Another Modification of Generalized Continued Fractions Associated with Recurrence Relations of Perron-Kreuser Type

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ABSTRACT

In this paper we present a new modification of a generalized continued fraction or n-fraction for which the underlying recurrence relation is of Perron-Kreuser type. If the characteristic equations for this recurrence relation have only simple roots with differing absolute values, then this new modification leads to convergence acceleration.

1. Introduction and notation

In a previous paper [2] we defined a modification of a convergent n-fraction associated with a recurrence relation of Perron-Kreuser type. This modification often leads to convergence acceleration. One of the disadvantages of the method given in [2] is that it requires the calculation of the roots of a certain polynomial equation. In this paper we present a new modification of n-fractions with the same convergence acceleration-properties for which no polynomial equations have to be solved.

The notation used will be that from [2]. Without going into details, we shall explain the machinery used in [2].
Let n be a fixed natural number ($n \geqslant 1$). Consider a sequence of complex $(n+1)$-tuples $(b_k, a_k^{(1)}, \ldots, a_k^{(n)})$, $a_k^{(1)} \neq 0$. Then the n-fraction associated with this sequence, written as

$$K_{k=1}^{\infty} \begin{pmatrix} a_k^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ a_k^{(n)} \\ b_k \end{pmatrix} \tag{1a}$$

is given by the sequence of approximants $\left\{ \frac{A_k^{(1)}}{B_k}, \ldots, \frac{A_k^{(n)}}{B_k} \right\}_{k=1}^{\infty}$ (if they exist), where the numerators and denominators satisfy the recurrence relation

$$X_k = b_k X_{k-1} + a_k^{(n)} X_{k-2} + \cdots + a_k^{(1)} X_{k-n-1} \qquad k = 1,2,\ldots \tag{1b}$$

with initial values

$$A_{-j}^{(i)} = \delta_{i+j,n+1}, \quad B_{-j} = \delta_{n+1+j,n+1} \quad (j = 0,\ldots,n) \tag{2}$$

for $i = 1,\ldots,n$. The n-fraction is said to converge in $\hat{\mathbb{C}}^n$ if the following limit exists in $\hat{\mathbb{C}}^n$:

$$\{\xi_0^{(1)}, \ldots, \xi_0^{(n)}\} = \lim_{k \to \infty} \left\{ \frac{A_k^{(1)}}{B_k}, \ldots, \frac{A_k^{(n)}}{B_k} \right\}. \tag{3}$$

For $k \geqslant 1$ we introduce the Moebius-transforms

$$s_k^{(1)}(w^{(1)}, \ldots, w^{(n)}) = \frac{a_k^{(1)}}{b_k + w^{(n)}}, \quad s_k^{(i)}(w^{(1)}, \ldots, w^{(n)}) = \frac{a_k^{(i)} + w^{(i-1)}}{b_k + w^{(n)}} \quad (i = 2,\ldots,n)$$

and for $i = 1,\ldots,n$

$$S_1^{(i)}(w^{(1)}, \ldots, w^{(n)}) = s_1^{(i)}(w^{(1)}, \ldots, w^{(n)})$$

$$S_k^{(i)}(w^{(1)}, \ldots, w^{(n)}) = S_{k-1}^{(i)}(s_k^{(1)}, \ldots, s_k^{(n)}) \quad (k \geqslant 2) \quad \text{(if they exist)} . \tag{4}$$

Then we have the connection formula

$$S_k^{(i)}(w^{(1)}, \ldots, w^{(n)}) = \frac{A_k^{(i)} + A_{k-1}^{(i)} w^{(n)} + A_{k-2}^{(i)} w^{(n-1)} + \cdots + A_{k-n}^{(i)} w^{(1)}}{B_k + B_{k-1} w^{(n)} + B_{k-2} w^{(n-1)} + \cdots + B_{k-n} w^{(1)}}. \tag{5}$$

From this formula it follows that

$$\frac{A_k^{(i)}}{B_k} = S_k^{(i)}(0, \ldots, 0) \quad \text{for } i = 1,\ldots,n, \tag{6}$$

and we see that the sequence of approximants of the n-fraction (1) may be expressed by the Moebius-transforms $S_k^{(i)}$. So as in the case of ordinary continued fractions, the approximants of the n-fraction (1) may be evaluated by replacing its tails, i.e.

$$K_{k=m+1}^{\infty} \begin{pmatrix} a_k^{(1)} \\ \cdot \\ \cdot \\ a_k^{(n)} \\ b_k \end{pmatrix} \tag{7}$$

by zero for successive values of m. If the tail (7) converges, we let $(\xi_m^{(1)}, \ldots, \xi_m^{(n)})$ denote its value.

