists a $\beta \in \mathbb{R}$ such that $H = \{x \in \mathbb{R}^n \mid a \cdot x = \beta\}$. Suppose there exist $i, j \in \{1, \ldots, n\}$ such that $a_i < 0$ and $a_j > 0$. See Figure 1.1.5.

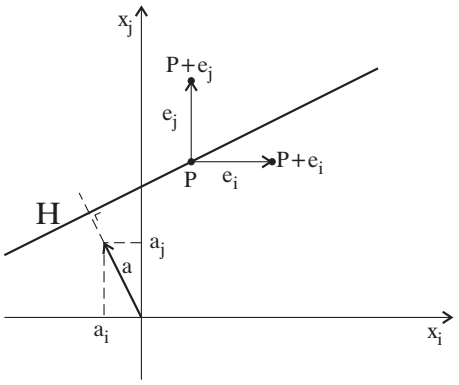


Figure 1.1.5

Take a point P in $H \cap \Gamma(f)$. Denote the k-th unit vector in $\mathbb{R}^n$ by e_k. Because e_i and e_j are directions of recession of $\Gamma(f)$, the points $P + e_i$ and $P + e_j$ are also contained in $\Gamma(f)$. But $(P + e_i) \cdot a = \beta + a_i < \beta$ and $(P + e_j) \cdot a = \beta + a_j > \beta$, which contradicts the fact that H is a supporting hyperplane of $\Gamma(f)$.

(ii) By [Roc70, Theorem 18.3], we know that τ is the convex hull of the set S, where S consists of the points in $\operatorname{supp}(f)$ that belong to τ and the directions of recession of τ from the set $\{e_1, \ldots, e_n\}$. Because τ is a facet, we may suppose that the supporting hyperplane H of $\Gamma(f)$ satisfying $\tau = H \cap \Gamma(f)$ is the affine hull of $\{t_1, \ldots, t_n\}$, where $t_1, \ldots, t_n$ are affinely independent vectors in $\mathbb{N}^n \cap \tau$. So, there exist an $a \in \mathbb{Z}^n \setminus \{0\}$ that is perpendicular to H. By (i), we may suppose that $a \in \mathbb{N}^n \setminus \{0\}$.

□

Recall that $\Delta_{\Gamma(f)} = \{0\}$. The following proposition gives the geometry of the other equivalence classes Δ_τ. Note that because a Newton polyhedron is a polyhedron, we can prove that every proper face τ of $\Gamma(f)$ is contained in a facet of $\Gamma(f)$. Moreover, every proper face τ of $\Gamma(f)$ is the (finite) intersection of the facets of $\Gamma(f)$ that contain τ.

Proposition 1.1.10. *Let f be as in Definition 0.1.4. Let τ be a proper face of $\Gamma(f)$ and let $\gamma_1, \ldots, \gamma_r$ be the facets of $\Gamma(f)$ that contain τ. Let $a_1, \ldots, a_r$*

be the unique primitive vectors in $\mathbb{N}^n \setminus \{0\}$ *that are perpendicular to* $\gamma_1, \ldots, \gamma_r$*, respectively. Then the cones* $\bar{\Delta}_\tau$ *and* Δ_τ *are the following convex cones:*

$$\bar{\Delta}_\tau = \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \right\} \quad \textit{and} \quad \Delta_\tau = \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0 \right\}.$$

Moreover, $\dim \Delta_\tau = \dim \bar{\Delta}_\tau = \operatorname{codim} \tau = n - \dim \tau$.

Proof. The fact that $\dim \bar{\Delta}_\tau = \operatorname{codim} \tau = n - \dim \tau$ follows from Lemma 1.1.7. Indeed, suppose that $\dim \tau = k$. Then there exist faces $\tau = \tau^k, \tau^{k+1}, \ldots, \tau^n = \Gamma(f)$ of $\Gamma(f)$, with $\dim \tau^i = i$, such that

$$\tau = \tau^k \subseteq \tau^{k+1} \subseteq \cdots \subseteq \tau^n = \Gamma(f).$$

Therefore,

$$\bar{\Delta}_\tau = \bar{\Delta}_{\tau^k} \supseteq \bar{\Delta}_{\tau^{k+1}} \supseteq \cdots \supseteq \bar{\Delta}_{\Gamma(f)} = \{0\}.$$

So, by Lemma 1.1.7(ii), we know that $\dim \tau + \dim \bar{\Delta}_\tau = \dim \tau^{k+1} + \dim \bar{\Delta}_{\tau^{k+1}} = \cdots = \dim \Gamma(f) + \dim\{0\} = n$. The assertion for $\bar{\Delta}_\tau$ follows now from Lemma 1.1.11, Lemma 1.1.7 and the fact that $\bar{\Delta}_{\gamma_i} = \mathbb{R}^+ a_i$ for $i = 1, \ldots, r$. The assertion for Δ_τ follows from the assertion for $\bar{\Delta}_\tau$, by Lemma 1.1.5, [Roc70, Theorem 6.2] and [Roc70, Theorem 6.9]. □

Lemma 1.1.11. *Let* C *be a polyhedral cone in* $\mathbb{R}^n$ *such that the origin is a face of* C*. Then the rays generated by a minimal set of generators are exactly the one-dimensional faces of* C*.*

Proof. [Ful93, Section 1.2]. □

1.2 Partition of a rational cone into rational simplicial cones or simple cones

Definition 1.2.1. If $a_1, \ldots, a_r \in \mathbb{R}^n \setminus \{0\}$, we call $\operatorname{cone}(a_1, \ldots, a_r) = \{\sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0\}$ the *cone strictly positively spanned by the vectors* $a_1, \ldots, a_r$. Fix a cone Δ. If there exist linearly independent vectors $a_1, \ldots, a_r \in \mathbb{R}^n \setminus \{0\}$ such that $\Delta = \operatorname{cone}(a_1, \ldots, a_r)$, then Δ is called a *simplicial cone*. If moreover you can choose $a_1, \ldots, a_r \in \mathbb{Z}^n$ (or in $\mathbb{Q}^n$), we say Δ is a *rational simplicial cone*. If one can choose $a_1, \ldots, a_r$ such that $\{a_1, \ldots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, we call Δ a *simple cone*.

Let τ be a proper face of $\Gamma(f)$ and let $a_1, \ldots, a_r$ be as in Proposition 1.1.10. Following from Proposition 1.1.10 and Lemma 1.2.2, one can partition the cone Δ_τ associated to the face τ into a finite number of rational simplicial cones such

that each Δ_i is spanned by vectors from the set $\{a_1, \ldots, a_r\}$. We will call such a decomposition a *rational simplicial decomposition of Δ_τ without introducing new rays*. By Lemma 1.2.3, one can even find a partition of Δ_τ into simple cones, but in general it will then be necessary to introduce new generators.

Lemma 1.2.2. *Let Δ be the cone strictly positively spanned by the vectors $a_1, \ldots, a_r \in (\mathbb{R}^+)^n \setminus \{0\}$. Then there exists a finite partition of Δ into cones Δ_i, such that each Δ_i is strictly positively spanned by some vectors from the set $\{a_1, \ldots, a_r\}$ that are linearly independent over $\mathbb{R}$.*

Proof. [Den95, Lemma 2]. We cite the proof here, because the algorithm is used in the proof of Proposition 3.3.3. Let $\bar{\Delta}$ be the closure of Δ. Thus $\bar{\Delta}$ is a closed convex polyhedral cone. If γ is a closed convex cone, we denote by $\operatorname{ri}\gamma$ the relative interior of γ (i.e., the interior in the linear space generated by γ). The proof is by induction on $\dim(\mathbb{R}a_1 + \cdots + \mathbb{R}a_r)$. We may suppose that $\mathbb{R}^+ a_1$ is a face of $\bar{\Delta}$. One can verify that Δ is the disjoint union of the cones of the form

$$W = \{\lambda_1 a_1 + b \mid \lambda_1 \in \mathbb{R}, \lambda_1 > 0, b \in \operatorname{ri}\gamma\},$$

where γ is a proper face of $\bar{\Delta}$ such that W is not contained in a facet of $\bar{\Delta}$.

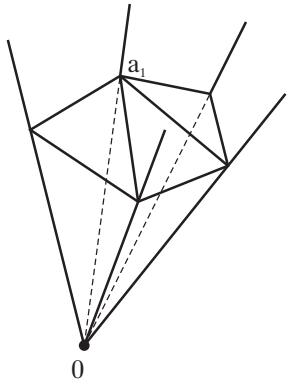


Figure 1.2.1. A partition into smaller cones

By induction, we have that $\operatorname{ri}\gamma$ is the disjoint union of cones w_i with the required property. But then, W is the disjoint union of cones

$$W_i = \{\lambda_1 a_1 + b \mid \lambda_1 \in \mathbb{R}, \lambda_1 > 0, b \in w_i\}.$$

□

Lemma 1.2.3. *Let Δ be a rational simplicial cone. Then there exists a finite partition of Δ into simple cones.*

Lemma 1.2.3 can be proved following the idea of [AVGZ86, Section 8, Lemma 7] and making use of the notion "multiplicity". For convenience of the reader, we shall prove this result in 1.2.6.

Definition 1.2.4. Let $a_1, \ldots, a_r$ be vectors in $\mathbb{Z}^n$ that are linearly independent over $\mathbb{Q}$. We define the *multiplicity* of $a_1, \ldots, a_r$ as the index of the lattice $\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_r$ in the group of the points with integral coordinates of the vector space generated by $a_1, \ldots, a_r$.

Remark. It is easy to verify that the multiplicity of $a_1, \ldots, a_r$ equals the number of elements in the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^{r} \lambda_i a_i \mid 0 \leq \lambda_i < 1 \text{ for } i = 1, \ldots, r \right\}.$$

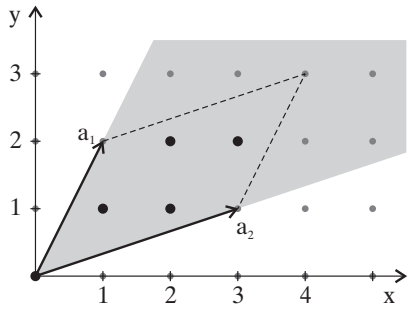


Figure 1.2.2. The multiplicity of a_1 and a_2 is 5.

Proposition 1.2.5. *Let $a_1, \ldots, a_r$ be vectors in $\mathbb{Z}^n$ that are linearly independent over $\mathbb{Q}$.*

(i) The multiplicity of $a_1, \ldots, a_r$ equals the greatest common divisor of the determinants of all $r \times r$-matrices obtained by omitting columns from the matrix with rows $a_1, \ldots, a_r$.

(ii) The multiplicity of $a_1, \ldots, a_r$ is 1 if and only if $\{a_1, \ldots, a_r\}$ is a subset of a basis of $\mathbb{Z}^n$ (as $\mathbb{Z}$-module).

(iii) The multiplicity of $a_1, \ldots, a_r$ equals the volume of the parallelopiped spanned by $a_1, \ldots, a_r$ with respect to ω_A, where ω_A is the volume form on the vector space $\operatorname{vct} A$ generated by $A = \{a_1, \ldots, a_r\}$, normalized such that the parallelopiped spanned by a lattice basis of $\mathbb{Z}^n \cap \operatorname{vct} A$ has volume 1.

Proof. Assertion (i) and (ii) can be obtained by using [AW92, Section 5.3, Theorem 3.1 and Lemma 3.11] or by the fact that there exist matrices $X \in GL_r(\mathbb{Z})$ and $Y \in GL_n(\mathbb{Z})$ such that XAY is a matrix with all matrix elements equal to zero except the ones on the diagonal, and with A the matrix with rows $a_1, \ldots, a_r$. Choose then a new basis for $\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_r$ and $\mathbb{Z}^n$ respectively.

Choose a lattice basis $V = \{v_1, \ldots, v_r\}$ for $\mathbb{Z}^n \cap \operatorname{vct} A$. Suppose that a_{i_V} are the coordinates of a_i with respect to the basis V. It is clear that the number of elements in the sets $\mathbb{Z}^n \cap \{\sum_{i=1}^r \lambda_i a_i \mid 0 \leq \lambda_i < 1 \text{ for } i = 1, \ldots, r\}$ and $\mathbb{Z}^r \cap \{\sum_{i=1}^r \lambda_i a_{i_V} \mid 0 \leq \lambda_i < 1 \text{ for } i = 1, \ldots, r\}$ are equal. Now it is clear by (i) that the multiplicity of $a_1, \ldots, a_r$ equals the determinant of the matrix with rows $a_{1_V}, \ldots, a_{r_V}$, which is the volume of the parallelopiped spanned by $a_{1_V}, \ldots, a_{r_V}$. This proves (iii). □

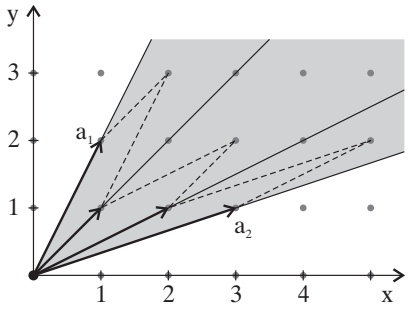


Figure 1.2.3. A partition into simple cones

1.2.6. Proof of Lemma 1.2.3.
Suppose that $\Delta = \{\lambda_1 a_1 + \cdots + \lambda_e a_e \mid \lambda_i \in \mathbb{R}, \lambda_i > 0\}$, with $a_1, \ldots, a_e$ primitive vectors that are linearly independent over $\mathbb{Q}$. If the multiplicity of $a_1, \ldots, a_e$ is 1, then Δ is a simple cone. Otherwise it suffices to prove that we can give a finite partition of Δ into simplicial cones which are strictly positively spanned by linearly independent vectors in $\mathbb{Z}^n$ with lower multiplicity. If the multiplicity of $a_1, \ldots, a_e$ is not 1, there exists a primitive vector b in $\mathbb{Z}^n$ and $\lambda_1, \ldots, \lambda_e \in \mathbb{R}$ with $0 \leq \lambda_i < 1$ for $i = 1, \ldots, e$, and not all λ_i zero, such that

$$b = \lambda_1 a_1 + \cdots + \lambda_e a_e.$$

Case 1.
Suppose that $b \in \Delta$ or equivalently, all λ_i are different from zero. Then the cones strictly positively spanned by b and $0, 1, \ldots$ or $e-1$ vectors from the set $\{a_1, \ldots, a_e\}$ give a partition of Δ into rational simplicial cones. From Proposition 1.2.5 and Lemma 1.2.7, it follows that the generators of these cones have lower multiplicity.

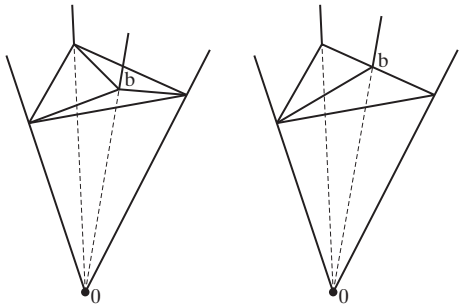


Figure 1.2.4

Case 2.
If some of the λ_i are equal to zero, then, without loss of generality, we may suppose that $b = \lambda_1 a_1 + \cdots + \lambda_d a_d$ with $\lambda_1, \ldots, \lambda_d > 0$. The same cones as in Case 1, except the cones in the boundary of Δ, form a partition of Δ. More concretely, the cones strictly positively spanned by b, the vectors from the set $\{a_{d+1}, \ldots, a_e\}$ and $0, 1, \ldots, d-1$ vectors from the set $\{a_1, \ldots, a_d\}$ form a partition of Δ into rational simplicial cones. From Proposition 1.2.5 and Lemma 1.2.7, it follows that the generators of these cones have lower multiplicity. □

Lemma 1.2.7. *Let $a_1, \ldots, a_n$ be linearly independent vectors in $\mathbb{Z}^n$.*

(i) The greatest common divisor of the determinants of all $(n-1)\times(n-1)$ matrices obtained by omitting columns from the matrix with rows $a_1, a_2, \ldots, a_{n-1}$ is a divisor of the determinant of the matrix with rows $a_1, a_2, \ldots, a_n$.

(ii) Let $b = \sum_{i=1}^{n} \lambda_i a_i$ with $\lambda_i \in \mathbb{R}$ and $0 < \lambda_1 < 1$. Then the absolute value of the determinant of the matrix with rows $b, a_2, \ldots, a_n$ is strictly less than the absolute value of the determinant of the matrix with rows $a_1, a_2, \ldots, a_n$.

Proof. An easy exercise. □

Chapter 2

An explicit formula for $Z_f(s,\chi)$

In Theorem 2.3.2, at the end of this chapter, we will give a formula for $Z_f(s,\chi)$ that holds if f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron and if the conductor c_χ of χ is equal to 1 (see also [DH01] and [Hoo01]). The conductor c_χ of the character χ of $\mathbb{Z}_p^\times$ is defined as the smallest $c \in \mathbb{N} \setminus \{0\}$ such that χ is trivial on $1 + p^c\mathbb{Z}_p$. So, $c_{\chi_{\text{triv}}}$ is also equal to 1. It is well-known (see [Den91, Theorem 3.3]) that $Z_f(s,\chi)$ is constant as a function of s for $p \gg 0$, if the conductor $c_\chi > 1$ and f is a polynomial over $\mathbb{Z}$. For the exact condition of a similar result for f a polynomial over $\mathbb{Z}_p$ and p a fixed prime number, we refer to the same reference. So, Theorem 2.3.2 gives a formula for $Z_f(s,\chi)$ for the relevant cases. The actors in this formula are the cones Δ_τ associated to the faces τ of $\Gamma(f)$ (see Definition 1.1.4) and, the numbers of elements N_τ of the sets $\{x \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(x) = 0\}$, if $\chi = \chi_{\text{triv}}$, or the sums $\sum_{a \in (\mathbb{F}_p^\times)^n} \chi(f_\tau(a))$, if $\chi \neq \chi_{\text{triv}}$. To calculate $Z_f(s)$ for a concrete polynomial f, we developed the computer program Polygusa [HL00], which is based on this formula; see also Appendix D. But first, we will calculate some useful integrals in Section 2.1 and Section 2.2.

In Section 5.3 of Chapter 5, we will give a similar formula for $Z_f(s,\chi)$ that holds if $f = f_\sigma$, where $\sigma \subset \Gamma(f)$ is an unstable convex set, and if f is non-degenerated over $\mathbb{F}_p$ with respect to a smaller set of faces of $\Gamma(f)$.

2.1 Computation of a well-known integral by Hensel's Lemma

Proposition 2.1.1. *Let $f(x) = f(x_1,\ldots,x_n) \in \mathbb{Z}_p[x_1,\ldots,x_n]$ and $a \in \mathbb{Z}_p^n$. Suppose that the set of congruences*

$$\begin{cases} f(x) & \equiv & 0 \bmod p \\ \frac{\partial f}{\partial x_i}(x) & \equiv & 0 \bmod p, \quad i = 1,\ldots,n \end{cases}$$

has no solution in the coset $a + (p\mathbb{Z}_p)^n$.
Then for s a complex variable with $\mathrm{Re}\,(s) > 0$ one has

$$\int\limits_{a+(p\mathbb{Z}_p)^n} |f(x)|^s |dx| = \begin{cases} p^{-n} & \text{if } f(a) \not\equiv 0 \bmod p \\ p^{-n}(p-1)\frac{p^{-(s+1)}}{1-p^{-(s+1)}} & \text{if } f(a) \equiv 0 \bmod p, \end{cases}$$

where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^n$ so normalized that $\mathbb{Z}_p^n$ has measure 1.

Remark. The proposition above is also a special case of a much more general result (see [Den87]), but in this section we give an elementary proof (see 2.1.6) based on Hensel's Lemma. By Proposition 2.1.1, we can easily prove Corollary 2.1.2. In the next section (see Proposition 2.2.2) we give the calculation of a similar integral over $(\mathbb{Z}_p^\times)^n$ for a non-trivial character. This results will help us to give a formula for Igusa's local zeta function for the trivial and the non-trivial characters.

Corollary 2.1.2. *Let p be a prime number and $f(x) = f(x_1,\ldots,x_n) \in \mathbb{Z}_p[x_1,\ldots,x_n]$. Denote by $\bar{f}$ the polynomial over $\mathbb{F}_p$ obtained by reducing each coefficient of f modulo $p\mathbb{Z}_p$. Let N be the number of elements in the set $\{a \in (\mathbb{F}_p^\times)^n \mid \bar{f}(a) = 0\}$. Suppose that the set of congruences*

$$\begin{cases} f(x) & \equiv & 0 \bmod p \\ \frac{\partial f}{\partial x_i}(x) & \equiv & 0 \bmod p, \quad i = 1,\ldots,n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^n$. Then for s a complex variable with $\mathrm{Re}\,(s) > 0$, one has

$$\int\limits_{(\mathbb{Z}_p^\times)^n} |f(x)|^s |dx| = p^{-n}\left((p-1)^n - pN\frac{(p^s-1)}{p^{s+1}-1}\right),$$

where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^n$ so normalized that $\mathbb{Z}_p^n$ has measure 1.

Proof. Because $\bigcup_{a\in\{1,\dots,p-1\}^n} a+(p\mathbb{Z}_p)^n$ is a partition of $(\mathbb{Z}_p^\times)^n$, we have

$$\begin{aligned}\int\limits_{(\mathbb{Z}_p^\times)^n} |f(x)|^s|dx| &= \sum_{\substack{a\in\{1,\dots,p-1\}^n\\ f(a)\not\equiv 0 \bmod p}} \int\limits_{a+(p\mathbb{Z}_p)^n} |f(x)|^s|dx| \\ &\quad + \sum_{\substack{a\in\{1,\dots,p-1\}^n\\ f(a)\equiv 0 \bmod p}} \int\limits_{a+(p\mathbb{Z}_p)^n} |f(x)|^s|dx|.\end{aligned}$$

From the condition in the statement of Corollary 2.1.2 and Proposition 2.1.1, it follows now that

$$\begin{aligned}\int\limits_{(\mathbb{Z}_p^\times)^n} |f(x)|^s|dx| &= ((p-1)^n - N)p^{-n} + Np^{-n}(p-1)\tfrac{p^{-(s+1)}}{1-p^{-(s+1)}} \\ &= p^{-n}\left((p-1)^n - N\tfrac{1-p^{-s}}{1-p^{-(s+1)}}\right).\end{aligned}$$

□

Lemma 2.1.3 (Hensel)**.** *Let $f(x)$ be a polynomial in one variable x over $\mathbb{Z}_p$ and $k\in\mathbb{N}\setminus\{0\}$. Let $a\in\mathbb{Z}_p$ such that $f(a)\equiv 0 \bmod p^k$ and $f'(a)\not\equiv 0 \bmod p$. Then there exists a unique $\xi\in\mathbb{Z}_p$ such that*

$$f(\xi)=0 \quad \textit{and} \quad \xi\equiv a \bmod p^k.$$

Proof. For the case $k=1$, we refer to [Kob77, pp. 16–17]. The case $k>1$ follows from the case $k=1$ applied to the polynomial $g(y)=p^{-(k-1)}f(a+p^{k-1}y)$, whose coefficients are in $\mathbb{Z}_p$ because $f(a)\equiv 0 \bmod p^k$. □

Corollary 2.1.4. *Let $f(x)\in\mathbb{Z}_p[x]$, $k\in\mathbb{N}\setminus\{0\}$. Let $a\in\mathbb{Z}_p$ such that $f(a)\equiv 0 \bmod p$ and $f'(a)\not\equiv 0 \bmod p$. Then there exists a $\xi\in\mathbb{Z}_p$ such that*

$$\xi+p^k\mathbb{Z}_p=\{x\in a+p\mathbb{Z}_p \mid f(x)\equiv 0 \bmod p^k\}.$$

Proof. By Lemma 2.1.3, it follows that there exists a unique $\xi\in\mathbb{Z}_p$ such that $f(\xi)=0$ and $\xi\equiv a \bmod p$. We claim that

$$\xi+p^k\mathbb{Z}_p=\{x\in a+p\mathbb{Z}_p \mid f(x)\equiv 0 \bmod p^k\}.$$

For the first inclusion "$\subset$", take $x\in\mathbb{Z}_p$ and suppose that $x\equiv\xi \bmod p^k$. Then $x\equiv\xi\equiv a \bmod p$ and $f(x)\equiv f(\xi)\equiv 0 \bmod p^k$.

For the second inclusion "$\supset$", take $x\in a+p\mathbb{Z}_p$ and suppose that $f(x)\equiv 0 \bmod p^k$. Because $x\equiv a \bmod p$, one has that $f'(x)\equiv f'(a)\not\equiv 0 \bmod p$. Then it follows by Lemma 2.1.3 that there exists a unique $\eta\in\mathbb{Z}_p$ such that $f(\eta)=0$ and $\eta\equiv x \bmod p^k$. Moreover, $\eta\equiv x\equiv a \bmod p$, and by the uniqueness of ξ, we thus have $\eta=\xi$. So, $x\equiv\xi \bmod p^k$. □

We generalize the preceding corollary to polynomials in many variables.

Lemma 2.1.5. *Let $f(x) = f(x_1,\dots,x_n) \in \mathbb{Z}_p[x_1,\dots,x_n]$ and $k \in \mathbb{N}\setminus\{0\}$. Suppose that $a = (a_1,\dots,a_n) \in \mathbb{Z}_p^n$ such that $f(a) \equiv 0 \bmod p$ and $(\partial f/\partial x_1)(a) \not\equiv 0 \bmod p$. Let $\xi_2,\dots,\xi_n \in \mathbb{Z}_p$ with $\xi_i \equiv a_i \bmod p$ for $i = 2,\dots,n$. Then there exists a $\xi_1 = \xi_1(\xi_2,\dots,\xi_n) \in \mathbb{Z}_p$ such that, for all $x_2,\dots,x_n \in \mathbb{Z}_p$ satisfying $x_i \equiv \xi_i \bmod p^k$ for $i = 2,\dots,n$, one has that*

$$\xi_1 + p^k\mathbb{Z}_p = \{x_1 \in a_1 + p\mathbb{Z}_p \mid f(x_1,\dots,x_n) \equiv 0 \bmod p^k\}.$$

This implies that the set $\{(x_1,\dots,x_n) \in a + (p\mathbb{Z}_p)^n \mid f(x_1,\dots,x_n) \equiv 0 \bmod p^k\}$ equals

$$\bigcup (\xi_1(\xi_2,\dots,\xi_n),\xi_2,\dots,\xi_n) + (p^k\mathbb{Z}_p)^n,$$

where the union is taken over all $(n-1)$-tuples $(\xi_2 + p^k\mathbb{Z}_p,\dots,\xi_n + p^k\mathbb{Z}_p)$ with $\xi_i \equiv a_i \bmod p$ for $i = 2,\dots,n$.

Remark. Of course, we have a similar result for the other variables x_i satisfying $(\partial f/\partial x_i)(a) \not\equiv 0 \bmod p$.

Proof. The result follows by Corollary 2.1.4, because $f(x_1,x_2,\dots,x_n) \equiv f(x_1,\xi_2,\dots,\xi_n) \bmod p^k$ for every $x_2,\dots,x_n \in \mathbb{Z}_p$ with $x_i \equiv \xi_i \bmod p^k$; $f(a_1,\xi_2,\dots,\xi_n) \equiv f(a_1,a_2,\dots,a_n) \equiv 0 \bmod p$ and $(\partial f/\partial x_1)(a_1,\xi_2,\dots,\xi_n) \equiv (\partial f/\partial x_1)(a_1,a_2,\dots,a_n) \not\equiv 0 \bmod p$. □

2.1.6. Proof of Proposition 2.1.1.
Case 1. $f(a) \not\equiv 0 \bmod p$.
For every $x \in a + (p\mathbb{Z}_p)^n$, one has $f(x) \equiv f(a) \not\equiv 0 \bmod p$. Hence, $\operatorname{ord} f(x) = 0$ and $|f(x)|^s = 1$ for every $x \in a + (p\mathbb{Z}_p)^n$. Thus,

$$\int_{a+(p\mathbb{Z}_p)^n} |f(x)|^s |dx| = \int_{a+(p\mathbb{Z}_p)^n} 1|dx| = p^{-n}.$$

Case 2. $f(a) \equiv 0 \bmod p$.
For every $x \in a + (p\mathbb{Z}_p)^n$, one has $f(x) \equiv f(a) \equiv 0 \bmod p$. Hence, $\operatorname{ord} f(x) \geq 1$ for every $x \in a + (p\mathbb{Z}_p)^n$. Thus,

$$\begin{aligned}\int_{a+(p\mathbb{Z}_p)^n} |f(x)|^s |dx| &= \sum_{k=1}^{\infty} \int_{\substack{x \in a+(p\mathbb{Z}_p)^n \\ \operatorname{ord} f(x) = k}} |f(x)|^s |dx| \\ &= \sum_{k=1}^{\infty} p^{-ks} \cdot \text{ measure of } \{x \in a + (p\mathbb{Z}_p)^n \mid \operatorname{ord} f(x) = k\}.\end{aligned}$$

We claim that the measure of $\{x \in a + (p\mathbb{Z}_p)^n \mid \operatorname{ord} f(x) = k\}$ equals $p^{-k-n+1} - p^{-k-n}$. Consequently,

$$\begin{aligned}\int_{a+(p\mathbb{Z}_p)^n} |f(x)|^s |dx| &= \sum_{k=1}^{\infty} p^{-ks}(p^{-k-n+1} - p^{-k-n}) \\ &= p^{-n}(p-1)\sum_{k=1}^{\infty}\left(p^{-(s+1)}\right)^k \\ &= p^{-n}(p-1)\frac{p^{-(s+1)}}{1-p^{-(s+1)}},\end{aligned}$$

because $\operatorname{Re}(s) > 0$ implies that the modulus of $p^{-(s+1)}$ is < 1.

We now prove that the measure of $\{x \in a + (p\mathbb{Z}_p)^n \mid \operatorname{ord} f(x) = k\}$ equals $p^{-k-n+1} - p^{-k-n}$. Since the set in the above formula equals the complement of $\{x \in a + (p\mathbb{Z}_p)^n \mid f(x) \equiv 0 \bmod p^{k+1}\}$ in $\{x \in a + (p\mathbb{Z}_p)^n \mid f(x) \equiv 0 \bmod p^k\}$, the additivity of the measure implies that it suffices to prove that the measure of $\{x \in a + (p\mathbb{Z}_p)^n \mid f(x) \equiv 0 \bmod p^k\}$ equals p^{-k-n+1} for $k \in \mathbb{N} \setminus \{0\}$.

From the condition in the statement of Proposition 2.1.1 and the fact that $f(a) \equiv 0 \bmod p$, it follows that there exists an $i \in \{1, \ldots, n\}$ such that $(\partial f/\partial x_i)(a) \not\equiv 0 \bmod p$. Without loss of generality, we may suppose that $(\partial f/\partial x_1)(a) \not\equiv 0 \bmod p$. Then by Lemma 2.1.5, it follows that the set $\{x \in a + (p\mathbb{Z}_p)^n \mid f(x) \equiv 0 \bmod p^k\}$ is equal to

$$\bigcup (\xi_1(\xi_2, \ldots, \xi_n), \xi_2, \ldots, \xi_n) + (p^k\mathbb{Z}_p)^n,$$

where $\xi_1(\xi_2, \ldots, \xi_n)$ is as in Lemma 2.1.5 and the union is taken over all $(n-1)$-tuples $(\xi_2 + p^k\mathbb{Z}_p, \ldots, \xi_n + p^k\mathbb{Z}_p)$ with $\xi_i \equiv a_i \bmod p$, for $i = 2, \ldots, n$. This union is disjoint and the measure of $\xi + (p^k\mathbb{Z}_p)^n$ is equal to p^{-kn}. Hence, the measure of $\{x \in a + (p\mathbb{Z}_p)^n \mid f(x) \equiv 0 \bmod p^k\}$ is equal to $p^{(k-1)(n-1)}p^{-kn} = p^{-k-n+1}$. □

2.2 Computation of the same integral for a non-trivial character

The aim of this section is to give a proof of Proposition 2.2.2, which will help us to give a formula for $Z_f(s, \chi)$ for a non-trivial character χ. Lemma 2.2.1 and Proposition 2.2.2 are special cases of more general results [Den91, Theorem 3.4], but for convenience of the reader, we will give a direct proof.

Lemma 2.2.1. *Let p be a prime number and $f(x) = f(x_1,\ldots,x_n) \in \mathbb{Z}_p[x_1,\ldots,x_n]$. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a non-trivial character of $\mathbb{Z}_p^\times$. Suppose that $a \in \mathbb{Z}_p^n$ such that $f(a) \equiv 0 \bmod p$ and such that the set of congruences $(\partial f/\partial x_i)(x) \equiv 0 \bmod p,\ i = 1,\ldots,n$ has no solution in $a + (p\mathbb{Z}_p)^n$. Then*

$$\int_{a+(p\mathbb{Z}_p)^n} \chi(\mathrm{ac}\, f(x))|f(x)|^s|dx| = 0.$$

Proof. From the condition in the statement of Lemma 2.2.1, it follows that there exists an $i \in \{1,\ldots,n\}$ such that $(\partial f/\partial x_i)(a) \not\equiv 0 \bmod p$. Put

$$\begin{array}{rccc} \psi: & \mathbb{Z}_p^n & \to & \mathbb{Z}_p^n \\ & (x_1,\ldots,x_n) & \mapsto & (v_1,\ldots,v_n), \end{array}$$

where

$$v_j = \begin{cases} p^{-1}(f(a+px) - f(a)) & \text{for } j = i \\ x_j & \text{for } j \neq i. \end{cases}$$

Then ψ is a measure-preserving bi-analytic map of $\mathbb{Z}_p^n$ to itself. (Therefore, we can use the change of variable formula for the p-adic case [Igu00, Proposition 7.4.1]). So, we get

$$\begin{aligned} \int_{a+(p\mathbb{Z}_p)^n} \chi(\mathrm{ac}\, f(x))|f(x)|^s|dx| &= p^{-n}\int_{\mathbb{Z}_p^n} \chi(\mathrm{ac}\, f(a+px))|f(a+px)|^s|dx| \\ &= p^{-n}\int_{\mathbb{Z}_p^n} \chi(\mathrm{ac}\,(f(a)+pv_i))|f(a)+pv_i|^s|dv| \\ &= p^{-n}\int_{\mathbb{Z}_p} \chi(\mathrm{ac}\,(f(a)+pv_i))|f(a)+pv_i|^s|dv_i| \\ &= p^{-n-s}\int_{\mathbb{Z}_p} \chi(\mathrm{ac}\,(v_i))|v_i|^s|dv_i|, \end{aligned}$$

because the translation over $f(a)/p \in \mathbb{Z}_p$ is a measure-preserving bi-analytic map of $\mathbb{Z}_p$ to itself. The last integral is equal to zero. Indeed, because χ is a non-trivial character, there exists an $u \in \mathbb{Z}_p^\times$, such that $\chi(u) \neq 1$. Hence,

$$\begin{aligned} \chi(u)\int_{\mathbb{Z}_p} \chi(\mathrm{ac}\,(v_i))|v_i|^s|dv_i| &= \int_{\mathbb{Z}_p} \chi(\mathrm{ac}\,(uv_i))|uv_i|^s|dv_i| \\ &= \int_{\mathbb{Z}_p} \chi(\mathrm{ac}\,(y_i))|y_i|^s|dy_i|. \end{aligned}$$

□

Proposition 2.2.2. *Let p be a prime number and $f(x) = f(x_1,\ldots,x_n) \in \mathbb{Z}_p[x_1,\ldots,x_n]$. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a non-trivial character of $\mathbb{Z}_p^\times$ and let c_χ be the conductor of χ. Suppose that the set of congruences*

$$\begin{cases} f(x) & \equiv 0 \bmod p \\ \frac{\partial f}{\partial x_i}(x) & \equiv 0 \bmod p, \quad i = 1,\ldots,n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^n$. *Then*

$$\int_{(\mathbb{Z}_p^\times)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| = p^{-nc_\chi} \sum_{\substack{a \in (\mathbb{Z}_p^\times)^n \\ a \bmod p^{c_\chi} \\ f(a) \not\equiv 0 \bmod p}} \chi(f(a)).$$

Proof. It is clear that

$$\begin{aligned}\int_{(\mathbb{Z}_p^\times)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| &= \sum_{\substack{a \in (\mathbb{Z}_p^\times)^n \\ a \bmod p^{c_\chi} \\ f(a) \not\equiv 0 \bmod p}} \int_{a+(p^{c_\chi}\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx| \\ &\quad + \sum_{\substack{a \in (\mathbb{Z}_p^\times)^n \\ a \bmod p \\ f(a) \equiv 0 \bmod p}} \int_{a+(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx|.\end{aligned}$$

By Lemma 2.2.1, we know that the integral $\int_{a+(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))|f(x)|^s|dx|$ is zero for every $a \in \mathbb{Z}_p^\times$ with $f(a) \equiv 0 \bmod p$, which implies the assertion. □

2.3 An explicit formula for $Z_f(s,\chi)$

In Theorem 2.3.2, we will give a formula for $Z_f(s,\chi)$ that holds if f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron and if the conductor c_χ of χ is equal to 1. Recall that the conductor c_χ of the character χ of $\mathbb{Z}_p^\times$ is defined as the smallest $c \in \mathbb{N} \setminus \{0\}$ such that χ is trivial on $1 + p^c\mathbb{Z}_p$. So, $c_{\chi_{\text{triv}}}$ is also equal to 1. In collaboration with Davy Loots, we developed the computer program Polygusa [HL00] to calculate Igusa's local zeta function $Z_f(s)$ for a concrete polynomial f. Polygusa is written in Maple and is based on the formula from Theorem 2.3.2; see also Appendix D.

Definition 2.3.1. For $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$, we define

$$\sigma(k) = \sum_{i=1}^n k_i.$$

Theorem 2.3.2. *Let p be a prime number. Let f be as in Definition 0.1.4. Suppose that f is non-degenerated over the finite field $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$. Let χ be a character of $\mathbb{Z}_p^\times$ with conductor $c_\chi = 1$. Denote for each face τ of $\Gamma(f)$ by N_τ the number of elements in the set*

$$\{a \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(a) = 0\}.$$

Let s be a complex variable with $\mathrm{Re}\,(s) > 0$. *Then*

$$Z_f(s,\chi) = \sum_{\substack{\tau \ face \\ of\ \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

with

$$L_\tau = \begin{cases} p^{-n}\left((p-1)^n - pN_\tau \frac{p^s-1}{p^{s+1}-1}\right) & \text{for } \chi = \chi_{\mathrm{triv}}\,, \\ p^{-n} \sum\limits_{a \in (\mathbb{F}_p^\times)^n} \chi\left(f_\tau(a)\right) & \text{for } \chi \neq \chi_{\mathrm{triv}}\,, \end{cases}$$

and

$$S_{\Delta_\tau} = \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s},$$

for each face τ of $\Gamma(f)$ (including $\tau = \Gamma(f)$).

We have $S_{\Delta_{\Gamma(f)}} = 1$ and the other S_{Δ_τ} can be calculated as follows. Take a partition of the cone Δ_τ associated to the proper face τ into rational simplicial cones Δ_i. Then clearly $S_{\Delta_\tau} = \sum_i S_{\Delta_i}$, where the summation is over the rational simplicial cones Δ_i and

$$S_{\Delta_i} = \sum_{k \in \mathbb{N}^n \cap \Delta_i} p^{-\sigma(k) - m(k)s}.$$

Let Δ_i be the cone strictly positively spanned by the linearly independent vectors $a_1, \ldots, a_r \in \mathbb{N}^n \setminus \{0\}$. Then

$$S_{\Delta_i} = \frac{\sum\limits_h p^{\sigma(h) + m(h)s}}{(p^{\sigma(a_1) + m(a_1)s} - 1) \cdots (p^{\sigma(a_r) + m(a_r)s} - 1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^r \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \ldots, r \right\}.$$

In the Theorem above, $m(\cdot)$ and $\sigma(\cdot)$ are defined as in Definition 1.1.1 and Definition 2.3.1.

Remarks.

1. The S_{Δ_τ} do only depend on the Newton polyhedron of f. On the other hand, the L_τ are dependent on the specific coefficients of the polynomial f. Note that S_{Δ_τ} and L_τ are rational functions in p^{-s} for $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) > 0$. We also denote their meromorphic continuation by S_{Δ_τ} and L_τ, respectively. So, Theorem 2.3.2 gives another proof of the fact that $Z_f(s,\chi)$ is a rational function of p^{-s} (for the case that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of $\Gamma(f)$).

2. By a similar argument as in the proof below, we can prove that

$$Z_{f,0}(s,\chi) = \sum_{\substack{\tau \text{ compact face} \\ \text{of } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

with L_τ and S_{Δ_τ} as in the statement of the theorem above. For this formula, we only need f to be non-degenerated over the finite field $\mathbb{F}_p$ with respect to the *compact* faces of $\Gamma(f)$.

Proof. We first prove the result for $\chi = \chi_{\text{triv}}$. Recall that the cones associated to the faces of $\Gamma(f)$ form a partition of $(\mathbb{R}^+)^n$. So, $(\mathbb{R}^+)^n$ is equal to the disjoint union

$$\bigcup_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \Delta_\tau,$$

where $\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}$ is the cone associated to τ. Therefore, one has

$$\begin{aligned} Z_f(s) &= \int_{\mathbb{Z}_p^n} |f(x)|^s |dx| \\ &= \sum_{k \in \mathbb{N}^n} \int_{\substack{x \in \mathbb{Z}_p^n \\ \text{ord } x = k}} |f(x)|^s |dx| \\ &= \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} \int_{\substack{x \in \mathbb{Z}_p^n \\ \text{ord } x = k}} |f(x)|^s |dx|. \end{aligned}$$

Suppose that τ is a face of $\Gamma(f)$, $k \in \mathbb{N}^n \cap \Delta_\tau$ and $x = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ with $\text{ord } x = k$. Write $x_j = p^{k_j} u_j$ with $u_j \in \mathbb{Z}_p^\times$. Then

$$\begin{aligned} |dx| &= p^{-\sigma(k)} |du| \quad \text{and} \\ x^\omega &= x_1^{\omega_1} \cdots x_n^{\omega_n} = p^{k \cdot \omega} u^\omega. \end{aligned}$$

We know that $F(k) = \tau$. Thus $k \cdot \omega = m(k)$ for every $\omega \in \text{supp}\,(f) \cap \tau$, and $k \cdot \omega \geq m(k) + 1$ for every $\omega \in \text{supp}\,(f) \setminus \tau$. It follows that

$$f(x) = p^{m(k)} (f_\tau(u) + p \tilde{f}_{\tau,k}(u)),$$

where $\tilde{f}_{\tau,k}(u)$ is a polynomial in the variables $u_1, \ldots, u_n$ over $\mathbb{Z}_p$ (depending on τ, k, and f. If $k = 0$, then $\tau = \Gamma(f)$, $m(k) = \sigma(k) = 0$, $f_\tau(u) = f(u)$ and $f_{\tau,k}(u) = 0$). Hence,

$$Z_f(s) = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s} \int_{(\mathbb{Z}_p^\times)^n} |f_\tau(u) + p \tilde{f}_{\tau,k}(u)|^s |du|.$$

Put

$$L_\tau = \int_{(\mathbb{Z}_p^\times)^n} |f_\tau(u) + p \tilde{f}_{\tau,k}(u)|^s |du|.$$

Because f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron, we know that the set of congruences

$$\begin{cases} f_\tau(u) + p\tilde{f}_{\tau,k}(u) & \equiv & 0 \bmod p \\ \frac{\partial(f_\tau + p\tilde{f}_{\tau,k})}{\partial u_i}(u) & \equiv & 0 \bmod p, \quad i = 1,\ldots,n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^n$. So, by Corollary 2.1.2, we have

$$L_\tau = p^{-n}\left((p-1)^n - pN_\tau\frac{(p^s-1)}{p^{s+1}-1}\right),$$

for every face τ of $\Gamma(f)$.

Note that L_τ does not depend on k. Thus

$$Z_f(s) = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} L_\tau S_{\Delta_\tau}, \quad \text{with} \quad S_{\Delta_\tau} = \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} p^{-\sigma(k)-m(k)s}.$$

It is clear that $S_{\Gamma(f)} = 1$. Now we only have to give a formula for S_{Δ_τ} with τ a proper face. By Proposition 1.1.10 and Lemma 1.2.2, we know that there exists a finite partition of Δ_τ into rational simplicial cones Δ_i. Then

$$S_{\Delta_\tau} = \sum_i S_{\Delta_i} \quad \text{with} \quad S_{\Delta_i} = \sum_{k\in\mathbb{N}^n\cap\Delta_i} p^{-\sigma(k)-m(k)s}.$$

Let Δ_i be the cone strictly positively spanned by the linearly independent vectors $a_1,\ldots,a_r \in \mathbb{N}^n \setminus \{0\}$. Fix now an element x_τ in τ. Then $m(k) = k\cdot x_\tau$ for every $k \in \Delta_\tau$. Hence,

$$S_{\Delta_i} = \sum_{k\in\mathbb{N}^n\cap\Delta_i} p^{-\sigma(k)-k\cdot x_\tau s}.$$

Special case. $\mathbb{N}^n \cap \Delta_i = \mathbb{N}\setminus\{0\}a_1 + \cdots + \mathbb{N}\setminus\{0\}a_r$ [1].
Because $a_1,\ldots,a_r$ are linearly independent, we obtain in this case that

$$\begin{aligned} S_{\Delta_i} &= \sum_{\lambda_1,\ldots,\lambda_r\in\mathbb{N}\setminus\{0\}} p^{-\sigma(\lambda_1 a_1+\cdots+\lambda_r a_r)-(\lambda_1 a_1+\cdots+\lambda_r a_r)\cdot x_\tau s} \\ &= \textstyle\sum_{\lambda_1=1}^\infty (p^{-\sigma(a_1)-a_1\cdot x_\tau s})^{\lambda_1}\cdots\sum_{\lambda_r=1}^\infty (p^{-\sigma(a_r)-a_r\cdot x_\tau s})^{\lambda_r}. \end{aligned}$$

Because $\mathrm{Re}\,(s) > 0$ and $p > 1$, we have $|p^{-\sigma(a_j)-a_j\cdot x_\tau s}| < 1$ for $j = 1,\ldots,r$ and

$$\begin{aligned} S_{\Delta_i} &= \tfrac{p^{-\sigma(a_1)-a_1\cdot x_\tau s}}{1-p^{-\sigma(a_1)-a_1\cdot x_\tau s}}\cdots\tfrac{p^{-\sigma(a_r)-a_r\cdot x_\tau s}}{1-p^{-\sigma(a_r)-a_r\cdot x_\tau s}} \\ &= \tfrac{1}{(p^{\sigma(a_1)+a_1\cdot x_\tau s}-1)\cdots(p^{\sigma(a_r)+a_r\cdot x_\tau s}-1)}. \end{aligned}$$

We know that the closure $\bar{\Delta}_\tau$ of Δ_τ is equal to $\{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$. So, $a_j \cdot x_\tau$ is equal to $m(a_j)$ for each j, because $a_j \in \bar{\Delta}_\tau$.

[1] This condition is equivalent to saying that Δ_i is a simple cone or that the multiplicity of $a_1,\ldots,a_r$ is 1.

General case.
It is clear that $\Delta_i \cap \mathbb{N}^n$ is the disjoint union of the sets $g + \mathbb{N}a_1 + \cdots + \mathbb{N}a_r$, where g runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r} \mu_j a_j \mid 0 < \mu_j \leq 1 \text{ for } j = 1, \ldots, r \right\}. \tag{2.3.1}$$

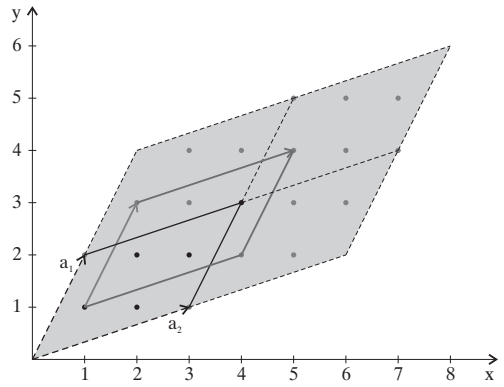


Figure 2.3.1

So, S_{Δ_i} equals

$$\sum_g p^{-\sigma(g)-g\cdot x_\tau s} \sum_{\lambda_1,\ldots,\lambda_r \in \mathbb{N}} p^{-\sigma(\lambda_1 a_1+\cdots+\lambda_r a_r)-(\lambda_1 a_1+\cdots+\lambda_r a_r)\cdot x_\tau s},$$

where g runs through the elements of the set (2.3.1).