Given a sequence of n-tuples $\{(w_k^{(1)}, \ldots, w_k^{(n)})\}_{k=1}^{\infty}$ of numbers from $\mathbb{C}$, a modification of (1) is given by the sequence of n-tuples

$$\{S_k^{(1)}(w_k^{(1)}, \ldots, w_k^{(n)}), \ldots, S_k^{(n)}(w_k^{(1)}, \ldots, w_k^{(n)})\}_{k=1}^{\infty}. \tag{8}$$

We assume $(w_k^{(1)}, \ldots, w_k^{(n)})$ to be chosen such that (8) is well defined.
Convergence acceleration is characterized by

$$\lim_{k \to \infty} \frac{\xi_0^{(i)} - S_k^{(i)}(w^{(1)}, \ldots, w^{(n)})}{\xi_0^{(i)} - S_k^{(i)}(0, \ldots, 0)} = 0 \quad (i = 1,\ldots,n) \;). \tag{9}$$

We will restrict ourselves to the case that the n-fraction converges in $\mathbb{C}^n$.

The recurrence relation

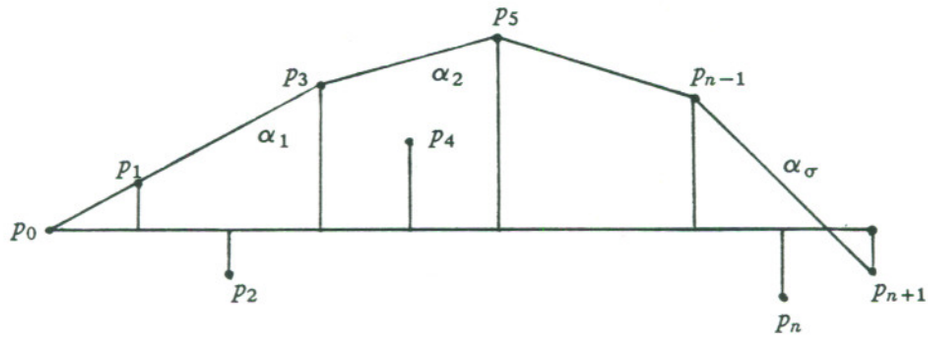
$$X_k = b_k X_{k-1} + a_k^{(n)} X_{k-2} + \cdots + a_k^{(1)} X_{k-n-1} \qquad k = 1,2,\ldots \tag{1b}$$

is said to be of Perron-Kreuser type if its coefficients satisfy

$$b_k = -s_{n+1} k^{m_{n+1}} (1 + o(1)) \quad (k \to \infty)$$

$$a_k^{(i)} = -s_i k^{m_i} (1 + o(1)) \quad (k \to \infty), \quad i = 1,\ldots,n \tag{10}$$

where the s_i are real or complex numbers and s_{n+2} is defined to be one. The m_i are real or $-\infty$ with $m_1 > -\infty$; m_{n+2} is defined to be zero. With such a recurrence relation we can associate a uniquely defined Newton-Puiseux polygon in a rectangular (x,y)-coordinate system :



Let the points $p_0, p_1, \ldots, p_{n+1}$ be defined by $x = i$, $y = m_{n+2-i}$ $(i=0,\ldots,n+1)$. Then some of the points $p_0, \ldots, p_{n+1}$ are connected with linear segments in such a way that the resulting polygon is concave downwards, each point p_i $(i=1,\ldots,n)$ being on or below the resulting figure, p_0 and p_{n+1} being the endpoints of the figure . If the polygon so constructed has σ distinct linear segments, their respective slopes are denoted by $\alpha_1, \ldots, \alpha_\sigma$ with $\alpha_1 > \alpha_2 \cdots > \alpha_\sigma$, and their abscissas are denoted by $0 = e_0 < e_1 < \cdots < e_\sigma = n+1$. It follows that

$$\alpha_\lambda = \frac{m_{n+2-e_\lambda} - m_{n+2-e_{\lambda-1}}}{e_\lambda - e_{\lambda-1}}\ , \quad \lambda = 1,\ldots,\sigma. \tag{11}$$