Because $\mathrm{Re}\,(s) > 0$, one then obtains that

$$\begin{aligned} S_{\Delta_i} &= \left(\sum_g p^{-\sigma(g)-g\cdot x_\tau s} \right) \sum_{\lambda_1,\ldots,\lambda_r \in \mathbb{N}} p^{-\sigma(\lambda_1 a_1+\cdots+\lambda_r a_r)-(\lambda_1 a_1+\cdots+\lambda_r a_r)\cdot x_\tau s} \\ &= \left(\sum_g p^{-\sigma(g)-g\cdot x_\tau s} \right) \frac{p^{\sigma(a_1+\cdots+a_r)+(a_1+\cdots+a_r)\cdot x_\tau s}}{(p^{\sigma(a_1)+a_1\cdot x_\tau s}-1)\cdots(p^{\sigma(a_r)+a_r\cdot x_\tau s}-1)} \\ &= \frac{\sum_g p^{\sigma(a_1+\cdots+a_r-g)+(a_1+\cdots+a_r-g)\cdot x_\tau s}}{(p^{\sigma(a_1)+a_1\cdot x_\tau s}-1)\cdots(p^{\sigma(a_r)+a_r\cdot x_\tau s}-1)}, \end{aligned}$$

where g runs through the elements of the set (2.3.1). The elements a_j and $(a_1 + \cdots + a_r - g)$ belong to the closure $\bar{\Delta}_\tau$ of Δ_τ. So, $a_j \cdot x_\tau$ equals $m(a_j)$ and $(a_1 + \cdots + a_r - g) \cdot x_\tau$ equals $m(a_1 + \cdots + a_r - g)$.

Therefore,

$$S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s}-1)\cdots(p^{\sigma(a_r)+m(a_r)s}-1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r} \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \ldots, r \right\}.$$

The proof for the case of $\chi \neq \chi_{\text{triv}}$ is completely similar by using Proposition 2.2.2 instead of Corollary 2.1.2. Note that Proposition 2.2.2 implies that the integral $\int_{(\mathbb{Z}_p^\times)^n} \chi(\text{ac}\,(f_\tau(u) + p\tilde{f}_{\tau,k}(u))|f_\tau(u) + p\tilde{f}_{\tau,k}(u)|^s|du|$ is independent of $\tilde{f}_{\tau,k}$ for $c_\chi = 1$, where $\tilde{f}_{\tau,k}(u)$ is the polynomial as defined in the proof above. □

2.4 Examples

We use Theorem 2.3.2 to calculate Igusa's local zeta function in the following example. To find the cones associated to the proper faces of the Newton polyhedron, we use Proposition 1.1.10. To calculate the S_{Δ_τ}, we divide these cones into simplicial cones. We denote the cone strictly positively spanned by $a_1, \ldots, a_r$ by $\text{cone}\{a_1, \ldots, a_r\}$. To find the number of the solutions of the set $\mathbb{Z}^n \cap \{\sum \lambda_i a_i \mid 0 \leq \lambda_i < 1\}$, one can use Proposition 1.2.5. One can also find some examples in Section 4.2, in Appendix D and in [HL00].

Example 2.4.1. Let $f(x,y,z) = x^4y^3z^2 + x^7y^4z + z^5$. One can verify that the Newton polyhedron $\Gamma(f)$ of f is defined by the system of linear inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 1 \\ x + 3z \geq 10 \\ 3x + 4z \geq 20 \\ y + z \geq 5. \end{cases}$$

We denote by $\tau_1, \ldots, \tau_6$ the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above. A picture of the Newton polyhedron of f is given in Figure 2.4.1.

There are 17 proper faces. One can verify (for example with Polygusa) that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron, for every prime $p \gg 0$. One can easily show that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron, for $p = 3$. So, we can use the formula for $Z_f(s)$ from Theorem 2.3.2 for those primes. More information about the cones associated to the proper faces is contained in Table 2.4.1.

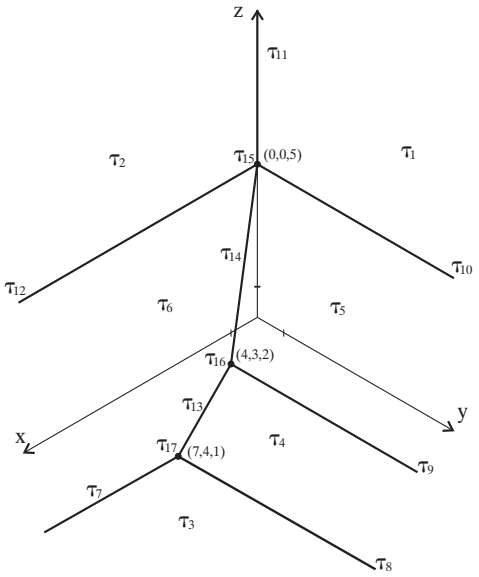


Figure 2.4.1. Newton polyhedron of $x^4y^3z^2 + x^7y^4z + z^5$

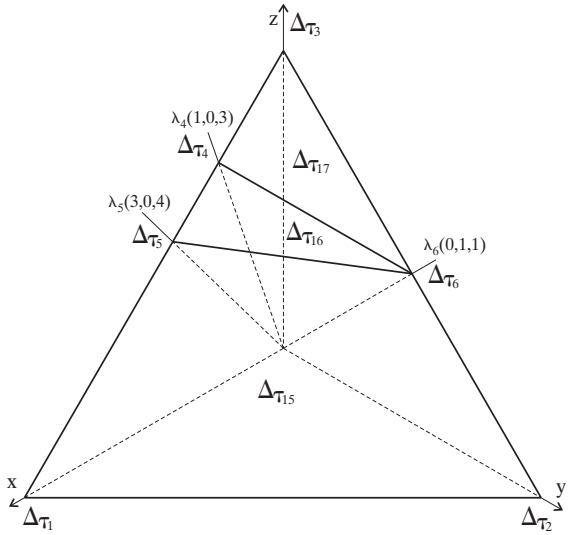


Figure 2.4.2. The associated cones of $\Gamma(x^4y^3z^2 + x^7y^4z + z^5)$

Proper face τ	Generators for Δ_τ	S_{Δ_τ}
τ_1	$(1,0,0)$	$\frac{1}{p-1}$
τ_2	$(0,1,0)$	$\frac{1}{p-1}$
τ_3	$(0,0,1)$	$\frac{1}{p^{1+s}-1}$
τ_4	$(1,0,3)$	$\frac{1}{p^{4+10s}-1}$
τ_5	$(3,0,4)$	$\frac{1}{p^{7+20s}-1}$
τ_6	$(0,1,1)$	$\frac{1}{(p^{2+5s}-1)}$
τ_7	$(0,0,1),(0,1,1)$	$\frac{1}{(p^{1+s}-1)(p^{2+5s}-1)}$
τ_8	$(0,0,1),(1,0,3)$	$\frac{1}{(p^{1+s}-1)(p^{4+10s}-1)}$
τ_9	$(1,0,3),(3,0,4)$	$\frac{1+p^{5+14s}+p^{8+22s}+p^{3+8s}+p^{6+16s}}{(p^{4+10s}-1)(p^{7+20s}-1)}$
τ_{10}	$(1,0,0),(3,0,4)$	$\frac{1+p^{2+5s}+p^{4+10s}+p^{6+15s}}{(p-1)(p^{7+20s}-1)}$
τ_{11}	$(1,0,0),(0,1,0)$	$\frac{1}{(p-1)^2}$
τ_{12}	$(0,1,0),(0,1,1)$	$\frac{1}{(p-1)(p^{2+5s}-1)}$
τ_{13}	$(1,0,3),(0,1,1)$	$\frac{1}{(p^{4+10s}-1)(p^{2+5s}-1)}$
τ_{14}	$(3,0,4),(0,1,1)$	$\frac{1}{(p^{7+20s}-1)(p^{2+5s}-1)}$
τ_{15}	$(1,0,0),(0,1,0),(3,0,4),(0,1,1)$	$\frac{1+p^{2+5s}+p^{4+10s}+p^{6+15s}}{(p-1)^2(p^{7+20s}-1)}$ $+\frac{1+p^{4+10s}+p^{6+10s}}{(p-1)(p^{2+5s}-1)(p^{7+20s}-1)}$ $+\frac{1}{(p-1)(p^{7+20s}-1)}$
τ_{16}	$(1,0,3),(3,0,4),(0,1,1)$	$\frac{1+p^{3+8s}+p^{5+14s}+p^{6+16s}+p^{8+22s}}{(p^{4+10s}-1)(p^{7+20s}-1)(p^{2+5s}-1)}$
τ_{17}	$(0,0,1),(1,0,3),(0,1,1)$	$\frac{1}{(p^{1+s}-1)(p^{4+10s}-1)(p^{2+5s}-1)}$

Table 2.4.1. The cones associated to the faces of $\Gamma(x^4y^3z^2+x^7y^4z+z^5)$

The cone Δ_{τ_9} is rational simplicial, but the multiplicity of $(1,0,3),(3,0,4)$ is 5 and

$$\begin{aligned}&\mathbb{Z}^3 \cap \{\lambda_1(1,0,3)+\lambda_2(3,0,4) \mid 0 \le \lambda_i < 1 \text{ for } i=1,2\}\\ &= \{(0,0,0),\\ &\quad (2,0,3)=1/5(1,0,3)+3/5(3,0,4),\ (3,0,5)=3/5(1,0,3)+4/5(3,0,4)\\ &\quad (1,0,2)=2/5(1,0,3)+1/5(3,0,4),\ (2,0,4)=4/5(1,0,3)+2/5(3,0,4)\}.\end{aligned}$$

So, $\sum_h p^{\sigma(h)+m(h)s}$, where h runs through the above set, is equal to $1+p^{5+14s}+p^{8+22s}+p^{3+8s}+p^{6+16s}$. One has to do a similar calculation for τ_{10}: the multiplicity of $(1,0,0),(3,0,4)$ is 4 and

$$\begin{aligned}&\mathbb{Z}^3 \cap \{\lambda_1(1,0,0)+\lambda_2(3,0,4) \mid 0 \le \lambda_i < 1 \text{ for } i=1,2\}\\ &= \{(0,0,0),(1,0,1),(2,0,2),(3,0,3)\}.\end{aligned}$$

The cone $\Delta_{\tau_{15}}$ is not rational simplicial. One can take the following partition of the cone $\Delta_{\tau_{15}}$ into rational simplicial cones:

$$\begin{aligned}\Delta_{\tau_{15}} &= \text{cone}\{(1,0,0),(0,1,0),(3,0,4)\} \cup \text{cone}\{(0,1,0),(0,1,1),(3,0,4)\}\\ &\quad \cup \text{cone}\{(0,1,0),(3,0,4)\}\end{aligned}$$

(see Figure 2.4.2).

The associated multiplicities are 4, 3 and 1, respectively and the associated sets of integer points are

$$\{(0,0,0),(1,0,1),(2,0,2),(3,0,3)\},\ \{(0,0,0),(1,1,2),(2,1,3)\} \text{ and } \{(0,0,0)\},$$

respectively. The associated set of integer points to $\Delta_{\tau_{16}}$ is $\{0,0,0),(1,0,2),(2,0,3),\ (2,0,4),(3,0,5)\}$. The calculation of the other S_{Δ_τ} is easy because their generators are linearly independent with multiplicity 1. We calculated the N_τ for $p=3$ and obtained (for $p=3$) that

$$Z_f(s) = \frac{N(3^s)}{(3^{s+1}-1)(3^{2+5s}-1)^2(3^{2+5s}+1)(3^{7+20s}-1)},$$

where $N(3^s)$ is a polynomial in 3^s, relatively prime with the denominator (see Example 2 in Appendix D). So the real poles of $Z_f(s)$ are $-7/20$, $-2/5$ and -1, with order equal to 1, 2 and 1, respectively.

Chapter 3

The candidate poles of $Z_f(s,\chi)$ and their expected orders

In this chapter, we study the poles of Igusa's local zeta function using the formula from Theorem 2.3.2. We obtain a set of candidate poles for $Z_f(s,\chi)$ together with their expected orders, i.e., upperbounds for the orders if these candidate poles are actual poles (see Proposition 3.1.1, Definition 3.1.2, Proposition 3.2.5 and Proposition 3.3.3). This can all be expressed in terms of the faces of $\Gamma(f)$ and their dimension. This formula also enables us to prove Theorem 3.4.1, which is the second main result. It states that $Z_f(s)$ has always at least one real pole. In particular, this theorem gives more information on the largest real pole of $Z_f(s)$: it is either -1 or $-1/t_0$, where $(t_0,t_0,\ldots,t_0)$ is the intersection point of the diagonal $D=\{(t,t,\ldots,t)\in\mathbb{R}^n\}$ with $\Gamma(f)$.

3.1 The candidate poles of $Z_f(s,\chi)$

The next proposition is a special case of [Zn99, Theorem A]. The case of $Z_f(s)$ is also a direct consequence of [Den95]. We show how to derive it from the material in Chapter 1 and Section 2.3. By taking the union of all the poles of the L_τ and the S_{Δ_i} from $S_{\Delta_\tau}=\sum_i S_{\Delta_i}$ (in the formula from Theorem 2.3.2), one obtains a set of all possible poles of $Z_f(s,\chi)$. Because one has to take the sum $\sum_\tau L_\tau S_{\Delta_\tau}$, it can occur that some of these possible poles are not actual poles. Nevertheless, one obtains in this way the only possible poles of $Z_f(s,\chi)$.

Proposition 3.1.1. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Let $\gamma_1,\ldots,\gamma_r$ be all the facets of $\Gamma(f)$ and let $a_1,\ldots,a_r$ be the unique primitive vectors in $\mathbb{N}^n \setminus \{0\}$ that are perpendicular to $\gamma_1,\ldots,\gamma_r$, respectively. Then the following holds:*

(i) If s_1 is a pole of $Z_f(s)$, then

$$\begin{aligned} s_1 &= -1 + i\tfrac{k2\pi}{\log_e p} \quad \text{with } k \in \mathbb{Z}, \text{ or} \\ s_1 &= -\tfrac{\sigma(a_j)}{m(a_j)} + i\tfrac{k2\pi}{m(a_j)\log_e p}, \end{aligned}$$

with $k \in \mathbb{Z}$ and $j \in \{1,\ldots,r\}$ such that $m(a_j) \neq 0$.

(ii) If s_1 is a pole of $Z_f(s)$, then $\mathrm{Re}\,(s_1)$ is -1 or $\mathrm{Re}\,(s_1)$ is of the form $-1/t_1$, where $(t_1,t_1,\ldots,t_1)$ is the intersection point of the diagonal $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$ with the supporting hyperplane of a facet of $\Gamma(f)$.

(iii) Suppose that s_1 is a pole of $Z_f(s,\chi)$ with $\chi \neq \chi_{\mathrm{triv}}$. Then

$$s_1 = -\frac{\sigma(a_j)}{m(a_j)} + i\frac{k2\pi}{m(a_j)\log_e p},$$

with $k \in \mathbb{Z}$ and $j \in \{1,\ldots,r\}$ such that $m(a_j) \neq 0$. Moreover, $\mathrm{Re}\,(s_1)$ is of the form $-1/t_1$, where $(t_1,t_1,\ldots,t_1)$ is the intersection point of the diagonal $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$ with the supporting hyperplane of a facet of $\Gamma(f)$.

Remarks.

1. The same is true for $Z_{f,0}(s)$ and $Z_{f,0}(s,\chi)$. Of course, then we only need f to be non-degenerated over $\mathbb{F}_p$ with respect to all the *compact* faces of $\Gamma(f)$ (see Remark 2 after Theorem 2.3.2).

2. We call the complex numbers in Proposition 3.1.1(i) and (iii) the *candidate poles* of $Z_f(s)$ and $Z_f(s,\chi)$, respectively. The first candidate poles in (i) (with real part equal to -1) come from the L_τ, the second ones in (i) and the ones in (iii) come from the S_{Δ_τ}. In the next section, we will give more information about the largest real candidate pole coming from the S_{Δ_τ}.

Proof. An equation of the supporting hyperplane of the facet γ_j is $a_j \cdot x = m(a_j)$. The intersection of this hyperplane with the diagonal D is the point $(t,t,\ldots,t)$, where $t = m(a_j)/\sigma(a_j)$.

Now the assertion follows directly from the formula for $Z_f(s,\chi)$ in Theorem 2.3.2, Proposition 1.1.10 and Lemma 1.2.2. □

Looking at the formula from Theorem 2.3.2, we can give upperbounds for the orders of the poles of $Z_f(s,\chi)$.

Definition 3.1.2. Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set of numbers described in Proposition 3.1.1. We define *the expected order of the candidate pole* s_1 for $Z_f(s,\chi)$ (according to the formula for $Z_f(s,\chi)$ in Theorem 2.3.2) as

$$\max\{\text{order of the pole } s_1 \text{ for } L_\tau S_{\Delta_\tau} \mid \tau \text{ a face of } \Gamma(f)\}, \tag{3.1.1}$$

where the order of the pole s_1 is defined as 0, if it is not a pole, and where L_τ and S_{Δ_τ} are defined as in Theorem 2.3.2. If $\mathrm{Re}\,(s_1) \neq -1$ or $\chi \neq \chi_{\mathrm{triv}}$, we omit L_τ in (3.1.1).

Remarks.

1. It is clear that the actual order of the pole s_1 for $Z_f(s,\chi)$ will not be larger than the expected order of s_1.
2. We can give a similar definition for the expected order of a candidate pole s_1 for $Z_{f,0}(s,\chi)$. Then τ runs through the *compact* faces of $\Gamma(f)$.

In Section 3.2 and Section 3.3, we give results that express the expected orders in terms of the dimensions of particular faces of $\Gamma(f)$.

3.2 The largest real candidate pole coming from the S_{Δ_τ} and its expected order

Lemma 3.2.1. *Let $f(x) = f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0) = 0$. The diagonal $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$ intersects the boundary of the Newton polyhedron $\Gamma(f)$ of f in one point $(t_0,t_0,\ldots,t_0)$.*

Proof. The diagonal D contains an element x of the relative interior $\mathrm{ri}\,\Gamma(f)$ of $\Gamma(f)$. Indeed, take an element $a = (a_1,\ldots,a_n)$ in $\Gamma(f)$. Then it is easy to see that D contains an element $x \in a + (\mathbb{R}^+ \setminus \{0\})^n$, because the set of equations $a_i + r_i = t$, $i = 1,\ldots,n$ has a solution $(r_1,\ldots,r_n) \in (\mathbb{R}^+ \setminus \{0\})^n$ if t is big enough.

The intersection of D with $\mathrm{ri}\,\Gamma(f)$ is an open convex subset of D and $(1,1,\ldots,1)$ is a direction of recession of $\mathrm{ri}\,\Gamma(f)$. Because the origin $0 \in D \setminus \Gamma(f)$ and $x \in D \cap \mathrm{ri}\,\Gamma(f)$, it follows that there exists a b such that $D \cap \mathrm{ri}\,\Gamma(f)$ is the open half-line $]bx$.

We prove now that b is contained in the boundary of $\Gamma(f)$. See Figure 3.2.1. Take an open ball $B(b,r)$ with center b and radius $r>0$. It is clear that this ball contains elements of $\mathrm{ri}\,\Gamma(f) \subset \Gamma(f)$. Take an element c of $B(b,r) \cap (D \setminus]bx)$. Then $c \notin \Gamma(f)$. Indeed, because $x \in \mathrm{ri}\,\Gamma(f)$, $c \in \Gamma(f)$ would imply that $b \in \mathrm{ri}\,\Gamma(f)$ [Roc70, Theorem 6.1].

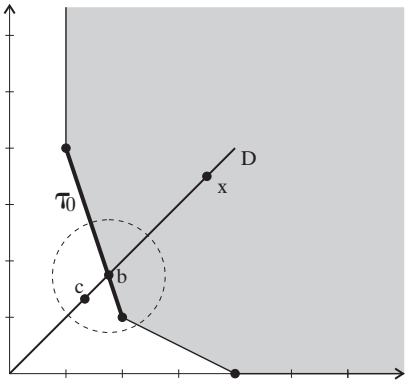


Figure 3.2.1

By a similar argument we obtain that b is the only element in the intersection of D with the boundary of $\Gamma(f)$. □

Definition 3.2.2. Let $f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0)=0$. We denote the unique intersection point of the diagonal $D=\{(t,t,\ldots,t)\in\mathbb{R}^n\}$ with the boundary of the Newton polyhedron $\Gamma(f)$ of f by $(t_0,t_0,\ldots,t_0)$. Denote the smallest face of $\Gamma(f)$ that contains $(t_0,t_0,\ldots,t_0)$ by τ_0 and its codimension in $\mathbb{R}^n$ by κ. Put $\rho=-1/t_0$.

Note that $\kappa \geq 1$, because $\dim \tau_0 \leq n-1$. Moreover, $\rho \in \mathbb{Q}$.

We search for more information about the largest real candidate pole of $Z_f(s)$ and $Z_f(s,\chi)$ coming from the S_{Δ_τ}. The next lemma implies that this is ρ and Proposition 3.2.5 shows that the expected order of ρ for $Z_f(s)$ and $Z_f(s,\chi)$ is either κ or $\kappa+1$.

Lemma 3.2.3. *Let $f(x)=f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0)=0$. For every $a\in(\mathbb{R}^+)^n$ we have $\sigma(a)-m(a)(1/t_0)\geq 0$ with equality if and only if $\tau_0 \subset F(a)$ (or equivalently $a\in\bar{\Delta}_{\tau_0}$).*

Proof. From the definition of $m(a)$, it follows that $(t_0,t_0,\ldots,t_0)\cdot a \geq m(a)$, because $(t_0,t_0,\ldots,t_0)\in\Gamma(f)$. Consequently, $\sigma(a)-m(a)(1/t_0)\geq 0$. Moreover, $\sigma(a)-m(a)(1/t_0)=0$ if and only if $(t_0,t_0,\ldots,t_0)\cdot a = m(a)$, i.e., if and only if $(t_0,t_0,\ldots,t_0)\in F(a)$, or equivalently, if and only if $\tau_0\subset F(a)$, because τ_0 is the smallest face of $\Gamma(f)$ that contains $(t_0,t_0,\ldots,t_0)$. □

Lemma 3.2.4. *Let $f(x_1,\dots,x_n)$ be a non-zero polynomial with $f(0)=0$. Suppose that τ is a proper face of $\Gamma(f)$ and let S_{Δ_τ} be as in Theorem 2.3.2. Then we have the following:*

(i) $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau}\geq 0$.

(ii) If $\tau\not\subset\tau_0$, then, if ρ is a pole of S_{Δ_τ}, its order is $<\kappa$. Otherwise, ρ is a pole of S_{Δ_τ} of order κ.

(iii) There exists a constant $c_0>0$, independent of p, such that

$$\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}}=c_0.$$

Define the half-space $H_1^-=\{x\in\mathbb{R}^n \mid \sum_{i=1}^n x_i\leq 1\}$ and define $\omega_{\bar\Delta_{\tau_0}}$ as the volume form on the vector space $\operatorname{vct}\bar\Delta_{\tau_0}$ generated by $\bar\Delta_{\tau_0}$, normalized such that the parallelopiped spanned by a lattice basis of $\mathbb{Z}^n\cap\operatorname{vct}\bar\Delta_{\tau_0}$ has volume 1. Then

$$c_0=\kappa!|\rho|^\kappa \operatorname{Vol}_{\omega_{\bar\Delta_{\tau_0}}}(\bar\Delta_{\tau_0}\cap H_1^-),$$

where the volume is taken with respect to $\omega_{\bar\Delta_{\tau_0}}$.

Proof. Suppose that $\cup_i\Delta_i$ is a finite partition of Δ_τ into rational simplicial cones. Then $S_{\Delta_\tau}=\sum_i S_{\Delta_i}$. We know by Lemma 3.2.3 that ρ is a pole of order 1 of $1/(p^{\sigma(a)+m(a)s}-1)$ if and only if $a\in\bar\Delta_{\tau_0}$. The maximal number of linearly independent vectors contained in $\bar\Delta_{\tau_0}$ is equal to κ by Proposition 1.1.10. From the formula for S_{Δ_i} in Theorem 2.3.2, one gets then the following: ρ is not a pole of S_{Δ_i} if there are no such "dangerous" a in the set of generators for Δ_i; otherwise, its order is equal to the number of those a. Moreover, the number of those a is at most κ. Therefore, Lemma 3.2.3 implies that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_i}\geq 0$ with equality if and only if the number of "dangerous" a in the set of generators for Δ_i is $<\kappa$. This proves (i).

If $\tau\not\subset\tau_0$, then the number of generators of Δ_i contained in $\bar\Delta_{\tau_0}$ is $\leq\dim(\bar\Delta_\tau\cap\bar\Delta_{\tau_0})<\dim\bar\Delta_{\tau_0}=\kappa$ by Proposition 1.1.10 and Lemma 1.1.7. This results to the first part of (ii).

If $\tau=\tau_0$, there is at least one cone Δ_i in the rational simplicial partition of Δ_{τ_0} that is spanned by κ linearly independent vectors (from $\bar\Delta_{\tau_0}$), because the dimension of $\Delta_{\tau_0}=\kappa$ (Proposition 1.1.10).

If $\tau\subsetneq\tau_0$, there is at least one cone Δ_i such that κ generators for Δ_i are elements of $\bar\Delta_{\tau_0}$. Indeed, we can write $\bar\Delta_{\tau_0}$ as $\bar\Delta_{\tau_0}\cap\bar\Delta_\tau=\cup_i\left(\bar\Delta_{\tau_0}\cap\bar\Delta_i\right)$ and because the last union is finite, there is at least one cone Δ_i such that $\dim(\bar\Delta_{\tau_0}\cap\bar\Delta_i)=\dim\bar\Delta_{\tau_0}$. Suppose that $\Delta_i=\{\sum_j\lambda_j a_j+\sum_k\mu_k b_k \mid \lambda_j,\mu_k>0\}$ with $a_j\in\bar\Delta_{\tau_0}$ and $b_k\in\bar\Delta_\tau\setminus\bar\Delta_{\tau_0}$. Then it is clear, by Lemma 1.1.3 and Lemma 1.1.5(ii), that $\bar\Delta_{\tau_0}\cap\bar\Delta_i=\{\sum_j\lambda_j a_j \mid \lambda_j\geq 0\}$, because τ_0 is not contained in $F(b_k)$. This yields to the second part of (ii).

Assertion (iii) is a special case of Lemma 3.3.2(ii). □

Proposition 3.2.5. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Then for every pole s_1 of $Z_f(s,\chi)$, one has $\mathrm{Re}\,(s_1) \leq \rho$ if $\chi \neq \chi_{\mathrm{triv}}$; otherwise $\mathrm{Re}\,(s_1) = -1$ or $\mathrm{Re}\,(s_1) \leq \rho$. Moreover, the expected order of the candidate pole ρ for $Z_f(s,\chi)$ equals $\kappa + 1$ if $\chi = \chi_{\mathrm{triv}}$, $\rho = -1$ and there is a face $\tau \subset \tau_0$ such that $N_\tau \neq 0$; otherwise, its expected order will be κ.*

Recall that the actual order is not larger than the expected order, which is defined in Definition 3.1.2.

Remark. If moreover $t_0 > 1$, it is well-known that ρ is a pole of order κ of $Z_f(s)$ (see Theorem 3.4.1). For a further study of the cases where $t_0 \leq 1$, see Chapter 4 and Chapter 5.

Proof. The assertion follows from the formula for $Z_f(s,\chi)$ in Theorem 2.3.2, Proposition 3.1.1, Lemma 3.2.3, and Lemma 3.2.4. □

3.3 The expected order of an arbitrary candidate pole of $Z_f(s,\chi)$

In Proposition 3.3.3, we also express the expected order of an arbitrary candidate pole of $Z_f(s,\chi)$ in terms of the dimensions of particular faces of $\Gamma(f)$. In order to do this, we first give Definition 3.3.1 and prove Lemma 3.3.2.

Definition 3.3.1. Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set described in Proposition 3.1.1. We say that a vector $a \in (\mathbb{R}^+)^n \setminus \{0\}$ or a facet $F(a)$ *contributes to the candidate pole* s_1 if $p^{\sigma(a)+m(a)s_1} = 1$, or equivalently, $\mathrm{Re}\,(s_1) = -\sigma(a)/m(a)$ and there exists a $k \in \mathbb{Z}$ such that $\mathrm{Im}\,(s_1) = k2\pi/(m(a)\log_e p)$. We say that a face of $\Gamma(f)$ is *a face of pure contribution to the candidate pole* s_1 if every facet that contains τ contributes to this candidate pole s_1. We define $\kappa(s_1)$ as the maximum of the codimensions of such faces. We call such a face of codimension $\kappa(s_1)$ *a face of* $\Gamma(f)$ *of maximal pure contribution to the candidate pole* s_1 .

Remarks.

1. It can occur that not all faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 are disjoint. See Example 5.4.2.
2. If $s_1 = \rho$, then $\kappa(s_1) = \kappa$ and τ_0 is the only face of $\Gamma(f)$ of maximal pure contribution to the candidate pole ρ of $Z_f(s,\chi)$.
3. It is easy to verify that $\kappa(s_1) \leq \kappa\,(\mathrm{Re}\,(s_1))$; see Example 3.5.1.

Lemma 3.3.2. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set described in Proposition 3.1.1. Suppose that μ is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1 and suppose that its codimension is k. Let S_{Δ_μ} be defined as in Theorem 2.3.2.*

(i) If $a \in \bar{\Delta}_\mu$, then a contributes to the pole s_1, i.e., $p^{\sigma(a)+m(a)s_1} = 1$.

(ii) There exists a constant $c_\mu > 0$, independent of p, such that

$$\lim_{s\to s_1} \left(p^{s-s_1}-1\right)^k S_{\Delta_\mu} = c_\mu.$$

Define the half-space $H_1^- = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq 1\}$ and define $\omega_{\bar{\Delta}_\mu}$ as the volume form on the vector space $\operatorname{vct}\bar{\Delta}_\mu$ generated by $\bar{\Delta}_\mu$, normalized such that the parallelopiped spanned by a lattice basis of $\mathbb{Z}^n \cap \operatorname{vct}\bar{\Delta}_\mu$ has volume 1. Then

$$c_\mu = k!|\operatorname{Re}(s_1)|^k \operatorname{Vol}_{\omega_{\bar{\Delta}_\mu}}(\bar{\Delta}_\mu \cap H_1^-),$$

where the volume is taken with respect to $\omega_{\bar{\Delta}_\mu}$.

Proof. Assertion (i) follows by the fact that all generators of $\bar{\Delta}_\mu$ contribute to the pole s_1 and the fact that σ and m are linear on $\bar{\Delta}_\mu$.

We now prove Assertion (ii). Suppose that $\cup_i \Delta_i$ is a finite partition of Δ_μ into rational simplicial cones. Then $S_{\Delta_\mu} = \sum_i S_{\Delta_i}$. From the formula for S_{Δ_i} in Theorem 2.3.2 and (i), it follows that $\lim_{s\to s_1}(p^{s-s_1}-1)^k S_{\Delta_i} > 0$ if and only if k generators of Δ_i belong to $\bar{\Delta}_\mu$, or equivalently, $\dim \Delta_i = k$; otherwise, this limit is zero. Moreover, if Δ_i is a cone from this partition with $\dim \Delta_i = k$, then $\operatorname{vct}\bar{\Delta}_i = \operatorname{vct}\bar{\Delta}_\mu$ and hence $\omega_{\bar{\Delta}_i} = \omega_{\bar{\Delta}_\mu}$. So, it suffices to prove that for such a cone Δ_i one has

$$\lim_{s\to s_1}(p^{s-s_1}-1)^k S_{\Delta_i} = k!|\operatorname{Re}(s_1)|^k \operatorname{Vol}_{\omega_{\bar{\Delta}_i}}(\bar{\Delta}_i \cap H_1^-),$$

where the volume is taken with respect to $\omega_{\bar{\Delta}_i}$. Suppose that Δ_i is the cone strictly positively spanned by $a_1,\ldots,a_k$, linearly independent vectors in $\mathbb{Z}^n \cap \bar{\Delta}_\mu$. Then we know from the formula for S_{Δ_i} in Theorem 2.3.2 that

$$S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s}-1)\cdots(p^{\sigma(a_k)+m(a_k)s}-1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^k \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1,\ldots,k \right\}. \tag{3.3.1}$$

We know that $p^{\sigma(h)+m(h)s_1} = 1$ for every h of the set (3.3.1), because every h belongs to $\bar{\Delta}_\mu$. Consequently,

$$\begin{aligned}\lim_{s\to s_1}(p^{s-s_1}-1)^k S_{\Delta_i} &= \frac{\text{multiplicity of } a_1,\dots,a_k}{m(a_1)\cdots m(a_k)} \\ &= |\mathrm{Re}\,(s_1)|^k \frac{\text{multiplicity of } a_1,\dots,a_k}{\sigma(a_1)\cdots\sigma(a_k)}.\end{aligned}$$

It follows by Proposition 1.2.5 that this equals $|\mathrm{Re}\,(s_1)|^k$ times the volume of the parallelopiped spanned by $a_1/\sigma(a_1),\dots,a_k/\sigma(a_k)$ with respect to $\omega_{\bar{\Delta}_i}$. Moreover, $\bar{\Delta}_i \cap H_1^-$ is the convex hull of $\{0, a_1/\sigma(a_1),\dots,a_k/\sigma(a_k)\}$. So, $\lim_{s\to s_1}(p^{s-s_1}-1)^k S_{\Delta_i}$ also equals $k!|\mathrm{Re}\,(s_1)|^k \mathrm{Vol}_{\omega_{\bar{\Delta}_i}}(\bar{\Delta}_i \cap H_1^-)$. □

Proposition 3.3.3. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set of numbers described in Proposition 3.1.1.*

(i) If τ is a face of $\Gamma(f)$ that is not contained in any face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1, then, if s_1 is a pole of S_{Δ_τ}, its order is $< \kappa(s_1)$.

(ii) If $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\mathrm{triv}}$, then the expected order of the candidate pole s_1 of $Z_f(s,\chi)$ is $\kappa(s_1)$. Otherwise, the expected order is either $\kappa(s_1)$ or $\kappa(s_1)+1$.

(iii) If s_1 is a pole of order k of $Z_f(s,\chi)$ with $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\mathrm{triv}}$, then there exists a face of codimension k that is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1. If s_1 is a pole of order k of $Z_f(s)$ with $p^{s_1+1} = 1$, then there exists a face of codimension $(k-1)$ that is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1.

Remark. Result (iii) also holds for $Z_{f,0}(s,\chi)$. We have to be more careful for an analogue of (ii) for the case of $Z_{f,0}(s,\chi)$. If s_1 is a candidate pole of $Z_{f,0}(s)$ with $p^{s_1+1} \neq 1$ and one of the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 is compact, then its expected order is $\kappa(s_1)$. Otherwise, we can only state that its expected order is $\leq \kappa(s_1)$. (A similar remark holds if $p^{s_1+1} = 1$ and $\chi = \chi_{\mathrm{triv}}$.) Indeed, suppose for example that s_1 is a real number that is not the largest real candidate pole of an S_{Δ_τ}. Then it can occur that $\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)} S_{\Delta_i} > 0$, for some cone Δ_i in a partition of Δ_τ into rational simplicial cones and $\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)} S_{\Delta_j} < 0$ for another cone Δ_j in this partition. So, in contrast to the case of ρ, it is not certain that s_1 is a pole of S_{Δ_τ} of order $\kappa(s_1)$, if τ is a face contained in one of the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1.

Proof. Because Lemma 3.3.2(ii) is true for a face μ of maximal pure contribution to the candidate pole s_1, for Assertion (ii), it suffices to verify that, if s_1 is a pole of S_{Δ_τ} with τ a face of $\Gamma(f)$, then its order is $\leq \kappa(s_1)$. Or it is sufficient to prove that there exists a partition of Δ_τ into rational simplicial cones Δ_i such that at most $\kappa(s_1)$ generators of Δ_i contribute to the candidate pole s_1. To prove Assertion (i), we have to verify that, if τ is not contained in a face of maximal pure contribution, there exists a partition of Δ_τ into rational simplicial cones Δ_i such that $< \kappa(s_1)$ generators of Δ_i contribute to s_1. We prove by induction on $\dim \Delta_\tau$ that we can even do this without introducing new rays. It is clear that the property is true if $\dim \Delta_\tau = 1$. Suppose now that $\dim \Delta_\tau > 1$ and that the property is true for cones of smaller dimension. If all generators of Δ_τ contribute to the pole s_1, then it is easy to see that a partition without introducing new rays is an appropriate partition of Δ_τ. If there exists a generator that does not contribute to the pole s_1, we then use this vector as a_1 in the algorithm described in the proof of Lemma 1.2.2. Then the property follows by the fact that it is true for cones of smaller dimension.

By the same argument, it is clear that we can find a partition of Δ_τ into rational simplicial cones Δ_i such that the set of generators of Δ_i that contribute to the candidate pole s_1 belong to some cone $\bar{\Delta}_{\mu_i}$, where μ_i is a face of $\Gamma(f)$ of pure contribution to the candidate pole s_1. This implies (iii). □

3.4 $Z_f(s)$ has always at least one real pole

Suppose that f is a non-zero polynomial over $\mathbb{Z}_p$. If $f(0) = 0$, all N_e in Definition 0.1.2 are > 0. So, in this case, Definition 0.1.2, the relation between $Z_f(s)$ and $P(t)$ from Lemma 0.1.3 and the rationality of both imply that $Z_f(s)$ has at least one pole. The following proposition shows that $Z_f(s)$ has always a *real* pole, if f satisfies also the conditions to use the formula for $Z_f(s)$ from Theorem 2.3.2. In particular, it gives more information on the largest real pole of $Z_f(s)$ and its order. Actually, Case (i) is well-known, but we give a proof in the framework of this thesis.

Theorem 3.4.1. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Let ρ, κ and τ_0 be as in Definition 3.2.2.*

(i) Suppose that $\rho > -1$.
Then ρ is the largest real pole of $Z_f(s)$ and its order is κ.

(ii) Suppose that $\rho < -1$.
If there exists a face τ of $\Gamma(f)$ such that $N_\tau \neq 0$, then -1 will be the largest real pole of $Z_f(s)$ and its order will be 1. Otherwise, ρ will be the largest real pole of $Z_f(s)$ and its order will be κ.

(iii) *Suppose that* $\rho = -1$.
Then $\rho = -1$ *will be the largest real pole of* $Z_f(s)$. *If there exists a face* $\tau \subset \tau_0$ *of* $\Gamma(f)$ *such that* $N_\tau \neq 0$, *then its order will be* $\kappa + 1$. *Otherwise, its order will be* κ.

Remark. A similar result holds for $Z_{f,0}(s)$: one has to replace "face" everywhere by "compact face".

Proof. First note that we can use the formula for $Z_f(s)$ from Theorem 2.3.2. Thus

$$Z_f(s) = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

with L_τ and S_{Δ_τ} as in this theorem. We know by Lemma 3.2.4 that

$$\begin{array}{rcll} \lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} & > & 0 & \text{for faces } \tau \subset \tau_0, \\ \lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} & = & 0 & \text{for faces } \tau \not\subset \tau_0. \end{array} \tag{3.4.1}$$

If $t_0 > 1$, then $(p^\rho - 1)/(p^{\rho+1} - 1) < 0$, because $0 > \rho = -1/t_0 > -1$. So, for each face τ one has

$$L_\tau(\rho) := p^{-n}\left((p-1)^n - pN_\tau \frac{p^\rho - 1}{p^{\rho+1}-1}\right) > 0.$$

Therefore $\lim_{s\to\rho} \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} L_\tau (p^{s-\rho}-1)^\kappa S_{\Delta_\tau} > 0$ if $\rho > -1$, which implies Assertion (i).

It is easy to see that

$$\begin{array}{rcll} L_\tau & = & p^{-n}(p-1)^n > 0, & \text{if } N_\tau = 0, \\ \lim_{s\to-1}(p^{s+1}-1)L_\tau & = & p^{-n}(1-\frac{1}{p})pN_\tau > 0, & \text{if } N_\tau \neq 0. \end{array} \tag{3.4.2}$$

By a similar argument as in the proof of Lemma 3.2.4(i), we can prove that $\lim_{s\to\rho}(p^{s-\rho}-1)^{\kappa-1} S_{\Delta_\tau} \geq 0$ for faces $\tau \not\subset \tau_0$. The second assertion for Case (ii) and the assertion for Case (iii) follow now from (3.4.2) and (3.4.1).

Suppose that $\rho < -1$. Fix a proper face τ of $\Gamma(f)$ and fix a partition of the cone Δ_τ into rational simplicial cones Δ_i. Then

$$\lim_{s\to-1} S_{\Delta_\tau} = \sum_i \lim_{s\to-1} S_{\Delta_i} > 0. \tag{3.4.3}$$

Indeed, let Δ_i be the cone strict positively spanned by linearly independent vectors $a_1, \ldots, a_r \in \mathbb{N}^n \setminus \{0\}$. Then we know that

$$S_{\Delta_i} = \frac{\sum_h p^{\sigma(h)+m(h)s}}{(p^{\sigma(a_1)+m(a_1)s}-1)\cdots(p^{\sigma(a_r)+m(a_r)s}-1)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r} \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \ldots, r \right\}.$$

Hence,

$$\lim_{s \to -1} S_{\Delta_i} = \frac{\sum_h p^{\sigma(h) - m(h)}}{(p^{\sigma(a_1) - m(a_1)} - 1) \cdots (p^{\sigma(a_r) - m(a_r)} - 1)} > 0,$$

because $\sigma(a_i) - m(a_i) > 0$ by Lemma 3.2.3 and the fact that $-1 > \rho$. The first assertion for Case (ii) follows now from (3.4.2) and (3.4.3). □

3.5 Examples

Example 3.5.1. Let $f(x, y, z) = x^4y^3z^2 + x^7y^4z + z^5$. We continue with Example 2.4.1. The Newton polyhedron $\Gamma(f)$ of f is defined by the system of linear inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 1 \\ x + 3z \geq 10 \\ 3x + 4z \geq 20 \\ y + z \geq 5. \end{cases}$$

We denote by $\tau_1, \ldots, \tau_6$ the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above. A picture of the Newton polyhedron of f is given in Figure 3.5.1.

The real candidate poles of $Z_f(s)$ are $-7/20$, $-2/5$ and -1. The information about the faces of pure contribution to these candidate poles and some other candidate poles is contained in Table 3.5.1; see also Table 2.4.1.

Note that the expected order of the candidate pole -1 of $Z_f(s)$ is 1 because $N_\tau = 0$ for every face $\tau \subset \tau_3$. So, the expected orders of the candidate poles s_1 of $Z_f(s, \chi)$ are all equal to $\kappa(s_1)$. In this example, all expected orders are equal to the actual orders of the poles of $Z_f(s, \chi)$ (for $p = 3$). This is not always the case: for instance, take Example 4.2.1, Example 4.2.2, Example 5.4.1 or Example 5.4.2.

One can find more examples in Appendix D and in [HL00].

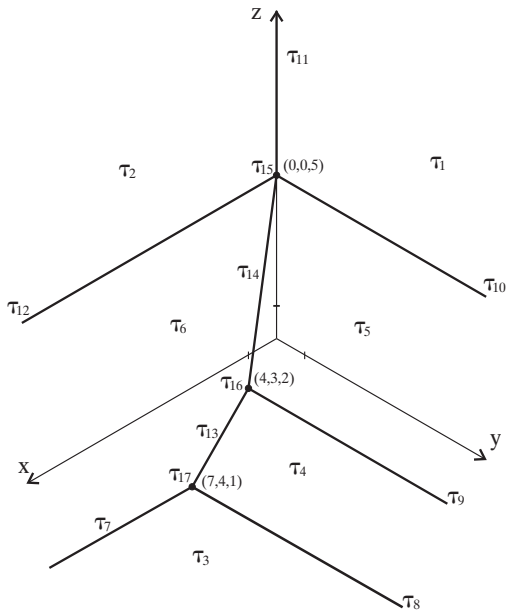


Figure 3.5.1. Newton polyhedron of $x^4y^3z^2 + x^7y^4z + z^5$

Candidate pole s_1	Faces of pure contribution	Faces of maximal pure contribution	$\kappa(s_1)$
$\rho = -7/20$	$\tau_0 = \tau_5$	$\tau_0 = \tau_5$	$\kappa = 1$
$-4/10 = -2/5$	$\tau_4,\ \tau_6,\ \tau_{13}$	τ_{13}	2
$-4/10 + i\,2\pi/10 \log_e p$	τ_4	τ_4	1
$-4/10 + i\,2.2\pi/10 \log_e p$ $= -2/5 + i\,2\pi/5 \log_e p$	$\tau_4,\ \tau_6,\ \tau_{13}$	τ_{13}	2
-1	τ_3	τ_3	1

Table 3.5.1. The faces of pure contribution to the candidate poles of $Z_f(s)$

Chapter 4

A sufficient condition to have $\rho = -1/t_0$ as a pole of $Z_f(s)$ or $Z_{f,0}(s)$ of the expected order (for $t_0 \leq 1$)

4.1 The results

Our next goal is to prove Theorem 4.1.11, which is the third main result of this thesis (see also [DH01]). We want to find a sufficient condition such that $\rho = -1/t_0$ is a pole of $Z_f(s)$ or $Z_{f,0}(s)$ of the expected order (see Proposition 3.2.5) in the case that $t_0 < 1$. Put

$$L_\tau(\rho) := p^{-n}\left((p-1)^n - pN_\tau \frac{p^\rho - 1}{p^{\rho+1}-1}\right)$$

for every face τ of $\Gamma(f)$. We know that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} \geq 0$. If $t_0 < 1$, then $(p^\rho - 1)/(p^{\rho+1}-1) > 0$. Therefore $L_\tau(\rho)$ can be < 0, $= 0$ or > 0 depending on how big pN_τ is. Then the question arises under which conditions $\lim_{s\to\rho}(p^{s-\rho} - 1)^\kappa Z_f(s)$ will still be different of 0. Note that we are only interested in pN_τ for faces τ that are contained in τ_0, because $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} = 0$ for the other τ. Proposition 4.1.7 will tell us more about pN_τ, but first we have to prove Lemma 4.1.3 and give a few definitions. Theorem 4.1.13 gives a sufficient condition such that $\rho = -1/t_0$ is a pole of $Z_f(s)$ or $Z_{f,0}(s)$ of the expected order in the case that $t_0 = 1$. For every face τ of $\Gamma(f)$ one has $\lim_{s\to-1}(p^{s+1}-1)L_\tau = (1-p^{-1})p^{-n}pN_\tau \geq 0$, with L_τ and N_τ as in Theorem 2.3.2. From Lemma 3.2.4 (or Theorem 3.4.1(iii)), it then follows that if this limit (or N_τ) is > 0 for at

least one $\tau \subset \tau_0$, then $\rho = -1$ will be a pole of order $\kappa + 1$. Indeed, then $\lim_{s \to -1}(p^{s+1} - 1)^{\kappa+1} Z_f(s) > 0$. So, also for this case, it will be useful to know more about the pN_τ.

Definition 4.1.1. Let k be an arbitrary field (e.g., $\mathbb{C}$ or $\mathbb{F}_p$) and $f = \sum_\omega a_\omega x^\omega \in k[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$ with $f(0) = 0$. The *Newton polyhedron* $\Gamma_\infty(f)$ *of* f *at infinity* is defined as the convex hull of $\text{supp}\,(f) \cup \{0\}$, where $\text{supp}\,(f) = \{\omega \in \mathbb{Z}^n \mid a_\omega \neq 0\}$ is the support of f. For a face τ of $\Gamma_\infty(f)$, set $f_\tau = \sum_{\omega \in \tau} a_\omega x^\omega$. We say that f is *non-degenerated over* $\bar{k}$ *with respect to* $\Gamma_\infty(f)$ if for every face τ of $\Gamma_\infty(f)$ that does not contain the origin, the set of equations $(\partial f_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\bar{k}^\times)^n$ (where $\bar{k}$ denotes an algebraic closure of k).

Remark. By a similar argument as in Remark 1 after Definition 0.1.5, we can prove that the condition "for every face τ of $\Gamma_\infty(f)$ that does not contain the origin, the set of equations $(\partial f_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$ has no solution in $(\bar{k}^\times)^n$" is equivalent to "for every face τ of $\Gamma_\infty(f)$ that does not contain the origin, the set of equations $f_\tau = 0, (\partial f_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$ has no solution in $(\bar{k}^\times)^n$", if $k = \mathbb{C}$ or $k = \mathbb{F}_p$ and $p \gg 0$.

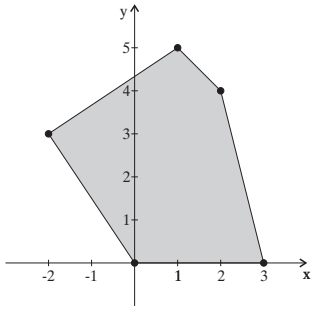


Figure 4.1.1. $\Gamma_\infty(f)$ for $f(x, y) = x^{-2}y^3 + 3xy^5 - 5x^2y^4 + 13x^3$

Definition 4.1.2. Let k be an arbitrary field (e.g., $\mathbb{C}$ or $\mathbb{F}_p$). A coordinate change T on $(\bar{k}^\times)^n$ is a change of variables of the form $(x_j) \mapsto \left(y_j = \prod_{i=1}^n x_i^{t_{i,j}}\right)$, where $T = (t_{i,j}) \in GL(n, \mathbb{Z})$. For $f \in k[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$, we define $f_T \in k[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$ by

$$f_T(x_1, \ldots, x_n) = f\left(\prod_{i=1}^n x_i^{t_{i,1}}, \ldots, \prod_{i=1}^n x_i^{t_{i,n}}\right).$$

Remark. [AS89, p. 388] Regarding T as a linear operator on $\mathbb{R}^n$ (i.e., $T(\omega_1, \ldots, \omega_n)$ is the result of the matrix product $T[\omega_1 \cdots \omega_n]^t$), we have $\Gamma_\infty(f_T) = T(\Gamma_\infty(f))$

and that τ is a face of $\Gamma_\infty(f)$ if and only if $T(\tau)$ is a face of $\Gamma_\infty(f_T)$. The chain rule for partial derivates implies that f is non-degenerated over $\bar{k}$ with respect to $\Gamma_\infty(f)$ if and only if f_T is non-degenerated over $\bar{k}$ with respect to $\Gamma_\infty(f_T)$. Moreover, $\dim \Gamma_\infty(f_T) = \dim \Gamma_\infty(f)$. It is also clear that $f_T \in \mathbb{Z}[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$ when $f \in \mathbb{Z}[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$.

Lemma 4.1.3. *Let k be an arbitrary field (e.g., $\mathbb{C}$ or $\mathbb{F}_p$) and $f \in k[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$. If $\dim \Gamma_\infty(f) < n$, then there exists a coordinate change T on $(\bar{k}^\times)^n : (x_j) \mapsto \left(y_j = \prod_{i=1}^n x_i^{t_{i,j}}\right)$ such that*

$$f(x_1, \ldots, x_n) = f_{T^{-1}}(y_1, \ldots, y_n) = h(y_1, \ldots, y_{\dim \Gamma_\infty(f)}),$$

where $h \in k[y_1, \ldots, y_{\dim \Gamma_\infty(f)}, (y_1 \cdots y_{\dim \Gamma_\infty(f)})^{-1}]$.

Proof. Take linearly independent vectors $a_1, \ldots, a_{\dim \Gamma_\infty(f)}$ of $\operatorname{supp}(f)$. There exists a partition of the cone strictly positively spanned by $a_1, \ldots, a_{\dim \Gamma_\infty(f)}$ into simple cones by Lemma 1.2.3. So, we can find vectors $t_1, \ldots, t_{\dim \Gamma_\infty(f)}$ in $\mathbb{Z}^n$ with multiplicity 1 that form a basis of the vector space generated by $\operatorname{supp}(f)$. By Proposition 1.2.5, it follows that we can add $n - \dim \Gamma_\infty(f)$ vectors in $\mathbb{Z}^n$ to obtain a basis $t_1, \ldots, t_{\dim \Gamma_\infty(f)}, \ldots, t_n$ of $\mathbb{Z}^n$ (as $\mathbb{Z}$ module). Let T be the matrix with columns $t_1, \ldots, t_{\dim \Gamma_\infty(f)}, \ldots, t_n$. One verifies that T satisfies the required condition. □

Definition 4.1.4. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $f(0) = 0$. We say that f is *non-degenerated over $\mathbb{C}$ with respect to its global Newton polyhedron* $\Gamma_{\mathrm{gl}}(f)$ if for every[1] face τ of $\Gamma_{\mathrm{gl}}(f)$, the locus of f_τ has no singularities in $(\mathbb{C}^\times)^n$.

Recall that the global Newton polyhedron of a polynomial is defined in Definition 0.1.4. Fix a global Newton polyhedron G. One proves that "almost all"[2] polynomials over $\mathbb{C}$ with G as global Newton polyhedron are non-degenerated over $\mathbb{C}$ with respect to their global Newton polyhedron (see Appendix C).

Definition 4.1.5. Let $f(p)$ and $g(p)$ be two complex valued functions of p and $|\cdot|$ the modulus on $\mathbb{C}$. The notation $f(p) = O(g(p))$ means that there exists a constant c independent of p such that $|f(p)| \leq c|g(p)|$ for $p \gg 0$. The notation $f(p) = o(g(p))$ means that $\lim_{p \to \infty} f(p)/g(p) = 0$.

Definition 4.1.6. Let f be a non-zero polynomial with $f(0) = 0$. For any face τ of the Newton polyhedron of f, we denote by $c(\tau)$ the convex hull of the vertices of τ.

Proposition 4.1.7. *Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Let τ be a face of $\Gamma(f)$. Suppose that f_τ is non-degenerated over $\mathbb{C}$ with respect to its global Newton polyhedron $\Gamma_{\mathrm{gl}}(f_\tau)$. Denote, for any prime p, by N_τ the number of elements in the set $\{a \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(a) = 0\}$.*

[1] Thus also for $\tau = \Gamma_{\mathrm{gl}}(f)$.

[2] In a similar sense as in Remark 4 after Definition 0.1.5.

(i) Then

$$pN_\tau = (p-1)^n + O\left(p^{n+1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right). \tag{4.1.1}$$

(ii) Suppose $t_0 < 1$ and put $L_\tau(\rho) := p^{-n}\left((p-1)^n - pN_\tau \frac{p^\rho - 1}{p^{\rho+1}-1}\right)$. Then

$$L_\tau(\rho) = -p^{\rho+1}(1+o(1)) + O\left(p^{1-(\dim c(\tau)+1)/2}\right). \tag{4.1.2}$$

Remarks.

1. For τ a compact face of $\Gamma(f)$, the convex sets $\Gamma_{\mathrm{gl}}(f_\tau)$ and $c(\tau)$ are both equal to τ. So, if τ is compact or $f_\tau = f_{\tau'}$ with τ' a compact face of $\Gamma(f)$, then the condition "f_τ is non-degenerated over $\mathbb{C}$ with respect to $\Gamma_{\mathrm{gl}}(f_\tau)$" is already satisfied if f is non-degenerated over $\mathbb{C}$ with respect to all the compact faces of $\Gamma(f)$.
2. Let $\tau_1 \subset \tau_2$ be faces of $\Gamma(f)$. If f_{τ_2} is non-degenerated over $\mathbb{C}$ with respect to $\Gamma_{\mathrm{gl}}(f_{\tau_2})$, then the same holds for τ_1. Indeed, then $\Gamma_{\mathrm{gl}}(f_{\tau_1})$ will be a face of $\Gamma_{\mathrm{gl}}(f_{\tau_2})$.
3. Actually, one can replace $c(\tau)$ by $\Gamma_{\mathrm{gl}}(f_\tau)$ in (4.1.2), but for later use we will only need (4.1.2).

Proof. An easy calculation shows that

$$\begin{aligned} pN_\tau &= \sum_{t\in\mathbb{F}_p} \sum_{x\in(\mathbb{F}_p^\times)^n} \exp\left(2\pi i \frac{t f_\tau(x)}{p}\right) \\ &= (p-1)^n + \sum_{(t,x)\in(\mathbb{F}_p^\times)^{n+1}} \exp\left(2\pi i \frac{t f_\tau(x)}{p}\right). \end{aligned} \tag{4.1.3}$$

Put $g(t,x) = t f_\tau(x)$. Then g is non-degenerated over $\mathbb{C}$ with respect to its Newton polyhedron $\Gamma_\infty(g)$ at infinity and $\dim \Gamma_\infty(g) = \dim \Gamma_{\mathrm{gl}}(f_\tau)+1$ (see Figure 4.1.2). By Lemma 4.1.3, we can change variables $t f_\tau(x) = g(t,x) = h(y_1, \ldots, y_{\dim \Gamma_{\mathrm{gl}}(f_\tau)+1})$ such that $h \in \mathbb{Z}[y_1, \ldots, y_{\dim \Gamma_{\mathrm{gl}}(f_\tau)+1}, (y_1 \cdots y_{\dim \Gamma_{\mathrm{gl}}(f_\tau)+1})^{-1}]$, h is non-degenerated over $\mathbb{C}$ with respect to its Newton polyhedron at infinity, and consequently, also over $\bar{\mathbb{F}}_p$ if $p \gg 0$. Moreover, $\dim \Gamma_\infty(h) = \dim \Gamma_{\mathrm{gl}}(f_\tau) + 1$. Therefore, from Theorem 4.1.8 below, it follows that

$$\begin{aligned} &\sum_{(t,x)\in(\mathbb{F}_p^\times)^{n+1}} \exp\left(2\pi i \frac{t f_\tau(x)}{p}\right) \\ &\quad = (p-1)^{n+1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)} \sum_{y\in(\mathbb{F}_p^\times)^{\dim \Gamma_{\mathrm{gl}}(f_\tau)+1}} \exp\left(2\pi i \frac{h(y)}{p}\right) \\ &\quad = (p-1)^{n+1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)} O\left(\sqrt{p}^{\dim \Gamma_{\mathrm{gl}}(f_\tau)+1}\right) \\ &\quad = O\left(p^{n+1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right). \end{aligned}$$

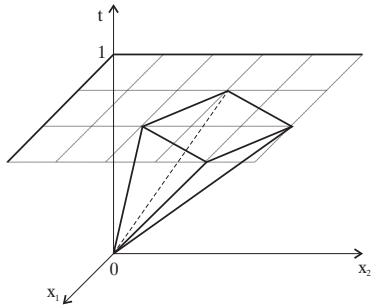


Figure 4.1.2

Assertion (i) follows now from (4.1.3).

We now prove (ii). It is clear that $\rho = -1/t_0 < -1$. Therefore, by Assertion(i), one gets

$$\begin{aligned} L_\tau(\rho) &= \left(\frac{p-1}{p}\right)^n \left(1 - \frac{p^\rho - 1}{p^{\rho+1}-1}\right) + O\left(p^{1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right) \frac{p^\rho - 1}{p^{\rho+1}-1} \\ &= (1+o(1))(-p^{\rho+1})(1+o(1)) + O\left(p^{1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right)(1+o(1)) \\ &= -p^{\rho+1}(1+o(1)) + O\left(p^{1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right). \end{aligned}$$

Now Assertion (ii) follows because $\dim \Gamma_{\mathrm{gl}}(f_\tau) \geq \dim c(\tau)$. □

Theorem 4.1.8 (Adolphson and Sperber [AS89, Corollary 4.3])**.** *Given an n-dimensional Newton polyhedron at infinity P in $\mathbb{R}^n$, there is a set $\mathcal{S}_P$ (which may be effectively determined) consisting of all but finitely many prime numbers, such that the following holds. If $p \in \mathcal{S}_P$, $f \in \mathbb{F}_p[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}]$ with $\Gamma_\infty(f) = P$ and f is non-degenerated over $\bar{\mathbb{F}}_p$ with respect to $\Gamma_\infty(f)$, then*

$$\left| \sum_{x \in (\mathbb{F}_p^\times)^n} \exp\left(2\pi i \frac{f(x)}{p}\right) \right| \leq n! \sqrt{p}^n V(f),$$

where $V(f)$ is the volume of $\Gamma_\infty(f) = P$ with respect to the Lebesgue measure on $\mathbb{R}^n$.

Remark. It follows from later results of Denef and Loeser [DL91] that one can actually take $\mathcal{S}_P$ equal to the set of all primes.

Lemma 4.1.9 (Denef and Sperber [DS99, Proposition 3.1 and Remark 3.2])**.** *Let $f(x_1, \ldots, x_n)$ be a non-zero polynomial with $f(0) = 0$. Let $k \in \mathbb{N}^n$ and suppose that no vertex of $\tau_0 \cap F(k)$ belongs to $\{0,1,2\}^n$. Let d be the dimension*

of the convex hull of the vertices of $\tau_0 \cap F(k)$. (*Thus* $d \leq \dim(\tau_0 \cap F(k))$.) *Then we have*

$$\sigma(k) \geq (m(k)+1)\frac{1}{t_0} - \frac{d+1}{3}. \tag{4.1.4}$$

If there is no condition on the vertices of $\tau_0 \cap F(k)$, *then the denominator 3 in* $(d+1)/3$ *should be replaced by 1. If no vertex of* $\tau_0 \cap F(k)$ *belongs to* $\{0,1\}^n$, *then the denominator should be replaced by 2.*

Remark. One can also find a proof of this lemma after Proposition 4.3.10.

Lemma 4.1.10. *Let* $f(x) = f(x_1, \ldots, x_n)$ *be a non-zero polynomial with* $f(0) = 0$ *and let* p *be a real number with* $p > 1$. *Let* τ *be a proper face of* $\Gamma(f)$ *and* S_{Δ_τ} *as in Theorem 2.3.2. Suppose that* $\tau \subsetneq \tau_0$ *and that no vertex of* τ_0 *belongs to* $\{0,1,2\}^n$. *Then there exists an* $\alpha_\tau > 0$, *independent of* p, *such that*

$$\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} = O(p^{-\alpha_\tau}) \tag{4.1.5}$$

$$= O(p^{\rho+(\dim c(\tau)+1)/3}). \tag{4.1.6}$$

Remarks.

1. Assertion (4.1.5) is even true without the condition on the vertices of τ_0.
2. If there is no condition on the vertices of τ_0, then the denominator 3 in $(\dim c(\tau)+1)/3$ from Assertion (4.1.6) should be replaced by 1. If no vertex of τ_0 belongs to $\{0,1\}^n$, then the denominator should be replaced by 2.

Proof. Take a partition of Δ_τ into a finite number of simple cones Δ_i. We know that $S_{\Delta_\tau} = \sum_i S_{\Delta_i}$. It suffices to prove the assertion for such a simple cone Δ_i. Take a set of linearly independent vectors in $\mathbb{N}^n \setminus \{0\}$ with multiplicity 1 that span Δ_i. We now can use the formula for S_{Δ_i} from Theorem 2.3.2. Recall that ρ is a pole of order 1 of $1/(p^{\sigma(a)+m(a)s}-1)$ if and only if $a \in \bar{\Delta}_{\tau_0}$ (Lemma 3.2.3). By Proposition 1.1.10, it follows that the number of generators for Δ_i that belong to $\bar{\Delta}_{\tau_0}$, is $\leq \kappa$. If this number is $< \kappa$, then $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_i} = 0$. So, we may suppose that the set of generators for Δ_i is $\{a_1, \ldots, a_\kappa, g_1, \ldots, g_r\}$, where $a_1, \ldots, a_\kappa \in \bar{\Delta}_{\tau_0}$ and $g_1, \ldots, g_r \in \bar{\Delta}_\tau \setminus \bar{\Delta}_{\tau_0}$. It is clear that $r \geq 1$, because the cone strictly positively spanned by $a_1, \ldots, a_\kappa$ is a subset of $\bar{\Delta}_{\tau_0}$, which is contained in the boundary of $\bar{\Delta}_\tau$ and thus not in Δ_τ (Lemma 1.1.5 and Lemma 1.1.7). Hence,

$$S_{\Delta_i} = \frac{1}{\begin{pmatrix} (p^{\sigma(a_1)+m(a_1)s}-1)\cdots(p^{\sigma(a_\kappa)+m(a_\kappa)s}-1) \\ (p^{\sigma(g_1)+m(g_1)s}-1)\cdots(p^{\sigma(g_r)+m(g_r)s}-1) \end{pmatrix}}$$

and $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_i}$ equals

$$\frac{1}{m(a_1)\cdots m(a_\kappa)(p^{\sigma(g_1)+m(g_1)\rho}-1)\cdots(p^{\sigma(g_r)+m(g_r)\rho}-1)}. \tag{4.1.7}$$

By Lemma 3.2.3 we know that $\sigma(g_j)+m(g_j)\rho>0$ for each j. Therefore, the denominator of (4.1.7) equals

$$\begin{aligned}&m(a_1)\cdots m(a_\kappa)p^{\sigma(g_1)+m(g_1)\rho\cdots+\sigma(g_r)+m(g_r)\rho}(1+o(1))\\&=m(a_1)\cdots m(a_\kappa)p^{\sigma(g_1+\cdots+g_r)+m(g_1+\cdots+g_r)\rho}(1+o(1)).\end{aligned}$$

The last equality follows from the fact that σ and m are linear on $\bar{\Delta}_\tau$. Thus, $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_i}$ equals

$$\begin{aligned}&\tfrac{1}{m(a_1)\cdots m(a_\kappa)}p^{-(\sigma(g_1+\cdots+g_r)+m(g_1+\cdots+g_r)\rho)}(1+o(1))\\&=O\left(p^{-(\sigma(g_1+\cdots+g_r)+m(g_1+\cdots+g_r)\rho)}\right).\end{aligned}$$

By Lemma 1.1.3, we know that $F(g_1+\cdots+g_r+a_1+\cdots+a_\kappa)=F(g_1+\cdots+g_r)\cap\tau_0=\tau$. Hence, Lemma 3.2.3 implies that $\sigma(g_1+\cdots+g_r)+m(g_1+\cdots+g_r)\rho>0$. This yields (4.1.5).

From Lemma 4.1.9, one gets that $\sigma(g_1+\cdots+g_r)+m(g_1+\cdots+g_r)\rho\geq-\rho-(\dim c(\tau)+1)/3$. Therefore, $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_i}=O\left(p^{\rho+(\dim c(\tau)+1)/3}\right)$. □

Theorem 4.1.11. *Let $f(x)=f(x_1,\ldots,x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0)=0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and suppose that f_{τ_0} is non-degenerated over $\mathbb{C}$ with respect to its global Newton polyhedron $\Gamma_{\mathrm{gl}}(f_{\tau_0})$. If $t_0<1$ and no vertex of τ_0 belongs to $\{0,1,2\}^n$, then ρ is a pole of $Z_f(s)$ of order κ, if $p\gg0$. Moreover,*

$$\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa Z_f(s)=-c_0p^{\rho+1}(1+o(1)), \tag{4.1.8}$$

where $c_0>0$ is as in Lemma 3.2.4(iii), and is independent of p.

Recall that τ_0, t_0, ρ, and κ are defined in Definition 3.2.2.

Remarks.

1. As we remarked after Proposition 4.1.7, the condition "f_{τ_0} is non-degenerated over $\mathbb{C}$ with respect to $\Gamma_{\mathrm{gl}}(f_{\tau_0})$" can be removed, if τ_0 is compact or if $f_{\tau_0}=f_{\tau_0'}$ with τ_0' a compact face of $\Gamma(f)$.
2. If no vertex of τ_0 belongs to $\{0,1,2\}^n$, then $t_0\geq3/n$. Indeed, because $(t_0,t_0,\ldots,t_0)\in\tau_0$, the point $(t_0,t_0,\ldots,t_0)$ is a convex combination of points P_i of τ_0 with $\sigma(P_i)\geq3$. Therefore, $nt_0=\sigma(t_0,t_0,\ldots,t_0)\geq3$. If moreover $t_0<1$, then $n\geq4$. Although there are many examples that satisfy the conditions of the theorem above, one only finds them if the number of variables of f is at least 4. For example, the polynomial $f(x_1,x_2,x_3,x_4)=x_1^3+x_2^3+x_3^3+x_4^3$ satisfies the conditions of Theorem 4.1.11.

3. Examples 4.2.1 and 4.2.2 in Section 4.2 show that, if the condition on the vertices of τ_0 is not satisfied, then (4.1.8) is not always true.

Proof. Note that the conditions to use the formula for $Z_f(s)$ in Theorem 2.3.2 are satisfied for $p \gg 0$. Because one has for every face τ that ρ is not a pole of L_τ, and that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} = 0$ for every face $\tau \not\subset \tau_0$ (Lemma 3.2.4), it suffices to investigate the limits $\lim_{s\to\rho} L_\tau(p^{s-\rho}-1)^\kappa S_{\Delta_\tau}$ for the faces $\tau \subset \tau_0$.

From Lemma 3.2.4(iii) and Proposition 4.1.7(ii), it follows that

$$\lim_{s\to\rho} L_{\tau_0}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}} = -c_0 p^{\rho+1}(1+o(1)) + O\left(p^{1-(\dim c(\tau_0)+1)/2}\right), \quad (4.1.9)$$

where $c_0 > 0$ is independent of p.

For $\tau \subsetneq \tau_0$, one has by Proposition 4.1.7(ii) and Lemma 4.1.10 that

$$\begin{aligned}
&\lim_{s\to\rho} L_\tau(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} \\
&= -p^{\rho+1}(1+o(1)) \lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} \\
&\quad + O\left(p^{1-(\dim c(\tau)+1)/2}\right) \lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} \\
&= -p^{\rho+1}(1+o(1)) O\left(p^{-\alpha_\tau}\right) \\
&\quad + O\left(p^{1-(\dim c(\tau)+1)/2}\right) O\left(p^{\rho+(\dim c(\tau)+1)/3}\right) \\
&= O\left(p^{\rho+1-\alpha_\tau}\right) + O\left(p^{\rho+1-(\dim c(\tau)+1)/6}\right), \qquad (4.1.10)
\end{aligned}$$

where $\alpha_\tau > 0$ is independent of p.

Lemma 4.1.9 implies that $\sigma(0) \geq (m(0)+1)(-\rho) - (\dim c(\tau_0)+1)/3$. Therefore,

$$\rho + 1 \geq 1 - \frac{\dim c(\tau_0)+1}{3} > 1 - \frac{\dim c(\tau_0)+1}{2}. \quad (4.1.11)$$

From (4.1.9), (4.1.10) and (4.1.11), it now follows that

$$\lim_{s\to\rho}\left(p^{s-\rho}-1\right)^\kappa Z_f(s) = -c_0 p^{\rho+1}(1+o(1)).$$

□

Remark. If no vertex of τ_0 belongs to $\{0,1\}^n$, (instead of $\{0,1,2\}^n$), one can prove, by using the suitable inequality in Lemma 4.1.10, that

$$\lim_{s\to\rho}\left(p^{s-\rho}-1\right)^\kappa Z_f(s) = O(p^{\rho+1}). \quad (4.1.12)$$

We conjecture that the same is true without a condition on the vertices of τ_0. With a similar argument as above, we know that $\lim_{s\to\rho}\left(p^{s-\rho}-1\right)^\kappa Z_f(s) = O(p^{\rho+1+(\dim c(\tau_0)+1)/2})$. We also expect that (4.1.12) is true without the condition of non-degeneracy, but with a different definition of ρ.

Theorem 4.1.12. *Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the compact faces of $\Gamma(f)$ and suppose that τ_0 is compact. If $t_0 < 1$ and no vertex of τ_0 belongs to $\{0,1,2\}^n$, then ρ is a pole of $Z_{f,0}(s)$ of order κ, if $p \gg 0$. Moreover,*

$$\lim_{s\to\rho}(p^{s-\rho}-1)^{\kappa} Z_{f,0}(s) = -c_0 p^{\rho+1}(1+o(1)),$$

where $c_0 > 0$ is as in Lemma 3.2.4(iii), and is independent of p.

Proof. First note that Proposition 4.1.7 holds for every compact face τ. Then the proof is similar to the proof of Theorem 4.1.11, because τ_0 is compact, which implies that $L_{\tau_0} S_{\Delta_{\tau_0}}$ occurs in the formula for $Z_{f,0}(s)$. □

Remark. One can find an alternative proof of this theorem in Subsection 4.3.3.

Theorem 4.1.13. *Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the faces of its Newton polyhedron $\Gamma(f)$ and suppose that f_{τ_0} is non-degenerated over $\mathbb{C}$ with respect to its global Newton polyhedron $\Gamma_{\mathrm{gl}}(f_{\tau_0})$. Suppose that $t_0 = 1$.*

(i) Suppose that $\dim \Gamma_{\mathrm{gl}}(f_{\tau_0}) > 1$. Then $\rho = -1$ is a pole of $Z_f(s)$ of order $\kappa + 1$ if $p \gg 0$.

(ii) If (moreover) no vertex of τ_0 belongs to $\{0,1,2\}^n$, then

$$\lim_{s\to -1}(p^{s+1}-1)^{\kappa+1} Z_f(s) = c_0 + o(1), \tag{4.1.13}$$

where $c_0 > 0$ is as in Lemma 3.2.4(iii) and is independent of p.

Remarks.

1. The easy example $f(x,y) = x^2 + y^2$ shows that the condition on the dimension of $\Gamma_{\mathrm{gl}}(f_{\tau_0})$ cannot be omitted. Indeed, in this case -1 is a pole of $Z_f(s)$ of order 2 when $p \equiv 1 \bmod 4$, but a pole of order 1 when $p \equiv 3 \bmod 4$.

2. Lemma 4.1.9 with $k = 0$ implies that the condition "$\dim \Gamma_{\mathrm{gl}}(f_{\tau_0}) > 1$" is satisfied if no vertex of τ_0 belongs to $\{0,1,2\}^n$ and $t_0 \leq 1$. Indeed, $\dim \Gamma_{\mathrm{gl}}(f_{\tau_0}) \geq \dim c(\tau_0)$.

3. Of course, we have a similar result for $Z_{f,0}(s)$ with a similar proof. Then we have to require that τ_0 is compact.

Proof. The conditions to use the formula for $Z_f(s)$ in Theorem 2.3.2 are satisfied for $p \gg 0$. For every face τ of $\Gamma(f)$, one has $\lim_{s\to -1}(p^{s+1}-1)L_\tau = (1 - p^{-1})p^{-n} p N_\tau \geq 0$ with L_τ and N_τ as in Theorem 2.3.2. From Lemma 3.2.4 or

Theorem 3.4.1, it follows then that, if this limit (or N_τ) is > 0 for at least one $\tau \subset \tau_0$, then $\rho = -1$ will be a pole of $Z_f(s)$ of order $\kappa + 1$.

From Proposition 4.1.7, it follows that for $\tau \subset \tau_0$

$$\begin{aligned}
\lim_{s\to -1}(p^{s+1}-1)L_\tau &= (1-p^{-1})p^{-n}\left((p-1)^n + O\left(p^{n+1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right)\right) \\
&= (1-p^{-1})\left(\frac{p-1}{p}\right)^n + O\left(p^{1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right) \\
&= 1 + o(1) + O\left(p^{1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2}\right) \qquad (4.1.14) \\
&= 1 + o(1) + O\left(p^{1-(\dim c(\tau)+1)/2}\right). \qquad (4.1.15)
\end{aligned}$$

Note that the error term $O(p^{1-(\dim \Gamma_{\mathrm{gl}}(f_\tau)+1)/2})$ in (4.1.14) is the smallest one for the face $\tau = \tau_0$. It is now clear that $\lim_{s\to -1}(p^{s+1}-1)L_{\tau_0} > 0$ for $p \gg 0$ if $\dim \Gamma_{\mathrm{gl}}(f_{\tau_0}) > 1$.

The proof of (ii) is similar to the proof of Theorem 4.1.11, by using (4.1.15) instead of Proposition 4.1.7(ii). □

4.2 Examples

We use Theorem 2.3.2 to calculate Igusa's local zeta function in the following examples. To find the cones associated to the proper faces of the Newton polyhedron, we use Proposition 1.1.10. To calculate the S_{Δ_τ}, we divide these cones into simplicial cones. We denote the cone strictly positively spanned by $a_1, \ldots, a_r$ by $\mathrm{cone}\{a_1, \ldots, a_r\}$. To find the number of the solutions of the set $\mathbb{Z}^n \cap \{\sum \lambda_i a_i \mid 0 \leq \lambda_i < 1\}$, one can use Proposition 1.2.5.

Example 4.2.1. Let $f_1(x,y,z) = x^2 + yz + z^2$. It is easy to verify that the Newton polyhedron $\Gamma(f_1)$ of f_1 is defined by the system of linear inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ x + 2z \geq 2 \\ x + y + z \geq 2. \end{cases}$$

If p is prime and $p \neq 2$, then it is easy to verify that f_1 is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron. A picture of the Newton polyhedron of f_1 and its proper faces $\tau_1, \ldots, \tau_{15}$ is given in Figure 4.2.1. More information about the cones associated to the proper faces is contained in Table 4.2.1.

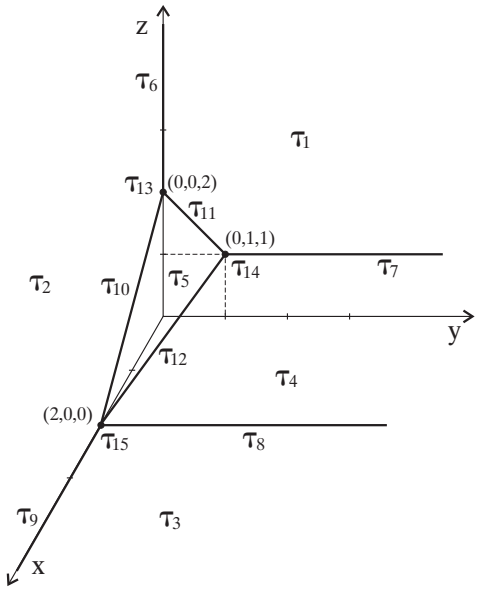


Figure 4.2.1. Newton polyhedron of $x^2 + yz + z^2$

The cone Δ_{τ_7} is rational simplicial, but the multiplicity of $(1,0,0),(1,0,2)$ is 2 and

$$\begin{aligned}\mathbb{Z}^3 \cap \{\lambda_1(1,0,0) + \lambda_2(1,0,2) \mid 0 \le \lambda_i < 1 \text{ for } i = 1,2\} \\ = \{(0,0,0), (1,0,1) = \frac{1}{2}(1,0,0) + \frac{1}{2}(1,0,2)\}.\end{aligned}$$

So, $\sum_h p^{\sigma(h)+m(h)s}$, where h runs through the above set, is equal to $1+p^{2+s}$. One has to do a similar calculation for τ_{14}. The cone $\Delta_{\tau_{15}}$ is not rational simplicial. One can take the following partition of the cone $\Delta_{\tau_{15}}$ into rational simplicial cones:

$$\begin{aligned}\Delta_{\tau_{15}} \quad = \quad & \text{cone}\{(0,1,0),(0,0,1),(1,1,1)\} \cup \text{cone}\{(0,0,1),(1,1,1)\} \\ & \cup \text{cone}\{(0,0,1),(1,1,1),(1,0,2)\}.\end{aligned}$$

The calculation of the other S_{Δ_τ} is easy, because the generators are linearly independent with multiplicity 1.

It is easy to see that $N_{\tau_3} = N_{\tau_8} = N_{\tau_9} = N_{\tau_{15}} = 0$; $N_{\tau_6} = N_{\tau_{13}} = 0$; $N_{\tau_7} = N_{\tau_{14}} = 0$; $N_{\tau_1} = N_{\tau_{11}} = \#\{(x,y,z) \in (\mathbb{F}_p^\times)^3 \mid y = -z\} = (p-1)^2$; $N_{\tau_4} = N_{\tau_{12}} = \#\{(x,y,z) \in (\mathbb{F}_p^\times)^3 \mid y = -x^2/z\} = (p-1)^2$; $N_{\Gamma(f_1)} = N_{\tau_5} = \#\{(x,y,z) \in (\mathbb{F}_p^\times)^3 \mid y = -(x^2+z^2)/z\} = (p-1)^2 - \#\{x,z) \in (\mathbb{F}_p^\times)^2 \mid x^2 + z^2 = 0\}$; $N_{\tau_2} = N_{\tau_{10}} = \#\{(x,y,z) \in (\mathbb{F}_p^\times)^3 \mid x^2 + z^2 = 0\}$.

Proper face τ	Generators for Δ_τ	S_{Δ_τ}
τ_1	$(1,0,0)$	$\frac{1}{p-1}$
τ_2	$(0,1,0)$	$\frac{1}{p-1}$
τ_3	$(0,0,1)$	$\frac{1}{p-1}$
τ_4	$(1,0,2)$	$\frac{1}{p^{3+2s}-1}$
τ_5	$(1,1,1)$	$\frac{1}{p^{3+2s}-1}$
τ_6	$(1,0,0),(0,1,0)$	$\frac{1}{(p-1)^2}$
τ_7	$(1,0,0),(1,0,2)$	$\frac{1+p^{2+s}}{(p-1)(p^{3+2s}-1)}$
τ_8	$(0,0,1),(1,0,2)$	$\frac{1}{(p-1)(p^{3+2s}-1)}$
τ_9	$(0,0,1),(0,1,0)$	$\frac{1}{(p-1)^2}$
τ_{10}	$(0,1,0),(1,1,1)$	$\frac{1}{(p-1)(p^{3+2s}-1)}$
τ_{11}	$(1,0,0),(1,1,1)$	$\frac{1}{(p-1)(p^{3+2s}-1)}$
τ_{12}	$(1,0,2),(1,1,1)$	$\frac{1}{(p^{3+2s}-1)^2}$
τ_{13}	$(1,0,0),(0,1,0),(1,1,1)$	$\frac{1}{(p-1)^2(p^{3+2s}-1)}$
τ_{14}	$(1,0,0),(1,1,1),(1,0,2)$	$\frac{1+p^{2+s}}{(p-1)(p^{3+2s}-1)^2}$
τ_{15}	$(0,1,0),(0,0,1),(1,1,1),(1,0,2)$	$\frac{1}{(p-1)^2(p^{3+2s}-1)}$
		$+\frac{1}{(p-1)(p^{3+2s}-1)}$
		$+\frac{1}{(p-1)(p^{3+2s}-1)^2}$

Table 4.2.1. The cones associated to the faces of $\Gamma(x^2+yz+z^2)$

If $p \equiv 1 \bmod 4$, then -1 is a square in $\mathbb{F}_p$. In this case, $N_{\Gamma(f_1)} = N_{\tau_5} = (p-1)(p-3)$ and $N_{\tau_2} = N_{\tau_{10}} = 2(p-1)^2$. If $p \equiv 3 \bmod 4$, then -1 is not a square in $\mathbb{F}_p$. In this case, $N_{\Gamma(f_1)} = N_{\tau_5} = (p-1)^2$ and $N_{\tau_2} = N_{\tau_{10}} = 0$. One can now calculate that in both cases (for $p \neq 2$), we obtain the following formula:

$$Z_{f_1}(s) = \frac{p^{2s}(p-1)(p^{3+s}-1)}{(p^{1+s}-1)(p^{3+2s}-1)}.$$

The real poles of $Z_{f_1}(s)$ are $s = -1$ and $s = \rho = -3/2$, both of order 1. Note that τ_{12} is of codimension 2 and is the smallest face of $\Gamma(f_1)$ that contains $(2/3, 2/3, 2/3)$. So, in this example, the order of the pole $-3/2$ is smaller than the expected order. Note that there is a vertex of τ_{12} that is contained in $\{0,1\}^3$, so that the condition in Theorem 4.1.11 is not satisfied. Example 4.2.2 will show that it is not enough to require in Theorem 4.1.11 that no vertex of τ_0 is contained in $\{0,1\}^n$.

Remark. One can also calculate $Z_{f_1}(s)$ very easily using Igusa's stationary phase formula [Igu94] or calculating the Igusa zeta functions for the trivial and the non-trivial characters of x^2 and $yz + z^2$. Alternatively, one can apply the formula in [Gol83] or use the method of resolution of singularities.

Example 4.2.2. Let $f_2(x,y,z,u) = zx^2 + zy^2 + u^3$. By using a computer program, one obtains that the Newton polyhedron $\Gamma(f_2)$ of f_2 is defined by the system of linear inequalities

$$\begin{cases} 3x + 3y + 2u \geq 6 \\ 3z + u \geq 3 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ u \geq 0. \end{cases}$$

One proves that f_2 is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron if $p \neq 2, 3$. We denote by $\tau_1, \ldots, \tau_6$ the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above. The vertices of $\Gamma(f_2)$ are $v_1 = (2,0,1,0), v_2 = (0,2,1,0)$ and $v_3 = (0,0,0,3)$. The candidate poles for $Z_{f_2}(s)$ (and $Z_{f_2,0}(s)$) are -1 and $-4/3$ with candidate order 1 and 2, respectively. Indeed, $\tau_0 = \tau_1 \cap \tau_2 = \operatorname{conv}\{v_1, v_2, v_3\}$ has codimension 2. Note that v_1 and v_2 are contained in $\{0,1,2\}^4$, but not in $\{0,1\}^4$. In this example, $-4/3$ will be a pole of $Z_{f_2}(s)$ of order 2 for $p \equiv 1 \bmod 4$, but for $p \equiv 3 \bmod 4$, it will be of order ≤ 1. To show this, by Lemma 3.2.4, it suffices to calculate $Z_{f_2,0}(s) = \sum_\tau L_\tau S_{\Delta_\tau}$, where the sum in this case runs over all the faces $\tau \subset \tau_0$ (see Table 4.2.2).

By using a computer program, we divided the cone $\Delta_{\tau_{13}}$ into rational simplicial

cones:

$$\begin{aligned}\Delta_{\tau_{13}} \;=\;& \operatorname{cone}\{(3,3,0,2),(0,0,3,1),(0,1,0,0),(0,0,1,0)\}\\ &\cup \operatorname{cone}\{(3,3,0,2),(0,0,3,1),(1,0,0,0),(0,0,1,0)\}\\ &\cup \operatorname{cone}\{(3,3,0,2),(1,0,0,0),(0,1,0,0),(0,0,1,0)\}\\ &\cup \operatorname{cone}\{(3,3,0,2),(0,0,3,1),(0,0,1,0)\}\\ &\cup \operatorname{cone}\{(3,3,0,2),(0,1,0,0),(0,0,1,0)\}\\ &\cup \operatorname{cone}\{(3,3,0,2),(1,0,0,0),(0,0,1,0)\}\\ &\cup \operatorname{cone}\{(3,3,0,2),(0,0,1,0)\}.\end{aligned}$$

Compact face τ	Generators for Δ_τ	S_{Δ_τ}
$\tau_7 = \operatorname{conv}\{v_1,v_2,v_3\}$	$(3,3,0,2),(0,0,3,1)$	$\frac{1+p^{4+3s}+p^{8+6s}}{(p^{8+6s}-1)(p^{4+3s}-1)}$
$\tau_8 = \operatorname{conv}\{v_1,v_2\}$	$(3,3,0,2),(0,0,3,1),e_4$	$\frac{\left(\begin{smallmatrix}1+p^{4+3s}+p^{8+6s}+2p^{3+2s}\\ +2p^{6+4s}+p^{7+5s}+p^{2+s}\end{smallmatrix}\right)}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)}$
$\tau_9 = \operatorname{conv}\{v_1,v_3\}$	$(3,3,0,2),(0,0,3,1),e_2$	$\frac{1+p^{4+3s}+p^{8+6s}}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)}$
$\tau_{10} = \operatorname{conv}\{v_2,v_3\}$	$(3,3,0,2),(0,0,3,1),e_1$	$\frac{1+p^{4+3s}+p^{8+6s}}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)}$
$\tau_{11} = v_1$	$(3,3,0,2),(0,0,3,1),$ e_2,e_4	$\frac{\left(\begin{smallmatrix}1+p^{4+3s}+p^{8+6s}+2p^{3+2s}\\ +2p^{6+4s}+p^{7+5s}+p^{2+s}\end{smallmatrix}\right)}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)^2}$
$\tau_{12} = v_2$	$(3,3,0,2),(0,0,3,1),$ e_1,e_4	$\frac{\left(\begin{smallmatrix}1+p^{4+3s}+p^{8+6s}+2p^{3+2s}\\ +2p^{6+4s}+p^{7+5s}+p^{2+s}\end{smallmatrix}\right)}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)^2}$
$\tau_{13} = v_3$	$(3,3,0,2),(0,0,3,1),$ e_1,e_2,e_3	$2\frac{1+p^{4+3s}+p^{8+6s}}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)^2}$ $+\frac{1+p^{5+3s}}{(p^{8+6s}-1)(p-1)^3}$ $+\frac{1+p^{4+3s}+p^{8+6s}}{(p^{8+6s}-1)(p^{4+3s}-1)(p-1)}$ $+2\frac{1}{(p^{8+6s}-1)(p-1)^2}$ $+\frac{1}{(p^{8+6s}-1)(p-1)}$

e_i denotes the i-th unit vector of $\mathbb{R}^4$

Table 4.2.2. The cones associated to the compact faces of $\Gamma(zx^2+zy^2+u^3)$

It is easy to see that $N_7 = (p-1)^3 - (p-1)\#\{(x,y) \in (\mathbb{F}_p^\times)^2 \mid x^2+y^2=0\}$, $N_8 = (p-1)^2\#\{(x,y) \in (\mathbb{F}_p^\times)^2 \mid x^2+y^2=0\}$, $N_9 = (p-1)\#\{(x,z,u) \in (\mathbb{F}_p^\times)^3 \mid z=-u^3/x^2\} = (p-1)^3$, $N_{10} = (p-1)^3$, $N_{11} = N_{12} = N_{13} = 0$.

For $p \equiv 1 \bmod 4$, we obtain for $Z_{f_2,0}(s)$

$$\frac{(-1+p)\left(\begin{array}{c} 1-6p^{4+2s}+p^{8+4s}-3p^{3+s}+2p^{4+s} \\ +3p^{3+2s}+3p^{5+2s}+2p^{4+3s}-3p^{5+3s} \end{array}\right)}{p^4(p^{1+s}-1)(p^{4+3s}-1)^2}.$$

For $p \equiv 3 \bmod 4$, we obtain for $Z_{f_2,0}(s)$

$$\frac{(-1+p)(1+p^{2+s})(-1+p^{2+s}+p^{3+s}-p^{3+2s}-p^{4+2s}+p^{6+3s})}{p^4(p^{1+s}-1)(p^{4+3s}-1)(p^{4+3s}+1)}.$$

This example shows that the condition of "no vertex of τ_0 is contained in $\{0,1,2\}^n$" in Theorem 4.1.11 can not be omitted or replaced by "no vertex of τ_0 is contained in $\{0,1\}^n$".

Remark. By using [Den91, Corollary 1.4.5], which gives a connection between exponential sums and Igusa's local zeta function, and by using separation of variables, one can also calculate the order of the pole $-4/3$ for $Z_{f_2}(s)$ and $Z_{f_2,0}(s)$. For $p \equiv 1 \bmod 4$, one obtains that $-4/3$ is a pole of order 2 of $Z_{f_2}(s)$ and $Z_{f_2,0}(s)$ and for $p \equiv 3 \bmod 4$, it is a pole of order 1. We also calculated $Z_{f_2}(s)$ with Polygusa; see [HL00] or Example 3 in Appendix D. The polyhedron $\Gamma(f_2)$ has 33 proper faces.

4.3 An alternative proof of Theorem 4.1.12 by using exponential sums

This section contains a small extension of the results of Denef and Sperber in [DS99]. The proofs are similar, but for convenience of the reader, we will give them here with more details. Some extra work had to be done for the case of $m = 1$ in the proof of Theorem 4.3.3. At the end of this section, we give an alternative proof of Theorem 4.1.12, now by using exponential sums.

Definition 4.3.1. Fix a prime number p. Let $f(x) = f(x_1,\ldots,x_n)$ be a non-zero polynomial over $\mathbb{Z}_p$ and $f(0) = 0$. For $m \in \mathbb{N}\setminus\{0\}$ and $u \in \mathbb{Z}_p^\times$, we consider the exponential sums

$$\begin{aligned} S_m(u) &= \frac{1}{p^{mn}} \sum_{x \bmod p^m} \exp\left(\tfrac{2\pi i}{p^m} u f(x)\right), \\ \text{and} \quad S_{0,m}(u) &= \frac{1}{p^{mn}} \sum_{\substack{x \bmod p^m \\ x \equiv 0 \bmod p}} \exp\left(\tfrac{2\pi i}{p^m} u f(x)\right). \end{aligned}$$

If $u = 1$, we write S_m, resp. $S_{0,m}$, instead of $S_m(1)$, resp. $S_{0,m}(1)$.

We recall here Definition 4.1.1.

Definition 4.3.2. Let p be a prime number. Let f be as in Definition 0.1.4. The *Newton polyhedron* $\Gamma_\infty(f)$ *of* f *at infinity* is defined as the convex hull of $\mathrm{supp}\,(f) \cup \{0\}$, where $\mathrm{supp}\,(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}$ is the support of f. For a face τ of $\Gamma_\infty(f)$, set $f_\tau = \sum_{\omega \in \tau} a_\omega x^\omega$. We say that f is *non-degenerated over* $\bar{\mathbb{F}}_p$ *with respect to the Newton polyhedron* $\Gamma_\infty(f)$ of f at infinity, if for every face τ of $\Gamma_\infty(f)$ which does not contain the origin, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$ has no solution in $(\bar{\mathbb{F}}_p^\times)^n$.

Remarks.

1. The following two conditions are equivalent, because a compact face is the convex hull of its vertices and a face of a face of a convex set C is a face of this convex set C.

   (1) For every compact face τ of the Newton polyhedron $\Gamma(f)$ of f, the polynomial f_τ is non-degenerated over $\bar{\mathbb{F}}_p$ with respect to the Newton polyhedron $\Gamma_\infty(f_\tau)$ of f_τ at infinity.

   (2) For every compact face τ of $\Gamma(f)$, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$ has no solution in $(\bar{\mathbb{F}}_p^\times)^n$.

   If we replace "compact face" by "face", the equivalence is not true. For example, let f be a polynomial over $\mathbb{Z}_p$ with the exponents of the monomials of f given by the points in Figure 4.3.1. Then there can be a problem for the face $\Gamma(f)$.

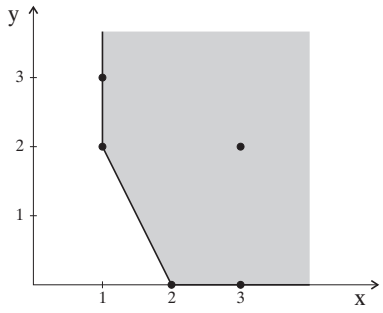


Figure 4.3.1

2. Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to the compact faces of its Newton polyhedron $\Gamma(f)$ (i.e., for every compact face τ of $\Gamma(f)$, the set of equations $(\partial f_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$ has no solution

in $(\mathbb{C}^\times)^n$). Then one proves that condition (2), and consequently condition (1), is satisfied for $p \gg 0$.

If moreover f is homogeneous, then for every face τ of $\Gamma(f)$, there exists a compact face $c(\tau)$ such that $f_\tau = f_{c(\tau)}$. So, for $p \gg 0$, the polynomials f_τ are non-degenerated over $\bar{\mathbb{F}}_p$ with respect to $\Gamma_\infty(f_\tau)$ for every face τ of $\Gamma(f)$.

In this section, we prove the following theorems. We will use Theorem 4.3.6 to give an alternative proof of Theorem 4.1.12 in Subsection 4.3.3.

Theorem 4.3.3. *Let Γ be a Newton polyhedron in $\mathbb{R}^n$. There exists a constant $c_\Gamma > 0$, only depending on Γ, such that the following holds. Let f be as in Definition 0.1.4 with $\Gamma(f) = \Gamma$ and let p be a prime number. Suppose that*

(i) for every[3] face τ of the Newton polyhedron $\Gamma(f)$ of f, the polynomial f_τ is non-degenerated over $\bar{\mathbb{F}}_p$ with respect to the Newton polyhedron $\Gamma_\infty(f_\tau)$ of f_τ at infinity;

(ii) for every (non-compact) face τ of $\Gamma(f)$, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\mathbb{F}_p^\times)^n$;

(iii) no vertex of τ_0 belongs to $\{0,1\}^n$.

Then

$$|S_m(u)| \leq c_\Gamma p^{\rho m} m^{\kappa - 1} \tag{4.3.1}$$

for all $m \in \mathbb{N} \setminus \{0\}$ and all $u \in \mathbb{Z}_p^\times$.

We recall that τ_0, ρ and κ are defined in Definition 3.2.2.

Remarks.

1. Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero homogeneous polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to the compact faces of its Newton polyhedron $\Gamma(f)$. Then the conditions (i) and (ii) are satisfied for $p \gg 0$.

2. Note that (4.3.1) implies Igusa's conjecture [Igu78, p. 170] for the toroidal resolutions of such f.

Theorem 4.3.4. *Let Γ be a Newton polyhedron in $\mathbb{R}^n$. There exists a constant $c_\Gamma > 0$, only depending on Γ, such that the following holds. Let f be as in Definition 0.1.4 with $\Gamma(f) = \Gamma$ and let p be a prime number. Suppose that*

(i) for every compact face τ of $\Gamma(f)$, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\bar{\mathbb{F}}_p^\times)^n$;

[3] Thus also for $\tau = \Gamma(f)$.

(ii) no vertex of τ_0 belongs to $\{0,1\}^n$.

Then

$$|S_{0,m}(u)| \leq c_\Gamma p^{\rho m} m^{\kappa-1} \tag{4.3.2}$$

for all $m \in \mathbb{N} \setminus \{0\}$ and all $u \in \mathbb{Z}_p^\times$.