To each linear segment corresponds a characteristic equation. The characteristic equation corresponding to the linear segment with slope α_λ is given by

$$r_0^{(\lambda)} x^{e_\lambda - e_{\lambda-1}} + r_1^{(\lambda)} x^{e_\lambda - e_{\lambda-1}-1} + \cdots + r_{e_\lambda - e_{\lambda-1}}^{(\lambda)} = 0, \tag{12}$$

where $r_i^{(\lambda)} = s_{n+2-e_{\lambda-1}-i}$ or 0 depending on whether the point $(e_{\lambda-1}+i, m_{n+2-e_{\lambda-1}-i})$ falls, respectively, on or below the $\lambda-$th side of the Newton-Puiseux polygon.
We will assume that for every λ $(\lambda = 1, \ldots, \sigma)$ the roots of (12) belonging to the $\lambda-$th class (corresponding to the linear segment with slope α_λ) are all different in modulus. Let u_i $(i=e_{\lambda-1}+1, \ldots, e_\lambda)$ be the roots of the characteristic equation for the $\lambda-$th class , with

$$|u_{e_{\lambda-1}+1}| > \cdots > |u_{e_\lambda}|\ . \tag{13}$$

Since $r_{e_\lambda - e_{\lambda-1}} = s_{n+2-e_\lambda} \neq 0$ we also have all $|u_i| > 0$.
In this case the solution space of the recurrence relation (1b) has a basis $\{\{D_k^{(1)}\}, \ldots, \{D_k^{(n+1)}\}\}$ which satisfies

$$\frac{D_{k+1}^{(i)}}{D_k^{(i)}} = k^{\alpha_\lambda} u_i (1 + o(1)) \ (k \to \infty) \quad \text{for } i = e_{\lambda-1}+1, \ldots, e_\lambda, \ \lambda = 1, \ldots, \sigma$$

$$= v_i(k)(1 + o(1)) \ (k \to \infty) \tag{14}$$

where we have put $v_i(k) = k^{\alpha_\lambda} u_i$.
In [2] we looked at two special types of Perron-Kreuser recurrence relations

Type I with $e_1 > 1$, $\alpha_1 = 0$ (15)

Type II with $e_1 = 1, \alpha_1 > \alpha_2 = 0.$ (16)

with the remark that using an equivalence transformation every recurrence relation of Perron-Kreuser type can be brought into one of these two forms.

The elementary symmetric functions for a set of m arbitrary complex numbers $x_1, \ldots, x_m$ are defined as

$$\begin{aligned} f_1(x_1, \ldots, x_m) &= x_1 + \cdots + x_m \\ f_2(x_1, \ldots, x_m) &= x_1x_2 + x_1x_3 + \cdots + x_{m-1}x_m \\ &\cdots \\ f_m(x_1, \ldots, x_m) &= x_1x_2 \cdots x_m . \end{aligned} \tag{17}$$

If we construct from these f_i the polynomial

$$P_{x_1, \ldots, x_m}(x) = x^m - f_1x^{m-1} + f_2x^{m-2} + \cdots + (-1)^m f_m \tag{18}$$

then this polynomial has the roots $x_1, \ldots, x_m$, and so we have

$$P_{x_1, \ldots, x_m}(x_i) = x_i^m - f_1x_i^{m-1} + f_2x_i^{m-2} + \cdots + (-1)^m f_m = 0 , \quad i = 1,\ldots,m. \tag{19}$$

It is easy to prove that if all x_i are different from each other, then

$$f_{m-i+1} = \frac{\det V^{(i)}}{\det V^{(m+1)}} \qquad i = 1,\ldots,m \tag{20}$$

where $V^{(i)}$ is the square matrix arising from the matrix V with

$$V = \begin{pmatrix} 1 & 1 & . & 1 \\ x_1 & x_2 & . & x_m \\ . & . & . & . \\ . & . & . & . \\ x_1^m & x_2^m & . & x_m^m \end{pmatrix}$$

by removing the i-th row. Further on we shall need the following lemma :

Lemma 1 *Let $x_1, \ldots, x_m$ $(m \geqslant 2)$ be complex numbers with $x_i \neq x_j$ for all $i \neq j$ and $x_1 \neq 0$. Then we have*

$$\frac{f_m(x_1, \ldots, x_m)}{x_1} = f_{m-1}(x_2, \ldots, x_m) \tag{21}$$

and if $m > 2$

$$\frac{f_i(x_1, \ldots, x_m) - f_i(x_2, \ldots, x_m)}{x_1} = f_{i-1}(x_2, \ldots, x_m) \tag{22}$$

for $i = 2,\ldots,m-1$.

Proof : Straightforward.

2. A new modification of an n-fraction

The choice of the modification used in this paper is based on the following observation : consider the special case of an $(n+1)$-th order recurrence relation (1b) of Poincaré-type. This means that in (10) $m_i = 0$ for $i = 1,\ldots,n+1$. In this case there is only one characteristic equation

$$x^{n+1} + s_{n+1}x^n + s_n x^{n-1} + \cdots + s_1 = 0. \tag{23}$$

We assume that the roots u_i $(i = 1,\ldots,n+1)$ of (23) satisfy

$$|u_1| > |u_2| > \cdots > |u_{n+1}| > 0.$$

In [1] it was shown that the use of modifying factors $w_k^{(i)} = w^{(i)}$ $(i = 1,\ldots,n,\ k = 1,2,\ldots)$ defined by

$$w^{(1)} = \frac{-s_1}{u_1}$$

$$w^{(i)} = \frac{-s_i + w^{(i-1)}}{u_1}, \quad i = 2,\ldots,n \tag{24}$$

often leads to convergence acceleration for the n-fraction associated with (23). To use this modification one has to know the value of u_1. Now from the Perron-Kreuser theorem ([2]) it follows that for any dominant solution $\{g_k\}$ of (1b) we have

$$\lim_{k\to\infty} \frac{g_{k+1}}{g_k} = u_1 \tag{25}$$

(a dominant solution of (1b) is a solution of (1b) of the form

$$d_1 D_k^{(1)} + d_2 D_k^{(2)} + \cdots + d_{n+1} D_k^{(n+1)}$$

with $d_1 \neq 0$ and the $\{D_k^{(i)}\}$ as in (14)). Therefore we propose the following modification of the n-fraction associated with (1b) :

$$w_k^{(1)} = \frac{a_{k+1}^{(1)}}{g_{k+1}/g_k}$$

$$w_k^{(i)} = \frac{a_{k+1}^{(i)} + w_k^{(i-1)}}{g_{k+1}/g_k}, \quad i = 2,\ldots,n.$$

Note that in this case

$$\lim_{k\to\infty} w_k^{(i)} = w^{(i)}$$

for $i = 1,\ldots,n$.

We have the following theorem :

Theorem *Given the n-fraction*

$$K_{k=1}^{\infty} \begin{pmatrix} a_k^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ a_k^{(n)} \\ b_k \end{pmatrix} \tag{1a}$$

satisfying the assumptions (10), (13) and (15) or (16) and such that for $j=1,\ldots,n$

$$\{A_k^{(j)}\} \notin \text{vect}\left\{ \{D_k^{(i)}\} \mid i=e_\rho+1,\ldots,n+1 \right\} \text{ with } \rho = \begin{cases} 1 & \text{for type I} \\ 2 & \text{for type II} \end{cases} \tag{26}$$

(for a set V vectV denotes the vectorspace spanned by the elements of V) .
Let $\{g_k\}$ *be a dominant solution of (1b), then there exists a natural number* k_0 *such that*

$$g_k \neq 0 \ , \ \forall k \geqslant k_0.$$

For $k \geqslant k_0$ *we define*

$$w_k^{(1)} = \frac{a_{k+1}^{(1)}}{g_{k+1}/g_k}$$

$$w_k^{(i)} = \frac{a_{k+1}^{(i)} + w_k^{(i-1)}}{g_{k+1}/g_k} \ , \ i=2,\ldots,n. \tag{27}$$