Theorem 4.3.5. *Let Γ be a Newton polyhedron in $\mathbb{R}^n$. There exist constants $c_\Gamma > 0$ and $M_\Gamma > 0$, only depending on Γ, such that the following holds. Let f be as in Definition 0.1.4 with $\Gamma(f) = \Gamma$ and let p be a prime number with $p > M_\Gamma$. Suppose that*

(i) for every[4] face τ of the Newton polyhedron $\Gamma(f)$ of f, the polynomial f_τ is non-degenerated over $\bar{\mathbb{F}}_p$ with respect to the Newton polyhedron $\Gamma_\infty(f_\tau)$ of f_τ at infinity;

(ii) for every non-compact face τ of $\Gamma(f)$, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\mathbb{F}_p^\times)^n$;

(iii) no vertex of τ_0 belongs to $\{0,1,2\}^n$.

Then

$$|S_m(u)| \geq c_\Gamma p^{\rho m} m^{\kappa-1} \tag{4.3.3}$$

for infinitely many $m \in \mathbb{N} \setminus \{0\}$ and all $u \in \mathbb{Z}_p^\times$.

Theorem 4.3.6. *Let Γ be a Newton polyhedron in $\mathbb{R}^n$. There exist constants $c_\Gamma > 0$ and $M_\Gamma > 0$, only depending on Γ, such that the following holds. Let f be as in Definition 0.1.4 with $\Gamma(f) = \Gamma$ and let p be a prime number with $p > M_\Gamma$. Suppose that*

(i) for every compact face τ of $\Gamma(f)$, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\bar{\mathbb{F}}_p^\times)^n$;

(ii) τ_0 is compact and no vertex of τ_0 belongs to $\{0,1,2\}^n$.

Then

$$|S_{0,m}(u)| \geq c_\Gamma p^{\rho m} m^{\kappa-1} \tag{4.3.4}$$

for infinitely many $m \in \mathbb{N} \setminus \{0\}$ and all $u \in \mathbb{Z}_p^\times$.

[4] Thus also for $\tau = \Gamma(f)$.

We compare the results above with the results of Denef and Sperber[DS99].

1. The constants c_Γ and M_Γ do only depend on $\Gamma(f)$, and not on the particular coefficients of f itself. As a result, one does not only obtain results for S_m and $S_{0,m}$, but also for $S_m(u)$ and $S_{0,m}(u)$. Indeed, if f satisfies the conditions of the theorems above, also uf satisfies the conditions and $\Gamma(uf) = \Gamma(f)$. It also suffices to prove the theorems above for S_m and $S_{0,m}$.

2. Denef and Sperber always work with polynomials over $\mathbb{Z}$ and require similar conditions over $\mathbb{C}$ (one obtains their conditions by replacing $\bar{\mathbb{F}}_p$ and $\mathbb{F}_p$ by $\mathbb{C}$ in the theorems above). Then they get bounds for $|S_m|$ and $|S_{0,m}|$, when $p \gg 0$. This is not surprising because for $p \gg 0$, the conditions in the theorems above are then satisfied.

3. Theorem 4.3.3 in [DS99] is only stated for homogeneous polynomials. Further, instead of conditions (i) and (ii), we find the following condition: for every compact face τ of Γ, the set of equations $(\partial \bar{f}_\tau/\partial x_i) = 0$, $i = 1, \ldots, n$ has no solution in $(\mathbb{C}^\times)^n$. One can possibly find non-homogeneous polynomials which satisfy the conditions of Theorem 4.3.3. Furthermore we added Theorem 4.3.5.

4.3.1 A formula for S_m

Proposition 4.3.7. *Let f be as in Definition 0.1.4. Suppose that for every[5] face τ of the Newton polyhedron $\Gamma(f)$ of f, the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\mathbb{F}_p^\times)^n$. Then we have for all $m \in \mathbb{N} \setminus \{0\}$ that*

$$S_m = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \left(p^{-n} E(f_\tau) \left(\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = m-1}} p^{-\sigma(k)} \right) + (1 - p^{-1})^n \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) \geq m}} p^{-\sigma(k)} \right),$$

where $E(f_\tau) = \sum_{a \in \{1, \ldots, p-1\}^n} \exp\left(\frac{2\pi i}{p} f_\tau(a)\right)$ and where Δ_τ and $\sigma(\cdot)$ are defined as in Definition 1.1.4 and 2.3.1.

Remark. By a similar argument as in the proof below, we can give an analogous formula for $S_{0,m}$. For $S_{0,m}$, the first sum is over the compact faces of $\Gamma(f)$ and we only need the condition that for every *compact* face τ of $\Gamma(f)$ the set of equations $(\partial \bar{f}_\tau/\partial x_i)(x) = 0$, $i = 1, \ldots, n$, has no solution in $(\mathbb{F}_p^\times)^n$.

[5] Thus also for $\tau = \Gamma(f)$.

Proof. We have

$$S_m = \int_{\mathbb{Z}_p^n} \exp\left(\frac{2\pi i}{p^m} f(x)\right) |dx|,$$

where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^n$, so normalized that $\mathbb{Z}_p^n$ has measure 1. Note that

$$S_{0,m} = \int_{(p\mathbb{Z}_p)^n} \exp\left(\frac{2\pi i}{p^m} f(x)\right) |dx|.$$

Recall that the cones associated to the faces of $\Gamma(f)$ form a partition of $(\mathbb{R}^+)^n$. So, $(\mathbb{R}^+)^n$ is equal to the disjoint union

$$\bigcup_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \Delta_\tau,$$

where $\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}$ is the cone associated to τ. Therefore, one has

$$\begin{aligned} S_m &= \sum_{k \in \mathbb{N}^n} \int_{\substack{x \in \mathbb{Z}_p^n \\ \operatorname{ord} x = k}} \exp\left(\frac{2\pi i}{p^m} f(x)\right) |dx| \\ &= \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} \int_{\substack{x \in \mathbb{Z}_p^n \\ \operatorname{ord} x = k}} \exp\left(\frac{2\pi i}{p^m} f(x)\right) |dx|. \end{aligned}$$

Suppose that τ is a face of $\Gamma(f)$, $k \in \mathbb{N}^n \cap \Delta_\tau$ and $x = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ with $\operatorname{ord} x = k$. Write $x_j = p^{k_j} u_j$ with $u_j \in \mathbb{Z}_p^\times$. Then

$$\begin{aligned} |dx| &= p^{-\sigma(k)} |du| \quad \text{and} \\ x^\omega &= x_1^{\omega_1} \cdots x_n^{\omega_n} = p^{k \cdot \omega} u^\omega. \end{aligned}$$

We know that $F(k) = \tau$. Thus, $k \cdot \omega = m(k)$ for every $\omega \in \operatorname{supp}(f) \cap \tau$, and $k \cdot \omega \geq m(k) + 1$ for every $\omega \in \operatorname{supp}(f) \setminus \tau$. It follows that

$$f(x) = p^{m(k)} (f_\tau(u) + p \tilde{f}_{\tau,k}(u)),$$

where $\tilde{f}_{\tau,k}(u)$ is a polynomial in the variables $u_1, \ldots, u_n$ over $\mathbb{Z}_p$ (depending on τ, k and f).

Hence

$$S_m = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma(f)}} \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k)} \int_{(\mathbb{Z}_p^\times)^n} \exp\left(\frac{2\pi i}{p^{m-m(k)}} (f_\tau(u) + p \tilde{f}_{\tau,k}(u))\right) |du|. \tag{4.3.5}$$

We examine the integral

$$\int\limits_{(\mathbb{Z}_p^\times)^n} \exp\left(\frac{2\pi i}{p^{m-m(k)}}(f_\tau(u)+p\tilde{f}_{\tau,k}(u))\right)|du|. \tag{4.3.6}$$

Case 1. $m-m(k)\leq 0$.
Then the integral (4.3.6) equals $\int\limits_{(\mathbb{Z}_p^\times)^n} 1|du| = (1-p^{-1})^n$.

Case 2. $m-m(k)=1$.
Then the integral (4.3.6) equals

$$\begin{aligned}&\sum_{a\in\{1,\dots,p-1\}^n}\ \int\limits_{a+(p\mathbb{Z}_p)^n} \exp\left(\tfrac{2\pi i}{p}(f_\tau(u)+p\tilde{f}_{\tau,k}(u))\right)|du|\\ &=\sum_{a\in\{1,\dots,p-1\}^n} p^{-n}\exp\left(\tfrac{2\pi i}{p}f_\tau(a)\right)\\ &=p^{-n}E(f_\tau).\end{aligned}$$

Case 3. $m-m(k)\geq 2$.
Put $e=m-m(k)$. Then the integral (4.3.6) equals

$$\begin{aligned}&\sum_{a\in\{1,\dots,p-1\}^n}\ \sum_{\substack{b\,\in\, a+(p\mathbb{Z}_p)^n\\ b \bmod p^e}}\ \int\limits_{b+(p^e\mathbb{Z}_p)^n} \exp\left(\tfrac{2\pi i}{p^e}(f_\tau(u)+p\tilde{f}_{\tau,k}(u))\right)|du|\\ &=\sum_{a\in\{1,\dots,p-1\}^n} p^{-en}\sum_{\substack{b\,\in\, a+(p\mathbb{Z}_p)^n\\ b \bmod p^e}} \exp\left(\tfrac{2\pi i}{p^e}(f_\tau(b)+p\tilde{f}_{\tau,k}(b))\right).\end{aligned}$$

From the conditions in the statement of Proposition 4.3.7, it follows that for each $a\in\{1,\dots,p-1\}^n$, there exists an $i\in\{1,\dots,n\}$ such that $(\partial f_\tau/\partial x_i)(a)\not\equiv 0 \bmod p$. Hence, one has $(\partial(f_\tau+p\tilde{f}_{\tau,k})/\partial x_i)(a)\not\equiv 0 \bmod p$ for this i. From Lemma 4.3.8 below, it follows now that

$$\begin{aligned}&\sum_{\substack{b\,\in\, a+(p\mathbb{Z}_p)^n\\ b \bmod p^e}} \exp\left(\frac{2\pi i}{p^e}(f_\tau(b)+p\tilde{f}_{\tau,k}(b))\right)\\ &\quad=(p^{e-1})^{n-1}\sum_{\substack{v\,\in\,\mathbb{Z}_p\\ v \bmod p^{e-1}}} \exp\left(\frac{2\pi i}{p^e}(pv+f_\tau(a)+p\tilde{f}_{\tau,k}(a)\right)\\ &\quad=(p^{e-1})^{n-1}\exp\left(\frac{2\pi i}{p^e}(f_\tau(a)+p\tilde{f}_{\tau,k}(a)\right)\sum_{\substack{v\,\in\,\mathbb{Z}_p\\ v \bmod p^{e-1}}} \exp\left(\frac{2\pi i}{p^{e-1}}v\right).\end{aligned}$$

Note that $e-1\geq 1$. The sum of all $(e-1)$-roots of unity is 0 and so, in this case the integral (4.3.6) is 0.

Proposition 4.3.7 follows now directly from (4.3.5). □

Lemma 4.3.8. *Let $g(x) \in \mathbb{Z}_p[x]$, $x = (x_1, \ldots, x_n)$, $a \in \mathbb{Z}_p^n$, $e \in \mathbb{N}\setminus\{0\}$. Suppose that $(\partial g/\partial x_1)(a) \not\equiv 0 \bmod p$. Define the function Ψ by*

$$\begin{array}{rccc} \Psi: & \mathbb{Z}_p^n & \to & \mathbb{Z}_p^n \\ & (x_1, \ldots, x_n) & \mapsto & (v_1, \ldots, v_n), \end{array}$$

where

$$v_j = \begin{cases} p^{-1}(g(a+px) - g(a)) & \text{for } j = 1, \\ x_j & \text{for } j \neq 1. \end{cases}$$

Then Ψ is one to one and onto. Moreover, $x \equiv x' \bmod p^e$ if and only if $\Psi(x) \equiv \Psi(x') \bmod p^e$.

Proof. This result is well-known, but for convenience of the reader, we will give a proof based on Hensel's Lemma.

Fix $v_1, x_2, \ldots, x_n \in \mathbb{Z}_p$. For the first assertion, it suffices to prove that there exists only one zero in $\mathbb{Z}_p$ of the polynomial k, where

$$k(x_1) := p^{-1}(g(a+px) - g(a)) - v_1. \tag{4.3.7}$$

We can write $g(x) = \alpha_0 + \sum_{i=1}^n \alpha_i(x_i - a_i) + h(x_1 - a_1, \ldots, x_n - a_n)$, where h is a polynomial over $\mathbb{Z}_p$ where the degree of each monomial of h is ≥ 2; $\alpha_0 = g(a)$ and $\alpha_i = (\partial g/\partial x_i)(a)$ for $i = 1, \ldots, n$. Therefore,

$$p^{-1}(g(a+px) - g(a)) = \sum_{i=1}^n \alpha_i x_i + p\tilde{h}(x_1, \ldots, x_n), \tag{4.3.8}$$

where $\tilde{h}$ is a polynomial over $\mathbb{Z}_p$ where the degree of each monomial of $\tilde{h}$ is ≥ 2; $\alpha_1 \not\equiv 0 \bmod p$. Therefore, (4.3.7) and (4.3.8) yields that there exists a solution $\tilde{x}_1 \in \mathbb{Z}_p$ of the congruence

$$k(x_1) \equiv 0 \bmod p, \tag{4.3.9}$$

and every other solution $x_1 \in \mathbb{Z}_p$ satisfies $x_1 \equiv \tilde{x}_1 \bmod p$.

Because we have also that $k'(\tilde{x}_1) = (\partial g/\partial x_1)(a+p\tilde{x}_1) \equiv (\partial g/\partial x_1)(a) \not\equiv 0 \bmod p$, Hensel's Lemma implies that there exists a unique $x_1 \in \mathbb{Z}_p$ such that $k(x_1) = 0$ and $x_1 \equiv \tilde{x}_1 \bmod p$. So, (4.3.9) yields that there exists a unique zero of k in $\mathbb{Z}_p$.

From (4.3.8), it follows that $x \equiv x' \bmod p^e$ if and only if $\Psi(x) \equiv \Psi(x') \bmod p^e$. Indeed, $x \equiv x' \bmod p^e$ implies that $\Psi(x) \equiv \Psi(x') \bmod p^e$. Moreover, because the map Ψ is onto, one then obtains that the map $\tilde{\Psi} = \mathbb{Z}_p^n/p^e\mathbb{Z}_p^n \to \mathbb{Z}_p^n/p^e\mathbb{Z}_p^n$: $x + p^e\mathbb{Z}_p^n \mapsto \Psi(x) + p^e\mathbb{Z}_p^n$ is one to one and onto. □

4.3.2 Proofs of the theorems

To proof Theorems 4.3.3, 4.3.4, 4.3.5 and 4.3.6, we will use Proposition 4.3.7 and search for upperbounds of $\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = m-1}} p^{-\sigma(k)}$ and $(1-p^{-1})^n \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) \geq m}} p^{-\sigma(k)}$. In order to do this, we first prove Lemma 4.3.9 and Proposition 4.1.9.

Lemma 4.3.9. *Let C be a polyhedral cone in $(\mathbb{R}^+)^n$ generated by vectors in $\mathbb{N}^n$ and let L be a linear form in n variables with coefficients in $\mathbb{N}$. We denote by $\operatorname{ri} C$ the relative interior of C. Let $\nu > 0$ and $\gamma \geq 0$ be real numbers and suppose that*

$$\sigma(k) \geq L(k)\nu + \gamma, \text{ for all } k \in \operatorname{ri} C \cap \mathbb{N}^n. \tag{4.3.10}$$

Put $e = \dim\{k \in C \mid \sigma(k) = L(k)\nu\}$. Then there exists a $c > 0$ such that for all $m \in \mathbb{N}$ and for all $p \in \mathbb{R}$, with $p \geq 2$, we have

$$P := \sum_{\substack{k \in \operatorname{ri} C \cap \mathbb{N}^n \\ L(k) = m}} p^{-\sigma(k)} \leq c p^{-m\nu-\gamma}(m+1)^{\max\{0,e-1\}}, \tag{4.3.11}$$

where $\sigma(\cdot)$ is defined as in Definition 2.3.1.

Remark. This lemma also holds for L equal to 0.

Proof. If $C = \{0\}$, then the assertion is trivial. So, let C be the cone $\{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_i \geq 0\}$ with $a_1, \ldots, a_r$ non-zero vectors in $\mathbb{N}^n$. Then $\operatorname{ri} C$ is the cone strictly positively spanned by $a_1, \ldots, a_r$. Since such a cone can be written as a finite disjoint union of simple cones (Lemma 1.2.3), we may suppose that the generators $a_1, \ldots, a_r$ are part of a $\mathbb{Z}$-module basis of $\mathbb{Z}^n$. Hence

$$\operatorname{ri} C \cap \mathbb{N}^n = \mathbb{N} \setminus \{0\} a_1 \oplus \cdots \oplus \mathbb{N} \setminus \{0\} a_r = w + \mathbb{N}a_1 \oplus \cdots \oplus \mathbb{N}a_r, \tag{4.3.12}$$

with $w = a_1 + \cdots + a_r$.

Put $A := \{k \in C \mid \sigma(k) = L(k)\nu\}$. A is a face[6] of C, because (4.3.10) implies that $\sigma(k) \geq L(k)\nu$ for all $k \in C$. Indeed, this follows from the fact that this is true for all the generators a_i of C. For example, from $\sigma(Ma_1 + a_2 + \cdots + a_r) \geq L(Ma_1 + a_2 + \cdots + a_r)\nu$ with $M \in \mathbb{N} \setminus \{0\}$ and $M \gg 0$, it follows that $\sigma(a_1) \geq L(a_1)\nu$. So, by [Roc70, Theorem 18.3], it follows that A is the origin or the convex cone spanned by the a_i that belong to A. Because $\dim A = e$, there are exactly e of such a_i. Without loss of generality, we may suppose that A is the cone $\{\lambda_1 a_1 + \cdots + \lambda_e a_e \mid \lambda_i \geq 0\}$ (if $e = 0$, then $A = \{0\}$). Put $B = \{\lambda_{e+1} a_{e+1} + \cdots + \lambda_r a_r \mid \lambda_i \geq 0\}$ (if $e = r$, then $B = \{0\}$). From (4.3.12), it follows that $\operatorname{ri} C \cap \mathbb{N}^n = w + (A \cap \mathbb{N}^n) \oplus (B \cap \mathbb{N}^n)$.

[6] A proper face or C itself.

There exist $\epsilon_{e+1}, \ldots, \epsilon_r > 0$ such that $\sigma(a_{e+1}) \geq L(a_{e+1})(\nu + \epsilon_{e+1}), \ldots, \sigma(a_r) \geq L(a_r)(\nu + \epsilon_r)$. Put $\epsilon = \min\{\epsilon_{e+1}, \ldots, \epsilon_r\} > 0$. Then $\sigma(b) \geq L(b)(\nu + \epsilon)$ for every $b \in B$.

Hence,

$$\begin{aligned}
P &= \sum_{\substack{a \in A \cap \mathbb{N}^n \\ b \in B \cap \mathbb{N}^n \\ L(w)+L(a)+L(b)=m}} p^{-\sigma(w)-\sigma(a)-\sigma(b)} \\
&= p^{-\sigma(w)} \sum_{\substack{\alpha,\beta \in \mathbb{N} \\ L(w)+\alpha+\beta=m}} \left(\sum_{\substack{a \in A \cap \mathbb{N}^n \\ L(a)=\alpha}} p^{-\sigma(a)} \right) \left(\sum_{\substack{b \in B \cap \mathbb{N}^n \\ L(b)=\beta}} p^{-\sigma(b)} \right) \\
&\leq p^{-L(w)\nu-\gamma} \times \\
&\qquad \sum_{\substack{\alpha,\beta \in \mathbb{N} \\ L(w)+\alpha+\beta=m}} \left(\sum_{\substack{a \in A \cap \mathbb{N}^n \\ \sigma(a)=\alpha\nu}} p^{-\sigma(a)} \right) \left(\sum_{\substack{b \in B \cap \mathbb{N}^n \\ \sigma(b) \geq \beta(\nu+\epsilon)}} p^{-\sigma(b)} \right) \\
&= p^{-L(w)\nu-\gamma} \times \\
&\qquad \sum_{\substack{\alpha,\beta \in \mathbb{N} \\ L(w)+\alpha+\beta=m}} \left(\sum_{\substack{a \in A \cap \mathbb{N}^n \\ \sigma(a)=\alpha\nu}} p^{-\alpha\nu} \right) \left(\sum_{\substack{j \in \mathbb{N} \\ j \geq \beta(\nu+\epsilon)}} \sum_{\substack{b \in B \cap \mathbb{N}^n \\ \sigma(b)=j}} p^{-j} \right).
\end{aligned}$$

If $e \neq 0$, each $a \in A \cap \mathbb{N}^n$ can be written in a unique way as $\lambda_1 a_1 + \cdots + \lambda_e a_e$, with $\lambda_i \in \mathbb{N}$. Then $\sigma(a) = \lambda_1 \sigma(a_1) + \cdots + \lambda_e \sigma(a_e)$, with $\sigma(a_i) \geq 1$. So, the number of elements $a \in A \cap \mathbb{N}^n$ which satisfy $\sigma(a) = \alpha\nu$, is $\leq (\alpha\nu+1)^{\max\{0,e-1\}}$. Similarly, the number of elements b in the set $B \cap \mathbb{N}^n$ which satisfy $\sigma(b) = j$ is, $\leq (j+1)^{\max\{0,r-e-1\}}$. Therefore,

$$P \leq p^{-L(w)\nu-\gamma} \times$$

$$\sum_{\substack{\alpha,\beta \in \mathbb{N} \\ L(w)+\alpha+\beta=m}} (\alpha\nu+1)^{\max\{0,e-1\}} p^{-\alpha\nu} \sum_{\substack{j \in \mathbb{N} \\ j \geq \beta(\nu+\epsilon)}} (j+1)^{\max\{0,r-e-1\}} p^{-j}.$$

We search for an upperbound for the last sum. For $k, t \in \mathbb{N}$ and $p \geq 2$, we have

$$\sum_{j \geq t} (j+1)^k p^{-j} = p^{-t} \sum_{x \geq 1} (t+x)^k p^{-(x-1)} \leq p^{-t} (t+1)^k \sum_{x \geq 1} x^k 2^{-(x-1)}. \quad (4.3.13)$$

Note that it can happen that $\beta(\nu+\epsilon) \notin \mathbb{N}$. So, by (4.3.13), it follows that

$$P \leq c_1 p^{-L(w)\nu-\gamma} \times$$

$$\sum_{\substack{\alpha,\beta \in \mathbb{N} \\ L(w)+\alpha+\beta=m}} (\alpha\nu+1)^{\max\{0,e-1\}} p^{-\alpha\nu} p^{-\beta(\nu+\epsilon)} (\beta(\nu+\epsilon)+2)^{\max\{0,r-e-1\}},$$

where c_1 is independent of m and p. Because $(\alpha\nu+1) \leq (\alpha+1)(\nu+1)$ and $\beta(\nu+\epsilon)+2 \leq (\beta+1)(\nu+\epsilon+2)$, we obtain

$$\begin{aligned} P &\leq c_2 p^{-m\nu-\gamma} \sum_{\substack{\alpha,\beta \in \mathbb{N} \\ L(w)+\alpha+\beta=m}} (\alpha+1)^{\max\{0,e-1\}} p^{-\beta\epsilon} (\beta+1)^{\max\{0,r-e-1\}} \\ &\leq c_2 p^{-m\nu-\gamma} (m+1)^{\max\{0,e-1\}} \sum_{\beta \geq 0} 2^{-\beta\epsilon} (\beta+1)^{\max\{0,r-e-1\}} \\ &\leq c p^{-m\nu-\gamma} (m+1)^{\max\{0,e-1\}}, \end{aligned}$$

where c_2 and c are independent of m and p. □

Proposition 4.3.10. *Let $f(x_1,\ldots,x_n)$ be a polynomial. Let $k \in \mathbb{N}^n$ and suppose that no vertex of $\tau_0 \cap F(k)$ belongs to $\{0,1\}^n$. Let d be the dimension of the convex hull of the vertices of $\tau_0 \cap F(k)$. (Thus $d \leq \dim(\tau_0 \cap F(k))$.) Then we have*

$$\sigma(k) \geq (m(k)+1)\frac{1}{t_0} - \frac{d+1}{2}. \tag{4.3.14}$$

Recall that $m(\cdot)$, $\sigma(\cdot)$, τ_0 and t_0 are defined as in Definition 1.1.1, 2.3.1 and 3.2.2.

Remark. The proof of this proposition also shows that, if there is no condition on the vertices of $\tau_0 \cap F(k)$, then the denominator 2 in $(d+1)/2$ should be replaced by 1. If no vertex of $\tau_0 \cap F(k)$ belongs to $\{0,1,2\}^n$, then the denominator should be replaced by 3.

Proof. Denote the i-th unit vector in $\mathbb{R}^n$ by e_i. Let τ be a face of $\Gamma(f)$, $v_1,\ldots,v_a$ the vertices of τ and let $e_{j_1},\ldots,e_{j_b}$ be the directions of recession of τ from the set $\{e_1,\ldots,e_n\}$. Then by [Roc70, Theorem 18.3], we know that

$$\tau = \left\{ \sum_{i=1}^{a} \lambda_i v_i + \sum_{i=1}^{b} \mu_{j_i} e_{j_i} (*) \mid \lambda_i, \mu_{j_i} \geq 0, \sum_{i=1}^{a} \lambda_i = 1 \right\}.$$

If $b \neq 0$, $(*)$ can also be written as

$$\lambda_1 v_1 + \cdots + \lambda_{a-1} v_{a-1} + (\lambda_a - \frac{\sum_{i=1}^{b} u_{j_i}}{L}) v_a + \frac{\mu_{j_1}}{L}(v_a + L e_{j_1}) + \cdots + \frac{\mu_{j_b}}{L}(v_a + L e_{j_b}), \tag{4.3.15}$$

where $L > 0$ is an arbitrary integer. So, each face of $\Gamma(f)$ is the convex hull of its integral points.

Let S be the intersection point $(t_0, \ldots, t_0)$ of the diagonal $D = \{(t, \ldots, t) \in \mathbb{R}^n\}$ with τ_0. Because S belongs to the convex hull of the integral points of τ_0, S belongs to the convex hull of some points $S_1, P_1, \ldots, P_r$, with

$$\begin{array}{l} S_1 \in \tau_0 \cap F(k), \\ P_1, \ldots, P_r \text{ integral points of } \tau_0 \setminus F(k). \end{array}$$

Remark: if $\tau_0 \cap F(k) = \emptyset$, then S belongs to the convex hull of $P_1, \ldots, P_r$ and $d = -1$. If $\tau_0 \subset F(k)$, then $S = S_1$. One can treat these cases in a similar way.

Choose $M > 0$ arbitrary large. Then S_1 belongs to the convex hull of the vertices of $\tau_0 \cap F(k)$ and (if $\tau_0 \cap F(k)$ is not compact) integral points of $\tau_0 \cap F(k)$ whose coordinates are not all smaller than M (see (4.3.15) with $L = M$). Hence, because d is the dimension of the convex hull of the vertices of $\tau_0 \cap F(k)$, S_1 belongs to the convex hull of some points $Q_1, \ldots, Q_{d+1}$ and $R_1, \ldots, R_s$ with

$$\begin{array}{ll} Q_1, \ldots, Q_{d+1} & \text{vertices of } \tau_0 \cap F(k), \text{ and} \\ R_1, \ldots, R_s & \text{integral points of } \tau_0 \cap F(k),\ 0 \leq s \leq n, \text{ and} \\ & \text{each } R_i \text{ has some coordinate} \geq M. \end{array}$$

Thus S belongs to the convex hull of $P_1, \ldots, P_r, Q_1, \ldots, Q_{d+1}, R_1, \ldots, R_s$. Therefore,

$$(1, 1, \ldots, 1) = \frac{1}{t_0} S = \sum_{i=1}^{r} \alpha_i P_i + \sum_{i=1}^{d+1} \beta_i Q_i + \sum_{i=1}^{s} \gamma_i R_i, \tag{4.3.16}$$

for some $\alpha_i, \beta_i, \gamma_i \geq 0$ in $\mathbb{R}$, and

$$\frac{1}{t_0} = \sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{d+1} \beta_i + \sum_{i=1}^{s} \gamma_i. \tag{4.3.17}$$

Some coordinate of each R_i is $\geq M$ and hence, (4.3.16) implies that $1 \geq \gamma_i M$ and

$$\gamma_i \leq \frac{1}{M}. \tag{4.3.18}$$

Because no vertex of $\tau_0 \cap F(k)$ belongs to $\{0, 1\}^n$, some coordinate of each Q_i is ≥ 2 and hence, (4.3.16) implies that $1 \geq \beta_i 2$ and

$$\beta_i \leq \frac{1}{2}. \tag{4.3.19}$$

We have, by (4.3.16), that

$$\begin{aligned}\sigma(k) = k \cdot (1, \ldots, 1) &= k \cdot \left(\sum_{i=1}^{r} \alpha_i P_i + \sum_{i=1}^{d+1} \beta_i Q_i + \sum_{i=1}^{s} \gamma_i R_i \right) \\ &= \sum_{i=1}^{r} \alpha_i k \cdot P_i + \sum_{i=1}^{d+1} \beta_i k \cdot Q_i + \sum_{i=1}^{s} \gamma_i k \cdot R_i. \end{aligned} \tag{4.3.20}$$

But $k \cdot P_i \geq m(k) + 1$, since $P_i \notin F(k)$ and $k \cdot P_i$, $m(k) \in \mathbb{N}$. Moreover, $k \cdot Q_i = k \cdot R_i = m(k)$, since $Q_i, R_i \in F(k)$. Thus (4.3.20) yields

$$\begin{aligned}\sigma(k) &\geq \sum_{i=1}^{r} \alpha_i (m(k)+1) + \sum_{i=1}^{d+1} \beta_i m(k) + \sum_{i=1}^{s} \gamma_i m(k) \\ &= \left(\sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{d+1} \beta_i + \sum_{i=1}^{s} \gamma_i \right) (m(k)+1) - \sum_{i=1}^{d+1} \beta_i - \sum_{i=1}^{s} \gamma_i \\ &= \frac{1}{t_0}(m(k)+1) - \sum_{i=1}^{d+1} \beta_i - \sum_{i=1}^{s} \gamma_i. \end{aligned} \tag{4.3.21}$$

Thus by (4.3.18), (4.3.19) and (4.3.21), it follows that

$$\sigma(k) \geq (m(k)+1)\frac{1}{t_0} - \frac{d+1}{2} - \frac{s}{M}.$$

Since $s \leq n$ and taking the limit for $M \to \infty$, we obtain

$$\sigma(k) \geq (m(k)+1)\frac{1}{t_0} - \frac{d+1}{2}.$$

□

4.3.11. Proof of Theorem 4.3.3.
We use the formula for S_m from Proposition 4.3.7. Indeed, this is possible (see Remark 1 after Definition 4.3.2). Let τ be a face of $\Gamma(f)$. We now give upperbounds for

$$|E(f_\tau)|, \quad \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = m-1}} p^{-\sigma(k)} \text{ and } (1-p^{-1})^n \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) \geq m}} p^{-\sigma(k)}.$$

Fix a vertex x_τ of τ. For $k \in \Delta_\tau$, one has $m(k) = k \cdot x_\tau$, because $F(k) = \tau$. The hypothesis of Lemma 4.3.9 for $C = \bar{\Delta}_\tau$, $\gamma = 0$, $\nu = 1/t_0 = -\rho$ and $L(k) = k \cdot x_\tau = m(k)$ is satisfied. Indeed, we know by Lemma 3.2.3 that $\sigma(k) \geq m(k)(1/t_0)$ for every $k \in (\mathbb{R}^+)^n$ and thus certainly for every $k \in \mathbb{N}^n \cap \Delta_\tau$. Furthermore,

$\dim\{k \in \bar{\Delta}_\tau \mid \sigma(k) = m(k)(1/t_0)\} = \dim(\bar{\Delta}_\tau \cap \bar{\Delta}_{\tau_0}) \leq \dim \bar{\Delta}_{\tau_0} = \kappa$. It follows then that

$$\begin{aligned}(1-p^{-1})^n \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) \geq m}} p^{-\sigma(k)} &\leq c_1 \sum_{l=m}^{\infty} p^{-l/t_0}(l+1)^{\kappa-1} \\ &\leq c_2 p^{m\rho} m^{\kappa-1}, \end{aligned} \tag{4.3.22}$$

where c_1 and c_2 do not depend on p and m. We obtain the last inequality (4.3.22) by a similar argument as in (4.3.13) of the proof of Lemma 4.3.9.

Denote the dimension of the Newton polyhedron $\Gamma_\infty(f_\tau)$ of f_τ at infinity by d_τ. By Lemma 4.1.3, we can change variables $f_\tau(x) = h(y_1, \ldots, y_{d_\tau})$ such that $h \in \mathbb{Z}_p[y_1, \ldots, y_{d_\tau}, (y_1 \cdots y_{d_\tau})^{-1}]$, h is non degenerated over $\bar{\mathbb{F}}_p$ with respect to its Newton polyhedron at infinity and $\dim \Gamma_\infty(h) = d_\tau$. Therefore, it follows from Theorem 4.1.8 that for $p \gg 0$ (or from [DL91] that for every p)

$$\begin{aligned}|E(f_\tau)| &:= \left| \sum_{a \in (\mathbb{F}_p^\times)^n} \exp\left(\frac{2\pi i}{p} f_\tau(a)\right) \right| \\ &= (p-1)^{n-d_\tau} \left| \sum_{y \in (\mathbb{F}_p^\times)^{d_\tau}} \exp\left(\frac{2\pi i}{p} h(y)\right) \right| \\ &\leq c_3 p^{n-d_\tau} \sqrt{p}^{d_\tau} \\ &= c_3 p^{n-\frac{d_\tau}{2}}, \end{aligned} \tag{4.3.23}$$

where c_3 does not depend on p (and of course not on m).

Note that $d_\tau = \dim \Gamma_\infty(f_\tau) = \dim c(\tau)$ (where $c(\tau)$ is defined as in Definition 4.1.6) if and only if $\{0\} \cup \operatorname{supp}(f_\tau)$ is contained in the affine hull of the vertices of τ. Otherwise, $\dim \Gamma_\infty(f_\tau) \geq \dim c(\tau) + 1$.

Case 1. $d_\tau = \dim \Gamma_\infty(f_\tau) \geq \dim c(\tau) + 1$.
Then (4.3.23) implies that

$$|E(f_\tau)| \leq c_3 p^{n-(\dim c(\tau)+1)/2}. \tag{4.3.24}$$

Moreover, the hypothesis of Lemma 4.3.9 is satisfied for $C = \bar{\Delta}_\tau$, $\nu = 1/t_0 = -\rho$, $L(k) = k \cdot x_\tau = m(k)$ and for $\gamma = \max\{0, 1/t_0 - (\dim c(\tau)+1)/2\}$, because of Proposition 4.3.10. Hence,

$$\begin{aligned}\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = m-1}} p^{-\sigma(k)} &\leq c_4 p^{-(m-1)/t_0 - 1/t_0 + (\dim c(\tau)+1)/2} m^{\kappa-1} \\ &= c_4 p^{m\rho + (\dim c(\tau)+1)/2} m^{\kappa-1}, \end{aligned} \tag{4.3.25}$$

where c_4 does not depend on m and p. From (4.3.22), (4.3.24), (4.3.25), it follows now that

$$\left| p^{-n} E(f_\tau) \left(\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = m-1}} p^{-\sigma(k)} \right) + (1-p^{-1})^n \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) \geq m}} p^{-\sigma(k)} \right| \leq (c_3 c_4 + c_2) p^{m\rho} m^{\kappa-1}. \tag{4.3.26}$$

Case 2. $d_\tau = \dim \Gamma_\infty(f_\tau) = \dim c(\tau)$.
Because the origin is contained in the affine hull of the vertices of τ, one proves subsequently that

(1) τ is not compact;

(2) τ is the intersection of some coordinate hyperplanes with $\Gamma(f)$ or is equal to $\Gamma(f)$ itself;

(3) $m(k) = 0$ for every $k \in \Delta_\tau$.

So, τ will only give a contribution to S_m for $m = 1$. Suppose that $m = 1$.

Case 2a. $\tau \neq \Gamma(f)$.
Proposition 4.3.10, (3) and the fact that the origin is not contained in the affine hull of the vertices of τ_0 imply that $\sigma(k) \geq 1/t_0 - (\dim c(\tau_0 \cap \tau) + 1)/2 \geq 1/t_0 - \dim c(\tau)/2$ for every $k \in \Delta_\tau$. From Lemma 4.3.9, it follows then that

$$\left(\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = 0}} p^{-\sigma(k)} \right) \leq c_5 p^{-1/t_0 + \dim c(\tau)/2}, \tag{4.3.27}$$

where c_5 is a constant independent of p.

Then (4.3.23) and (4.3.27) imply that

$$\left| p^{-n} E(f_\tau) \left(\sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k)} \right) \right| \leq c_3 c_5 p^\rho. \tag{4.3.28}$$

Case 2b. $\tau = \Gamma(f)$.
From (4.3.23) and the fact that $\Delta_{\Gamma(f)} = \{0\}$, it follows that

$$\begin{aligned} \left| p^{-n} E(f_\tau) \left(\sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k)} \right) \right| &\leq c_3 p^{-\dim c(\tau)/2}.1 \\ &\leq c_3 p^\rho. \end{aligned} \tag{4.3.29}$$

Indeed, by Proposition 4.3.10 and the fact that the origin is not contained in the affine hull of the vertices of τ_0, we know that $\rho = -1/t_0 \geq -(\dim c(\tau_0)+1)/2 \geq -\dim c(\tau)/2$.

We remark that the constants c_1, c_2, c_3, c_4, c_5 only depend on τ. Because there are only a finite number of faces, Theorem 4.3.3 follows now from (4.3.26), (4.3.28), (4.3.29) and Proposition 4.3.7. □

4.3.12. Proof of Theorem 4.3.4.
Similar to the proof above, but now use the formula for $S_{0,m}$ given in the remark after Proposition 4.3.7. Note that here Case 2 does not occur. Moreover, $\tau = c(\tau)$ and $\dim \Gamma_\infty(f_\tau) = \dim c(\tau) + 1$ for every compact face τ of $\Gamma(f)$. □

4.3.13. Proof of Theorem 4.3.5.
We can use the formula for S_m of Proposition 4.3.7. Indeed, this is possible (see Remark 1 after Definition 4.3.2). We may suppose that $m > 1$. Then only the faces τ with $\dim \Gamma_\infty(f_\tau) \geq \dim c(\tau) + 1$ give a contribution to S_m.

By Proposition 1.1.10 and Lemma 3.2.3, we know that the cone Δ_{τ_0} has dimension κ and is contained in the set $\{k \in (\mathbb{R}^+)^n | m(k)(1/t_0) = \sigma(k)\}$. Hence,

$$\begin{aligned}(1-p^{-1})^n \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_{\tau_0} \\ m(k)=m}} p^{-\sigma(k)} & \\ = (1-p^{-1})^n p^{-m/t_0} \#\{k \in \mathbb{N}^n \cap \Delta_{\tau_0} | \sigma(k) = m/t_0\} & \\ \geq c_{\tau_0} p^{m\rho} m^{\kappa-1} \qquad \text{for infinitely many } m, & \qquad (4.3.30)\end{aligned}$$

where $c_{\tau_0} > 0$ is independent of p and m. In fact, c_{τ_0} only depends on τ_0. Indeed, suppose that Δ_{τ_0} is strictly positively spanned by non-zero vectors $a_1, \ldots, a_r$ from $\mathbb{N}^n$. Then the dimension of the vector space generated by $a_1, \ldots, a_r$ is κ. So, Δ_{τ_0} contains a cone Δ_i that is strictly positively spanned by κ linearly independent vectors from the set $\{a_1, \ldots, a_r\}$, say $a_1, \ldots, a_\kappa$. Then

$$\begin{aligned}&\#\{k \in \mathbb{N}^n \cap \Delta_{\tau_0} | \sigma(k) = m/t_0\} \\ &\quad \geq \#\{(\lambda_1, \ldots, \lambda_\kappa) \in (\mathbb{N} \setminus \{0\})^\kappa | \lambda_1 \sigma(a_1) + \cdots + \lambda_\kappa \sigma(a_\kappa) = m/t_0\}.\end{aligned}$$

Take for m a multiple of $\sigma(a_1), \ldots, \sigma(a_\kappa)$ and the denominator of $1/t_0$. Search then for solutions λ_i which are a multiple of the numerator of $1/t_0$ and the $\sigma(a_j)$ with $j \neq i$. Then the assertion is clear, since

$$\#\{(k_1, \ldots, k_\kappa) \in (\mathbb{N} \setminus \{0\})^\kappa | k_1 + \cdots + k_\kappa = t\} \geq \tilde{c} t^{\kappa-1} \qquad \text{for } t \gg 0,$$

where $\tilde{c} > 0$ is independent of t. On the other hand, using the remark after Proposition 4.3.10, the estimate (4.3.25) can be replaced by

$$\begin{aligned}\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k)=m-1}} p^{-\sigma(k)} &\leq c'' p^{-(m-1)/t_0 - 1/t_0 + (\dim c(\tau)+1)/3} m^{\kappa-1} \\ &= c'' p^{m\rho + (\dim c(\tau)+1)/3} m^{\kappa-1}, \qquad (4.3.31)\end{aligned}$$

where c'' is independent of m and p. Together with (4.3.24), this yields

$$|p^{-n}E(f_\tau) \sum_{\substack{k \in \mathbb{N}^n \cap \Delta_\tau \\ m(k) = m-1}} p^{-\sigma(k)}| \leq c'' c_3 p^{m\rho} p^{-1/6} m^{\kappa-1}. \tag{4.3.32}$$

We also remark that the constants c'', c_3 only depend on τ. Choose now an M_Γ such that

$$c_{\tau_0} > p^{-1/6} \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \dim \Gamma_\infty(f_\tau) \geq \dim c(\tau)+1}} c''_\tau c_{3,\tau} \qquad \text{for } p \geq M_\Gamma$$

and take $0 < c_\Gamma < c_{\tau_0} - \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \dim \Gamma_\infty(f_\tau) \geq \dim c(\tau)+1}} c''_\tau c_{3,\tau} M_\Gamma^{-1/6}$.

Theorem 4.3.5 follows now directly from Proposition 4.3.7, (4.3.30) and (4.3.32). □

4.3.14. Proof of Theorem 4.3.6.
The proof is similar as above. Now use the formula for $S_{0,m}$. Note that $\sum_{\substack{k \in \mathbb{N}^n \cap \Delta_{\tau_0} \\ m(k) = m}} p^{-\sigma(k)}$ occurs in this formula, because τ_0 is compact. □

Note that the statement (4.3.4) in Theorem 4.3.6 can be replaced by

$$\text{“} \left| \sum_u S_{0,m}(u) \right| \geq c_\Gamma p^{\rho m} m^{\kappa-1}, \tag{4.3.33}$$

for infinitely many $m \in \mathbb{N} \setminus \{0\}$ and where u runs through the elements of a finite subset of $\mathbb{Z}_p^\times$”.

4.3.3 An alternative proof of Theorem 4.1.12

Proposition 4.3.15. *Let $f(x) = f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$ and let $p \gg 0$. Then for m big enough and $u \in \mathbb{Z}_p^\times$, the exponential sum $S_{0,m}(u)$ is a finite $\mathbb{C}$-linear combination of functions of the form*

$$\chi(u) p^{m\lambda} m^\beta$$

with coefficients independent of m and u; $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ a character of $\mathbb{Z}_p^\times$; $\lambda \in \mathbb{C}$ a pole of $(p^{s+1}-1)Z_{f,0}(s, \chi_{triv})$ if $\chi = \chi_{triv}$ or of $Z_{f,0}(s,\chi)$ if $\chi \neq \chi_{triv}$ and with $\beta \in \mathbb{N}$, $\beta \leq$ (order of the pole λ) $- 1$. Moreover, all poles λ appear effectively in this linear combination (with maximal β).

Proof. This is a special case of [Den91, Corollary 1.4.5], [Igu74b] or [Igu74a]. Indeed, one can prove that the condition $\{x \in (p\mathbb{Z}_p)^n \mid (\partial f/\partial x_i)(x) = 0, i = 1, \ldots, n\} \subset f^{-1}(0)$ is satisfied for $p \gg 0$. □

We repeat here the statement of Theorem 4.1.12 that we shall now prove by using exponential sums.

Theorem 4.3.16. *Let $f(x_1, \ldots, x_n)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the compact faces of $\Gamma(f)$. Suppose that τ_0 is compact, no vertex of τ_0 belongs to $\{0,1,2\}^n$ and $t_0 < 1$. Then ρ is a pole of $Z_{f,0}(s)$ of order κ if $p \gg 0$.*

Proof. Fix $p \gg 0$. By Proposition 4.3.15, we can write $S_{0,m}(u)$ for m big enough and $u \in \mathbb{Z}_p^\times$ as a finite $\mathbb{C}$-linear combination of functions of the form $\chi(u)p^{m\lambda}m^\beta$ with coefficients independent of m and u. Take r equal to the maximum of the conductors of the characters that occur in this linear combination. Then $\sum_{u \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \chi(u)$ is well-defined for every occurring character χ and vanishes there except for the trivial character. Therefore, one has for m big enough that

$$\begin{aligned} \sum_{u \in (\mathbb{Z}/p^r\mathbb{Z})^\times} S_{0,m}(u) &= \#(\mathbb{Z}/p^r\mathbb{Z})^\times \sum_j c_{\lambda_j,\beta_j} p^{m\lambda_j} m^{\beta_j} \\ &= \#(\mathbb{Z}/p^r\mathbb{Z})^\times p^{m\rho} m^{\kappa-1} \sum_j c_{\lambda_j,\beta_j} \frac{p^{m\lambda_j} m^{\beta_j}}{p^{m\rho} m^{\kappa-1}}, \end{aligned} \tag{4.3.34}$$

where the sum is finite, $\lambda_j \in \mathbb{C}$ is a pole of $(p^{s+1} - 1)Z_{f,0}(s, \chi_{\text{triv}})$, β_j satisfies $\beta_j \leq$ (order of the pole λ_j) $- 1$ and the $c_{\lambda_j,\beta_j} \in \mathbb{C}$ are independent of m. By Remark (4.3.33) after the proof of Theorem 4.3.6, we know that there exists a $c_{\Gamma(f)} > 0$ such that for infinitely many $m \in \mathbb{N} \setminus \{0\}$

$$\left| \sum_{u \in (\mathbb{Z}/p^r\mathbb{Z})^\times} S_{0,m}(u) \right| \geq c_{\Gamma(f)} p^{\rho m} p^{\kappa-1}. \tag{4.3.35}$$

It suffices to show that the coefficient $c_{\rho,\kappa-1}$ in (4.3.34) is different from 0. Because ρ is the largest candidate pole of $Z_{f,0}(s)$ for which c_{λ_j,β_j} occurs and the order of ρ is $\leq \kappa$, it is clear that for $(\lambda_j, \beta_j) \neq (\rho, \kappa - 1)$

$$\lim_{m \to \infty} \frac{p^{m\lambda_j} m^{\beta_j}}{p^{m\rho} m^{\kappa-1}} = 0.$$

So, if $c_{\rho,\kappa-1} = 0$, then for m big enough, one has

$$\left| \sum_{u \in (\mathbb{Z}/p^r\mathbb{Z})^\times} S_{0,m}(u) \right| < c_{\Gamma(f)} p^{\rho m} m^{\kappa-1},$$

which is in contradiction with (4.3.35). □

Chapter 5

A sufficient condition to have that the actual order of a pole of $Z_f(s,\chi)$ or $Z_{f,0}(s,\chi)$ is smaller than expected

The main purpose of Chapter 5 (see also [Hoo02]) is to prove one of the conjectures of Denef and Sargos [DS92, Section 6.2] for non-degenerated polynomials in the p-adic case:

> *if τ_0 is an unstable face of $\Gamma(f)$ over $\mathbb{F}_p$ with respect to f, and if $-1/t_0 \neq -1$ is a pole of $Z_f(s,\chi)$, then its order will be smaller than expected, i.e., $< \kappa$.*

The notion "unstable" (see Definition 5.3.1) was first established by Denef and Sargos in [DS92]. The analogue of the above conjecture for the real case has been proved by Denef, Laeremans and Sargos [DLS97].

Inspired by the proof for the real case, we tried to reduce our question about the polynomial f to the same question about the polynomial f_{τ_0}. Namely Theorem 5.1.3, our fourth main result, states that $-1/t_0$ is an actual pole of $Z_f(s,\chi)$ of the expected order if and only if the same holds for $Z_{f_{\tau_0}}(s,\chi)$. In particular, if $-1/t_0$ is not a pole of $Z_{f_{\tau_0}}(s,\chi)$ of the expected order, the same holds for $Z_f(s,\chi)$. Theorem 5.2.1 extends this result to arbitrary poles. In Theorem 5.3.3, we will give a formula for $Z_f(s,\chi)$ that holds if $f = f_\sigma$, where $\sigma \subset \Gamma(f)$ is an

unstable convex set over $\mathbb{F}_p$ with respect to f (see Definition 5.3.1), and if f is non-degenerated over $\mathbb{F}_p$ with respect to a particular set of faces of $\Gamma(f)$. The formula in this theorem, which is our fifth main result, is similar to the one in Theorem 2.3.2, but can provide a smaller set of candidate poles for $Z_f(s,\chi)$ or smaller upperbounds for their orders (see Example 5.4.1). Finally, Theorem 5.3.6 uses this formula together with Theorem 5.1.3 to obtain a proof of the mentioned conjecture of Denef and Sargos. Theorem 5.3.8 gives an analogous result for arbitrary poles.

5.1 Checking if $\rho = -1/t_0$ is a pole of the expected order: reduction from f to f_{τ_0}

Definition 5.1.1. Let $V = \{v_1, \ldots, v_r\}$ be a finite subset of $\mathbb{Z}^n$ such that its elements are linearly independent over $\mathbb{Q}$. Define

$$H_V = H_{\{v_1,\ldots,v_r\}} = \mathbb{Z}^n \cap \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid 0 \leq \lambda_i < 1 \text{ for } i = 1, \ldots, r \right\}. \qquad (5.1.1)$$

Note that all λ_i belong to $\mathbb{Q}$.

Recall that we can choose a unique representative in $H_{\{v_1,\ldots,v_r\}}$ for a coset of the lattice $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r$ in the group of the points with integral coordinates of the vector space generated by $v_1, \ldots, v_r$. So, the number of elements in $H_{\{v_1,\ldots,v_r\}}$ equals the multiplicity of $v_1, \ldots, v_r$.

In the formula for Igusa's local zeta function from Theorem 2.3.2, we have to calculate sums of the form $\sum_h p^{\sigma(h)+m(h)s}$, where h runs through the elements of a set of the form $H_{\{v_1,\ldots,v_r\}}$. The next lemma gives a relation between those sets if some of the vectors from $\{v_1, \ldots, v_r\}$ are the same and the others generate the same vector space. We use this relation in the proof of Theorem 5.1.3 which shows that $\rho \neq -1$ is a pole of $Z_f(s,\chi)$ of the expected order (see Proposition 3.2.5) if and only if the same holds for $Z_{f_{\tau_0}}(s,\chi)$. This is very surprising because the cones Δ_τ, and hence the S_{Δ_τ} for $\tau \subset \tau_0$ in the formulas for $Z_f(s,\chi)$ and $Z_{f_{\tau_0}}(s,\chi)$ from Theorem 2.3.2, can be different. For example, let $f(x,y,z) = xy + xz^2 + yz^2 + z^5 + y^7$. Then τ_0 is the convex hull of $(1,1,0)$, $(1,0,2)$ and $(0,1,2)$. So, $f_{\tau_0} = xy + xz^2 + yz^2$ and

$$\begin{aligned} \Delta_{(0,1,2)}(f) &= \text{cone}\,((1,0,0),(2,2,1),(3,3,1),(6,1,3)), \\ \Delta_{(0,1,2)}(f_{\tau_0}) &= \text{cone}\,((1,0,0),(2,2,1),(1,1,0),(2,0,1)). \end{aligned}$$

See also Figure 5.1.1 and Figure 5.4.2. In the case that the codimension of τ_0 is 1, one can easily prove Theorem 5.1.3. The real difficulties only occur when its codimension is larger than 1.

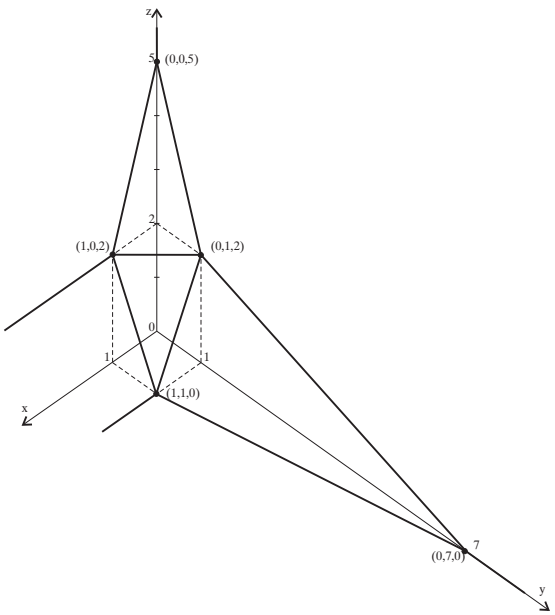


Figure 5.1.1. Newton polyhedron of $xy + xz^2 + yz^2 + z^5 + y^7$

Lemma 5.1.2. *Let A, A' and B be finite subsets of $\mathbb{Z}^n$ such that each of the following three sets A, A' and $A \cup B$ consists of linearly independent elements over $\mathbb{Q}$ and such that A and A' generate the same vector space. Let π be the natural projection of* $\operatorname{vct} A + \operatorname{vct} B$ *onto* $\operatorname{vct} B$ *(here "vct" denotes "the vector space generated by").* *Then we have*

(i) $\#\left(\pi^{-1}(\{b\}) \cap H_{A\cup B}\right) = \#H_A$ *for any* $b \in \pi(H_{A\cup B})$, *and*

(ii) $\pi(H_{A\cup B}) = \pi(H_{A'\cup B})$.

Proof. Let $A = \{a_1, \ldots, a_u\}$, $A' = \{a'_1, \ldots, a'_u\}$ and $B = \{b_1, \ldots, b_v\}$.

(i) Suppose that $h = \sum_{i=1}^{u} \alpha_i a_i + b \in H_{A\cup B}$ with $b = \sum_{j=1}^{v} \beta_j b_j$. Suppose that $g = \sum_{i=1}^{u} \gamma_i a_i + b \in \left(\pi^{-1}(\{b\}) \cap H_{A\cup B}\right)$. Then $g - h \in \mathbb{Z}^n \cap \operatorname{vct}\{a_1, \ldots, a_u\}$. So, there exists a unique $k \in H_A$ and unique elements $z_1, \ldots, z_u \in \mathbb{Z}$ such that $g = h + k + \sum_{i=1}^{u} z_i a_i$ because $a_1, \ldots, a_u \in \mathbb{Z}^n$ are linearly independent. Conversely, if $k \in H_A$, there exists a unique $g \in \left(\pi^{-1}(\{b\}) \cap H_{A\cup B}\right)$ and unique elements $z_1, \ldots, z_u \in \mathbb{Z}$ such that $g = h + k + \sum_{i=1}^{u} z_i a_i$. This shows that $\#\left(\pi^{-1}(\{b\}) \cap H_{A\cup B}\right) = \#H_A$.

(ii) By the symmetry of the problem, it suffices to prove that $\pi(H_{A\cup B}) \subset \pi(H_{A'\cup B})$. Suppose that $h = \sum_{i=1}^{u} \alpha_i a_i + b \in H_{A\cup B}$ with $b = \sum_{j=1}^{v} \beta_j b_j$.

We now show that there also exists an element $h' = \sum_{i=1}^{u} \alpha_i' a_i' + b \in H_{A' \cup B}$, such that $\pi(h') = b = \pi(h)$. Because $a_1', \ldots, a_u'$ generate the same vector space over $\mathbb{Q}$ as $a_1, \ldots, a_u$, there exist $\gamma_1, \ldots, \gamma_u \in \mathbb{Q}$ such that $\sum_{i=1}^{u} \alpha_i a_i = \sum_{i=1}^{u} \gamma_i a_i'$. For each i, there exists a unique $0 \leq \alpha_i' < 1$ and a $z_i \in \mathbb{Z}$ such that $\gamma_i = \alpha_i' + z_i$. Because the elements $a_1', \ldots, a_u'$ belong to $\mathbb{Z}^n$, it is clear that $h' = \sum_{i=1}^{u} \alpha_i' a_i' + b = h - \sum_{i=1}^{u} z_i a_i'$ is an element of $\mathbb{Z}^n$ and hence, also of $H_{A' \cup B}$.

□

Theorem 5.1.3. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Let τ_0, ρ and κ be defined as in Definition 3.2.2. Then*

$$\lim_{s \to \rho} (p^{s-\rho} - 1)^k Z_f(s,\chi) = c \lim_{s \to \rho} (p^{s-\rho} - 1)^k Z_{f_{\tau_0}}(s,\chi),$$

where

$$k = \begin{cases} \kappa + 1 & \text{if } \rho = -1 \text{ and } \chi = \chi_{\text{triv}}\,, \\ \kappa & \text{otherwise} \end{cases}$$

and $c > 0$ is a constant real number, independent of p. In fact,

$$c = \frac{\text{Vol}\,(\bar{\Delta}_{\tau_0}(f) \cap H_1^-)}{\text{Vol}\,(\bar{\Delta}_{\tau_0}(f_{\tau_0}) \cap H_1^-)},$$

where the volume is the one on the vector space generated by $\bar{\Delta}_{\tau_0}$, induced by the standard metric on $\mathbb{R}^n$, and H_1^- is the half-space $\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1\}$.

Remarks.

1. If it is important to know whether we examine the cone Δ_τ associated to the polynomial f or to the polynomial f_{τ_0}, we will indicate this by $\Delta_\tau(f)$ and $\Delta_\tau(f_{\tau_0})$, respectively.

2. The proof below also shows that a similar result holds for $Z_{f,0}(s,\chi)$, even if τ_0 is not compact. Then f has to be non-degenerated over $\mathbb{F}_p$ with respect to all the *compact* faces of its Newton polyhedron.

3. If $\chi = \chi_{\text{triv}}\,$, $\rho = -1$ and the expected order of the candidate pole ρ for $Z_f(s)$ equals κ, we know by Theorem 3.4.1(iii) that ρ is an actual pole of $Z_f(s)$ of order κ. So, Theorem 5.1.3 will be useful for the other cases.

Proof. Note that also f_{τ_0} is non-degenerated over the finite field $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f_{\tau_0})$. Indeed, suppose that τ is a face of $\Gamma(f_{\tau_0})$. Then $(f_{\tau_0})_\tau = f_{\tau_0 \cap \tau}$ and the face $\tau_0 \cap \tau$ of τ_0 is also a face of the Newton polyhedron $\Gamma(f)$, by [Roc70, p. 165]. So, we can use the formula for $Z_f(s,\chi)$ and $Z_{f_{\tau_0}}(s,\chi)$ from Theorem 2.3.2.

We know from Lemma 3.2.4 that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} = 0$ for faces $\tau \not\subset \tau_0$ and that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau} > 0$ for faces $\tau \subset \tau_0$. Moreover, $L_\tau(f) = L_\tau(f_{\tau_0})$ for faces $\tau \subset \tau_0$. So, it suffices to prove that

$$\frac{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau(f)}}{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f)}} = \frac{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau(f_{\tau_0})}}{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f_{\tau_0})}} \tag{5.1.2}$$

for faces $\tau \subset \tau_0$, and that

$$\frac{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f)}}{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f_{\tau_0})}} = \frac{\mathrm{Vol}\,(\bar{\Delta}_{\tau_0}(f)\cap H_1^-)}{\mathrm{Vol}\,(\bar{\Delta}_{\tau_0}(f_{\tau_0})\cap H_1^-)}. \tag{5.1.3}$$

The last equality follows by Lemma 3.2.4(iii) because $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}}$ equals $\kappa!|\rho|^\kappa$ times the volume of $\bar{\Delta}_{\tau_0}\cap H_1^-$ with respect to the volume form $\omega_{\bar{\Delta}_{\tau_0}}$, where $\omega_{\bar{\Delta}_{\tau_0}}$ is the volume form on $\mathrm{vct}\,\bar{\Delta}_{\tau_0}$, normalized such that the parallelopiped spanned by a lattice basis of $\mathbb{Z}^n\cap \mathrm{vct}\,\bar{\Delta}_{\tau_0}$ has volume 1. Because $\bar{\Delta}_{\tau_0}(f)$ and $\bar{\Delta}_{\tau_0}(f_{\tau_0})$ generate the same vector space, it is clear that we can also take the quotient of the volumes induced by the standard metric on $\mathbb{R}^n$.

Suppose that $\cup_i\Delta_i$ is a finite partition of Δ_τ into rational simplicial cones. Then $S_{\Delta_\tau} = \sum_i S_{\Delta_i}$. From the formula for S_{Δ_i} in Theorem 2.3.2, Lemma 3.2.3 and the fact that $\dim \bar{\Delta}_{\tau_0} = \kappa$, it follows that $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_i} > 0$ if and only if (exactly) κ generators of Δ_i belong to $\bar{\Delta}_{\tau_0}$; otherwise, this limit is zero. So, to prove equality (5.1.2), it is only important to know which cones in the partition have κ generators in $\bar{\Delta}_{\tau_0}$. We will choose partitions of $\Delta_\tau(f)$ and $\Delta_\tau(f_{\tau_0})$, respectively that are closely related to some chosen partitions of $\Delta_{\tau_0}(f)$ and $\Delta_{\tau_0}(f_{\tau_0})$, respectively.

Fix a finite partition of $\Delta_{\tau_0}(f)$ into rational simplicial cones and let

$$\mathrm{cone}\,A_i,\ \text{for } i = 1,\dots,\ell,$$

be the only cones of dimension κ (i.e., with κ generators in $\bar{\Delta}_{\tau_0}(f)$) in this partition. Here, each A_i is a set of κ vectors in $\mathbb{N}^n$ that are linearly independent over $\mathbb{Q}$. Similarly, fix a finite partition of $\Delta_{\tau_0}(f_{\tau_0})$ into rational simplicial cones, and let

$$\mathrm{cone}\,A'_j,\ \text{for } j = 1,\dots,m,$$

be the only cones of dimension κ in this partition and each A'_j a set of κ vectors in $\mathbb{N}^n$ that are linearly independent over $\mathbb{Q}$. Note that the sets A_i and the sets A'_j all generate the same vector space.

It is clear that $\Delta_\tau(f)\subset\Delta_\tau(f_{\tau_0})$ for every face $\tau\subset\tau_0$. Indeed, if $H_a = \{x\in\mathbb{R}^n \mid a\cdot x = b\}$ is a supporting hyperplane of $\Gamma(f)$ such that $H_a\cap\Gamma(f) = \tau$, then the same property holds if we replace $\Gamma(f)$ by $\Gamma(f_{\tau_0})$. Fix now for each face $\tau\subsetneq\tau_0$ a primitive vector

$$b_\tau\in\Delta_\tau(f)\subset\Delta_\tau(f_{\tau_0}). \tag{5.1.4}$$

From Lemma 1.1.7 and Lemma 5.1.5 below, it follows by induction on d that the following assertion is true for any d in $\mathbb{N}\setminus\{0\}$.

Assertion 5.1.4. *Suppose that the face $\tau \subsetneq \tau_0$ and that $\dim\tau = \dim\tau_0 - d$. Then there exists a partition of $\Delta_\tau(f)$ into rational simplicial cones, such that the only cones in this partition that have κ generators in $\bar{\Delta}_{\tau_0}(f)$ are precisely the cones of the form*

$$\operatorname{cone}(A_i \cup B_\tau),$$

where $i \in \{1,\ldots,\ell\}$ and B_τ is a set of the form $\{b_{\tau_1},\ldots,b_{\tau_{v-1}},b_\tau\}$, where $v \leq d$ and $v = d$ for the cones in this partition of dimension equal to $\dim\Delta_\tau(f)$, $\tau \subsetneq \tau_{v-1} \subsetneq \cdots \subsetneq \tau_1 \subsetneq \tau_0$ are faces of τ_0 and $b_{\tau_1},\ldots,b_{\tau_{v-1}},b_\tau$ are defined in (5.1.4). Similarly, there exists a partition of $\Delta_\tau(f_{\tau_0})$ into rational simplicial cones such that the only cones in this partition that have κ generators in $\bar{\Delta}_{\tau_0}(f_{\tau_0})$ are precisely the cones of the form

$$\operatorname{cone}(A'_j \cup B_\tau),$$

where $j \in \{1,\ldots,m\}$ and B_τ is defined as above.

Note that one adds the same sets B_τ to the sets A'_j, as one adds to the sets A_i. Also note that $\tau \subsetneq \tau_{v-1} \subsetneq \cdots \subsetneq \tau_1 \subsetneq \tau_0$ for faces of τ_0 is equivalent to $\bar{\Delta}_\tau \supsetneq \bar{\Delta}_{\tau_{v-1}} \supsetneq \cdots \supsetneq \bar{\Delta}_{\tau_1} \supsetneq \bar{\Delta}_{\tau_0}$ for faces of $\bar{\Delta}_\tau$ (Lemma 1.1.7), and that $\dim\tau = \dim\tau_0 - d$ is equivalent to $\dim\bar{\Delta}_\tau = \dim\bar{\Delta}_{\tau_0} + d$ (Proposition 1.1.10).

From the formula for $S_{\operatorname{cone}(A_i\cup B_\tau)}$ in Theorem 2.3.2, it follows that

$$S_{\operatorname{cone}(A_i\cup B_\tau)} = \frac{\sum_{h\in H_{A_i\cup B_\tau}} p^{\sigma(h)+m(h)s}}{\prod_{x\in A_i\cup B_\tau}(p^{\sigma(x)+m(x)s}-1)}.$$

We know that m is linear on $\bar{\Delta}_\tau(f)$ (Lemma 1.1.5(iii)) and so, σ and m are linear on $H_{A_i\cup B_\tau}$. It is also clear that $\sigma(a)+m(a)\rho = 0$ for every $a \in \operatorname{cone} A_i \subset \bar{\Delta}_{\tau_0}(f)$. Let π be the natural projection of $\operatorname{vct} A_i + \operatorname{vct} B_\tau$ onto $\operatorname{vct} B_\tau$. Then

$$\begin{aligned}\sum_{h\in H_{A_i\cup B_\tau}} p^{\sigma(h)+m(h)\rho} &= \sum_{h\in H_{A_i\cup B_\tau}} p^{\sigma(\pi(h))+m(\pi(h))\rho} \\ &= \#H_{A_i} \sum_{b\in\pi(H_{A_i\cup B_\tau})} p^{\sigma(b)+m(b)\rho},\end{aligned}$$

by Lemma 5.1.2(i). Consequently, $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\operatorname{cone}(A_i\cup B_\tau)}$ equals

$$\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa \frac{\#H_{A_i}}{\prod_{x\in A_i}(p^{\sigma(x)+m(x)s}-1)} \frac{\sum_{b\in\pi(H_{A_i\cup B_\tau})} p^{\sigma(b)+m(b)\rho}}{\prod_{x\in B_\tau}(p^{\sigma(x)+m(x)\rho}-1)}.$$

To finish the proof, we define $S(P,V,g,s)$ for a finite subset P of $\mathbb{Q}^n$, a finite subset V of $\mathbb{N}^n$, a polynomial g and $s \in \mathbb{C}$ as

$$S(P,V,g,s) = \frac{\sum_{x\in P} p^{\sigma(x)+m(x)s}}{\prod_{x\in V}(p^{\sigma(x)+m(x)s}-1)},$$

and where the function m is completely determined by the Newton polyhedron $\Gamma(g)$ of the polynomial g. Note that if cone V is a rational simplicial cone contained in $(\mathbb{R}^+)^n$ (i.e., V is a finite subset of $\mathbb{N}^n$ such that its elements are linearly independent) and cone V is contained in a cone associated to a face of $\Gamma(g)$, then

$$S_{\text{cone}\,V} = S(H_V, V, g, s),$$

where $S_{\text{cone}\,V}$ is defined as in Theorem 2.3.2. Using this notation, $\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\text{cone}\,(A_i\cup B_\tau)}$ equals

$$S(\pi(H_{A_i\cup B_\tau}), B_\tau, f, \rho)\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\text{cone}\,A_i}.$$

Note that the sets A_i all generate the same vector space. So, Lemma 5.1.2(ii) implies that $\pi(H_{A_i\cup B_\tau}) = \pi(H_{A_1\cup B_\tau})$ for every $i \in \{1,\dots,\ell\}$. Therefore,

$$\begin{aligned}
&\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau(f)}\\
&\quad= \sum_{B_\tau}\sum_{i=1}^{\ell}\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\text{cone}\,(A_i\cup B_\tau)}\\
&\quad= \left(\sum_{B_\tau} S(\pi(H_{A_1\cup B_\tau}), B_\tau, f, \rho)\right)\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f)},
\end{aligned}$$

where B_τ runs through all the sets described in Assertion 5.1.4. Similarly,

$$\begin{aligned}
&\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau(f_{\tau_0})}\\
&\quad= \left(\sum_{B_\tau} S(\pi(H_{A_1'\cup B_\tau}), B_\tau, f_{\tau_0}, \rho)\right)\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f_{\tau_0})}.
\end{aligned}$$

Note that A_1' generates the same vector space as A_1. So, the sets $\pi(H_{A_1'\cup B_\tau})$ and $\pi(H_{A_1\cup B_\tau})$ are equal, by Lemma 5.1.2(ii). Moreover, the $m(b)$ for $b \in \pi(H_{A_1\cup B_\tau})\cup B_\tau$ are also the same, because $m(b) = b\cdot x_\tau$ for every $b \in \bar{\Delta}_\tau$, where x_τ is a fixed element of $\tau \subset \tau_0$. Now, it is clear that $S(\pi(H_{A_1\cup B_\tau}), B_\tau, f, \rho) = S(\pi(H_{A_1'\cup B_\tau}), B_\tau, f_{\tau_0}, \rho)$ and that

$$\begin{aligned}
\frac{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau(f)}}{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f)}} &= \textstyle\sum_{B_\tau} S(\pi(H_{A_1\cup B_\tau}), B_\tau, f, \rho)\\
&= \frac{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_\tau(f_{\tau_0})}}{\lim_{s\to\rho}(p^{s-\rho}-1)^\kappa S_{\Delta_{\tau_0}(f_{\tau_0})}},
\end{aligned}$$

which proves (5.1.2). □

One easily proves the following lemma.

Lemma 5.1.5. *Suppose that $\Delta = \text{cone}\,\{a_1,\dots,a_r\}$ is the cone strictly positively spanned by the vectors $a_1,\dots,a_r \in (\mathbb{R}^+)^n \setminus \{0\}$. Fix an element $b \in \Delta$. Then Δ is the disjoint union of the cones of the form*

$$W = \{\beta_1 b + f \mid \beta_1 \in \mathbb{R},\ \beta_1 > 0,\ f \in \text{ri}\ F\},$$

where F is a proper face of $\bar{\Delta}$ and where ri F *is its relative interior.*

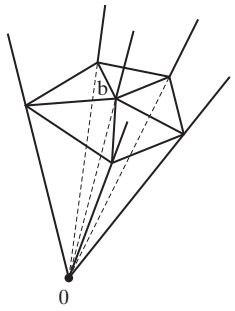


Figure 5.1.2

5.2 Checking if an arbitrary candidate pole is a pole of the expected order: reduction from f to the f_μ

Theorem 5.2.1 is a generalization of Theorem 5.1.3 to arbitrary candidate poles s_1 of $Z_f(s,\chi)$. Because the faces of $\Gamma(f)$ of maximal pure contribution to a candidate pole s_1 with $\mathrm{Re}\,(s_1) \neq \rho$ do not intersect the diagonal $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$, the situation can be more difficult.

Theorem 5.2.1. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set described in Proposition 3.1.1. Suppose that for every face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 of $Z_f(s,\chi)$, the face μ is the only face of $\Gamma(f_\mu)$ of maximal pure contribution to the candidate pole s_1 of $Z_{f_\mu}(s,\chi)$. Then*

$$\lim_{s\to s_1}(p^{s-s_1}-1)^k Z_f(s,\chi) = \sum_\mu c_\mu \lim_{s\to s_1}(p^{s-s_1}-1)^k Z_{f_\mu}(s,\chi),$$

where μ runs through the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1,

$$k = \begin{cases} \kappa(s_1)+1 & \text{if } p^{s_1+1}=1 \text{ and } \chi=\chi_{\mathrm{triv}}\,, \\ \kappa(s_1) & \text{otherwise,} \end{cases}$$

and the $c_\mu > 0$ are constant real numbers, independent of p. In fact,

$$c_\mu = \frac{\mathrm{Vol}\,(\bar{\Delta}_\mu(f)\cap H_1^-)}{\mathrm{Vol}\,(\bar{\Delta}_\mu(f_\mu)\cap H_1^-)},$$

where the volume is the one on the vector space generated by $\bar{\Delta}_\mu$, induced by the standard metric on $\mathbb{R}^n$ and H_1^- is the half-space $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq 1\}$.

We recall that $\kappa(s_1)$ and a face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 are defined in Definition 3.3.1.

Remarks.

1. If it is important to know whether we examine the cone Δ_τ associated to the polynomial f or to the polynomials f_μ, we will indicate this by $\Delta_\tau(f)$ and $\Delta_\tau(f_\mu)$, respectively.

2. Suppose that μ is a face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1. If the dimension of μ is 0 or 1, then μ is the only face of $\Gamma(f_\mu)$ of maximal pure contribution to the candidate pole s_1 of $Z_{f_\mu}(s,\chi)$. If the dimension of μ is > 1, this condition on μ will be satisfied in most examples. Nevertheless, it is possible that μ is not the only face of $\Gamma(f_\mu)$ of maximal pure contribution to the candidate pole s_1. For example, let $f(x,y,z) = x^7y^3z + x^4y^8 + x^8y^4 + z^7$. Then the Newton polyhedron $\Gamma(f)$ is given by the system of linear inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ x + y + 2z \geq 12 \\ y + z \geq 4 \\ 2y + z \geq 7 \\ 27x + 25y + 44z \geq 308 \\ 7x + 4z \geq 28 \end{cases}$$

(see also Figure 5.2.1). We denote the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above by $\tau_1, \ldots, \tau_8$. Face τ_4 is the only face of $\Gamma(f)$ of maximal pure contribution to the candidate pole $-1/3$.

The Newton polyhedron $\Gamma(f_{\tau_4})$ of $f_{\tau_4} = x^7y^3z + x^4y^8 + x^8y^4$ is given by the system of linear inequalities

$$\begin{cases} x \geq 4 \\ y \geq 3 \\ z \geq 0 \\ x + y + 2z \geq 12 \\ y + z \geq 4 \\ 5x + 3y \geq 44 \end{cases}$$

(see also Figure 5.2.2). We denote the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above by

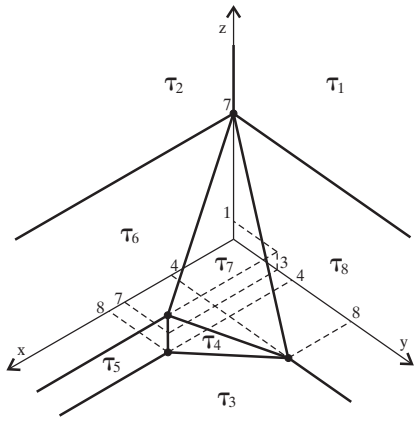


Figure 5.2.1. Newton polyhedron of $x^7y^3z + x^4y^8 + x^8y^4 + z^7$

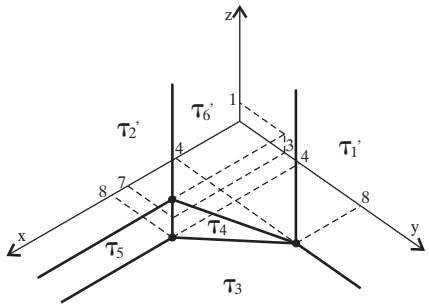


Figure 5.2.2. Newton polyhedron of $x^7y^3z + x^4y^8 + x^8y^4$

$\tau_1', \tau_2', \tau_3, \tau_4, \tau_5, \tau_6'$. The face τ_4 is not the only face of $\Gamma(f_{\tau_4})$ of maximal pure contribution to the candidate pole $-1/3$: also τ_2' is such a face.

We refer to Example 4 of Appendix D for the calculation of Igusa's local zeta function of f and f_{τ_4}, respectively, for some prime numbers p.

3. The proof below also shows that a similar result holds for $Z_{f,0}(s,\chi)$, even if the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 are not compact. Then f has to be non-degenerated over $\mathbb{F}_p$ with respect to all the *compact* faces of its Newton polyhedron.

Proof. By the same argument as in the proof of Theorem 5.1.3, we can show that also f_μ is non-degenerated over the finite field $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f_\mu)$ if μ is a face of $\Gamma(f)$. So, we can use the formulas for $Z_f(s,\chi)$ and $Z_{f_\mu}(s,\chi)$ from Theorem 2.3.2.

We know from Proposition 3.3.3 that $\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\tau}=0$ for faces τ not contained in any face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1. It is clear that $L_\tau(f)=L_\tau(f_\mu)$ for faces τ contained in a face μ of $\Gamma(f)$. For faces τ contained in at least one face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole, we will write

$$\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\tau(f)}=\sum_\mu R_\mu, \tag{5.2.1}$$

where μ runs through all the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 and where the R_μ, which depend on τ, will be specified later. The R_μ for $\mu\not\supset\tau$ will vanish. It suffices then to show that

$$\frac{R_\mu}{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f)}}=\frac{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\tau(f_\mu)}}{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f_\mu)}} \tag{5.2.2}$$

for faces $\tau\subset\mu$, and that

$$\frac{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f)}}{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f_\mu)}}=\frac{\mathrm{Vol}\,(\bar\Delta_\mu(f)\cap H_1^-)}{\mathrm{Vol}\,(\bar\Delta_\mu(f_\mu)\cap H_1^-)}, \tag{5.2.3}$$

where μ is a face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1. The last equality follows by Lemma 3.3.2(ii) and the fact that $\bar\Delta_\mu(f)$ and $\bar\Delta_\mu(f_\mu)$ generate the same vector space.

Suppose that $\cup_i\Delta_i$ is a finite partition of Δ_τ into rational simplicial cones such that each Δ_i has at most $\kappa(s_1)$ generators contributing to the candidate pole s_1. This is always possible; see below or the proof of Proposition 3.3.3(ii). Then $S_{\Delta_\tau}=\sum_i S_{\Delta_i}$. From the formula for S_{Δ_i} in Theorem 2.3.2, it follows that $\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_i}=0$ if less than $\kappa(s_1)$ generators of Δ_i contribute to the candidate pole s_1. So, to prove equality (5.2.2), it is only important to know the other cones. We will choose partitions of $\Delta_\tau(f)$ and $\Delta_\tau(f_\mu)$, respectively that are closely related to some chosen partitions of $\Delta_\mu(f)$ and $\Delta_\mu(f_\mu)$, respectively.

Fix a finite partition of $\Delta_\mu(f)$ into rational simplicial cones, and let

$$\mathrm{cone}\,A_{i,\mu},\ \text{for } i=1,\dots,\ell_\mu,$$

be the only cones of dimension $\kappa(s_1)$, i.e., with $\kappa(s_1)$ generators contributing to the candidate pole s_1, in this partition. Here, each $A_{i,\mu}$ is a set of $\kappa(s_1)$ vectors in $\mathbb{N}^n$ that are linearly independent over $\mathbb{Q}$. Similarly, fix a finite partition of $\Delta_\mu(f_\mu)$ into rational simplicial cones, and let

$$\mathrm{cone}\,A'_{j,\mu},\ \text{for } j=1,\dots,m_\mu,$$

be the only cones of dimension $\kappa(s_1)$ in this partition and each $A'_{j,\mu}$ a set of $\kappa(s_1)$ vectors in $\mathbb{N}^n$ that are linearly independent over $\mathbb{Q}$. Note that the sets $A_{i,\mu}$ and the sets $A'_{j,\mu}$ all generate the same vector space.

It is clear that $\Delta_\tau(f) \subset \Delta_\tau(f_\mu)$ for every face $\tau \subset \mu$. Fix now for each face $\tau \subsetneq \mu$ a primitive vector

$$b_\tau \in \Delta_\tau(f) \subset \Delta_\tau(f_\mu) \tag{5.2.4}$$

that does not contribute to the candidate pole s_1. This is always possible because σ and m are linear on $\bar{\Delta}_\tau(f)$ and not all generators of $\Delta_\tau(f)$ contribute to the candidate pole s_1. From Lemma 1.1.7 and Lemma 5.1.5, it follows by induction on d that the following assertion is true for any d in $\mathbb{N} \setminus \{0\}$.

Assertion 5.2.2. *Suppose that the face τ is a face of $\Gamma(f)$ that is contained in at least one face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 and that $\dim \tau = n - \kappa(s_1) - d$. Then there exists a partition of $\Delta_\tau(f)$ into rational simplicial cones such that the only cones in this partition that have $\kappa(s_1)$ generators contributing to the candidate pole s_1 are precisely the cones of the form*

$$\operatorname{cone}(A_{i,\mu} \cup B_\tau),$$

where $\mu \supset \tau$ is a face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1, $i \in \{1, \ldots, \ell_\mu\}$, and B_τ is a set of the form $\{b_{\tau_1}, \ldots, b_{\tau_{v-1}}, b_\tau\}$, where $v \leq d$ and $v = d$ for the cones in this partition of dimension equal to $\dim \Delta_\tau(f)$, $\tau \subsetneq \tau_{v-1} \subsetneq \cdots \subsetneq \tau_1 \subsetneq \mu$ are faces of μ and $b_{\tau_1}, \ldots, b_{\tau_{v-1}}, b_\tau$ are defined in (5.2.4). The other cones in the partition have less than $\kappa(s_1)$ generators contributing to the candidate pole s_1. Similarly, there exists a partition of $\Delta_\tau(f_\mu)$ into rational simplicial cones such that the only cones in this partition that have $\kappa(s_1)$ generators contributing to the candidate pole s_1 are precisely the cones of the form

$$\operatorname{cone}(A'_{j,\mu} \cup B_\tau),$$

where $j \in \{1, \ldots, m_\mu\}$ and B_τ is defined as above. The other cones in the partition have less than $\kappa(s_1)$ generators contributing to the candidate pole s_1.

From the formula for $S_{\operatorname{cone}(A_{i,\mu} \cup B_\tau)}$ in Theorem 2.3.2, it follows that

$$S_{\operatorname{cone}(A_{i,\mu} \cup B_\tau)} = \frac{\sum_{h \in H_{A_{i,\mu} \cup B_\tau}} p^{\sigma(h) + m(h)s}}{\prod_{x \in A_{i,\mu} \cup B_\tau} (p^{\sigma(x) + m(x)s} - 1)}.$$

We know that m is linear on $\bar{\Delta}_\tau$ (Lemma 1.1.5(iii)) and so, σ and m are linear on $H_{A_{i,\mu} \cup B_\tau}$. It is also clear that $p^{\sigma(a) + m(a)s_1} = 1$ for every $a \in \operatorname{cone} A_{i,\mu} \subset \bar{\Delta}_\mu(f)$ because all generators of $\Delta_\mu(f)$ contribute to the candidate pole s_1. Let π be the natural projection of $\operatorname{vct} A_{i,\mu} + \operatorname{vct} B_\tau$ onto $\operatorname{vct} B_\tau$. Then

$$\begin{aligned} \sum_{h \in H_{A_{i,\mu} \cup B_\tau}} p^{\sigma(h) + m(h)s_1} &= \sum_{h \in H_{A_{i,\mu} \cup B_\tau}} p^{\sigma(\pi(h)) + m(\pi(h))s_1} \\ &= \#H_{A_{i,\mu}} \sum_{b \in \pi(H_{A_{i,\mu} \cup B_\tau})} p^{\sigma(b) + m(b)s_1}, \end{aligned}$$

by Lemma 5.1.2(i). Consequently, $\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\mathrm{cone}\,(A_{i,\mu}\cup B_\tau)}$ equals

$$\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}\frac{\#H_{A_{i,\mu}}}{\prod_{x\in A_{i,\mu}}(p^{\sigma(x)+m(x)s}-1)}\frac{\sum_{b\in\pi(H_{A_{i,\mu}\cup B_\tau})}p^{\sigma(b)+m(b)s_1}}{\prod_{x\in B_\tau}(p^{\sigma(x)+m(x)s_1}-1)}.$$

Using the same notation as in the proof of Theorem 5.1.3, we have

$$\begin{aligned}&\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\mathrm{cone}\,(A_{i,\mu}\cup B_\tau)}\\ &\quad= S(\pi(H_{A_{i,\mu}\cup B_\tau}),B_\tau,f,s_1)\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\mathrm{cone}\,A_{i,\mu}}.\end{aligned}$$

Note that the sets $A_{i,\mu}$ all generate the same vector space. So, Lemma 5.1.2(ii) implies that $\pi(H_{A_{i,\mu}\cup B_\tau})=\pi(H_{A_{1,\mu}\cup B_\tau})$ for every $i\in\{1,\dots,\ell_\mu\}$. Therefore,

$$\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\tau(f)}=\sum_\mu R_\mu,$$

where μ runs through the faces of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1 (that contain τ) and

$$\begin{aligned}R_\mu &= \sum_{B_\tau}\sum_{i=1}^{\ell_\mu}\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\mathrm{cone}\,(A_{i,\mu}\cup B_\tau)}\\ &= \left(\sum_{B_\tau}S(\pi(H_{A_{1,\mu}\cup B_\tau}),B_\tau,f,s_1)\right)\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f)},\end{aligned}$$

where B_τ runs through all the sets described in Assertion 5.2.2. Similarly, for μ a face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1, we have

$$\begin{aligned}&\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\tau(f_\mu)}\\ &\quad=\left(\sum_{B_\tau}S(\pi(H_{A'_{1,\mu}\cup B_\tau}),B_\tau,f_\mu,s_1)\right)\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f_\mu)}\end{aligned}$$

because μ is the only face of $\Gamma(f_\mu)$ of maximal pure contribution to the candidate pole s_1.

Note that $A'_{1,\mu}$ generates the same vector space as $A_{1,\mu}$. So, the sets $\pi(H_{A'_{1,\mu}\cup B_\tau})$ and $\pi(H_{A_{1,\mu}\cup B_\tau})$ are equal, by Lemma 5.1.2(ii). Moreover, the $m(b)$ for $b\in\pi(H_{A_{1,\mu}\cup B_\tau})\cup B_\tau$ are also the same because $m(b)=b\cdot x_\tau$ for every $b\in\bar{\Delta}_\tau$, where x_τ is a fixed element of $\tau\subset\mu$. Now, it is clear that $S(\pi(H_{A_{1,\mu}\cup B_\tau}),B_\tau,f,s_1)=S(\pi(H_{A'_1\cup B_\tau}),B_\tau,f_\mu,s_1)$ and that

$$\begin{aligned}\frac{R_\mu}{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f)}} &= \textstyle\sum_{B_\tau}S(\pi(H_{A_{1,\mu}\cup B_\tau}),B_\tau,f,s_1)\\ &= \frac{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\tau(f_\mu)}}{\lim_{s\to s_1}(p^{s-s_1}-1)^{\kappa(s_1)}S_{\Delta_\mu(f_\mu)}},\end{aligned}$$

which proves (5.2.2). □

5.3 Another explicit formula for $Z_f(s,\chi)$, when $f = f_\sigma$ where $\sigma \subset \Gamma(f)$ is an unstable convex set, and proof of the conjecture of Denef and Sargos

We refer to [DS92] for an explanation of the choice of the name "unstable", which is used in Definition 5.3.1. In contrast to [DS92], we do not only define the notion of an unstable face of $\Gamma(f)$, but also the notion of an unstable convex set. Moreover, for the following results, it is not necessary to exclude the case where $\sigma \subset \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_j = 0\}$. Of course, the face τ_0 can not be contained in the set $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_j = 0\}$ because τ_0 must intersect the diagonal and the origin does not belong to the Newton polyhedron.

Definition 5.3.1. Let f be as in Definition 0.1.4 and let p be a prime number. A convex set σ is called *unstable over $\mathbb{F}_p$, relatively to x_j and with respect to f*, if the following two conditions are satisfied:

(i) $\sigma \subset \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid 0 \leq x_j \leq 1\}$ and

(ii) for every face τ of σ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_j = 1\}$, the polynomial $\bar{f}_\tau$ does not vanish on $(\mathbb{F}_p^\times)^n$.

A convex set σ is called unstable over $\mathbb{F}_p$ with respect to f if there exists an index j $(1 \leq j \leq n)$ such that σ is unstable over $\mathbb{F}_p$ relatively to x_j and with respect to f.

We have a similar definition for a convex set σ that is unstable over $\mathbb{C}$ with respect to f. Then f has to be a polynomial over $\mathbb{C}$, and in (ii) the polynomial f_τ does not vanish on $(\mathbb{C}^\times)^n$.

In Theorem 5.3.3, we will give a formula for $Z_f(s,\chi)$ that holds if $f = f_\sigma$ where $\sigma \subset \Gamma(f)$ is an unstable convex set over $\mathbb{F}_p$ with respect to f and if f is non-degenerated over $\mathbb{F}_p$ with respect to a particular set of faces of $\Gamma(f)$. This formula will be similar to the one in Theorem 2.3.2. But first, we state Lemma 5.3.2.

Lemma 5.3.2. *Let p be a prime number and $f(x) = f(x_1,\ldots,x_n) \in \mathbb{Z}_p[x_1,\ldots,x_n]$. Indicate by $\bar{f}$ the polynomial over $\mathbb{F}_p$ obtained by reducing each coefficient of f modulo $p\mathbb{Z}_p$. Let s be a complex variable with $Re(s) > 0$ and let χ be a character of $\mathbb{Z}_p^\times$ with conductor c_χ.*

(i) Let $\tilde{N}$ be the number of elements in the set $\{a \in (\mathbb{F}_p^\times)^{n-1} \times \mathbb{F}_p \mid \bar{f}(a) = 0\}$. Suppose that the set of congruences

$$\begin{cases} f_\tau(x) & \equiv & 0 \bmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) & \equiv & 0 \bmod p, \qquad i = 1,\ldots,n \end{cases} \tag{5.3.1}$$

has no solution in $(\mathbb{Z}_p^\times)^{n-1} \times \mathbb{Z}_p$. *Then one has*

$$\int\limits_{(\mathbb{Z}_p^\times)^{n-1}\times\mathbb{Z}_p} \chi(\operatorname{ac} f(x))\,|f(x)|^s|dx| = \begin{cases} p^{-n}\left((p-1)^{n-1}p - p\tilde{N}\frac{p^s-1}{p^{s+1}-1}\right) & \text{for } \chi = \chi_{\text{triv}}\,, \\ p^{-nc_\chi} \sum\limits_{\substack{a \in (\mathbb{Z}_p^\times)^{n-1}\times\mathbb{Z}_p \\ a \bmod p^{c_\chi} \\ f(a) \not\equiv 0 \bmod p}} \chi(f(a)) & \text{for } \chi \neq \chi_{\text{triv}}\,. \end{cases}$$

(ii) Let N' be the number of elements in the set $\{a \in (\mathbb{F}_p^\times)^{n-1} \times \{0\} \mid \bar{f}(a) = 0\}$. Suppose that the set of congruences (5.3.1) *has no solution in $(\mathbb{Z}_p^\times)^{n-1} \times p\mathbb{Z}_p$. Then one has*

$$\int\limits_{(\mathbb{Z}_p^\times)^{n-1}\times p\mathbb{Z}_p} \chi(\operatorname{ac} f(x))\,|f(x)|^s|dx| = \begin{cases} p^{-n}\left((p-1)^{n-1} - pN'\frac{p^s-1}{p^{s+1}-1}\right) & \text{for } \chi = \chi_{\text{triv}}\,, \\ p^{-nc_\chi} \sum\limits_{\substack{a \in (\mathbb{Z}_p^\times)^{n-1}\times p\mathbb{Z}_p \\ a \bmod p^{c_\chi} \\ f(a) \not\equiv 0 \bmod p}} \chi(f(a)) & \text{for } \chi \neq \chi_{\text{triv}}\,. \end{cases}$$

Proof. The proof is similar to the proof of Corollary 2.1.2 and Proposition 2.2.2. □

Theorem 5.3.3. *Let f be as in Definition 0.1.4 and let p be a prime number. Suppose that f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of $\Gamma(f)$ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and suppose that $f = f_\sigma$ where $\sigma \subset \Gamma(f)$ is an unstable convex set over $\mathbb{F}_p$ relatively to x_n and with respect to f. Let s be a complex variable with $\operatorname{Re}(s) > 0$ and let χ be a character of $\mathbb{Z}_p^\times$ with conductor $c_\chi = 1$. Then*

$$Z_f(s,\chi) = \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ e_n \in \operatorname{rec}\tau}} \tilde{L}_\tau S_{\Delta_\tau}, \tag{5.3.2}$$

with

$$\begin{aligned} \tilde{L}_\tau &= \begin{cases} p^{-n}\left((p-1)^{n-1}p - p\tilde{N}_\tau\frac{p^s-1}{p^{s+1}-1}\right) & \text{for } \chi = \chi_{\text{triv}}\,, \\ p^{-n} \sum\limits_{a\in(\mathbb{F}_p^\times)^{n-1}\times\mathbb{F}_p} \chi(f_\tau(a)) & \text{for } \chi \neq \chi_{\text{triv}}\,, \end{cases} \\ \tilde{N}_\tau &= \{a \in (\mathbb{F}_p^\times)^{n-1} \times \mathbb{F}_p \mid \bar{f}_\tau(a) = 0\}, \end{aligned} \tag{5.3.3}$$

S_{Δ_τ} as in Theorem 2.3.2, $e_n = (0,0,\ldots,1)$ and $\operatorname{rec}\tau$ the set of directions of recession[1] *of τ. Recall that $\Gamma(f)$ is also a face of $\Gamma(f)$ with e_n as direction of recession.*

[1] See Definition B.1.8.

Moreover, suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set described in Proposition 3.1.1. If $e_n \notin \operatorname{rec}\mu$ for each face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1, then the following holds: If $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\text{triv}}$ and s_1 is a pole of $Z_f(s,\chi)$, its order is $< \kappa(s_1)$; if $p^{s_1+1} = 1$ and s_1 is a pole of $Z_f(s)$, its order is $< \kappa(s_1)+1$.

Remarks.

1. Note that the formula (5.3.2) holds, if f is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of $\Gamma(f)$ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and if $\Gamma_{\text{gl}}(f)$ is an unstable convex set over $\mathbb{F}_p$ relatively to x_n and with respect to f.

   The condition "$f = f_\sigma$ where $\sigma \subset \Gamma(f)$ is an unstable convex set over $\mathbb{F}_p$ with respect to f" is in fact too strong. It is already enough that there exists a convex set σ such that the following holds: for each face τ of $\Gamma(f)$ with $e_n \in \operatorname{rec}\tau$, the polynomial $\bar{f}_{\sigma\cap\tau\cap H_1}$ does not vanish on $(\mathbb{F}_p^\times)^n$, with $H_1 = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 1\}$. Moreover, the proof below shows that formula (5.3.2) for $Z_f(s,\chi)$ is even true when one has, for every face τ of $\Gamma(f)$ for which e_n is a direction of recession, that the set of congruences

   $$\begin{cases} f_\tau(x) & \equiv & 0 \bmod p \\ \frac{\partial f_\tau(x)}{\partial x_i} & \equiv & 0 \bmod p, \quad i = 1,\ldots,n \end{cases}$$

   has no solution in $(\mathbb{Z}_p^\times)^{n-1} \times \mathbb{Z}_p$.

2. Let $f(x)$ be a non-zero polynomial over $\mathbb{Z}$ with $f(0) = 0$. Suppose that f is non-degenerated over $\mathbb{C}$ with respect to all the faces of $\Gamma(f)$ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and suppose that $f = f_\sigma$ where $\sigma \subset \Gamma(f)$ is an unstable convex set over $\mathbb{C}$ relatively to x_n and with respect to f. Then the conditions of Theorem 5.3.3 are satisfied for $p \gg 0$.