If the n-fraction converges to a finite value $\{\xi_0^{(1)}, \ldots, \xi_0^{(n)}\} \in \mathbb{C}^n$ *, we have*

$$\lim_{k\to\infty} \frac{\xi_0^{(i)} - S_k^{(i)}(w_k^{(1)}, \ldots, w_k^{(n)})}{\xi_0^{(i)} - S_k^{(i)}(0, \ldots, 0)} = 0 \quad (i=1,\ldots,n). \tag{28}$$

Proof :
If $\{g_k\}$ is a dominant solution of (1b), then, as a consequence of the Perron-Kreuser theorem, we have

$$\frac{g_{k+1}}{g_k} = v_1(k)\,(1+o(1)) \quad (k\to\infty) \tag{29}$$

and

$$g_k \neq 0$$

for k sufficiently large.
We now prove the following lemma :

Lemma 2 *The coefficients* $a_k^{(i)}$ *of the recurrence relation (1b) satisfy :*

$$a_k^{(i)} = (-1)^{n+1-i} \frac{\det \mathbb{T}_k^{(i)}}{\det \mathbb{T}_k^{(n+1)}} \ , \ i=1,\ldots,n \tag{30}$$

where $\mathbb{T}_k^{(i)}$ *is the square matrix arising from the matrix* $\mathbb{T}_k$ *with*

$$\mathbb{T}_k = \begin{pmatrix} D_{k-n-1}^{(1)} & D_{k-n-1}^{(2)} & . & D_{k-n-1}^{(n+1)} \\ D_{k-n}^{(1)} & D_{k-n}^{(2)} & . & D_{k-n}^{(n+1)} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ D_k^{(1)} & D_k^{(2)} & . & D_k^{(n+1)} \end{pmatrix}$$

by removing the i-th row.

Proof of lemma 2 :
We use the recurrence relation (1b) to rewrite the last row of $\mathrm{I\!T}_k^{(i)}$ in (30) :

$$D_k^{(m)} = b_k D_{k-1}^{(m)} + a_k^{(n)} D_{k-2}^{(m)} + \cdots + a_k^{(1)} D_{k-n-1}^{(m)} , \qquad m = 1,\ldots,n+1.$$

Using some elementary properties of determinants gives the desired result.

If we divide the l-th column of $\mathrm{I\!T}_k^{(i)}$ and $\mathrm{I\!T}_k^{(n+1)}$ by $D_{k-n-1}^{(l)}$ for $l = 1,\ldots,n+1$, we get, using (14) and (20) :

$$a_{k+1}^{(i)} = (-1)^{n+1-i} f_{n+2-i}(v_1(k), \ldots, v_{n+1}(k)) \, (1+o(1)) \quad (k \to \infty)$$

for $i = 1,\ldots,n$. Now we can prove that

$$w_k^{(i)} = (-1)^{n+1-i} f_{n+1-i}(v_2(k), \ldots, v_{n+1}(k)) \, (1+o(1)) \quad (k \to \infty) \tag{31}$$

for $i = 1,\ldots,n$. From (27) and (29) we have for $i = 1$

$$w_k^{(1)} = \frac{(-1)^n f_{n+1}(v_1, \ldots, v_{n+1})}{v_1} \, (1+o(1)) \quad (k \to \infty)$$

$$= (-1)^n f_n(v_2, \ldots, v_{n+1}) \, (1+o(1)) \quad (k \to \infty)$$

using lemma 1. If (31) is true for $i = m-1$, then we have for $i = m$:

$$w_k^{(m)} = \frac{(-1)^{n+1-m} f_{n+2-m}(v_1, \ldots, v_{n+1}) + (-1)^{n+2-m} f_{n+2-m}(v_2, \ldots, v_{n+1})}{v_1} \, (1+o(1))$$

$$= (-1)^{n+1-m} f_{n+1-m}(v_2, \ldots, v_{n+1}) \, (1+o(1)) \quad (k \to \infty)$$

again using lemma 1. From the proof of Theorem I in [2] it follows that

$$\lim_{k \to \infty} w_k^{(i)} = \lim_{k \to \infty} \xi_k^{(i)}$$

for $i = 1,\ldots,n$. Using this it is easy to see that the proof of theorem II in [2] remains valid for the modification defined by (27). This proves (28).

Remark : It is interesting to note that this theorem gives a new method for choosing modifying factors for ordinary continued fractions satisfying the conditions of the theorem.