Proof. Put $H_0 = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and $H_1 = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 1\}$. Recall that the cones associated to the faces of $\Gamma(f)$ form a partition of $(\mathbb{R}^+)^n$. Moreover, for every $(a_1,\ldots,a_n) \in (\mathbb{R}^+)^n$ one has $a_n = 0$ if and only if e_n is a direction of recession of $F(a_1,\ldots,a_n)$. So, if $\Delta_\tau \cap H_0 \neq \emptyset$, then $\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}$ is contained completely in H_0. Consequently, $H_0 \cap (\mathbb{R}^+)^n$ is equal to the following disjoint union

$$\bigcup_{\tau \text{ face of } \Gamma(f)} (\Delta_\tau \cap H_0) = \bigcup_{\substack{\tau \text{ face of } \Gamma(f) \\ \Delta_\tau \subset H_0}} \Delta_\tau = \bigcup_{\substack{\tau \text{ face of } \Gamma(f) \\ e_n \in \operatorname{rec}\tau}} \Delta_\tau.$$

Therefore, $Z_f(s,\chi)$ equals

$$\sum_{(k_1,\ldots,k_{n-1})\in\mathbb{N}^{n-1}} \int_{\substack{(x_1,\ldots,x_n)\in\mathbb{Z}_p^n \\ \operatorname{ord}(x_1,\ldots,x_{n-1})=(k_1,\ldots,k_{n-1})}} \chi(\operatorname{ac} f(x))\,|f(x)|^s|dx|$$

$$= \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ e_n \in \operatorname{rec}\tau}} \sum_{(k_1,\ldots,k_{n-1},0)\in\mathbb{N}^n\cap\Delta_\tau} \int_{\substack{(x_1,\ldots,x_n)\in\mathbb{Z}_p^n \\ \operatorname{ord}(x_1,\ldots,x_{n-1})=(k_1,\ldots,k_{n-1})}} \chi(\operatorname{ac} f(x))\,|f(x)|^s|dx|.$$

Suppose that τ is a face of $\Gamma(f)$ such that e_n is a direction of recession of τ (or equivalently $\Delta_\tau \subset H_0$). Suppose that $k = (k_1,\ldots,k_{n-1},0) \in \mathbb{N}^n \cap \Delta_\tau$, $x = (x_1,\ldots,x_n) \in \mathbb{Z}_p^n$ with $\operatorname{ord}(x_1,\ldots,x_{n-1}) = (k_1,\ldots,k_{n-1})$. Write $x_j = p^{k_j}u_j$ with $u_j \in \mathbb{Z}_p^\times$, for $j = 1,\ldots,n-1$. Then

$$\begin{aligned} |dx| &= p^{-\sigma(k)}|du||dx_n| \quad \text{and} \\ x^\omega &= x_1^{\omega_1}\cdots x_{n-1}^{\omega_{n-1}}x_n^{\omega_n} \\ &= p^{(k_1,\ldots,k_{n-1})\cdot(\omega_1,\ldots,\omega_{n-1})}u_1^{\omega_1}\cdots u_{n-1}^{\omega_{n-1}}x_n^{\omega_n} \\ &= p^{k\cdot\omega}u_1^{\omega_1}\cdots u_{n-1}^{\omega_{n-1}}x_n^{\omega_n}. \end{aligned}$$

We know that $F(k) = \tau$. Thus $k\cdot\omega = m(k)$ for every $\omega \in \operatorname{supp}(f)\cap\tau$, and $k\cdot\omega \geq m(k)+1$ for every $\omega \in \operatorname{supp}(f)\setminus\tau$. It follows then that

$$f(x) = p^{m(k)}\left(f_\tau(u_1,\ldots,u_{n-1},x_n) + p\tilde{f}_{\tau,k}(u_1,\ldots,u_{n-1},x_n)\right),$$

where $\tilde{f}_{\tau,k}$ is a polynomial in the variables $u_1,\ldots,u_{n-1},x_n$ over $\mathbb{Z}_p$ (depending on τ, k and f). Hence, $Z_f(s,\chi)$ equals

$$\sum_{\substack{\tau \text{ face of } \Gamma(f) \\ e_n \in \operatorname{rec}\tau}} \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} p^{-\sigma(k)-m(k)s} \int_{(\mathbb{Z}_p^\times)^{n-1}\times\mathbb{Z}_p} \chi\left(\operatorname{ac}\left(f_\tau(u_1,\ldots,u_{n-1},x_n) + p\tilde{f}_{\tau,k}(u_1,\ldots,u_{n-1},x_n)\right)\right) |f_\tau(u_1,\ldots,u_{n-1},x_n) + p\tilde{f}_{\tau,k}(u_1,\ldots,u_{n-1},x_n)|^s|du||dx|.$$

Denote the last integral by $\tilde{L}_\tau$. By the following assertion and Lemma 5.3.2(i), we obtain equality (5.3.3) of Theorem 5.3.3.

Assertion 5.3.4. *For every face τ of $\Gamma(f)$ with $e_n \in \operatorname{rec}\tau$, one has that the set of congruences*

$$\begin{cases} f_\tau(x) & \equiv 0 \bmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) & \equiv 0 \bmod p, \quad i = 1,\ldots,n \end{cases}$$

has no solution in $(\mathbb{Z}_p^\times)^{n-1}\times\mathbb{Z}_p$.

Note that $\tilde{L}_\tau$ does not depend on k. Thus,

$$Z_f(s,\chi) = \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ e_n \in \operatorname{rec}\tau}} \tilde{L}_\tau S_{\Delta_\tau}, \tag{5.3.4}$$

with

$$S_{\Delta_\tau} = \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k)-m(k)s}.$$

Suppose moreover that $e_n \notin \operatorname{rec}\mu$ for each face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1. Then the faces τ contained in such a μ do not occur in the sum (5.3.4). We know from Proposition 3.3.3(i) that, if s_1 is a pole of S_{Δ_τ} for τ not contained in such a μ, its order is $< \kappa(s_1)$. Now, the assertion about the order follows.

We now prove Assertion 5.3.4. Fix a face τ of $\Gamma(f)$ with $e_n \in \operatorname{rec}\tau$. Note that because $\sigma \subset \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid 0 \leq x_n \leq 1\}$, one has $\operatorname{supp}(f_\tau) \subset H_0 \cup H_1$ and $f_\tau = f_{\tau\cap H_0} + f_{\tau \cap H_1}$.

Case 1. $\operatorname{supp}(f_\tau) \cap H_1 = \emptyset$. In this case $f_\tau = f_{\tau\cap H_0}$ is independent of x_n and $\tau \cap H_0$ is a face of $\Gamma(f)$. Because f is non-degenerated with respect to all the faces of $\Gamma(f)$ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$, the assertion now follows.

Case 2. $\operatorname{supp}(f_\tau) \cap H_1 \neq \emptyset$. In this case $(\partial f_\tau/\partial x_n) = x_n^{-1} f_{\tau\cap H_1} = x_n^{-1} f_{\sigma\cap\tau\cap H_1}$ is independent of x_n and non-zero on $(\mathbb{F}_p^\times)^{n-1}$ because $\sigma \cap \tau \cap H_1$ is a face of σ (see below) contained in H_1 and σ is unstable over $\mathbb{F}_p$ relatively to x_n and with respect to f. Now the assertion follows.

We explain that $\sigma \cap \tau \cap H_1$ is a face of σ. Suppose that H is a supporting hyperplane of $\Gamma(f)$ such that $H \cap \Gamma(f) = \tau$. Then H is also a supporting hyperplane of σ such that $H \cap \sigma = \tau \cap \sigma$ because $\sigma \subset \Gamma(f)$. Consequently, $\tau \cap \sigma$ is a face of σ. Also $\sigma \cap H_1$ is a face of σ because σ is unstable. Due to the fact that the intersection of two faces is again a face, it is now clear that $\sigma \cap \tau \cap H_1$ is a face of σ. □

A similar result holds for $Z_{f,0}(s,\chi)$.

Theorem 5.3.5. *Let f be as in Definition 0.1.4 and let p be a prime number. Suppose that f is non-degenerated over $\mathbb{F}_p$ with respect to all the compact faces of $\Gamma(f)$ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and suppose that $f = f_\sigma$ where $\sigma \subset \Gamma(f)$ is an unstable convex set over $\mathbb{F}_p$ relatively to x_n and with respect to f. (In fact one only has to require condition* (ii) *in Definition 5.3.1 for the* compact *faces of σ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 1\}$). Let s be a complex variable with $\operatorname{Re}(s) > 0$ and let χ be a character of $\mathbb{Z}_p^\times$ with conductor $c_\chi = 1$. Then*

$$Z_{f,0}(s,\chi) = \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \{e_n\} = \operatorname{rec}\tau \cap \{e_1,\ldots,e_n\}}} L'_\tau S_{\Delta_\tau}$$

with

$$\begin{aligned} L'_\tau &= \begin{cases} p^{-n}\left((p-1)^{n-1} - pN'_\tau \frac{p^s-1}{p^{s+1}-1}\right) & \text{for } \chi = \chi_{\text{triv}}\,, \\ p^{-n} \sum\limits_{a \in (\mathbb{F}_p^\times)^{n-1}\times\{0\}} \chi(f_\tau(a)) & \text{for } \chi \neq \chi_{\text{triv}}\,, \end{cases} \\ N'_\tau &= \{a \in (\mathbb{F}_p^\times)^{n-1} \times \{0\} \mid \bar{f}_\tau(a) = 0\}, \end{aligned} \tag{5.3.5}$$

S_{Δ_τ} as in Theorem 2.3.2, $e_n = (0,0,\ldots,1)$ and $\operatorname{rec}\tau$ the set of directions of recession of τ.

Proof. The proof is almost similar to the proof of Theorem 5.3.3. We only give a few important steps. It is clear that $Z_{f,0}(s,\chi)$ equals

$$\begin{aligned} &\int\limits_{(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x))\,|f(x)|^s |dx| \\ &= \sum_{(k_1,\ldots,k_{n-1}) \in (\mathbb{N}\setminus\{0\})^{n-1}} \int\limits_{\substack{(x_1,\ldots,x_n) \in (p\mathbb{Z}_p)^n \\ \operatorname{ord}(x_1,\ldots,x_{n-1}) = (k_1,\ldots,k_{n-1})}} \chi(\operatorname{ac} f(x))\,|f(x)|^s |dx| \\ &= \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \{e_n\} = \operatorname{rec}\tau \cap \{e_1,\ldots,e_n\}}} \sum_{(k_1,\ldots,k_{n-1},0) \in \mathbb{N}^n \cap \Delta_\tau} \int\limits_{\substack{(x_1,\ldots,x_n) \in (p\mathbb{Z}_p)^n \\ \operatorname{ord}(x_1,\ldots,x_{n-1}) = (k_1,\ldots,k_{n-1})}} \chi(\operatorname{ac} f(x))\,|f(x)|^s |dx| \\ &= \sum_{\substack{\tau \text{ face of } \Gamma(f) \\ \{e_n\} = \operatorname{rec}\tau \cap \{e_1,\ldots,e_n\}}} \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k)-m(k)s} \\ &\qquad \int\limits_{(\mathbb{Z}_p^\times)^{n-1}\times p\mathbb{Z}_p} \chi\left(\operatorname{ac}\left(f_\tau(u_1,\ldots,u_{n-1},x_n) + p\tilde{f}_{\tau,k}(u_1,\ldots,u_{n-1},x_n)\right)\right) \\ &\qquad\qquad |f_\tau(u_1,\ldots,u_{n-1},x_n) + p\tilde{f}_{\tau,k}(u_1,\ldots,u_{n-1},x_n)|^s |du||dx|, \end{aligned}$$

where $\tilde{f}_{\tau,k}$ is as in the proof of Theorem 5.3.3. Denote the last integral by L'_τ. By a similar argument as in Theorem 5.3.3, we obtain equality (5.3.5) of Theorem 5.3.5. Note that $\tau \cap H_0$ is a compact face of $\Gamma(f)$ contained in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and $\sigma \cap \tau \cap H_1$ is a compact face of σ if τ is a face of $\Gamma(f)$ with $\{e_n\} = \operatorname{rec}\tau \cap \{e_1,\ldots,e_n\}$. □

We are now able to prove one of the conjectures of Denef and Sargos [DS92, Section 6.2] for the p-adic case.

Theorem 5.3.6. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. If τ_0 is unstable over $\mathbb{F}_p$ with respect to f, then the following holds:*

(i) *If $\rho \neq -1$ or $\chi \neq \chi_{\text{triv}}$ and ρ is a pole of $Z_f(s,\chi)$, then its order is $< \kappa$.*

(ii) *If $\rho = -1$ is a pole of $Z_f(s)$, then its order is $< \kappa + 1$.*

Proof. One proves by the same argument as in the proof of Theorem 5.1.3 that f_{τ_0} is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron $\Gamma(f_{\tau_0})$. Then Theorem 5.3.6 follows immediately by Theorem 5.3.3 and Theorem 5.1.3. Indeed, if τ_0 is unstable over $\mathbb{F}_p$ relatively to x_j, the unit vector e_j does not belong to $\operatorname{rec}\tau$ for faces $\tau \subset \tau_0$. □

We can prove Theorem 5.3.7, a similar result for $Z_{f,0}(s)$, by using Theorem 5.3.5 and Remark 2 after Theorem 5.1.3. Note that τ_0 does not have to be compact.

Theorem 5.3.7. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Remark 2 after Theorem 2.3.2. If τ_0 is unstable over $\mathbb{F}_p$ with respect to f, then the following holds:*

(i) *If $\rho \neq -1$ or $\chi \neq \chi_{\text{triv}}$ and ρ is a pole of $Z_{f,0}(s,\chi)$, then its order is $< \kappa$.*

(ii) *If $\rho = -1$ is a pole of $Z_{f,0}(s)$, then its order is $< \kappa + 1$.*

By using Theorem 5.2.1 instead of Theorem 5.1.3, we can prove the following result for an arbitrary candidate pole of $Z_f(s,\chi)$.

Theorem 5.3.8. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Theorem 2.3.2. Suppose that s_1 is a candidate pole of $Z_f(s,\chi)$, i.e., a number from the set described in Proposition 3.1.1. Suppose that for every face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1, the face μ is the only face of $\Gamma(f_\mu)$ of maximal pure contribution to the candidate pole s_1, and μ is unstable over $\mathbb{F}_p$ with respect to f. Then the following holds:*

(i) *If $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\text{triv}}$ and s_1 is a pole of $Z_f(s,\chi)$, then its order is $< \kappa(s_1)$.*

(ii) *If $p^{s_1+1} = 1$ and s_1 is a pole of $Z_f(s)$, then its order is $< \kappa(s_1) + 1$.*

Remark. The proof of Theorem 5.2.1 shows that the condition "μ is the only face of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1" in Theorem 5.3.8 can be replaced by a weaker condition. Examples show that even this condition might be omitted; see Example 4 in Appendix D. We would like to search for a proof or a counterexample.

A similar property holds for $Z_{f,0}(s)$.

Theorem 5.3.9. *Suppose that the prime number p, the polynomial f and the character χ satisfy the conditions of Remark 2 after Theorem 2.3.2. Suppose*

that s_1 is a candidate pole of $Z_{f,0}(s,\chi)$, i.e., a number from the set of numbers described in Proposition 3.1.1. Suppose that for every face μ of $\Gamma(f)$ of maximal pure contribution to the candidate pole s_1, the face μ is the only face of $\Gamma(f_\mu)$ of maximal pure contribution to the candidate pole s_1, and μ is unstable over $\mathbb{F}_p$ with respect to f. Then the following holds:

(i) If $p^{s_1+1} \neq 1$ or $\chi \neq \chi_{\text{triv}}$ and s_1 is a pole of $Z_{f,0}(s,\chi)$, then its order is $< \kappa(s_1)$.

(ii) If $p^{s_1+1} = 1$ and s_1 is a pole of $Z_{f,0}(s,\chi)$, then its order is $< \kappa(s_1)+1$.

Remark. In the very special case that there is only one facet of pure contribution to s_1 that is moreover an unstable simplex, J. Denef obtained the following related result.

Theorem 5.3.10. *Let f be a polynomial over $\mathbb{Z}_p$ in n variables. Suppose that f is non-degenerated over $\mathbb{Q}_p$ with respect to the compact faces of its Newton polyhedron. Suppose that $s_1 \neq 1$ is a real candidate pole of $Z_f(s)$ such that there is only one facet μ of pure contribution to s_1. Suppose that μ is a simplex with $n-1$ vertices contained in a coordinate hyperplane and one vertex at distance 1 from that coordinate hyperplane. Then there is no pole of*

$$\int_{p^k\mathbb{Z}_p^n} |f(x)|^s |dx|$$

with real part equal to s_1 for $k \gg 0$.

5.4 Examples

We use Theorem 2.3.2 and Theorem 5.3.3 to calculate Igusa's local zeta function in the following examples. To find the cones associated to the proper faces of the Newton polyhedron, we use Proposition 1.1.10. To calculate the S_{Δ_τ}, we divide these cones into rational simplicial cones. To find the number of the solutions of the set $\mathbb{Z}^n \cap \{\sum \lambda_i a_i \mid 0 \leq \lambda_i < 1\}$, Proposition 1.2.5(i) can be useful. More examples can be found in [HL00].

Example 5.4.1. Let $f_1(x,y) = x^2y + x^8$. It is easy to verify that the Newton polyhedron $\Gamma(f_1)$ of f_1 is defined by the system of linear inequalities

$$\begin{cases} x \geq 2 \\ y \geq 0 \\ x + 6y \geq 8. \end{cases}$$

A picture of the Newton polyhedron of f_1 and its proper faces $\tau_1, \ldots, \tau_5$ is given in Figure 5.4.1. If p is prime, then it is easy to verify that f_1 is non-degenerated

over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron. So, we can use the formula for $Z_{f_1}(s)$ from Theorem 2.3.2. More information about the cones associated to the proper faces is contained in Table 5.4.1.

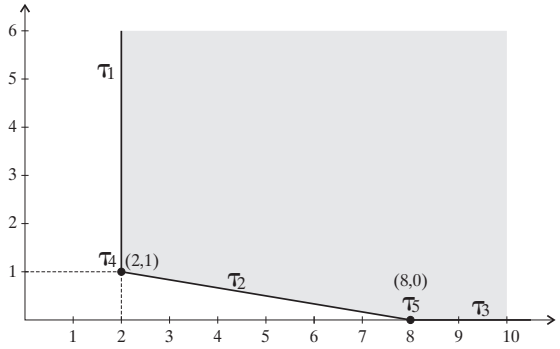


Figure 5.4.1. Newton polyhedron of $x^2y + x^8$

Proper face τ	Generators for Δ_τ	S_{Δ_τ}
τ_1	$(1,0)$	$\frac{1}{p^{1+2s}-1}$
τ_2	$(1,6)$	$\frac{1}{p^{7+8s}-1}$
τ_3	$(0,1)$	$\frac{1}{p-1}$
τ_4	$(1,0),(1,6)$	$\frac{1+p^{2+3s}+p^{3+4s}+p^{4+5s}+p^{5+6s}+p^{6+7s}}{(p^{1+2s}-1)(p^{7+8s}-1)}$
τ_5	$(0,1),(1,6)$	$\frac{1}{(p-1)(p^{7+8s}-1)}$

Table 5.4.1. The cones associated to the faces of $\Gamma(x^2y + x^8)$

The cone Δ_{τ_4} is rational simplicial, but the multiplicity of $(1,0),(1,6)$ is 6 and

$$\begin{aligned}&\mathbb{Z}^2 \cap \{\lambda_1(1,0) + \lambda_2(1,6) \mid 0 \leq \lambda_i < 1, i = 1,2\}\\ =\ &\{(0,0),(1,1),(1,2),(1,3),(1,4),(1,5)\}.\end{aligned}$$

The other cones Δ_τ are all simple cones. It is easy to see that $N_{\tau_2} = N_{\Gamma(f_1)} = \{(x,y) \in (\mathbb{F}_p^\times)^2 \mid x^2y + x^8\} = p-1$ and $N_{\tau_1} = N_{\tau_3} = N_{\tau_4} = N_{\tau_5} = 0$. According to the formula for $Z_{f_1}(s)$ in Theorem 2.3.2, the real candidate poles of $Z_{f_1}(s)$ are -1, $-1/2$ and $-7/8$, all with expected order 1. However, $\Gamma_{\mathrm{gl}}(f_1)$ is unstable over $\mathbb{F}_p$ relatively to y and with respect to f_1. So, if we use the formula for

$Z_{f_1}(s)$ from Theorem 5.3.3, we get

$$Z_{f_1}(s) = \tilde{L}_{\Gamma(f_1)} + \tilde{L}_{\tau_1} S_{\Delta_{\tau_1}}.$$

According to this formula, it is not surprising that $-7/8$ is not an actual pole of $Z_{f_1}(s)$. Indeed,

$$Z_{f_1}(s) = \frac{p^{3s}(p-1)^2}{(p^{1+2s}-1)(p^{1+s}-1)}.$$

Let $f_2(x,y) = x^2y + x^8 + x^{10}$ and $f_3(x,y) = x^2y + x^8 + x^{10}y^9$. If p is prime and $p \neq 2$, then it is easy to verify that f_2 is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron. If p is prime and $p \gg 0$, then f_3 is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron. So, we can use the formulas for $Z_{f_2}(s)$ and $Z_{f_3}(s)$ from Theorem 2.3.2 for those primes. The Newton polyhedra of f_1, f_2 and f_3 are all equal and so, also the S_{Δ_τ}. Again according to the formulas for $Z_{f_2}(s)$ and $Z_{f_3}(s)$ from Theorem 2.3.2, the number $-7/8$ is a real candidate pole of $Z_{f_2}(s)$ and $Z_{f_3}(s)$ with expected order 1. But by the same argument as for f_1, it is not surprising that the real number $-7/8$ is not an actual pole of $Z_{f_2}(s)$ because $\Gamma_{\mathrm{gl}}(f_2)$ is unstable over $\mathbb{F}_p$, relatively to y and with respect to f_2. Indeed,

$$Z_{f_2}(s) = \frac{p^{3s}(p-1)^2}{(p^{1+2s}-1)(p^{1+s}-1)}.$$

The set $\Gamma_{\mathrm{gl}}(f_3)$ is not unstable, but

$$\lim_{s\to -7/8}(p^{s+7/8}-1)Z_{f_3}(s) = \lim_{s\to -7/8}(p^{s+7/8}-1)Z_{f_1}(s) \qquad (5.4.1)$$

because $L_{\tau_2}(f_3) = L_{\tau_2}(f_1)$, $L_{\tau_4}(f_3) = L_{\tau_4}(f_1)$, $L_{\tau_5}(f_3) = L_{\tau_5}(f_1)$. One may also obtain (5.4.1) by applying Theorem 5.2.1 to f_3 and f_1. So, $-7/8$ is not an actual pole of $Z_{f_3}(s)$ either.

Remark. The zero sets of the polynomials f_1, f_2 and f_3 are normal crossing divisors. By the work of Igusa [Igu74a], one can also obtain that $-1/2$ and -1 are the only possible poles. One can also easily obtain the results above by the formula of Denef [Den87].

Example 5.4.2. Let $f_4(x,y,z) = xy + xz^2 + yz^2$. The Newton polyhedron $\Gamma(f_4)$ of f_4 is defined by the system of linear inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ 2x + 2y + z \geq 4 \\ 2y + z \geq 2 \\ 2x + z \geq 2 \\ x + y \geq 1. \end{cases}$$

We denote by $\tau_1, \ldots, \tau_7$ the facets with supporting hyperplanes given by the corresponding equalities in the inequalities above. A picture of the Newton polyhedron of f_4 is given in Figure 5.4.2.

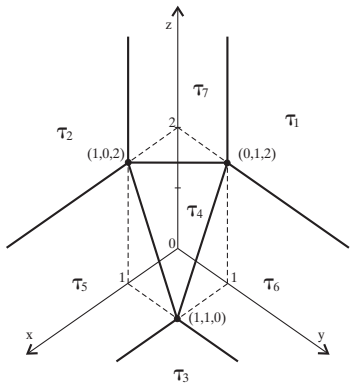


Figure 5.4.2. Newton polyhedron of $xy + xz^2 + yz^2$

There are 19 proper faces. One easily verifies that f_4 is non-degenerated over $\mathbb{F}_p$ with respect to all the faces of its Newton polyhedron, for every prime $p \neq 2$. So, we can use the formula for $Z_{f_4}(s)$ from Theorem 2.3.2 for those primes. According to this formula, the candidate real poles of $Z_{f_4}(s)$ are -1, $-5/4$, $-3/2$ and -2, all with expected order 1. The set of faces of $\Gamma(f_4)$ of maximal pure contribution to the candidate poles $-5/4$, $-3/2$, -2 are $\{\tau_4\}$, $\{\tau_5, \tau_6\}$, $\{\tau_7\}$ respectively (see Definition 3.1.2, Definition 3.3.1 and Proposition 3.3.3). Note that this example shows that two faces of $\Gamma(f_4)$ of maximal pure contribution to the same candidate pole do not have to be disjoint. However, $-3/2$ and -2 are not actual poles of $Z_{f_4}(s)$. Indeed, by using the formula for $Z_{f_4}(s)$ from Theorem 2.3.2 (see [HL00]), we get

$$Z_{f_4}(s) = \frac{p^{2s}(p-1)(p^{5+3s} + p^{2+s} - p^{2+2s} - p^{1+s} + p - 1)}{(p^{1+s}-1)(p^{5+4s}-1)}.$$

Theorem 5.3.8 explains why this happens. For example, $-3/2$ is not an actual pole because the faces τ_5, τ_6 of $\Gamma(f_4)$ of maximal pure contribution to the candidate pole $-3/2$ are unstable and in addition the following property is satisfied. The face τ_5 (resp. τ_6) is the only face of $\Gamma((f_4)_{\tau_5})$ (resp. $\Gamma((f_4)_{\tau_6})$) of maximal pure contribution to $-3/2$. A similar argument holds for -2.

Chapter 6

Determination of the real poles of maximal order

6.1 The case of the trivial character

From the formula for $Z_f(s)$ (resp. $Z_{f,0}(s)$) in Theorem 2.3.2 we can derive that, if s_0 is a candidate pole of $Z_f(s)$ (resp. $Z_{f,0}(s)$), then its order is $\leq n$ and $\mathrm{Re}\,(s_0)$ is also a candidate pole. Indeed, if s_0 is a pole of an S_{Δ_τ}, then $\mathrm{Re}\,(s_0)$ is also a pole of S_{Δ_τ} and the order of the pole s_0 is not larger than the order of the pole $\mathrm{Re}\,(s_0)$. Consequently, the maximal order of a pole of an S_{Δ_τ} is equal to n, and in this case, $\dim \Delta_\tau$ will be equal to n or equivalently, τ will be a vertex. Furthermore, if s_0 is a pole of an L_τ, then $\mathrm{Re}\,(s_0) = -1$ is also a pole of L_τ and they are both poles of order equal to 1. Moreover, if -1 is a pole of order n of an S_{Δ_τ}, then τ is a vertex. Consequently, -1 is in that case not a pole of L_τ because $N_\tau = 0$.

We shall prove the following results about the real poles of order n. The method we follow is similar to the method and the proofs of [LV99] or [Lae97, Chapter 6] in which similar results are derived for the topological zeta function.

Proposition 6.1.1. *Suppose that the prime number p and the polynomial f satisfy the conditions of Remark 2 after Theorem 2.3.2. Then*

(i) $Z_{f,0}(s)$ has at most one real pole of order n, and

(ii) if $Z_{f,0}(s)$ has in s_0 a real pole of order n, then $s_0 = -1/t_0$ and $t_0 \in \mathbb{N}\setminus\{0\}$.

Note that t_0 is defined in Definition 3.2.2.

Proposition 6.1.2. *Suppose that the prime number p and the polynomial f satisfy the conditions of Theorem 2.3.2. Then*

(i) $Z_f(s)$ has at most two real poles of order n, and

(ii) if $Z_f(s)$ has in s_0 a real pole of order n, then $s_0 = -1$ or $s_0 = -1/t_0$ and $t_0 \in \mathbb{N} \setminus \{0\}$.

Remark. If $s_0 = -1/t_0$ is a pole with $t_0 \in \mathbb{N} \setminus \{0\}$, then s_0 is the largest pole because it is the largest candidate pole.

Remark. The following result is known for general polynomials over $\mathbb{Z}_p$ [LV99].

Theorem 6.1.3. *Let f be a polynomial over $\mathbb{Z}_p$ in n variables and χ a character of $\mathbb{Z}_p^\times$. If s_1 is a pole of order n of $Z_f(s,\chi)$ or $Z_{f,0}(s,\chi)$, then $\mathrm{Re}\,(s_1) = -1/N$ with $N \in \mathbb{N} \setminus \{0\}$.*

Assertion (i) in Proposition 6.1.1 is a conjecture for general polynomials f over $\mathbb{Z}_p$, which Veys proved for $n = 2$ (with f a polynomial over $\mathbb{Z}$ and $p \gg 0$) in [Vey95]. We should remark that Assertion (i) in Proposition 6.1.2 is not true for general polynomials f over $\mathbb{Z}_p$.

Lemma 6.1.4. *Let $f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0) = 0$. Let V be a vertex of the Newton polyhedron $\Gamma(f)$ of f such that S_{Δ_V} has in s_0 a real pole of order n. Then $V = \{(-1/s_0, -1/s_0, \ldots, -1/s_0)\} = \{(t_0, t_0, \ldots, t_0)\}$.*

Proof. By Proposition 1.1.10 and Lemma 1.2.2, it follows that we can find a partition of Δ_V into rational simplicial cones Δ_i such that each Δ_i is strictly positively spanned by primitive linearly independent vectors from $\mathbb{N}^n \setminus \{0\}$ which are perpendicular to facets that contain V. Take then a Δ_i such that s_0 is a pole of order n of S_{Δ_i} and let $a_1,\ldots,a_n$ be such generators for Δ_i. Since S_{Δ_i} has in s_0 a real pole of order n, we know that $s_0 = -\sigma(a_j)/m(a_j)$ for $j = 1,\ldots,n$. An equation of the supporting hyperplane H_j of the facet $F(a_j)$ is $a_j \cdot x = m(a_j)$. Letting $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$ be the diagonal, we thus obtain that $H_j \cap D = \{(-1/s_0, -1/s_0, \ldots, -1/s_0)\}$ for $j = 1,\ldots,n$. From $V \subseteq \cap_{j=1}^n F(a_j) \subseteq \cap_{j=1}^n H_j$ and the fact that $a_1,\ldots,a_n$ are linearly independent, we can then derive that $V = \cap_{j=1}^n F(a_j) = \cap_{j=1}^n H_j$ and $V = \{(-1/s_0, -1/s_0, \ldots, -1/s_0)\}$. Now the result follows, because the diagonal D intersects the Newton polyhedron $\Gamma(f)$ only in the point $(t_0, t_0, \ldots, t_0)$. □

Lemma 6.1.5. *Let $f(x_1,\ldots,x_n)$ be a non-zero polynomial with $f(0) = 0$. Let τ be a 1-dimensional face of $\Gamma(f)$ such that S_{Δ_τ} has in -1 a pole of order $n-1$. Then $(1,1,\ldots,1)$ is contained in the affine line $\mathrm{aff}\,(\tau)$ through τ.*

Proof. As in the proof of Lemma 6.1.4, we can find a partition of Δ_τ into rational simplicial cones Δ_i such that each Δ_i is strictly positively spanned by primitive

linearly independent vectors from $\mathbb{N}^n \setminus \{0\}$ which are perpendicular to facets that contain τ. Take then a Δ_i such that -1 is a pole of order $n-1$ of S_{Δ_i} and let $a_1, \ldots, a_{n-1}$ such generators for Δ_i. Since S_{Δ_i} has in -1 a pole of order $n-1$, it follows that $\sigma(a_j) = m(a_j)$ for $j = 1, \ldots, n-1$. Letting H_j denote the supporting hyperplane of the facet $F(a_j)$, we thus have that $(1, 1, \ldots, 1) \in H_j$, for $j = 1, \ldots, n-1$. Since $\tau = \cap_{j=1}^{n-1} F(a_j) \subset \cap_{j=1}^{n-1} H_j$ and $a_1, \ldots, a_{n-1}$ are linearly independent, we obtain that $(1, 1, \ldots, 1) \in \cap_{j=1}^{n-1} H_j = \text{aff}\,(\tau)$. □

Lemma 6.1.6. *Let $f(x_1, \ldots, x_n)$ be a non-zero polynomial with $f(0) = 0$. Let τ be a 1-dimensional compact face of $\Gamma(f)$ such that the affine line* aff (τ) *through τ contains $(1, 1, \ldots, 1)$. Then $(1, 1, \ldots, 1) \in \tau$.*

Proof. [Lae97, Lemma 6.2.3] or [LV99, Lemma 2.3]. □

Remark. The condition "compact" cannot be omitted (see Figure 6.1.1).

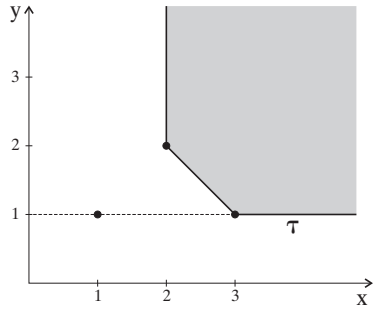


Figure 6.1.1

6.1.7. Proof of Proposition 6.1.1.
We can use the formula for $Z_{f,0}(s)$ from Remark 2 after Theorem 2.3.2. Suppose that s_0 is a real candidate pole of $Z_{f,0}(s)$ of order n. Suppose that there exists a vertex V of $\Gamma(f)$ such that S_{Δ_V} has in s_0 a real pole of order n. It then follows from Lemma 6.1.4 that $V = \{(-1/s_0, -1/s_0, \ldots, -1/s_0)\} = \{(t_0, t_0, \ldots, t_0)\}$. This implies that $t_0 \in \mathbb{N} \setminus \{0\}$. Suppose that there does not exist any vertex V of $\Gamma(f)$ such that S_{Δ_V} has in s_0 a pole of order n. Then s_0 must be equal to -1 and there exists a compact 1-dimensional face τ of $\Gamma(f)$ such that S_{Δ_τ} has in -1 a pole of order $n-1$. From Lemma 6.1.5 and Lemma 6.1.6, it follows then that $(1, 1, \ldots, 1) \in \tau$. Because the diagonal $\{(t, t, \ldots, t) \in \mathbb{R}^n\}$ intersects the boundary of $\Gamma(f)$ in $(t_0, t_0, \ldots, t_0)$, we obtain that $t_0 = 1$. This proves Assertion (ii). Assertion (i) follows immediately from Assertion (ii). □

6.1.8. Proof of Proposition 6.1.2.
Suppose that $s_0 \neq -1$ is a pole of order n of $Z_f(s)$. Then it follows from the formula for $Z_f(s)$ in Theorem 2.3.2 that there exists a vertex V of $\Gamma(f)$ such that S_{Δ_V} has in s_0 a pole of order n. As in the proof of Theorem 6.1.1, the Assertions (ii) and (i) then follow. □

6.2 The case of a non-trivial character

By a similar argument as in Section 6.1, we obtain the following result.

Proposition 6.2.1. *Suppose that the prime number p, the polynomial f and the non-trivial character χ satisfy the conditions of Theorem 2.3.2. Then*

(i) $Z_f(s,\chi)$ has at most one real pole of order n, and

(ii) $Z_f(s,\chi)$ has in s_0 a real pole of order n if and only if $s_0 = -1/t_0$, $t_0 \in \mathbb{N} \setminus \{0\}$ and $\{(t_0,\ldots,t_0)\}$ is a vertex of $\Gamma(f)$.

A similar result holds for $Z_{f,0}(s,\chi)$ if the prime number p, the polynomial f and the non-trivial character χ satisfy the conditions of Remark 2 after Theorem 2.3.2. Veys [Vey95] proved that $Z_{f,0}(s,\chi)$ has at most one real pole of order n for general polynomials f over $\mathbb{Z}$, $p \gg 0$ and $n = 2$.

Appendix A

The Haar measure of $\mathbb{Q}_p$

We refer to [Kob77] for some preliminaries about the p-adic numbers $\mathbb{Q}_p$. The set of p-adic numbers $\mathbb{Q}_p$ is a locally compact group. Indeed, $\mathbb{Q}_p$ is a topological group and the topology on $\mathbb{Q}_p$ determined by the p-adic valuation makes $\mathbb{Q}_p$ into a Hausdorff space. Moreover, the compact open sets of the form $x + p^k\mathbb{Z}_p$ with $x \in \mathbb{Q}_p$ and $k \in \mathbb{N}$ are a base for the open sets. A Haar measure on a locally compact and commutative group G is a map from the set of measurable sets of G to $[0, \infty]$ that satisfies a list of properties. One proves that there exists a unique Haar measure on G up to a normalization factor.

We give more details about the Haar measure on $\mathbb{Q}_p$. The open sets of $\mathbb{Q}_p$ are examples of measurable sets of $\mathbb{Q}_p$, but there are also other measurable sets. In fact, the set of measurable sets of $\mathbb{Q}_p$ is the smallest set of subsets of $\mathbb{Q}_p$ that contains the open sets of $\mathbb{Q}_p$ and satisfies the following three properties:

(M1) if a set E is measurable, then also its complement $\mathbb{Q}_p \setminus E$ must be measurable;

(M2) if $(E_n)_{n\geq 1}$ is a series of measurable sets, then also its union $\bigcup_{n\geq 1} E_n$ must be measurable.

This implies that also finite intersections of measurable sets, the closed sets of $\mathbb{Q}_p$ and so, also the compact sets of $\mathbb{Q}_p$ are measurable.

A Haar measure μ on $\mathbb{Q}_p$ is a (non-identically zero) map from the set of measurable sets of $\mathbb{Q}_p$ to $[0, \infty]$ that satisfies the following properties:

(H1) $\mu(\emptyset) = 0$;

(H2) if $(E_n)_{n\geq 1}$ is a series of disjoint measurable sets, then

$$\mu\left(\bigcup_{n\geq 1} E_n\right) = \sum_{n\geq 1} \mu(E_n);$$

(H3) for every compact set C of $\mathbb{Q}_p$ one has $\mu(C) < \infty$;

(H4) μ is invariant under translations, i.e., for every measurable set E of $\mathbb{Q}_p$ and every $x \in \mathbb{Q}_p$ one has $\mu(x + E) = \mu(E)$;

and for completeness also

(H5) for every measurable set E, one has

$$\mu(E) = \inf\{\mu(O) \mid O \text{ open in } \mathbb{Q}_p, O \supseteq E\};$$

(H6) for every open set O of $\mathbb{Q}_p$, one has

$$\mu(O) = \sup\{\mu(C) \mid C \text{ compact in } \mathbb{Q}_p, C \subset O\}.$$

The Haar measure μ on $\mathbb{Q}_p$ is completely determined by supposing that the measure of the compact set $\mathbb{Z}_p$ is equal to 1, because all Haar measures on $\mathbb{Q}_p$ are equal up to a normalization factor.

We show that the fact that $\mu(\mathbb{Z}_p) = 1$ implies that $\mu(x + p^k\mathbb{Z}_p) = p^{-k}$ for every $x \in \mathbb{Q}_p$ and every $k \in \mathbb{N}$. The set $\mathbb{Z}_p$ is equal to the disjoint union $\bigcup_{a\in\{0,\ldots,p^k-1\}} a + p^k\mathbb{Z}_p$. So,

$$\begin{aligned}
1 = \mu(\mathbb{Z}_p) &= \sum_{a\in\{0,\ldots,p^k-1\}} \mu(a + p^k\mathbb{Z}_p) && \text{by (H2)} \\
&= \sum_{a\in\{0,\ldots,p^k-1\}} \mu(p^k\mathbb{Z}_p) && \text{by (H4)} \\
&= p^k\mu(p^k\mathbb{Z}_p) \\
&= p^k\mu(x + p^k\mathbb{Z}_p) && \text{by (H4).}
\end{aligned}$$

Now it is clear that $\mu(x + p^k\mathbb{Z}_p) = p^{-k}$.

From the properties of the valuation of $\mathbb{Q}_p$, it follows that every open set of $\mathbb{Q}_p$ is the disjoint union of sets of the form $x + p^k\mathbb{Z}_p$ with $x \in \mathbb{Q}_p$ and $k \in \mathbb{N}$. Now the measure of every measurable set of $\mathbb{Q}_p$ is determined by (H2) and (H5). The Haar measure on $\mathbb{Q}_p^n$ is the product of n Haar measures of $\mathbb{Q}_p$.

Once the measure is defined, the integral is defined as in the classical case.

Appendix B

Convex geometry

This chapter is based on [Roc70], [MS71] and [Zie95] and recalls some notions and results from convex geometry.

B.1 Convex sets

Definition B.1.1. A subset C of $\mathbb{R}^n$ is said to be *convex* if and only if for all $x, y \in C$, and $0 \leq \lambda \leq 1$, the point $\lambda x + (1-\lambda)y$ also belongs to C.

We define the *dimension of a convex set* $C(\subset \mathbb{R}^n)$ as the dimension of the affine hull of C. The *affine hull* $\operatorname{aff}(C)$ of C is the smallest affine subset of $\mathbb{R}^n$ that contains C.

The following two properties are easy to verify.

Property B.1.2. *A subset C of $\mathbb{R}^n$ is convex if and only if C contains all the* convex combinations *of its elements, i.e.,*

$$\forall k \in \mathbb{N} \setminus \{0\}, \forall x_1, \ldots, x_k \in C, \forall \lambda_1, \ldots, \lambda_k \geq 0 \text{ with } \sum_{i=1}^{k} \lambda_i = 1 : \ \sum_{i=1}^{k} \lambda_i x_i \in C.$$

Property B.1.3. *The intersection of an arbitrary collection of convex sets is convex.*

Definition B.1.4. Let $X \subset \mathbb{R}^n$. The intersection of all the convex sets containing X is called the *convex hull* of X and is denoted by $\operatorname{conv} X$. It is a convex set by Property B.1.3 and by Property B.1.2 we have

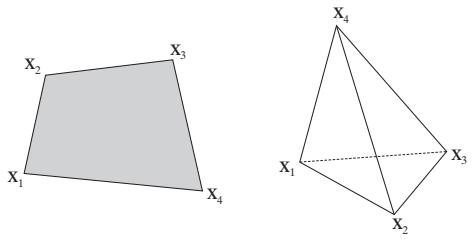
$$\operatorname{conv} X = \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbb{N} \setminus \{0\}, x_i \in X, \lambda_i \geq 0 \text{ with } \sum_{i=1}^{k} \lambda_i = 1 \right\}. \quad \text{(B.1.1)}$$

Actually, it suffices to take convex combinations involving $\dim(\operatorname{conv} X) + 1$ elements of X at a time in (B.1.1) [Roc70, Theorem 17.1].

The convex hull of finitely many points is called a *polytope*. If these points are affinely independent, its convex hull is called a *simplex*.

A *pyramid* is the convex hull of a polytope (the base) and a point (the top) that is not contained in the affine hull of this polytope.

Example B.1.5. $\operatorname{conv}\{x_1, x_2, x_3, x_4\} = \{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \mid \lambda_i \geq 0 \text{ with } \sum_{i=1}^{4} \lambda_i = 1\}$.



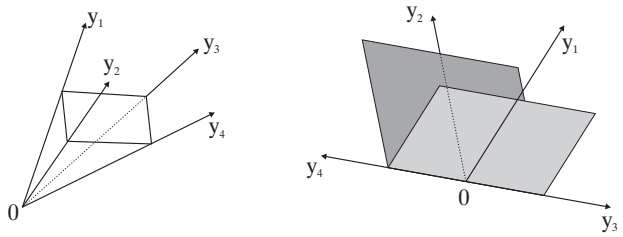
Definition B.1.6. A convex subset K of $\mathbb{R}^n$ such that $\forall x \in K, \forall \lambda > 0 : \lambda x \in K$ is called a *convex cone*.

Let $Y \subset \mathbb{R}^n$. The intersection of all convex cones in $\mathbb{R}^n$ that contain Y and the origin is denoted by CONE Y. One proves that this is a convex cone that equals

$$\text{CONE } Y = \left\{ \sum_{i=1}^{l} \mu_i y_i \mid l \in \mathbb{N} \setminus \{0\}, y_i \in Y, \mu_i \geq 0 \right\}.$$

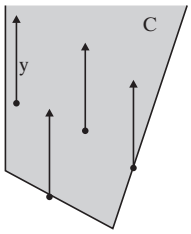
If Y is a finite set of points, CONE Y is called a *polyhedral cone*. If moreover Y is a finite subset of $\mathbb{Q}^n$, CONE Y is called a *rational polyhedral cone*.

Example B.1.7. $\text{CONE } \{y_1, y_2, y_3, y_4\} = \{\mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 + \mu_4 y_4 \mid \mu_i \geq 0\}$.



One proves that the origin is a face (see Definition B.4.1) of a polyhedral cone if and only if C contains no (non-zero) linear subspace.

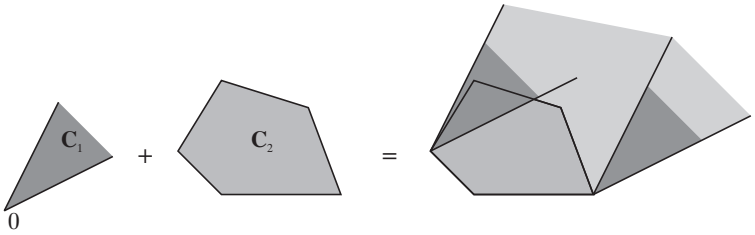
Definition B.1.8. Let C be a non-empty convex set in $\mathbb{R}^n$ and y a non-zero vector in $\mathbb{R}^n$. We say that C *recedes* in the direction of y if and only if $x+\lambda y \in C$ for every $\lambda \geq 0$ and every $x \in C$. In this case we call y a *direction of recession* of C.



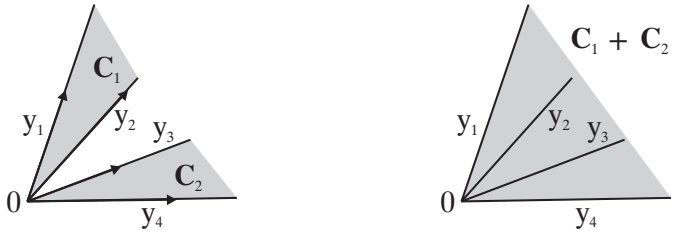
Property B.1.9. *Let C be a non-empty closed convex set (e.g., a polyhedron; see Section B.2), and let $y \neq 0$. If there exists even one $x \in C$ such that the half-line $\{x + \lambda y \mid \lambda \geq 0\}$ is contained in C, then this is true for every $x \in C$, i.e., y is a direction of recession of C.*

Proof. [Roc70, Theorem 8.3]. □

Property B.1.10. *Let C_1, C_2 be two convex sets. Then their sum $C_1 + C_2 = \{x_1 + x_2 \mid x_1 \in C_1, x_2 \in C_2\}$ is also convex.*



The sum of two polyhedral cones is again a polyhedral cone.



B.2 Polyhedra

Definition B.2.1. The following two definitions of a *polyhedron* or a *polyhedral convex set* are equivalent (see [Zie95, Chapter 1] or [Roc70, Theorem 19.1]).

1. A polyhedron in $\mathbb{R}^n$ is the intersection of some finite collection of closed half-spaces, i.e., the set of solutions to some finite system of inequalities of the form
$$x \cdot a_i \geq b_i \qquad i = 1, \ldots, m,$$
where $x \cdot a_i$ denotes the scalar product of x and a_i.

2. A polyhedron in $\mathbb{R}^n$ is the convex hull of a finite number of points $x_1, \ldots, x_k \in \mathbb{R}^n$ and a finite number of directions of recession $y_1, \ldots, y_l \in \mathbb{R}^n$, i.e., the intersection of all convex sets that contain these points $x_1, \ldots, x_k$ and recede in these directions $y_1, \ldots, y_l$:
$$\begin{aligned} &\operatorname{conv}\{x_1, \ldots, x_k\} + \operatorname{CONE}\{y_1, \ldots, y_l\} \\ &= \left\{ \textstyle\sum_{i=1}^k \lambda_i x_i + \sum_{i=1}^l \mu_i y_i \mid \lambda_i, \mu_i \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\}. \end{aligned}$$

Special cases:

A *polytope* is

1. the bounded intersection of some finite collection of closed half-spaces, or equivalently,

2. the convex hull of a finite number of points.

A *polyhedral cone* is

1. the set of solutions to some finite system of inequalities of the form
$$x \cdot a_i \geq 0, \ i = 1 \ldots m,$$
or equivalently,

2. the convex hull of the origin and a finite number of directions of recession.

B.3 Some topological notions

Definition B.3.1. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Denote the open ball in $\mathbb{R}^n$ with radius ϵ and center x by $B(x, \epsilon)$.

Let C be a set in $\mathbb{R}^n$. We say:

- x is contained in the *closure* $\operatorname{cl} C$ of C if and only if $\forall \epsilon > 0 : B(x, \epsilon) \cap C \neq \emptyset$;
- x is contained in the *interior* $\operatorname{int} C$ of C if and only if $\exists \epsilon > 0 : B(x, \epsilon) \subset C$;
- x is contained in the *relative interior* $\operatorname{ri} C$ of C if and only if $x \in \operatorname{aff} C$ and $\exists \epsilon > 0 : B(x, \epsilon) \cap \operatorname{aff} C \subset C$;
- x is contained in the *boundary* $\operatorname{bd} C$ of C if and only if $x \in \operatorname{cl} C \setminus \operatorname{ri} C$, or equivalently, if and only if $\forall \epsilon > 0 : B(x, \epsilon) \cap \operatorname{aff} C$ contains an element in C and an element outside C;
- the set C is *relatively open* if and only if $\operatorname{ri} C = C$.

Property B.3.2. *Let $x_1, \ldots, x_k, y_1, \ldots, y_l$ be elements of $\mathbb{R}^n$. Then we have*

(i) $\operatorname{ri}(\operatorname{conv}\{x_1, \ldots, x_k\}) = \left\{ \sum_{i=1}^k \lambda_i x_i \mid \lambda_i > 0,\ \sum_i^k \lambda_i = 1 \right\}$,

(ii) $\operatorname{ri}(\text{CONE } \{y_1, \ldots, y_l\}) = \left\{ \sum_{i=1}^l \mu_i y_i \mid \mu_i > 0 \right\}$ *and*

(iii) $\operatorname{ri}(\operatorname{conv}\{x_1, \ldots, x_k\} + \text{CONE } \{y_1, \ldots, y_l\})$
$= \left\{ \sum_{i=1}^k \lambda_i x_i + \sum_{i=1}^l \mu_i y_i \mid \mu_i, \lambda_i > 0,\ \sum_i^k \lambda_i = 1 \right\}$.

Remark.
We denote $\left\{ \sum_{i=1}^l \mu_i y_i \mid \mu_i > 0 \right\}$ by $\operatorname{cone}\{y_1, \ldots, y_l\}$ (see Definition 1.2.1).

Proof. [Roc70, Corollary 6.6.2 and Theorem 6.9]. □

Property B.3.3. *Let C be a convex set in $\mathbb{R}^n$. Then we have*

$$\operatorname{ri} C \subset C \subset \operatorname{cl} C \subset \operatorname{aff}(C) = \operatorname{aff}(\operatorname{ri} C) = \operatorname{aff}(\operatorname{cl} C).$$

Consequently, $\operatorname{ri} C$ *and* $\operatorname{cl} C$ *have the same dimension as C. In particular,* $\operatorname{ri} C \neq \emptyset$*, if $C \neq \emptyset$.*

Proof. [Roc70, Theorem 6.2]. □

B.4 Faces of a convex set C

Definition B.4.1. Let C be a convex set of $\mathbb{R}^n$.

(i) A *face* of the convex set C is a non-empty convex subset F of C such that every (closed) line segment in C with a relatively interior point in F has both endpoints in F.

(ii) Let H be the hyperplane $\{x \in \mathbb{R}^n \mid x \cdot a = b\}$. Then H determines two closed half-spaces:

$$\begin{aligned} H^+ &= \{x \in \mathbb{R}^n \mid x \cdot a \geq b\} \quad \text{and} \\ H^- &= \{x \in \mathbb{R}^n \mid x \cdot a \leq b\}. \end{aligned}$$

We say that H is a *supporting hyperplane* of the convex set C, if $H \cap C \neq \emptyset$ and C is contained in one of the closed half-spaces determined by H. The intersection $H \cap C$ is called an *exposed face* of C.

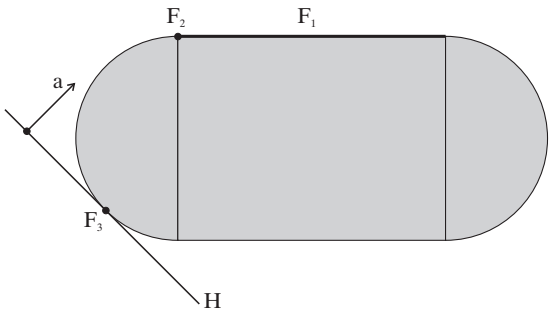


Figure B.4.1. F_1, F_2 and F_3 are faces; F_1 and F_3 are exposed faces of the set.

Remarks.

1. The convex set C is called an *improper face* of C. The other faces of C are said to be *proper*. Some authors also call the empty set an (improper) face of C, but in this thesis we shall not call the empty set a face of C.

2. The faces of C of dimension 0 are called *vertices*, the faces of C of dimension 1 are called *edges*. (Sometimes, we also call the unique element in a vertex *a vertex*.) The faces of C of codimension 1 (i.e., of dimension equal to $\dim C - 1$) are called *facets*.

3. If F is an exposed face of C, then F is also a face of C. On the other hand, not all faces of a convex set C are exposed faces. For instance, take

the face F_2 in Figure B.4.1. Following the ideas of [MS71], one can prove that for a polyhedron P, all proper faces of P are exposed. Nevertheless, some properties about faces of a polyhedron are more easy to verify with Definition B.4.1(i).

Properties B.4.2.

1. *The intersection of a finite collection of faces of C is again a face of C or is empty.*

2. *If F_1 is a face of C and F_2 is a face of F_1, then F_2 is also a face of C [Roc70, p. 163]. In general this is not true if one replaces "face" by "exposed face". For instance, look at Example B.4.1.*

3. *Let F be a face of C. If D is a convex set in C such that $\operatorname{ri} D$ meets F, then $D \subset F$ [Roc70, Theorem 18.1].*

   Consequently, if F_1 and F_2 are faces of C such that $\operatorname{ri} F_1$ and $\operatorname{ri} F_2$ have a point in common, then actually $F_1 = F_2$ [Roc70, Corollary 18.1.2].

   This also implies that a proper face of C is contained in the boundary $\operatorname{bd} C$ of C [Roc70, Corollary 18.1.3].

   If C is a closed convex set, then $\operatorname{bd} C = \bigcup_{\substack{F \text{ face} \\ \text{of } C}} F$ *[Roc70, Theorem 18.2].*

   If P is a polyhedron, then $\operatorname{bd} C = \bigcup_{\substack{F \text{ facet} \\ \text{of } P}} F$.

4. *Let C be the convex hull of a set S, where S is a set of points and directions of recession, and let F be a face of C. Then F is the convex hull of S', where S' consists of the points in S which belong to F and the directions of recession in S which are directions of recession of F [Roc70, Theorem 18.3].*

   Consequently, a face of a polyhedron/polytope/polyhedral cone is again a polyhedron/polytope/polyhedral cone. Moreover, a polyhedron has only a finite number of faces.

Following the ideas of [MS71, Chapter 2], one proves the following properties.

Properties B.4.3. *Let P be a polyhedron in $\mathbb{R}^n$.*

(i) A proper face of P is contained in at least one facet of P.

(ii) A face of P of codimension 2 is the intersection of two facets of P. Consequently, every face τ of P is the intersection of all facets of P that contain τ.

(iii) *Let $\tau^j \subset \tau^k$ be faces of P of dimension j and k, respectively, with $0 \leq j < k$. Then for $i = j+1, \ldots, k-1$, there exist faces τ^i of P such that*

$$\tau^j \subsetneq \tau^{j+1} \subsetneq \ldots \subsetneq \tau^{k-1} \subsetneq \tau^k.$$

Appendix C

The condition of non-degeneracy

C.1 The results

Proposition C.1.1. *Fix a Newton polyhedron P in $\mathbb{R}^n$. Almost all polynomials $f(x_1, \ldots, x_n)$ over $\mathbb{C}$ with Newton polyhedron equal to P are non-degenerated over $\mathbb{C}$ with respect to all the faces of $\Gamma(f) = P$.*

Corollary C.1.2. *Fix a Newton polyhedron P in $\mathbb{R}^n$. Almost all polynomials $f(x_1, \ldots, x_n)$ over $\mathbb{Q}_p$ with Newton polyhedron equal to P are non-degenerated over $\mathbb{Q}_p$ with respect to all the faces of $\Gamma(f) = P$.*

We explain what we mean by "almost all". Put $T_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \sum_{i=1}^n x_i \leq \lambda\}$ for each λ (where λ is big enough such that T_λ contains all vertices of P). Denote the set of vertices of P by V and the set of other points in $\mathbb{Z}^n \cap T_\lambda$ that are contained in P by F_λ . Note that V and F_λ are finite. See Figure C.1.1.

There exists a one-to-one and onto map between the set $\prod_{k\in V} \mathbb{C}^\times \prod_{k\in F_\lambda} \mathbb{C}$ and the polynomials f over $\mathbb{C}$ with Newton polyhedron $\Gamma(f)$ equal to P and supp (f) contained in T_λ. By "almost all" in Proposition C.1.1 we mean that there exists a non-zero polynomial g over $\mathbb{C}$ such that the elements in $\prod_{k\in V} \mathbb{C}^\times \prod_{k\in F_\lambda} \mathbb{C}$ that correspond to polynomials that are degenerated over $\mathbb{C}$, are contained in the zero-set of g. The meaning of "almost all" in Corollary C.1.2 is similar, one simply has to replace $\mathbb{C}$ by $\mathbb{Q}_p$.

Proposition C.1.3. *Fix a Newton polyhedron P in $\mathbb{R}^n$ and a proper face τ_0 of P. Let G be the convex hull of the vertices of τ_0 and a finite number of other fixed points in $\tau_0 \cap \mathbb{Z}^n$. Almost all polynomials $f(x_1, \ldots, x_n)$ over $\mathbb{C}$ (resp. $\mathbb{Q}_p$)*

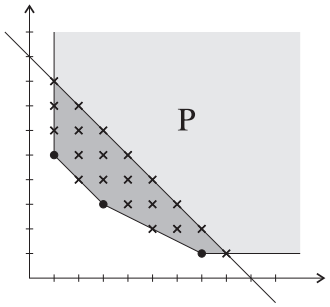


Figure C.1.1. The points marked with $\cdot$ (resp. $\times$) are contained in V (resp. F_λ).

with Newton polyhedron $\Gamma(f)$ equal to P and global Newton polyhedron of f_{τ_0} equal to G are non-degenerated over $\mathbb{C}$ (resp. $\mathbb{Q}_p$) with respect to all the faces of $\Gamma(f)$ and are such that f_{τ_0} is non-degenerated over $\mathbb{C}$ (resp. $\mathbb{Q}_p$) with respect to $G = \Gamma_{\mathrm{gl}}(f_{\tau_0})$.

We explain what we mean by "almost all" for the case of the complex numbers. For the case of $\mathbb{Q}_p$, one simply has to replace $\mathbb{C}$ by $\mathbb{Q}_p$. Put again $T_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \sum_{i=1}^n x_i \leq \lambda\}$ for each λ (where λ is now big enough such that T_λ contains G and all vertices of P). Denote the set of vertices of G or P by W and the set of other points in $\mathbb{Z}^n \cap T_\lambda$ that belong to G or to $P \setminus \tau_0$ by H_λ. There exists a one-to-one and onto map between the set $\prod_{k\in W} \mathbb{C}^\times \prod_{k\in H_\lambda} \mathbb{C}$ and the polynomials f over $\mathbb{C}$ with Newton polyhedron $\Gamma(f)$ equal to P, global Newton polyhedron of f_{τ_0} equal to G and $\mathrm{supp}\,(f)$ contained in T_λ. By "almost all" we mean that there exists a non-zero polynomial g over $\mathbb{C}$ such that the elements of $\prod_{k\in W} \mathbb{C}^\times \prod_{k\in H_\lambda} \mathbb{C}$ for which the corresponding polynomial does not satisfy the required condition are contained in the zero-set of g.

C.2 The proofs

The proofs of Proposition C.1.1 and Proposition C.1.3 are similar to [AVGZ86, p. 157]. For convenience of the reader, we give all details.

Theorem C.2.1. *Let $f : U \subset \mathbb{C}^m \to \mathbb{C}^n$ with U open, $n \leq m$ and $f = (f_1, \ldots, f_n)$, where each f_i is a quotient of polynomials over $\mathbb{C}$ whose denominator does not vanish on U. Let $C = \{x \in U \mid Df(x)$ is not onto $\}$, where $Df(x)$ is the total derivative of f in the element x. Then $f(C)$, the set of critical values of f, has measure 0.*

Proof. This result is well-known. For convenience of the reader, we include a proof. We show that this theorem follows from the analogous result for $\mathbb{C}$

replaced by $\mathbb{R}$, which is a special case of the Theorem of Sard [Mil65, p. 10].

Because the partial derivatives of the f_i exist and are continuous on U, the total derivative of f exists for every element of U. Define

$$\tilde{U} = \{(x_1, y_1, \ldots, x_m, y_m) \in \mathbb{R}^{2m} \mid (x_1 + iy_1, \ldots, x_m + iy_m) \in U\}$$

and

$$\begin{array}{rccl} \tilde{f}: & \tilde{U} & \to & \mathbb{R}^{2n} \\ & (x_1, y_1, \ldots, x_m, y_m) & \mapsto & (\operatorname{Re} f_1(x_1 + iy_1, \ldots, x_m + iy_m), \\ & & & \operatorname{Im} f_1(x_1 + iy_1, \ldots, x_m + iy_m), \\ & & & \ldots, \\ & & & \operatorname{Re} f_n(x_1 + iy_1, \ldots, x_m + iy_m), \\ & & & \operatorname{Im} f_n(x_1 + iy_1, \ldots, x_m + iy_m)). \end{array}$$

Clearly $\tilde{U}$ is open and if $Df(x_1 + iy_1, \ldots, x_m + iy_m)$ is the linear map with matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix},$$

then $D\tilde{f}(x_1, y_1, \ldots, x_m, y_m)$ is the linear map with matrix

$$\tilde{A} = \begin{pmatrix} \operatorname{Re}(a_{11}) & -\operatorname{Im}(a_{11}) & \cdots & \operatorname{Re}(a_{1m}) & -\operatorname{Im}(a_{1m}) \\ \operatorname{Im}(a_{11}) & \operatorname{Re}(a_{11}) & \cdots & \operatorname{Im}(a_{1m}) & \operatorname{Re}(a_{1m}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \operatorname{Re}(a_{n1}) & -\operatorname{Im}(a_{n1}) & \cdots & \operatorname{Re}(a_{nm}) & -\operatorname{Im}(a_{nm}) \\ \operatorname{Im}(a_{n1}) & \operatorname{Re}(a_{n1}) & \cdots & \operatorname{Im}(a_{nm}) & \operatorname{Re}(a_{nm}) \end{pmatrix}.$$

From the condition on the f_i, it follows that each coordinate function of $\tilde{f}$ is a quotient of polynomials over $\mathbb{R}$ in $x_1, y_1, \ldots, x_m, y_m$ whose denominator does not vanish on $\tilde{U}$. So, for $\tilde{f}$, the conditions of the Theorem of Sard are satisfied. Put $\tilde{C} = \{x_1, y_1, \ldots, x_m, y_m) \in \tilde{U} \mid D\tilde{f}(x_1, y_1, \ldots, x_m, y_m)$ is not onto$\}$. Then $\tilde{f}(\tilde{C}) \subset \mathbb{R}^{2n}$ has measure 0. Note that $D\tilde{f}(x_1, y_1, \ldots, x_m, y_m)$ is not onto if and only if $Df(x_1 + iy_1, \ldots, x_m + iy_m)$ is not onto. As a result, also $f(C)$ has measure 0. □

C.2.2. Proof of Proposition C.1.1.
Suppose $V = \{\omega_1, \ldots, \omega_s\}$ and $F_\lambda = \{\omega_{s+1}, \ldots, \omega_{s+t}\}$, where V and F_λ are defined on page 111. Let A be the complement in $(\mathbb{C}^\times)^s \times \mathbb{C}^t$ of the set $\{(a_1, \ldots, a_s, a_{s+1}, \ldots, a_{s+t}) \in (\mathbb{C}^\times)^s \times \mathbb{C}^t \mid f(x) = \sum a_i x^{\omega_i}$ is non-degenerated over $\mathbb{C}$ with respect to all the faces of $\Gamma(f) = P\}$.
Then

$$A = \bigcup_{\substack{\tau \text{ face} \\ \text{of } P}} A_\tau,$$

where $A_\tau = \{(a_1,\dots,a_s,a_{s+1},\dots,a_{s+t}) \in (\mathbb{C}^\times)^s \times \mathbb{C}^t \mid \exists x_1,\dots,x_n \in \mathbb{C}^\times : f_\tau(a_1,\dots,a_{s+t},x_1,\dots,x_n) = 0,\ (\partial f_\tau/\partial x_i)(a_1,\dots,a_{s+t},x_1,\dots,x_n) = 0, i = 1,\dots,n\}$, where $f_\tau(a_1,\dots,a_{s+t},x_1,\dots,x_n) = \sum_{\omega_i\in\tau} a_i x^{\omega_i}$. Because the projection of a constructible set of $\mathbb{C}^{s+t+n}$ is constructible by [Mum76, Proposition 2.31] or [Hod93, Theorem 2.7.3], it suffices to prove that the measure of each set A_τ in this union in zero.

Fix a face τ of P. Without loss of generality, we may suppose that ω_1 is a vertex of τ. Consider the mapping $\pi : (\mathbb{C}^\times)^{s-1} \times \mathbb{C}^t \times (\mathbb{C}^\times)^n \to \mathbb{C}^{s+t} : (a_2,\dots,a_{s+t},x_1\dots x_n) \mapsto$

$$\left(\tilde{a}_1(a_2,\dots,a_{s+t},x_1,\dots,x_n) = \frac{-\sum\limits_{\omega_i\in\tau\setminus\{\omega_1\}} a_i x^{\omega_i}}{x^{\omega_1}}, a_2,\dots,a_{s+t}\right).$$

Then π satisfies the conditions of Theorem C.2.1. Therefore, the set of the critical values of π has measure zero. It suffices now to prove that every element of A_τ is a critical value of π. Take an arbitrary element $(a_1,\dots,a_{s+t})$ from A_τ and $x_1,\dots,x_n \in \mathbb{C}^\times$ such that

$$\left\{\begin{array}{rcll} f_\tau(a_1,\dots,a_{s+t},x_1,\dots,x_n) & = & 0 & \\ \frac{\partial f_\tau}{\partial x_i}(a_1,\dots,a_{s+t},x_1,\dots,x_n) & = & 0, & i = 1,\dots,n \end{array}\right. \tag{C.2.1}$$

The Jacobian $D_\pi(a_2,\dots,a_{s+t},x_1,\dots,x_n)$ equals

$$\left(\begin{array}{ccccccc} * & \cdots & * & \frac{\partial\tilde{a}_1}{\partial x_1}(a_2,\dots,a_{s+t},x_1,\dots,x_n) & \cdots & \frac{\partial\tilde{a}_1}{\partial x_n}(a_2,\dots,a_{s+t},x_1,\dots,x_n) \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right).$$

Recall that x^{ω_1} is a notation for $x_1^{\omega_{11}}\cdots x_n^{\omega_{1n}}$ with $\omega_1 = (\omega_{11},\dots,\omega_{1n})$. One calculates by (C.2.1) that

$$\begin{array}{ll} \frac{\partial\tilde{a}_1}{\partial x_i} & (a_2,\dots,a_{s+t},x_1,\dots,x_n) \\ & = -\frac{x_i^{\omega_{1i}}}{x^{\omega_1}} \frac{(\partial f_\tau/\partial x_i)(a_1,\dots,a_{s+t},x_1,\dots,x_n) x_i^{\omega_{1i}} - \omega_{1i} x_i^{\omega_{1i}-1} f_\tau(a_1,\dots,a_{s+t},x_1,\dots,x_n)}{(x_i^{\omega_{1i}})^2} = 0. \end{array}$$

Consequently, $D_\pi(a_2,\dots,a_{s+t},x_1,\dots,x_n)$ is not onto. Note that from $f_\tau(a_1,\dots,a_{s+t},x_1,\dots,x_n) = 0$, it follows that

$$-\frac{\sum\limits_{\omega_i\in\tau\setminus\{\omega_1\}} a_i x^{\omega_i}}{x^{\omega_1}} = a_1.$$

So, $\pi(a_2,\dots,a_{s+t},x_1,\dots,x_n) = (a_1,a_2,\dots,a_{s+t})$ is a critical value of π. $\square$

C.2.3. Proof of Corollary C.1.2.
Because $\bar{\mathbb{Q}}_p$ and $\mathbb{C}$ are algebraically closed fields of the same cardinality (bigger than the cardinality of $\mathbb{N}$), one proves that the fields $\bar{\mathbb{Q}}_p$ and $\mathbb{C}$ are isomorphic fields. From Proposition C.1.1, it follows then that there exists a non-zero polynomial g over $\bar{\mathbb{Q}}_p$ such that the elements in $\prod_{k\in V}\mathbb{Q}_p^{\times}\prod_{k\in F_\lambda}\mathbb{Q}_p$ that correspond to a polynomial that is degenerated over $\mathbb{Q}_p$ are contained in the zero-set of g. The coefficients of g are contained in a finite extension of $\mathbb{Q}_p$ and therefore in a finite normal extension K of $\mathbb{Q}_p$. Let $\sigma_1,\ldots,\sigma_k$ be the elements in the Galois group of K with respect to $\mathbb{Q}_p$. Then the zero's of g in $\mathbb{Q}_p$ are contained in the zero-set of the polynomial $g^{\sigma_1}\cdots g^{\sigma_k}$ over $\mathbb{Q}_p$, where g^{σ_i} is the polynomial obtained by replacing each coefficient of g by its image under σ_i. □

C.2.4. Proof of Proposition C.1.3.
The proof is similar to the proof of Proposition C.1.1 and Corollary C.1.2. □

Appendix D

Polygusa

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Nederlandse samenvatting

Igusa's lokale zetafunctie is een rationale functie die geassocieerd wordt met een veelterm f met coëfficiënten in de p-adische gehele getallen en waarbij p een priemgetal is. De bedoeling van dit proefschrift is Igusa's lokale zetafunctie te bestuderen voor veeltermen f die niet-gedegenereerd zijn t.o.v. hun Newtonpolyheder. Voor zulke veeltermen f stellen we een expliciete formule voor Igusa's lokale zetafunctie op. Met behulp van deze formule bewijzen we vervolgens resultaten die ons informatie geven over de polen van deze rationale functie.

Definitie van Igusa's lokale zetafunctie, Newtonpolyheder en niet-gedegenereerdheid

Zij p een priemgetal. Noteer het veld van de p-adische getallen met $\mathbb{Q}_p$, de ring van de p-adische gehele getallen met $\mathbb{Z}_p$ en het eindig veld met p elementen met $\mathbb{F}_p$. Als R een commutatieve ring met eenheid is, noteren we de verzameling van zijn eenheden met $R^\times$.

Definitie N.0.1. Zij $f(x) = f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ met p een priemgetal. Voor $z \in \mathbb{Q}_p$, is $\operatorname{ord} z \in \mathbb{Z} \cup \{\infty\}$ de notatie voor zijn valuatie, $|z| = p^{-\operatorname{ord} z}$ en zijn angulaire component is $\operatorname{ac}(z) = p^{-\operatorname{ord} z} z$. Zij $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ een karakter op $\mathbb{Z}_p^\times$, d.w.z. een groepsmorfisme met eindig beeld. We spreken af dat $\chi(0) = 0$. Aan de bovenvermelde gegevens associëren we de volgende twee Igusa lokale zetafuncties (de globale en de lokale):

$$\begin{aligned} Z_f(s, \chi) &= \int_{\mathbb{Z}_p^n} \chi(\operatorname{ac} f(x)) |f(x)|^s |dx|, \\ \text{en } Z_{f,0}(s, \chi) &= \int_{(p\mathbb{Z}_p)^n} \chi(\operatorname{ac} f(x)) |f(x)|^s |dx|, \end{aligned}$$

voor $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, waarbij $|dx|$ de Haarmaat is op $\mathbb{Q}_p^n$, zodanig genormalizeerd dat $\mathbb{Z}_p^n$ maat 1 heeft. Als χ gelijk is aan het triviale karakter χ_{triv}, dan

noteren we de zetafunctie met $Z_f(s)$ (resp. $Z_{f,0}(s)$) in plaats van met $Z_f(s, \chi_{\text{triv}})$ (resp. $Z_{f,0}(s, \chi_{\text{triv}})$).

Igusa's lokale zetafunctie $Z_f(s)$ voor het triviale karakter bevat alle informatie betreffende het aantal oplossingen modulo p^e van de congruenties $f(x) \equiv 0 \bmod p^e$ voor $e \in \mathbb{N} \setminus \{0\}$; zie Lemma N.0.3 of [Igu78, pp. 97–98], [Den91, Paragraaf 1.2].

Definitie N.0.2. Zij $f(x) = f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ met p een priemgetal. We definiëren de Poincaré-reeks van f als de formele machtreeks

$$P(t) = \sum_{e=0}^{\infty} N_e p^{-ne} t^e,$$

waarbij N_e het aantal elementen is van de verzameling

$$\{x + p^e \mathbb{Z}^n \mid x \in \mathbb{Z}^n \text{ en } f(x) \equiv 0 \bmod p^e\}$$

voor $e \geq 1$ en $N_0 = 1$.

Tussen de Poincaré-reeks van f en Igusa's lokale zetafunctie $Z_f(s)$ geassocieerd aan f bestaat er de volgende relatie:

Lemma N.0.3. *Zij $f(x) = f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$, p een priemgetal en $t = p^{-s}$ met $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$. Dan geldt*

$$Z_f(s) = P(t) - \frac{P(t) - 1}{t} \qquad \text{en} \qquad P(t) = \frac{1 - tZ_f(s)}{1 - t}.$$

Igusa [Igu74a] bewees met behulp van resoluties van singulariteiten dat $Z_f(s, \chi)$ (en dus ook $P(t)$) een rationale functie is in p^{-s} (zie ook [Igu78]). Denef [Den84] gaf tien jaar later een alternatief bewijs met behulp van p-adische celdecompositie. We noteren de meromorfe uitbreiding van $Z_f(s, \chi)$ opnieuw met $Z_f(s, \chi)$.

Merk op dat het feit dat $Z_f(s)$ een rationale functie is in p^{-s} impliceert dat oneindig veel informatie, namelijk alle getallen N_e voor $e \in \mathbb{N}$, bevat is in het eindige aantal coëfficiënten van teller en noemer van de rationale zetafunctie. De polen van $Z_f(s)$ en hun ordes bepalen hoe het aantal oplossingen van $f(x) \equiv 0 \bmod p^e$ zich gedraagt als e groot wordt. Bovendien geven de polen met het grootste reële deel de grootste bijdrage tot deze aantallen. Over de polen bestaan er verscheidene vermoedens: namelijk een verband met topologische invarianten van de singulariteiten van het hyperoppervlak $f(x) = 0$. Op dit ogenblik heeft men echter geen enkel idee hoe deze vermoedens in het algemeen bewezen kunnen worden. De bedoeling van dit proefschrift is de polen van Igusa's lokale zetafunctie te bestuderen voor een speciale maar belangrijke klasse van veeltermen, namelijk de veeltermen die niet-gedegenereerd zijn t.o.v. hun Newtonpolyheder. Het is de befaamde Russische school van V. Arnold die in de jaren

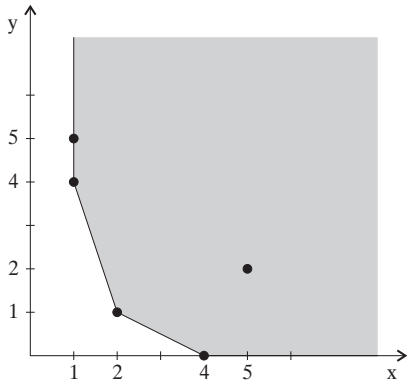
'70 tot het belangrijke inzicht kwam dat voor deze klasse van veeltermen vele aspecten afgelezen kunnen worden uit de meetkundige eigenschappen van hun Newtonpolyheder. Lichtin en Meuser [LM85] bestudeerden als eersten Igusa's lokale zetafunctie voor niet-gedegenereerde veeltermen t.o.v. hun Newtonpolyheder, en dit voor veeltermen in twee veranderlijken.

Wij bestuderen niet-gedegenereerde veeltermen in een willekeurig aantal veranderlijken. Voor zulke veeltermen stellen we een expliciete formule voor Igusa's lokale zetafunctie op die ons zal helpen om informatie te verkrijgen over haar polen. Voor de basisnoties van convexe meetkunde, verwijzen we naar [Roc70] of Appendix B.

Definitie N.0.4. Zij $f(x) = f(x_1, \ldots, x_n) = \sum_{\omega \in \mathbb{N}^n} a_\omega x_1^{\omega_1} \cdots x_n^{\omega_n}$ een niet-nulveelterm over $\mathbb{Z}_p$ met $f(0) = 0$. Zij $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ en $\mathrm{supp}\,(f) = \{\omega \in \mathbb{N}^n \mid a_\omega \neq 0\}$ de drager van f. De *Newtonpolyheder* $\Gamma(f)$ van f is gedefinieerd als de convexe omhullende in $(\mathbb{R}^+)^n$ van de verzameling

$$\bigcup_{\omega \in \mathrm{supp}\,(f)} \omega + (\mathbb{R}^+)^n.$$

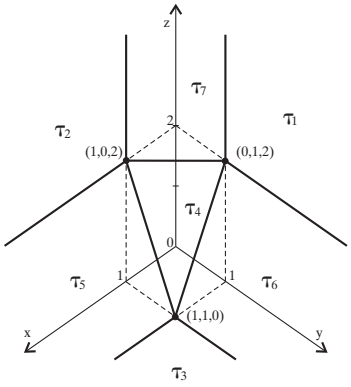
De *globale Newtonpolyheder* $\Gamma_{\mathrm{gl}}(f)$ van f is gedefinieerd als de convexe omhullende van $\mathrm{supp}\,(f)$. Men kan nagaan dat $\Gamma(f) = \Gamma_{\mathrm{gl}}(f) + (\mathbb{R}^+)^n$.



Figuur N.0.1. De Newtonpolyheder van $xy^4 + xy^5 + x^2y + x^4 + x^5y^2$

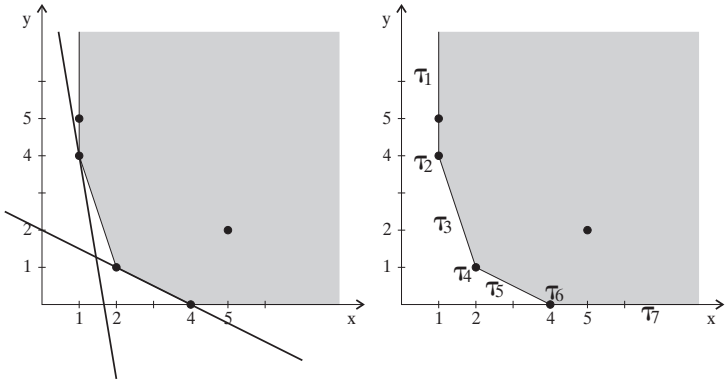
Omdat een Newtonpolyheder een polyheder is, is elke echte zijde van $\Gamma(f)$ de doorsnede van $\Gamma(f)$ met een dragend hypervlak. Zij H het hypervlak $\{x \in \mathbb{R}^n \mid x \cdot a = b\}$, waarbij we met $x \cdot a$ het scalair product van x met a bedoelen. Dan bepaalt H twee gesloten halfruimten:

$$\begin{array}{rcl} H^+ & = & \{x \in \mathbb{R}^n \mid x \cdot a \geq b\} \quad \text{en} \\ H^- & = & \{x \in \mathbb{R}^n \mid x \cdot a \leq b\}. \end{array}$$



Figuur N.0.2. De Newtonpolyheder van $xy + xz^2 + yz^2$

We zeggen dat H een *dragend hypervlak* is van $\Gamma(f)$, als $H \cap \Gamma(f) \neq \emptyset$ en $\Gamma(f)$ bevat is in één van de gesloten halfruimten bepaald door H. **Met de zijden van $\Gamma(f)$ bedoelen we de echte zijden van $\Gamma(f)$ en de Newtonpolyheder $\Gamma(f)$ zelf.**



Figuur N.0.3. De echte zijden van een Newtonpolyheder

In Definitie N.0.6 geven we een partitie van $(\mathbb{R}^+)^n$ in verzamelingen die nauw samenhangen met de Newtonpolyheder van een veelterm f. Eerst hebben we Definitie N.0.5 nodig.

Definitie N.0.5. Voor $a \in (\mathbb{R}^+)^n$, definiëren we

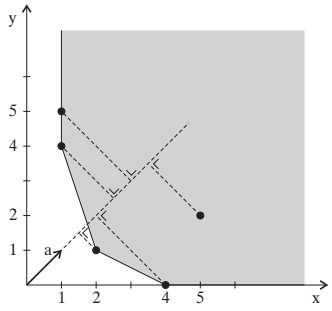
$$m(a) = \inf_{x \in \Gamma(f)} \{a \cdot x\}$$

en de *eerste contactzijde van* a als

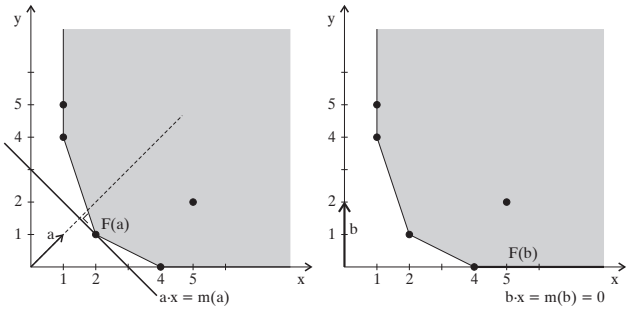
$$F(a) = \{x \in \Gamma(f) \mid a \cdot x = m(a)\}.$$

Opmerking.

$$\begin{aligned} m(a) &= \inf_{x \in \Gamma_{\mathrm{gl}}(f)} \{a \cdot x\} \quad (\Gamma(f) = \Gamma_{\mathrm{gl}}(f) + (\mathbb{R}^+)^n) \\ &= \min_{x \in \Gamma_{\mathrm{gl}}(f)} \{a \cdot x\} \quad (\Gamma_{\mathrm{gl}}(f) \text{ is gesloten en begrensd}) \\ &= \min_{x \in \operatorname{supp} f} \{a \cdot x\} \quad (\Gamma_{\mathrm{gl}}(f) = \operatorname{conv}(\operatorname{supp} f)). \end{aligned}$$



Figuur N.0.4. Berekening van $m(a)$



Figuur N.0.5. De eerste contactzijde van $a = (1, 1)$ en $b = (0, 2)$.

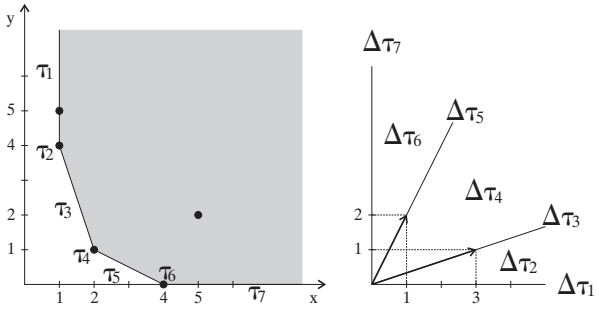
Men kan gemakkelijk aantonen dat $F(a)$, de eerste contactzijde van a, een zijde is van $\Gamma(f)$. In het bijzonder geldt dat $F(0) = \Gamma(f)$ en dat $F(a)$ een echte zijde is van $\Gamma(f)$ indien $a \neq 0$. Bovendien is $F(a)$ een compacte zijde als en slechts als $a \in (\mathbb{R}^+ \setminus \{0\})^n$.

Definitie N.0.6. We definiëren de volgende equivalentierelatie op $(\mathbb{R}^+)^n$:

$$a \sim a' \qquad \text{als en slechts als} \qquad F(a) = F(a').$$

Als τ een zijde is van $\Gamma(f)$, definiëren we de *kegel geassocieerd aan* τ als

$$\Delta_\tau = \{a \in (\mathbb{R}^+)^n \mid F(a) = \tau\}.$$



Figuur N.0.6. De kegels geassocieerd aan de zijden van een Newtonpolyheder

Duidelijk is $\Delta_{\Gamma(f)} = \{0\}$. We geven in Propositie N.0.7 een interessante beschrijving van de andere equivalentieklassen Δ_τ. Men kan aantonen dat de sluiting $\bar{\Delta}_\tau$ van Δ_τ gelijk is aan $\{a \in (\mathbb{R}^+)^n \mid F(a) \supset \tau\}$ en dat Δ_τ het relatief inwendige is van $\bar{\Delta}_\tau$.

Een vector $a \in \mathbb{R}^n$ noemt men *primitief* als de componenten gehele getallen zijn die onderling ondeelbaar zijn. Men kan aantonen dat er voor elk facet[1] van $\Gamma(f)$ een unieke primitieve vector in $\mathbb{N}^n \setminus \{0\}$ bestaat die loodrecht op dit facet staat.

Propositie N.0.7. *Zij τ een echte zijde van $\Gamma(f)$ en $\gamma_1, \ldots, \gamma_r$ de facetten van $\Gamma(f)$ die τ bevatten. Zij $a_1, \ldots, a_r$ de unieke primitieve vectoren in $\mathbb{N}^n \setminus \{0\}$ die loodrecht staan op respectievelijk $\gamma_1, \ldots, \gamma_r$. Dan geldt*

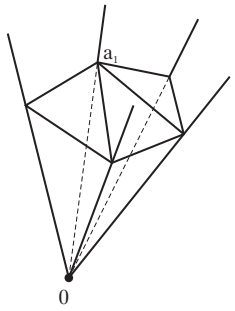
$$\bar{\Delta}_\tau = \left\{ \sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \right\} \quad \textit{en} \quad \Delta_\tau = \left\{ \sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0 \right\}.$$

Bovendien geldt dat $\dim \Delta_\tau = \dim \bar{\Delta}_\tau = \operatorname{codim} \tau = n - \dim \tau$.

Definitie N.0.8. Als $a_1, \ldots, a_r \in \mathbb{R}^n \setminus \{0\}$, dan noemen we $\operatorname{cone}(a_1, \ldots, a_r) = \{\sum_{i=1}^r \lambda_i a_i \mid \lambda_i \in \mathbb{R}, \lambda_i > 0\}$ *de kegel die strikt positief opgespannen wordt door de vectoren* $a_1, \ldots, a_r$. Fixeer een kegel Δ. Als er lineair onafhankelijke vectoren $a_1, \ldots a_r \in \mathbb{R}^n \setminus \{0\}$ bestaan zodat $\Delta = \operatorname{cone}(a_1, \ldots, a_r)$, dan zeggen we dat Δ een *simpliciale kegel* is. Indien we bovendien $a_1, \ldots, a_r \in \mathbb{Z}^n$ (of in $\mathbb{Q}^n$) kunnen nemen, dan zeggen we dat Δ een *rationale simpliciale kegel* is.

[1] Een facet is een zijde met codimensie 1.

Zij τ een echte zijde van $\Gamma(f)$ en neem $a_1, \ldots, a_r$ zoals in Propositie N.0.7. Men kan aantonen dat het altijd mogelijk is om een partitie van Δ_τ in rationale simpliciale kegels Δ_i te geven zodat elke kegel Δ_i voortgebracht wordt door een stel lineair onafhankelijke vectoren uit de verzameling $\{a_1, \ldots, a_r\}$. Men noemt dit een opdeling zonder nieuwe generatoren te introduceren. Zie Figuur N.0.7.



Figuur N.0.7. Een partitie in kleinere kegels

Definitie N.0.9. Zij p een priemgetal. Zij f zoals in Definitie N.0.4. Voor elke zijde τ van de Newtonpolyheder $\Gamma(f)$ van f definiëren we

$$f_\tau(x) = \sum_{\omega \in \tau} a_\omega x^\omega,$$

en de veelterm $\bar{f}_\tau(x)$ met coëfficiënten in $\mathbb{F}_p$ door elke coëfficiënt van f_τ te reduceren modulo $p\mathbb{Z}_p$.

We zeggen dat f *niet-gedegenereerd is over* $\mathbb{F}_p$ *t.o.v. alle zijden van zijn Newtonpolyheder* $\Gamma(f)$, als voor elke[2] zijde τ van $\Gamma(f)$ de locus van de veelterm $\bar{f}_\tau$ geen singulariteiten in $(\mathbb{F}_p^\times)^n$ heeft, of nog als de verzameling congruenties

$$\begin{cases} f_\tau(x) & \equiv & 0 \bmod p \\ \frac{\partial f_\tau}{\partial x_i}(x) & \equiv & 0 \bmod p, \quad i = 1, \ldots, n \end{cases}$$

geen oplossing heeft in $(\mathbb{Z}_p^\times)^n$.

We zeggen dat f *niet-gedegenereerd is over* $\mathbb{F}_p$ *t.o.v. de compacte zijden van zijn Newtonpolyheder*, indien net dezelfde voorwaarden voldaan zijn, maar enkel voor de compacte zijden τ van $\Gamma(f)$.

[2] Dus ook voor $\Gamma(f)$.

Opmerkingen.

1. Zij $f(x) = f(x_1, \ldots, x_n)$ een niet-nulveelterm over $\mathbb{Z}$ met $f(0) = 0$. Veronderstel dat f niet-gedegenereerd is over $\mathbb{C}$ t.o.v. alle zijden van zijn Newtonpolyheder (d.w.z. voor elke zijde τ van $\Gamma(f)$ geldt dat de locus van f_τ geen singulariteiten in $(\mathbb{C}^\times)^n$ heeft). Dan is f niet-gedegenereerd over $\mathbb{F}_p$ t.o.v. alle zijden van zijn Newtonpolyheder voor elk priemgetal $p \gg 0$. De notatie $p \gg 0$ betekent dat p groot genoeg moet zijn.

2. Fixeer een Newtonpolyheder P in $\mathbb{R}^n$. "Bijna alle" veeltermen $f(x_1, \ldots, x_n)$ over $\mathbb{C}$ met Newtonpolyheder gelijk aan P zijn niet-gedegenereerd over $\mathbb{C}$ t.o.v. alle zijden van $\Gamma(f)$. Aan een veelterm f met Newtonpolyheder P (en waarbij de drager $\operatorname{supp}(f)$ bevat is in een gegeven begrensde verzameling) kunnen we een n_P-tal associëren bestaande uit de coëfficiënten van f. Met "bijna alle" bedoelen we dat de n_P-tallen van de veeltermen die gedegeneerd zijn, bevat zijn in de nulpuntsverzameling van een niet-nulveelterm g over $\mathbb{C}$. Het bewijs is analoog met [AVGZ86, p. 157]; zie ook Appendix C.

In de volgende paragrafen geven we een overzicht van de hoofdresulaten uit dit proefschrift.

N.1 Een expliciete formule voor $Z_f(s, \chi)$

Stelling N.1.2, het eerste hoofdresultaat, geeft een expliciete formule voor $Z_f(s, \chi)$ (zie ook [DH01] en [Hoo01]). Deze formule is geldig indien f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van zijn Newtonpolyheder en indien de conductor c_χ van χ gelijk is aan 1. De conductor c_χ van het karakter χ van $\mathbb{Z}_p^\times$ is gedefinieerd als het kleinste natuurlijk getal $c \in \mathbb{N} \setminus \{0\}$ zodat χ triviaal is op $1 + p^c\mathbb{Z}_p$. In het bijzonder is $c_{\chi_{\text{triv}}}$ gelijk aan 1. Een gekend resultaat [Den91, Theorem 3.3] zegt dat $Z_f(s, \chi)$ een constante functie is in s voor $p \gg 0$, indien de conductor $c_\chi > 1$ en f een veelterm is over $\mathbb{Z}$. Voor de exacte voorwaarde van een gelijkaardig resultaat voor f een veelterm over $\mathbb{Z}_p$ met p een vast priemgetal verwijzen we naar dezelfde referentie. Dus Stelling N.1.2 geeft een formule voor $Z_f(s, \chi)$ voor de relevante gevallen.

Definitie N.1.1. Voor $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$, definiëren we

$$\sigma(k) = \sum_{i=1}^{n} k_i.$$

Stelling N.1.2. *Zij p een priemgetal. Zij f zoals in Definitie N.0.4. Veronderstel dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van zijn Newtonpolyheder $\Gamma(f)$. Zij χ een karakter op $\mathbb{Z}_p^\times$ met conductor $c_\chi = 1$. Noteer voor elke*

zijde τ van $\Gamma(f)$ met N_τ het aantal elementen in de verzameling

$$\{a \in (\mathbb{F}_p^\times)^n \mid \bar{f}_\tau(a) = 0\}.$$

Zij s een complexe variabele met $\mathrm{Re}\,(s) > 0$. Dan geldt dat

$$Z_f(s,\chi) = \sum_{\substack{\tau \text{ zijde} \\ \text{van } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

met

$$L_\tau = \begin{cases} p^{-n}\left((p-1)^n - pN_\tau \frac{p^s-1}{p^{s+1}-1}\right) & \text{voor } \chi = \chi_{\mathrm{triv}}, \\ p^{-n} \sum\limits_{a \in (\mathbb{F}_p^\times)^n} \chi\left(f_\tau(a)\right) & \text{voor } \chi \neq \chi_{\mathrm{triv}}, \end{cases}$$

en

$$S_{\Delta_\tau} = \sum_{k \in \mathbb{N}^n \cap \Delta_\tau} p^{-\sigma(k) - m(k)s},$$

voor elke zijde τ van $\Gamma(f)$ (dus ook voor $\tau = \Gamma(f)$).

Duidelijk is $S_{\Delta_{\Gamma(f)}} = 1$. De overige S_{Δ_τ} kunnen als volgt berekend worden. Neem een partitie van de kegel Δ_τ geassocieerd aan de echte zijde τ in rationale simpliciale kegels Δ_i. Dan is $S_{\Delta_\tau} = \sum_i S_{\Delta_i}$, waarbij de som loopt over de rationale simpliciale kegels Δ_i en

$$S_{\Delta_i} = \sum_{k \in \mathbb{N}^n \cap \Delta_i} p^{-\sigma(k) - m(k)s}.$$

Zij Δ_i de kegel die strikt positief opgespannen wordt door de lineair onafhankelijke vectoren $a_1, \ldots, a_r \in \mathbb{N}^n \setminus \{0\}$. Dan is

$$S_{\Delta_i} = \frac{\sum\limits_h p^{\sigma(h) + m(h)s}}{(p^{\sigma(a_1) + m(a_1)s} - 1) \cdots (p^{\sigma(a_r) + m(a_r)s} - 1)},$$

waarbij h loopt over de elementen in de verzameling

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^r \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ voor } j = 1, \ldots, r \right\}.$$

Opmerkingen.

1. De S_{Δ_τ} hangen enkel af van de Newtonpolyheder van f. De L_τ daarentegen zijn afhankelijk van de concrete coëfficiënten van f. Merk op dat S_{Δ_τ} en L_τ rationale functies in p^{-s} zijn, voor $s \in \mathbb{C}$ met $\mathrm{Re}\,(s) > 0$. We noteren hun meromorfe uitbreiding opnieuw met respectievelijk S_{Δ_τ} en L_τ. Stelling N.1.2 toont dus nog eens opnieuw aan dat $Z_f(s,\chi)$ een rationale functie is in p^{-s}, en dit voor het speciaal geval dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van zijn Newtonpolyheder.

2. We tonen bovendien aan dat

$$Z_{f,0}(s,\chi) = \sum_{\substack{\tau \text{ compacte zijde} \\ \text{van } \Gamma(f)}} L_\tau S_{\Delta_\tau},$$

met L_τ en S_{Δ_τ} zoals in de bovenstaande stelling. Voor deze formule hebben we enkel nodig dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. de *compacte* zijden van $\Gamma(f)$.

In samenwerking met Davy Loots hebben we software ontwikkeld, gebaseerd op de bovenstaande formule, om $Z_f(s)$ uit te rekenen voor een concrete veelterm f. Dit programma, Polygusa [HL00], is geschreven in Maple en geeft bovendien nog allerlei andere relevante informatie, zoals bijvoorbeeld, een tekening van $\Gamma(f)$, de kegels en de N_τ geassocieerd aan de zijden τ van $\Gamma(f)$, de topologische zetafunctie van f. Polygusa kan nuttig zijn om voorbeelden te genereren en wordt reeds gebruikt door verscheidene wiskundigen die Igusa zetafuncties bestuderen.

N.2 De kandidaat-polen van $Z_f(s,\chi)$

In deze paragraaf bestuderen we de polen van Igusa's lokale zetafunctie door gebruik te maken van de formule uit Stelling N.1.2. Zo verkrijgen we een lijst van mogelijke polen van $Z_f(s,\chi)$ (die we kandidaat-polen noemen), tezamen met bovengrenzen voor hun ordes (die we verwachte ordes noemen). Dit alles kan uitgedrukt worden in termen van bepaalde zijden van $\Gamma(f)$ en hun dimensie. Bovendien kunnen we m.b.v. de formule uit Stelling N.1.2 een tweede hoofdresultaat aantonen. Dit resultaat zegt dat $Z_f(s)$ steeds een reële pool heeft (in de veronderstelling dat f niet-gedegenereerd is t.o.v. zijn Newtonpolyheder). We kunnen zelfs informatie geven over de grootste reële pool en zijn orde.

De lijst van kandidaat-polen van $Z_f(s,\chi)$ verkrijgen we door de unie te nemen van de polen van de L_τ en de polen van de S_{Δ_i} in de $S_{\Delta_\tau} = \sum_i S_{\Delta_i}$. Omdat je de som

$$\sum_{\substack{\tau \text{ zijde} \\ \text{van } \Gamma(f)}} L_\tau S_{\Delta_\tau}$$

neemt, kan het in speciale gevallen gebeuren dat sommige kandidaat-polen geen echte polen van $Z_f(s,\chi)$ zijn. Na op gelijke noemer te zetten en te vereenvoudigen kan het immers gebeuren dat sommige polen wegvallen. Toch verkrijg je zo wel de enige *mogelijke* polen.