3. Examples

To use the modification defined in (27) we need a dominant solution of the recurrence relation (1b). If the n-fraction associated with (1b) converges, then $\{B_k\}$ is such a dominant solution of (1b). For $k \geqslant k_0$ we may calculate $z_k = \dfrac{B_{k+1}}{B_k}$ from

$$z_k = b_{k+1} + \frac{a_{k+1}^{(n)}}{z_{k-1}} + \frac{a_{k+1}^{(n-1)}}{z_{k-1} z_{k-2}} + \cdots + \frac{a_{k+1}^{(1)}}{\prod_{i=1}^{n} z_{k-i}}. \tag{32}$$

In many cases k_0 will be equal to one. Note that evaluating B_k for successive values of k

often leads to overflow.

In the examples we compare three methods for the calculation of the value of an n-fraction :

1. the usual method of replacing the tails by zero (first column in tables 1,3 and 5) ;
2. using the modification defined in [2] (second column) ;
3. using the modification defined by (27) with $g_k = B_k$ (third column) .

Methods 2 and 3 are both improvements of method 1. If we compare methods 2 and 3, the examples show that in some cases convergence is faster using method 3 (see examples 1 and 2), while in other cases method 2 is best (example 3).

Example 1 :

Consider the 3-fraction

$$\begin{cases} a_k^{(1)} = \dfrac{1}{k^2} \\ a_k^{(2)} = -1.1 - \dfrac{1}{100k} \\ b_k = 2.1 + \dfrac{1}{50k(k+1)}. \end{cases}$$

The exact value of the 3-fraction is (all digits correct) :

$$\xi_0^{(1)} = 0.78140324769506 \ , \ \xi_0^{(2)} = -0.70697896674506 \ .$$

In table 1 we compare successive approximations to the value of the 3-fraction using the three methods given above. In table 2 the exact values of the tails of the 3-fraction are compared with the approximations of the tails used in methods 2 and 3.

ν	$S_\nu^{(1)}(0, 0)$	$S_\nu^{(1)}(w_\nu^{(1)}, w_\nu^{(2)})$	$S_\nu^{(1)}(\tilde{w}_\nu^{(1)}, \tilde{w}_\nu^{(2)})$
50	0.781322484437	0.781404314117	0.781400465790
100	0.781402609218	0.781403247261	0.781403245962
150	0.781403242178	0.781403247681	0.781403247693
158	0.781403245112	0.781403247688	0.781403247694
159	0.781403245346	0.781403247689	0.781403247695
180	0.781403247374	0.781403247694	0.781403247695
184	0.781403247476	0.781403247694	0.781403247695
185	0.781403247495	0.781403247695	0.781403247695
200	0.781403247647	0.781403247695	0.781403247695
240	0.781403247694	0.781403247695	0.781403247695
246	0.781403247694	0.781403247695	0.781403247695
247	0.781403247695	0.781403247695	0.781403247695

Table 1a

ν	$S_\nu^{(2)}(0, 0)$	$S_\nu^{(2)}(w_\nu^{(1)}, w_\nu^{(2)})$	$S_\nu^{(2)}(\tilde{w}_\nu^{(1)}, \tilde{w}_\nu^{(2)})$
50	-0.706935500252	-0.706979540689	-0.706977469534
100	-0.706978623119	-0.706978966511	-0.706978965812
150	-0.706978963776	-0.706978966738	-0.706978966744
158	-0.706978965355	-0.706978966741	-0.706978966745
159	-0.706978965481	-0.706978966742	-0.706978966745
180	-0.706978966572	-0.706978966745	-0.706978966745
184	-0.706978966627	-0.706978966745	-0.706978966745
185	-0.706978966638	-0.706978966745	-0.706978966745
200	-0.706978966719	-0.706978966745	-0.706978966745
240	-0.706978966744	-0.706978966745	-0.706978966745
246	-0.706978966745	-0.706978966745	-0.706978966745
247	-0.706978966745	-0.706978966745	-0.706978966745