Propositie N.2.1. *Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Zij $\gamma_1,\ldots,\gamma_r$ alle facetten van $\Gamma(f)$ en zij $a_1,\ldots,a_r$ de unieke primitieve vectoren in $\mathbb{N}^n \setminus \{0\}$ die loodrecht staan op respectievelijk $\gamma_1,\ldots,\gamma_r$. Dan geldt het volgende:*

(i) Als s_1 een pool is van $Z_f(s)$, dan is

$$\begin{aligned} s_1 &= -1 + i\tfrac{k2\pi}{\log_e p} \quad \textit{met } k \in \mathbb{Z}, \textit{ of} \\ s_1 &= -\tfrac{\sigma(a_j)}{m(a_j)} + i\tfrac{k2\pi}{m(a_j)\log_e p}, \end{aligned}$$

met $k \in \mathbb{Z}$ en $j \in \{1,\ldots,r\}$ zodat $m(a_j) \neq 0$.

(ii) Als s_1 een pool is van $Z_f(s)$, dan is $\mathrm{Re}\,(s_1)$ gelijk aan -1 of is $\mathrm{Re}\,(s_1)$ van de vorm $-1/t_1$, waarbij $(t_1,t_1,\ldots,t_1)$ het snijpunt is van de diagonaal $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$ met het dragend hypervlak van een facet van $\Gamma(f)$.

(iii) Veronderstel dat s_1 een pool is van $Z_f(s,\chi)$ met $\chi \neq \chi_{\mathrm{triv}}$, dan is

$$s_1 = -\frac{\sigma(a_j)}{m(a_j)} + i\frac{k2\pi}{m(a_j)\log_e p},$$

met $k \in \mathbb{Z}$ en $j \in \{1,\ldots,r\}$ zodat $m(a_j) \neq 0$. Bovendien is $\mathrm{Re}\,(s_1)$ van de vorm $-1/t_1$, waarbij $(t_1,t_1,\ldots,t_1)$ het snijpunt is van de diagonaal $D = \{(t,t,\ldots,t) \in \mathbb{R}^n\}$ met het dragend hypervlak van een facet van $\Gamma(f)$.

In dit proefschrift bewijzen we de bovenstaande propositie m.b.v. de formule uit Stelling N.1.2. Deze propositie is ook een speciaal geval van[Zn99, Theorem A] en het geval van het triviale karakter is ook een onmiddellijk gevolg van [Den95].

Opmerkingen.

1. Hetzelfde resultaat geldt voor $Z_{f,0}(s)$ en $Z_{f,0}(s,\chi)$. Natuurlijk moet f dan enkel niet-gedegenereerd zijn over $\mathbb{F}_p$ t.o.v. de *compacte* zijden van $\Gamma(f)$ (zie Opmerking 2 na Stelling N.1.2).

2. We noemen de complexe getallen in Propositie N.2.1(i) en (iii) de *kandidaat-polen* van respectievelijk $Z_f(s)$ en $Z_f(s,\chi)$. De eerste kandidaat-polen in (i) (met reëel deel gelijk aan -1) zijn afkomstig van de L_τ, de tweede in (i) en degenen in (iii) zijn afkomstig van de S_{Δ_τ}. In de volgende paragraaf geven we meer informatie over de de grootste reële kandidaat-pool die afkomstig is van de S_{Δ_τ}.

Als men de formule uit Stelling N.1.2 bestudeert, kan men de volgende bovengrenzen voor de ordes van de polen van $Z_f(s,\chi)$ opstellen:

Definitie N.2.2. Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Veronderstel dat s_1 een kandidaat-pool is van $Z_f(s,\chi)$, d.w.z. een getal uit de verzameling beschreven in Propositie N.2.1. We definiëren *de verwachte orde van de kandidaat-pool* s_1 van $Z_f(s,\chi)$ als

$$\max\{\text{orde van de pool } s_1 \text{ voor } L_\tau S_{\Delta_\tau} \mid \tau \text{ een zijde van } \Gamma(f)\}, \qquad \text{(N.2.1)}$$

waarbij we zeggen dat de orde van s_1 gelijk is aan 0, als s_1 geen pool is en waarbij L_τ en S_{Δ_τ} gedefinieerd zijn zoals in Stelling N.1.2. Als $s_1 \neq -1$ of $\chi \neq \chi_{\text{triv}}$, dan laten we L_τ weg in (N.2.1).

In Paragraaf N.2.1 en Paragraaf N.2.2 geven we resultaten die de verwachte ordes uitdrukken in termen van de dimensies van bepaalde zijden van $\Gamma(f)$.

N.2.1 De grootste reële kandidaat-pool die afkomstig is van de S_{Δ_τ} en zijn verwachte orde

Definitie N.2.3. Zij $f(x_1, \ldots, x_n)$ een niet-nulveelterm met $f(0) = 0$. Noteer het unieke snijpunt van de diagonaal $D = \{(t, t, \ldots, t) \in \mathbb{R}^n\}$ met de rand van de Newtonpolyheder $\Gamma(f)$ van f met $(t_0, t_0, \ldots, t_0)$. Noteer de kleinste zijde van $\Gamma(f)$ die $(t_0, t_0, \ldots, t_0)$ bevat met τ_0 en zijn codimensie met κ. Verder definiëren we $\rho = -1/t_0$.

We zoeken meer informatie over de grootste reële kandidaat-pool van $Z_f(s, \chi)$ die afkomstig is van de S_{Δ_τ}. Omdat men kan aantonen dat $\sigma(a) - m(a)(1/t_0) \geq 0$ voor elke $a \in (\mathbb{R}^+)^n$ (met gelijkheid als en slechts als $a \in \bar{\Delta}_{\tau_0}$), volgt uit Propositie N.2.1 dat deze kandidaat-pool gelijk is aan ρ. Uit het volgende resultaat kan men halen dat zijn verwachte orde gelijk is aan κ of $\kappa + 1$:

> *Zij τ een echte zijde van $\Gamma(f)$. Als $\tau \not\subset \tau_0$ en ρ een pool is van S_{Δ_τ}, dan is zijn orde $< \kappa$. In het andere geval is ρ een pool van S_{Δ_τ} van orde κ.*

Het bewijs van dit resultaat steunt op de studie van het aantal (lineair onafhankelijke) generatoren a van de Δ_i die tot $\bar{\Delta}_{\tau_0}$ behoren (met $\cup_i \Delta_i$ een partitie van Δ_τ), d.w.z. het aantal generatoren a waarvoor $-\sigma(a)/m(a) = -1/t_0$. Merk op dat indien $\rho = -1$ en $\chi = \chi_{\text{triv}}$, het getal ρ ook een pool kan zijn van de L_τ. Samengevat verkrijgen we dus het volgende bekende resultaat als een gevolg van de formule uit Stelling N.1.2:

Propositie N.2.4. *Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Dan heeft men voor elke pool s_1 van $Z_f(s, \chi)$ dat $\text{Re}(s_1) \leq \rho$, indien $\chi \neq \chi_{\text{triv}}$; anders is $\text{Re}(s_1) = -1$ of $\text{Re}(s_1) \leq \rho$. Bovendien is de verwachte orde van de kandidaat-pool ρ voor $Z_f(s, \chi)$ gelijk aan $\kappa + 1$, als $\chi = \chi_{\text{triv}}$, $\rho = -1$ en er een zijde $\tau \subset \tau_0$ is zodat $N_\tau \neq 0$; anders is de verwachte orde gelijk aan κ.*

Opmerking. Indien bovendien $t_0 > 1$, dan zegt een reeds gekend resultaat dat ρ een pool is van orde κ van $Z_f(s)$ (zie ook Stelling N.2.7). In Paragraaf N.3 en Paragraaf N.4 bestuderen we het geval dat $t_0 \leq 1$.

N.2.2 De verwachte orde van een willekeurige kandidaat-pool van $Z_f(s,\chi)$

In Propositie N.2.6 drukken we ook de verwachte orde van een willekeurige kandidaat-pool van $Z_f(s,\chi)$ uit in termen van de dimensie van bepaalde zijden van $\Gamma(f)$. Hiervoor formuleren we eerst de volgende definitie:

Definitie N.2.5. Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Veronderstel dat s_1 een kandidaat-pool is van $Z_f(s,\chi)$, d.w.z. een getal uit de verzameling beschreven in Propositie N.2.1. We zeggen dat de vector $a \in (\mathbb{R}^+)^n \setminus \{0\}$ of het facet $F(a)$ *bijdraagt tot de kandidaat-pool* s_1 als $p^{\sigma(a)+m(a)s_1} = 1$, d.w.z. als $\mathrm{Re}\,(s_1) = -\sigma(a)/m(a)$ en er een $k \in \mathbb{Z}$ bestaat zodat $\mathrm{Im}\,(s_1) = k2\pi/(m(a)\log_e p)$. We zeggen dat een zijde van $\Gamma(f)$ een *zijde is van zuivere bijdrage tot de kandidaat-pool* s_1, als elk facet dat τ bevat, bijdraagt tot deze kandidaat-pool s_1. We definiëren $\kappa(s_1)$ als het maximum van de codimensies van dergelijke zijden. We noemen zo'n zijde van codimensie $\kappa(s_1)$ *een zijde van* $\Gamma(f)$ *van maximale zuivere bijdrage tot de kandidaat-pool* s_1.

Opmerking. Voor $s_1 = \rho$, is $\kappa(s_1) = \kappa$ en is τ_0 de enige zijde van $\Gamma(f)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1.

Propositie N.2.6. *Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Veronderstel dat s_1 een kandidaat-pool is van $Z_f(s,\chi)$, d.w.z. een getal uit de verzameling beschreven in Propositie N.2.1.*

Als $p^{s_1+1} \neq 1$ of $\chi \neq \chi_{\mathrm{triv}}$, dan is de verwachte orde van de kandidaat-pool s_1 van $Z_f(s,\chi)$ gelijk aan $\kappa(s_1)$. Anders is de verwachte orde ofwel $\kappa(s_1)$ ofwel $\kappa(s_1)+1$.

Het bewijs van deze propositie is gebaseerd op een optimale keuze van een partitie van de Δ_τ in kegels Δ_i zodat het aantal (lineair onafhankelijke) generatoren a van Δ_i waarvoor $p^{\sigma(a)+m(a)s_1} = 1$ zo klein mogelijk is. Bovendien zal s_1 een echte pool van S_{Δ_τ} van orde k zijn, indien τ een zijde van $\Gamma(f)$ van zuivere bijdrage tot de kandidaat-pool s_1 met codimensie k is.

N.2.3 $Z_f(s)$ heeft steeds een reële pool

Indien f een veelterm is over $\mathbb{Z}_p$ met $f(0) = 0$, zullen de aantallen N_e van de verzamelingen $\{x + p^e\mathbb{Z}_p^n \mid x \in \mathbb{Z}_p^n \text{ en } f(x) \equiv 0 \bmod p^e\}$ steeds > 0 zijn. Uit Definitie N.0.2, het verband tussen $Z_f(s)$ en $P(t)$ uit Lemma N.0.3 en de rationaliteit van beide volgt dat $Z_f(s)$ minstens één pool heeft in dit geval. In het geval dat f bovendien niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van

zijn Newtonpolyheder, kunnen we m.b.v. de formule uit Stelling N.1.2 nog een sterker resultaat bewijzen. Stelling N.2.7, het tweede hoofdresultaat, zegt dat $Z_f(s)$ dan minstens één *reële* pool heeft. Deze stelling geeft zelfs informatie over de grootste reële pool of $Z_f(s)$ en zijn orde. De grootste reële pool van $Z_f(s)$ is ofwel -1 ofwel $\rho = -1/t_0$. Geval (i) was zoals vermeld reeds gekend.

Stelling N.2.7. *Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2.*

(i) Veronderstel dat $\rho > -1$.
Dan is ρ de grootste reële pool van $Z_f(s)$ en zijn orde is κ.

(ii) Veronderstel dat $\rho < -1$.
Als er een zijde τ van $\Gamma(f)$ bestaat zodat $N_\tau \neq 0$, dan is -1 de grootste reële pool van $Z_f(s)$ en zijn orde is 1. Anders is ρ de grootste reële pool van $Z_f(s)$ en zijn orde is dan κ.

(iii) Veronderstel dat $\rho = -1$.
Dan is $\rho = -1$ de grootste reële pool of $Z_f(s)$. Als er een zijde $\tau \subset \tau_0$ van $\Gamma(f)$ bestaat zodat $N_\tau \neq 0$, dan is de orde van ρ gelijk aan $\kappa + 1$. Anders is zijn orde κ.

Opmerking. Een gelijkaardig resultaat geldt voor $Z_{f,0}(s)$: men moet dan "zijde" overal vervangen door "compacte zijde".

N.3 Een voldoende voorwaarde opdat $\rho = -1/t_0$ een pool is van $Z_f(s)$ of $Z_{f,0}(s)$ van de verwachte orde (voor $t_0 \leq 1$)

De polen van $Z_f(s)$ en hun ordes bepalen hoe het aantal oplossingen van $f(x) \equiv 0 \bmod p^e$ zich gedraagt als e groot wordt. Bovendien geven de polen met het grootste reële deel dan de grootste bijdrage tot deze aantallen. We trachten zoveel mogelijk informatie te verkrijgen over de grootste reële kandidaat-pool van $Z_f(s, \chi)$ verschillend van -1. Zoals reeds vermeld is deze kandidaat-pool gelijk aan $-1/t_0$, indien $t_0 \neq 1$, en is zijn verwachte orde gelijk aan κ. Een gekend resultaat zegt dat $-1/t_0$ een echte pool is van $Z_f(s)$ van orde κ, indien $t_0 > 1$ (en f niet-gedegenereerd is). Stelling N.3.3, het derde hoofdresultaat, zegt dat dit ook nog zo is indien $t_0 < 1$, voor veeltermen f over $\mathbb{Z}$ die voldoende niet-gedegenereerd zijn en voor $p \gg 0$. Dit alles geldt onder de voorwaarde dat geen enkel hoekpunt van τ_0 tot de verzameling $\{0, 1, 2\}^n$ behoort. Het bewijs van deze stelling is gebaseerd op onze expliciete formule voor $Z_f(s)$ en een zeer diepgaand resultaat van Adolphson en Sperber over bovengrenzen voor exponentiële

sommen. Met gelijkaardige technieken kunnen we Stelling N.3.5 aantonen. Dit is een analoog resultaat voor het geval dat $t_0 = 1$. Onder dezelfde voorwaarden als in Stelling N.3.3 (of zelfs een iets zwakkere voorwaarde), zal $-1/t_0 = -1$ de grootste reële pool zijn van $Z_f(s)$ met orde gelijk aan $\kappa + 1$. (Zie ook [DH01].)

Definitie N.3.1. Zij $f \in \mathbb{C}[x_1, \ldots, x_n]$ met $f(0) = 0$. We zeggen dat f *niet-gedegenereerd is over* $\mathbb{C}$ *t.o.v. zijn globale Newtonpolyheder* $\Gamma_{\mathrm{gl}}(f)$ als voor elke[3] zijde τ van $\Gamma_{\mathrm{gl}}(f)$, de locus van f_τ geen singulariteiten in $(\mathbb{C}^\times)^n$ heeft.

Fixeer een globale Newtonpolyheder G. Men kan aantonen dat "bijna alle"[4] veeltermen over $\mathbb{C}$ met G als globale Newtonpolyheder niet-gedegenereerd zijn over $\mathbb{C}$ t.o.v. hun globale Newtonpolyheder $\Gamma_{\mathrm{gl}}(f)$.

Definitie N.3.2. Zij $g(p)$ een functie van p die complexe waarden aanneemt. De notatie $g(p) = o(1)$ betekent dat $\lim_{p\to\infty} g(p) = 0$.

Stelling N.3.3. *Zij* $f(x) = f(x_1, \ldots, x_n)$ *een niet-nulveelterm over* $\mathbb{Z}$ *met* $f(0) = 0$. *Veronderstel dat* f *niet-gedegenereerd is over* $\mathbb{C}$ *t.o.v. alle zijden van zijn Newtonpolyheder* $\Gamma(f)$ *en veronderstel dat* f_{τ_0} *niet-gedegenereerd is over* $\mathbb{C}$ *t.o.v. zijn globale Newtonpolyheder* $\Gamma_{\mathrm{gl}}(f_{\tau_0})$. *Als* $t_0 < 1$ *en geen enkel hoekpunt van* τ_0 *tot de verzameling* $\{0,1,2\}^n$ *behoort, dan is* ρ *een pool van* $Z_f(s)$ *van orde* κ *voor* $p \gg 0$. *Bovendien is*

$$\lim_{s\to\rho} (p^{s-\rho} - 1)^\kappa Z_f(s) = -c_0 p^{\rho+1}(1 + o(1)), \tag{N.3.1}$$

waarbij $c_0 > 0$ *onafhankelijk van* p *is.*

Opmerkingen.

1. De conditie "f_{τ_0} is niet-gedegenereerd over $\mathbb{C}$ t.o.v. $\Gamma_{\mathrm{gl}}(f_{\tau_0})$" kan weggelaten worden indien τ_0 compact is of indien $f_{\tau_0} = f_{\tau_0'}$ met τ_0' een compacte zijde van $\Gamma(f)$.

2. Hoewel er vele voorbeelden voldoen aan de voorwaarden van de bovenstaande stelling kan men zo'n voorbeeld enkel vinden indien het aantal variabelen van f ten minste 4 is. Zo voldoet bijvoorbeeld de veelterm $f(x_1, x_2, x_3, x_4) = x_1^3 + x_2^3 + x_3^3 + x_4^3$ aan de voorwaarden van Stelling N.3.3.

3. Voorbeelden 4.2.1 en 4.2.2 uit Paragraaf 4.2 tonen aan dat, indien de voorwaarde op de hoekpunten van τ_0 niet voldaan is, (N.3.1) niet altijd waar is.

[3] Dus ook voor $\tau = \Gamma_{\mathrm{gl}}(f)$.
[4] In een gelijkaardige betekenis als in Opmerking 2 na Definitie N.0.9.

Er geldt een analoog resultaat voor $Z_{f,0}(s)$.

Stelling N.3.4. *Zij $f(x) = f(x_1, \ldots, x_n)$ een niet-nulveelterm over $\mathbb{Z}$ met $f(0) = 0$. Veronderstel dat f niet-gedegenereerd is over $\mathbb{C}$ t.o.v. de compacte zijden van $\Gamma(f)$ en veronderstel dat τ_0 compact is. Als $t_0 < 1$ en geen enkel hoekpunt van τ_0 tot $\{0,1,2\}^n$ behoort, dan is ρ een pool van $Z_{f,0}(s)$ van orde κ voor $p \gg 0$. Bovendien is*

$$\lim_{s \to \rho} (p^{s-\rho} - 1)^{\kappa} Z_{f,0}(s) = -c_0 p^{\rho+1}(1 + o(1)),$$

waarbij $c_0 > 0$ onafhankelijk van p is.

We hebben een gelijkaardig resultaat voor het geval dat $t_0 = 1$.

Stelling N.3.5. *Zij $f(x) = f(x_1, \ldots, x_n)$ een niet-nulveelterm over $\mathbb{Z}$ met $f(0) = 0$. Veronderstel dat f niet-gedegenereerd is over $\mathbb{C}$ t.o.v. alle zijden van zijn Newtonpolyheder $\Gamma(f)$ en veronderstel dat f_{τ_0} niet-gedegenereerd is over $\mathbb{C}$ t.o.v. zijn globale Newtonpolyheder $\Gamma_{\mathrm{gl}}(f_{\tau_0})$. Veronderstel dat $t_0 = 1$.*

(i) Veronderstel dat $\dim \Gamma_{\mathrm{gl}}(f_{\tau_0}) > 1$. Dan is $\rho = -1$ een pool van $Z_f(s)$ van orde $\kappa + 1$ voor $p \gg 0$.

(ii) Indien bovendien geen enkel hoekpunt van τ_0 tot $\{0,1,2\}^n$ behoort, dan is

$$\lim_{s \to -1} (p^{s+1} - 1)^{\kappa+1} Z_f(s) = c_0 + o(1), \tag{N.3.2}$$

waarbij $c_0 > 0$ onafhankelijk van p is.

Opmerkingen.

1. Het eenvoudige voorbeeld $f(x,y) = x^2 + y^2$ toont aan dat de voorwaarde op de dimensie van $\Gamma_{\mathrm{gl}}(f_{\tau_0})$ niet kan weggelaten worden. Inderdaad, in dit geval is -1 een pool van $Z_f(s)$ van orde 2 voor $p \equiv 1 \bmod 4$, maar slechts een pool van orde 1 voor $p \equiv 3 \bmod 4$.

2. Men kan aantonen dat de voorwaarde "$\dim \Gamma_{\mathrm{gl}}(f_{\tau_0}) > 1$" voldaan is indien geen enkel hoekpunt van τ_0 tot de verzameling $\{0,1,2\}^n$ behoort en $t_0 \leq 1$.

3. We hebben een gelijkaardig resultaat voor $Z_{f,0}(s)$. Men moet dan wel eisen dat τ_0 compact is.

N.4 Een voldoende voorwaarde opdat de echte orde van een pool van $Z_f(s,\chi)$ of $Z_{f,0}(s,\chi)$ kleiner is dan verwacht

Het hoofddoel van de volgende twee hoofdresultaten (zie ook [Hoo02]) is een bewijs te leveren voor één van de vermoedens van Denef en Sargos [DS92, Paragraaf 6.2] voor niet-gedegenereerde veeltermen in het p-adisch geval:

> *Als τ_0 een onstabiele zijde is van $\Gamma(f)$ over $\mathbb{F}_p$ t.o.v. f en $-1/t_0 \neq -1$ is een pool van $Z_f(s,\chi)$, dan is zijn orde kleiner dan verwacht, d.w.z $< \kappa$.*

De notie "onstabiel" werd voor het eerst gebruikt door Denef en Sargos in [DS92].

Definitie N.4.1. Zij f zoals in Definitie N.0.4 en zij p een priemgetal. Een convexe verzameling σ noemt men *onstabiel over $\mathbb{F}_p$ t.o.v. x_j en f*, indien de volgende twee voorwaarden voldaan zijn:

(i) $\sigma \subset \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid 0 \leq x_j \leq 1\}$ en

(ii) voor elke zijde τ van σ die bevat is in $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_j = 1\}$ geldt dat de veelterm $\bar{f}_\tau$ niet nul wordt op $(\mathbb{F}_p^\times)^n$.

Een convexe verzameling σ noemt men onstabiel over $\mathbb{F}_p$ t.o.v. f als er een index j bestaat (met $1 \leq j \leq n$) zodat σ onstabiel is over $\mathbb{F}_p$ t.o.v x_j en f.

We hebben een gelijkaardige definitie voor een convexe verzameling σ die onstabiel is over $\mathbb{C}$ t.o.v. f. Dan moet f een veelterm over $\mathbb{C}$ zijn, en in (ii) wordt de veelterm f_τ niet nul op $(\mathbb{C}^\times)^n$.

De analoge uitspraak van het bovenvermelde vermoeden voor het reële geval werd bewezen door Denef, Laeremans en Sargos [DLS97]. Geïnspireerd door het bewijs voor het reëel geval, probeerden we de vraag voor de veelterm f te reduceren tot dezelfde vraag voor de veelterm f_{τ_0}. Stelling N.4.2, ons vierde hoofdresultaat, zegt dat $-1/t_0$ een pool is van $Z_f(s,\chi)$ van de verwachte orde als en slechts als hetzelfde geldt voor $Z_{f_{\tau_0}}(s,\chi)$. In het bijzonder geldt het volgende: als $-1/t_0$ geen pool is van $Z_{f_{\tau_0}}(s,\chi)$ van de verwachte orde, dan geldt hetzelfde voor $Z_f(s,\chi)$. Stelling N.4.7 breidt dit resultaat uit naar willekeurige polen. In Stelling N.4.4, geven we een formule voor $Z_f(s,\chi)$ die geldig is in het geval dat $f = f_\sigma$, waarbij $\sigma \subset \Gamma(f)$ een onstabiele convexe verzameling over $\mathbb{F}_p$ t.o.v. f is en f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. een bepaalde verzameling zijden van $\Gamma(f)$. De formule uit deze stelling, die ons vijfde hoofdresultaat is, lijkt zeer sterk op de formule uit Stelling N.1.2. Ze kan echter een kleinere verzameling van kandidaat-polen voor $Z_f(s,\chi)$ opleveren of kleinere bovengrenzen voor hun

ordes. Ten slotte gebruiken we in Stelling N.4.6 deze formule tezamen met Stelling N.4.2 om een bewijs te verkrijgen van het bovenvermelde vermoeden van Denef en Sargos. Stelling N.4.8 geeft een analoog resultaat voor willekeurige polen.

N.4.1 Het geval van de kandidaat-pool $\rho = -1/t_0$

De volgende stelling reduceert de vraag of $\rho = -1/t_0$ een pool is van de verwachte orde van de Igusa zetafunctie van f tot dezelfde vraag voor f_{τ_0}.

Stelling N.4.2. *Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Dan geldt dat*

$$\lim_{s\to\rho}(p^{s-\rho}-1)^k Z_f(s,\chi) = c\lim_{s\to\rho}(p^{s-\rho}-1)^k Z_{f_{\tau_0}}(s,\chi),$$

waarbij

$$k = \begin{cases} \kappa+1 & \text{als } \rho=-1 \text{ en } \chi=\chi_{\text{triv}}\,, \\ \kappa & \text{anders} \end{cases}$$

en $c>0$ een constant reëel getal is dat onafhankelijk is van p.

Opmerkingen.

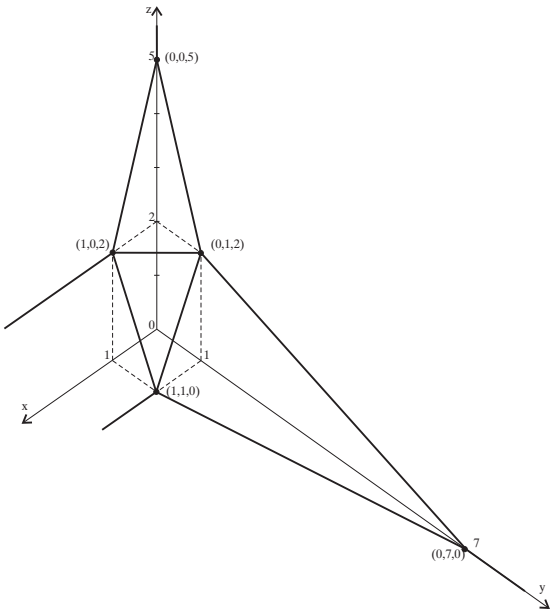
1. Een gelijkaardig resultaat geldt voor $Z_{f,0}(s,\chi)$, zelfs indien τ_0 niet compact is. Dan moet f niet-gedegenereerd zijn over $\mathbb{F}_p$ t.o.v. de *compacte* zijden van zijn Newtonpolyheder.

2. Als $\chi=\chi_{\text{triv}}$, $\rho=-1$ en de verwachte orde van de kandidaat-pool ρ voor $Z_f(s)$ gelijk is aan κ, dan weten we door Stelling N.2.7(iii) dat ρ een echte pool is van $Z_f(s)$ van orde κ. Stelling N.4.2 kan dus nuttig zijn voor de andere gevallen.

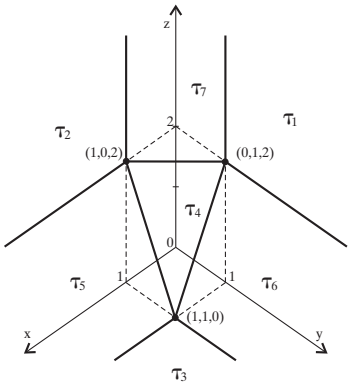
Bovenstaand resultaat is zeer verrassend, omdat de kegels Δ_τ voor $\tau\subset\tau_0$ (en dus ook de S_{Δ_τ}) in de formules voor $Z_f(s,\chi)$ en $Z_{f_{\tau_0}}(s,\chi)$ uit Stelling N.1.2 zeer verschillend kunnen zijn. Neem bijvoorbeeld de veelterm $f(x,y,z)=xy+xz^2+yz^2+z^5+y^7$. Dan is τ_0 de convexe omhullende van $(1,1,0)$, $(1,0,2)$ en $(0,1,2)$. Dus is $f_{\tau_0}=xy+xz^2+yz^2$ en is

$$\begin{aligned}\Delta_{(0,1,2)}(f) &= \text{cone}\,((1,0,0),(2,2,1),(3,3,1),(6,1,3)),\\ \Delta_{(0,1,2)}(f_{\tau_0}) &= \text{cone}\,((1,0,0),(2,2,1),(1,1,0),(2,0,1));\end{aligned}$$

zie ook Figuur N.4.1 en Figuur N.4.2. In het geval dat de codimensie van τ_0 gelijk is aan 1, kan men Stelling N.4.2 nog gemakkelijk aantonen. De echte moeilijkheden treden pas op indien de codimensie groter is dan 1. Het bewijs gebruikt een partitie van $\Delta_\tau(f)$ en $\Delta_\tau(f_{\tau_0})$ in rationale simpliciale kegels door



Figuur N.4.1. Newtonpolyheder of $xy + xz^2 + yz^2 + z^5 + y^7$



Figuur N.4.2. Newton polyhedron of $xy + xz^2 + yz^2$

de introductie van nieuwe generatoren die hetzelfde zijn voor beide. Bovendien vertrekken we hierbij van een vast gekozen partitie van respectievelijk $\Delta_{\tau_0}(f)$ en $\Delta_{\tau_0}(f_{\tau_0})$.

Stelling N.4.4 geeft een formule voor $Z_f(s,\chi)$ die geldig is in het geval dat $f = f_\sigma$, waarbij $\sigma \subset \Gamma(f)$ een onstabiele convexe verzameling over $\mathbb{F}_p$ t.o.v. f is en f

niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. een bepaalde verzameling zijden van $\Gamma(f)$. Deze formule lijkt zeer sterk op de formule uit Stelling N.1.2. Geven we eerst de definitie van een recessierichting van een zijde.

Definitie N.4.3. Zij τ een zijde van de Newtonpolyheder van een veelterm f en e_i de i-de eenheidsvector van $\mathbb{R}^n$. We zeggen dat e_i een recessierichting is van τ, indien τ doorloopt in de richting van e_i, d.w.z.

$$\forall x \in \tau, \forall \lambda \in \mathbb{R} \text{ met } \lambda \geq 0 : x + \lambda e_i \in \tau.$$

Stelling N.4.4. *Zij f zoals in Definitie N.0.4 en zij p een priemgetal. Veronderstel dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van $\Gamma(f)$ die bevat zijn in de verzameling $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ en veronderstel dat $f = f_\sigma$, waarbij $\sigma \subset \Gamma(f)$ een onstabiele convexe verzameling is over $\mathbb{F}_p$ t.o.v. x_n en f. Zij s een complexe variable met $\mathrm{Re}(s) > 0$ en zij χ een karakter op $\mathbb{Z}_p^\times$ met conductor $c_\chi = 1$. Dan is*

$$Z_f(s, \chi) = \sum_{\substack{\tau \text{ zijde van } \Gamma(f) \\ e_n \in \mathrm{rec}\,\tau}} \tilde{L}_\tau S_{\Delta_\tau}, \tag{N.4.1}$$

met

$$\begin{aligned} \tilde{L}_\tau &= \begin{cases} p^{-n}\left((p-1)^{n-1}p - p\tilde{N}_\tau \frac{p^s-1}{p^{s+1}-1}\right) & \text{voor } \chi = \chi_{\mathrm{triv}}, \\ p^{-n} \sum\limits_{a \in (\mathbb{F}_p^\times)^{n-1} \times \mathbb{F}_p} \chi(f_\tau(a)) & \text{voor } \chi \neq \chi_{\mathrm{triv}}, \end{cases} \\ \tilde{N}_\tau &= \{a \in (\mathbb{F}_p^\times)^{n-1} \times \mathbb{F}_p \mid \bar{f}_\tau(a) = 0\}, \end{aligned} \tag{N.4.2}$$

S_{Δ_τ} zoals in Stelling N.1.2, $e_n = (0, 0, \ldots, 1)$ en $\mathrm{rec}\,\tau$ de verzameling recessierichtingen van τ. Merk op dat e_n ook een recessierichting is van $\Gamma(f)$.

Veronderstel dat s_1 een kandidaat-pool is van $Z_f(s, \chi)$, d.w.z. een getal uit de verzameling beschreven in Propositie N.2.1. Als $e_n \notin \mathrm{rec}\,\mu$ voor elke zijde μ van $\Gamma(f)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1, dan geldt het volgende: Als $p^{s_1+1} \neq 1$ of $\chi \neq \chi_{\mathrm{triv}}$ en s_1 een pool is van $Z_f(s, \chi)$, dan is zijn orde $< \kappa(s_1)$; als $p^{s_1+1} = 1$ en s_1 is een pool van $Z_f(s)$, dan is zijn orde $< \kappa(s_1) + 1$.

Een gelijkaardig resultaat geldt voor $Z_{f,0}(s, \chi)$.

Stelling N.4.5. *Zij f zoals in Definitie N.0.4 en zij p een priemgetal. Veronderstel dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. de compacte zijden van $\Gamma(f)$ die bevat zijn in de verzameling $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ en veronderstel dat $f = f_\sigma$, waarbij $\sigma \subset \Gamma(f)$ een onstabiele convexe verzameling is over $\mathbb{F}_p$ t.o.v. x_n en f. (In feite moet voorwaarde* (ii) *in Definitie N.4.1 slechts geëist worden voor de* compacte *zijden van σ die bevat zijn in $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 1\}$). Zij s*

een complexe variable met $\mathrm{Re}\,(s) > 0$ *en zij* χ *een karakter op* $\mathbb{Z}_p^\times$ *met conductor* $c_\chi = 1$. *Dan is*

$$Z_{f,0}(s,\chi) = \sum_{\substack{\tau \text{ zijde van } \Gamma(f) \\ \{e_n\} = \mathrm{rec}\,\tau \cap \{e_1,\ldots,e_n\}}} L'_\tau S_{\Delta_\tau},$$

met

$$\begin{aligned} L'_\tau &= \begin{cases} p^{-n}\left((p-1)^{n-1} - pN'_\tau \frac{p^s-1}{p^{s+1}-1}\right) & \text{voor } \chi = \chi_{\mathrm{triv}}\,, \\ p^{-n} \sum\limits_{a \in (\mathbb{F}_p^\times)^{n-1}\times\{0\}} \chi(f_\tau(a)) & \text{voor } \chi \neq \chi_{\mathrm{triv}}\,, \end{cases} \qquad \text{(N.4.3)} \\ N'_\tau &= \{a \in (\mathbb{F}_p^\times)^{n-1} \times \{0\} \mid \bar{f}_\tau(a) = 0\}, \end{aligned}$$

S_{Δ_τ} *zoals in Stelling N.1.2,* $e_n = (0,0,\ldots,1)$ *en* $\mathrm{rec}\,\tau$ *de verzameling recessierichtingen van* τ.

Het bewijs van het vermoeden van Denef en Sargos volgt nu uit Stelling N.4.2 en Stelling N.4.4.

Stelling N.4.6. *Veronderstel dat het priemgetal* p, *de veelterm* f *en het karakter* χ *voldoen aan de voorwaarden van Stelling N.1.2. Als* τ_0 *onstabiel is over* $\mathbb{F}_p$ *t.o.v.* f, *dan geldt het volgende:*

(i) Als $\rho \neq -1$ *of* $\chi \neq \chi_{\mathrm{triv}}$ *en* ρ *een pool is van* $Z_f(s,\chi)$, *dan is zijn orde* $< \kappa$.

(ii) Als $\rho = -1$ *een pool is van* $Z_f(s)$, *dan is zijn orde* $< \kappa + 1$.

Hetzelfde resultaat geldt voor $Z_{f,0}(s,\chi)$. Dan moet het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Opmerking 2 na Stelling N.1.2. Dit resultaat volgt uit Stelling N.4.5 en Opmerking 1 na Stelling N.4.2.

N.4.2 Het geval van een willekeurige kandidaat-pool

Stelling N.4.7 is een veralgemening van Stelling N.4.2 naar een willekeurige kandidaat-pool s_1 van $Z_f(s,\chi)$. Deze stelling reduceert de vraag of s_1 een pool is van de Igusa zetafunctie van f van kleinere orde dan verwacht, tot dezelfde vraag voor de f_μ.

Stelling N.4.7. *Veronderstel dat het priemgetal* p, *de veelterm* f *en het karakter* χ *voldoen aan de voorwaarden van Stelling N.1.2. Veronderstel dat* s_1 *een kandidaat-pool is van* $Z_f(s,\chi)$, *d.w.z. een getal uit de verzameling beschreven in Propositie N.2.1. Veronderstel dat voor elke zijde* μ *van* $\Gamma(f)$ *van maximale*

zuivere bijdrage tot de kandidaat-pool s_1 van $Z_f(s,\chi)$ geldt dat de zijde μ de enige zijde van $\Gamma(f_\mu)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1 van $Z_{f_\mu}(s,\chi)$ is. Dan is

$$\lim_{s\to s_1}(p^{s-s_1}-1)^k Z_f(s,\chi) = \sum_\mu c_\mu \lim_{s\to s_1}(p^{s-s_1}-1)^k Z_{f_\mu}(s,\chi),$$

waarbij μ over de zijden van $\Gamma(f)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1 loopt,

$$k = \begin{cases} \kappa(s_1)+1 & \text{als } p^{s_1+1}=1 \text{ en } \chi=\chi_{\text{triv}}\,, \\ \kappa(s_1) & \text{anders,} \end{cases}$$

en de $c_\mu > 0$ constante reële getallen zijn, onafhankelijk van p.

Opmerking. Veronderstel dat μ een zijde is van $\Gamma(f)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1. Als de dimensie van μ gelijk is aan 0 of 1, dan is μ de enige zijde van $\Gamma(f_\mu)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1 van $Z_{f_\mu}(s,\chi)$. Als de dimensie van μ groter dan 1 is, dan is het mogelijk dat deze voorwaarde op μ niet voldaan is. Toch zal in de meeste voorbeelden de voorwaarde op μ wél voldaan zijn.

Uit Stelling N.4.7 en Stelling N.4.4 volgt de volgende veralgemening van Stelling N.4.6 naar een willekeurige kandidaat-pool van $Z_f(s,\chi)$:

Stelling N.4.8. *Veronderstel dat het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Veronderstel dat s_1 een kandidaat-pool is van $Z_f(s,\chi)$, d.w.z. een getal uit de verzameling beschreven in Propositie N.2.1. Veronderstel dat voor elke zijde μ van $\Gamma(f)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1, de zijde μ de enige zijde van $\Gamma(f_\mu)$ is van maximale zuivere bijdrage tot de kandidaat-pool s_1, en μ onstabiel is over $\mathbb{F}_p$ t.o.v. f. Dan geldt het volgende:*

(i) Als $p^{s_1+1}\neq 1$ of $\chi\neq\chi_{\text{triv}}$ en s_1 een pool is van $Z_f(s,\chi)$, dan is zijn orde $<\kappa(s_1)$.

(ii) Als $p^{s_1+1}=1$ en s_1 een pool is van $Z_f(s)$, dan is zijn orde $<\kappa(s_1)+1$.

Hetzelfde resultaat geldt voor $Z_{f,0}(s,\chi)$. Dan moet het priemgetal p, de veelterm f en het karakter χ voldoen aan de voorwaarden van Opmerking 2 na Stelling N.1.2.

N.5 De polen van maximale orde

Het laatste resultaat geeft informatie over de reële polen van maximale orde van $Z_f(s,\chi)$ en $Z_{f,0}(s,\chi)$ voor een karakter χ met conductor c_χ gelijk aan 1. We zeggen dat een pool maximale orde heeft, als zijn orde gelijk is aan het aantal variabelen van f. Als de veelterm f voldoende niet-gedegenereerd is, kunnen we de volgende resultaten bewijzen m.b.v. de formule uit Stelling N.1.2: Igusa's lokale zetafuncties $Z_f(s,\chi)$ en $Z_{f,0}(s,\chi)$ voor $\chi \neq \chi_{\text{triv}}$ en $Z_{f,0}(s)$ hebben ten hoogste één reële pool van maximale orde. Aan de andere kant heeft Igusa's lokale zetafunctie $Z_f(s)$ ten hoogste twee reële polen van maximale orde. We geven de resultaten hieronder. De methode die we gebruiken is volledig analoog met de methode en de bewijzen uit [LV99] of [Lae97, Hoofdstuk 6] waarin gelijkaardige resultaten worden bewezen voor de topologische zetafunctie.

Proposition N.5.1. *Veronderstel dat het priemgetal p, de veelterm f en het niet-triviale karakter χ voldoen aan de voorwaarden van Stelling N.1.2. Dan geldt het volgende:*

(i) *$Z_f(s,\chi)$ heeft ten hoogste één reële pool van orde n, en*

(ii) *$Z_f(s,\chi)$ heeft in s_0 een reële pool van orde n als en slechts als $s_0 = -1/t_0$, $t_0 \in \mathbb{N} \setminus \{0\}$ en $(t_0,\ldots,t_0)$ een hoekpunt is van $\Gamma(f)$.*

Een gelijkaardig resultaat geldt voor $Z_f(s,\chi)$, indien het priemgetal p, de veelterm f en het niet-triviale karakter χ voldoen aan de voorwaarden van Opmerking 2 na Stelling N.1.2.

Proposition N.5.2. *Veronderstel dat het priemgetal p en de veelterm f voldoen aan de voorwaarden van Opmerking 2 na Stelling N.1.2. Dan geldt het volgende:*

(i) *$Z_{f,0}(s)$ heeft ten hoogste één reële pool van orde n, en*

(ii) *als $Z_{f,0}(s)$ in s_0 een reële pool van orde n heeft, dan is $s_0 = -1/t_0$ en $t_0 \in \mathbb{N} \setminus \{0\}$.*

Proposition N.5.3. *Veronderstel dat het priemgetal p en de veelterm f voldoen aan de voorwaarden van Stelling N.1.2. Dan geldt het volgende:*

(i) *$Z_f(s)$ heeft ten hoogste twee reële polen van orde n, en*

(ii) *als $Z_f(s)$ in s_0 een reële pool van orde n heeft, dan is $s_0 = -1$ of is $s_0 = -1/t_0$ en $t_0 \in \mathbb{N} \setminus \{0\}$.*

N.6 Open problemen voor toekomstig onderzoek

Veronderstel dat de veelterm $f \in \mathbb{Z}[x_1, \ldots, x_n]$ niet-gedegenereerd is over $\mathbb{C}$ t.o.v. alle zijden van zijn Newtonpolyheder $\Gamma(f)$ en dat $p \gg 0$. Denef en Sargos poneerden het vermoeden dat er een karakter χ bestaat zodat $-1/t_0$ een pool is van $Z_f(s, \chi)$ van orde κ als en slechts als τ_0 niet onstabiel is over $\mathbb{C}$ t.o.v. f [DS92, Paragraaf 6.2]. We bewezen reeds één deel van dit vermoeden in het proefschrift; zie Stelling N.4.6 en Opmerking 1 na Definitie N.0.9. Het volgende overblijvende deel moet nog bewezen worden.

Vermoeden N.6.1. *Zij $f(x_1, \ldots, x_n)$ een veelterm over $\mathbb{Z}_p$ met $f(0) = 0$ en p een priemgetal. Veronderstel dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van zijn Newtonpolyheder $\Gamma(f)$ en veronderstel dat τ_0 niet onstabiel is over $\mathbb{F}_p$ t.o.v. f. Veronderstel dat de oorsprong een singulier punt is van f. Als $t_0 < 1$, dan is $-1/t_0$ een pool van $Z_f(s, \chi)$ van orde κ voor ten minste één karakter χ.*

Dit vermoeden is misschien zelfs waar voor $\chi = \chi_{\text{triv}}$. Tot nu toe hebben we hiervoor nog geen tegenvoorbeeld gevonden.

Vermoeden N.6.2. *Zij $f(x_1, \ldots, x_n)$ een veelterm over $\mathbb{Z}$ met $f(0) = 0$. Veronderstel dat f niet-gedegenereerd is over $\mathbb{C}$ t.o.v. alle zijden van zijn Newtonpolyheder $\Gamma(f)$ en dat de oorsprong een singulier punt is van f.*

(i) Als $t_0 < 1$, dan is $-1/t_0$ een pool van $Z_f(s)$ voor $p \gg 0$.

Met Stelling N.3.3 bewezen we dit vermoeden in het speciaal geval dat geen enkel hoekpunt van τ_0 tot $\{0, 1, 2\}^n$ behoort. In dit geval is $-1/t_0$ zelfs een pool van $Z_f(s)$ van orde κ. Deze extra voorwaarde is niet voldaan voor de veelterm $f(x, y, z) = x^2 + yz + z^2$ en $-1/t_0$ is dan geen pool van $Z_f(s)$ van orde κ. Niettemin is $-1/t_0$ nog steeds een pool van $Z_f(s)$ in dit voorbeeld. Ook voor de veelterm $f(x, y, z, u) = zx^2 + zy^2 + u^3$ is er een hoekpunt van τ_0 bevat in $\{0, 1, 2\}^n$. Voor $p \equiv 3 \bmod 4$ is $-1/t_0$ geen pool van $Z_f(s)$ van orde κ, maar blijft nog steeds een pool van $Z_f(s)$. Voor de berekeningen van deze twee voorbeelden verwijzen we naar Paragraaf 4.2.

Indien men een tegenvoorbeeld zou vinden voor Vermoeden N.6.2, kan men Bewering (i) misschien vervangen door één van de volgende twee zwakkere beweringen:

(ii) Als $t_0 < 1$, dan is $-1/t_0$ een pool van $Z_f(s)$ voor oneindig veel p.

(ii) Als $t_0 < 1$, dan bestaat er ten minste één karakter χ zodat $-1/t_0$ een pool is van $Z_f(s, \chi)$ voor oneindig veel p.

Verder wensen we te onderzoeken of het volgende vermoeden waar is.

Vermoeden N.6.3. *Zij $f(x_1, \ldots, x_n)$ een veelterm over $\mathbb{Z}_p$ met $f(0) = 0$ en p een priemgetal. Veronderstel dat f niet-gedegenereerd is over $\mathbb{F}_p$ t.o.v. alle zijden van zijn Newtonpolyheder $\Gamma(f)$ en dat s_1 een kandidaat-pool van $Z_f(s)$ is. Veronderstel dat alle facetten van $\Gamma(f)$ van zuivere bijdrage tot de kandidaat-pool s_1 onstabiel zijn t.o.v. dezelfde variabele. Veronderstel dat de oorsprong een singulier punt is van f. Als $s_1 \neq -1$, dan is s_1 geen echte pool van $Z_f(s)$.*

Zie Voorbeeld 3 en Voorbeeld 6 uit Appendix D voor een illustratie van dit vermoeden. De analoge uitspraak voor reële analytische functies werd bewezen door Denef en Sargos [DS92, Théorème 1] in het speciaal geval dat $s_1 = -1/t_0$. Als eerste stap zou men kunnen proberen of de ideeën uit dit bewijs ook bruikbaar zijn voor Igusa's lokale zetafunctie. Verbonden met Vermoeden N.6.3 is het volgende resultaat van Denef:

Stelling N.6.4. *Zij f een veelterm over $\mathbb{Z}_p$ in n variabelen. Veronderstel dat f niet-gedegenereerd is over $\mathbb{Q}_p$ t.o.v. de compacte zijden van zijn Newtonpolyheder. Veronderstel dat $s_1 \neq 1$ een reële kandidaat-pool is van $Z_f(s)$ zodat er slechts één facet μ van zuivere bijdrage tot s_1 is. Veronderstel dat μ een simplex is met $n-1$ hoekpunten in een coördinaathypervlak en één hoekpunt op afstand 1 van dit coördinaathypervlak. Dan heeft de functie*

$$\int_{p^k \mathbb{Z}_p^n} |f(x)|^s |dx|$$

geen enkele pool met reëel deel gelijk aan s_1 voor $k \gg 0$.

Ten slotte toont het bewijs van Stelling N.4.2 aan dat de voorwaarde "μ is de enige zijde van $\Gamma(f)$ van maximale zuivere bijdrage tot de kandidaat-pool s_1" in Stelling N.4.8 vervangen kan worden door een iets zwakkere voorwaarde. Voorbeelden doen vermoeden dat zelfs deze voorwaarde wellicht weggelaten kan worden (zie Voorbeeld 4 uit Appendix D). We wensen onderzoek te leveren naar een bewijs of een tegenvoorbeeld voor deze bewering.