Table 1b

ν	$\xi_\nu^{(1)}$	$w_\nu^{(1)}$	$\tilde{w}_\nu^{(1)}$
50	0.000349247186	0.000000000000	0.000348064572
100	0.000089130337	0.000000000000	0.000089108418
150	0.000039880298	0.000000000000	0.000039879058
158	0.000035968255	0.000000000000	0.000035967472
159	0.000035520066	0.000000000000	0.000035519328
180	0.000027756016	0.000000000000	0.000027755849
184	0.000026568719	0.000000000000	0.000026568608
185	0.000026283798	0.000000000000	0.000026283699

Table 2a

ν	$\xi_\nu^{(2)}$	$w_\nu^{(2)}$	$\tilde{w}_\nu^{(2)}$
50	-0.999104055048	-1.000000000000	-0.995709951081
100	-1.000150991047	-1.000000000000	-0.999903473988
150	-1.000266167964	-1.000000000000	-1.000234601268
158	-1.000269677836	-1.000000000000	-1.000247513271
159	-1.000269986170	-1.000000000000	-1.000248823546
180	-1.000271598245	-1.000000000000	-1.000265290830
184	-1.000271110918	-1.000000000000	-1.000266649750
185	-1.000270959678	-1.000000000000	-1.000266924356

Table 2b

Example 2 :

Consider the 3-fraction

$$\begin{cases} a_k^{(1)} = 0.9975 + \dfrac{1}{15(k+10^5)} \\ a_k^{(2)} = -2.9975 - \dfrac{1}{10(k+10^5)^2} \\ b_k = 3. \end{cases}$$

The exact value of the 3-fraction is (all digits correct) :

$$\xi_0^{(1)} = 0.950313196405813\ ,\ \xi_0^{(2)} = -1.95034675538094.$$

The results are given in table 3 and 4.

ν	$S_\nu^{(1)}(0, 0)$	$S_\nu^{(1)}(w_\nu^{(1)}, w_\nu^{(2)})$	$S_\nu^{(1)}(\tilde{w}_\nu^{(1)}, \tilde{w}_\nu^{(2)})$
100	0.949609727594	0.950307981763	0.950301812020
200	0.950307344701	0.950313152531	0.950313195619
250	0.950312660132	0.950313192387	0.950313196401
260	0.950312863890	0.950313193914	0.950313196404
268	0.950312969549	0.950313194706	0.950313196405
269	0.950312980136	0.950313194785	0.950313196406
350	0.950313191901	0.950313196372	0.950313196406
400	0.950313195993	0.950313196403	0.950313196406
447	0.950313196362	0.950313196405	0.950313196406
448	0.950313196364	0.950313196406	0.950313196406
450	0.950313196368	0.950313196406	0.950313196406
500	0.950313196402	0.950313196406	0.950313196406
550	0.950313196405	0.950313196406	0.950313196406
551	0.950313196406	0.950313196406	0.950313196406

Table 3a.

ν	$S_\nu^{(2)}(0, 0)$	$S_\nu^{(2)}(w_\nu^{(1)}, w_\nu^{(2)})$	$S_\nu^{(2)}(\tilde{w}_\nu^{(1)}, \tilde{w}_\nu^{(2)})$
100	-1.949605819498	-1.950341263678	-1.950334766240
200	-1.950340593607	-1.950346709181	-1.950346754552
250	-1.950346190692	-1.950346751149	-1.950346755376
260	-1.950346405247	-1.950346752757	-1.950346755380
268	-1.950346516503	-1.950346753591	-1.950346755381
269	-1.950346527652	-1.950346753675	-1.950346755381
350	-1.950346750638	-1.950346755345	-1.950346755381
400	-1.950346754946	-1.950346755378	-1.950346755381
447	-1.950346755335	-1.950346755381	-1.950346755381
448	-1.950346755337	-1.950346755381	-1.950346755381
450	-1.950346755341	-1.950346755381	-1.950346755381
500	-1.950346755377	-1.950346755381	-1.950346755381
550	-1.950346755381	-1.950346755381	-1.950346755381
551	-1.950346755381	-1.950346755381	-1.950346755381

Table 3b.

ν	$\xi_\nu^{(1)}$	$w_\nu^{(1)}$	$\tilde{w}_\nu^{(1)}$
100	0.950312880275	0.950000000000	0.949574816998
200	0.950312564781	0.950000000000	0.950306626187
250	0.950312407272	0.950000000000	0.950312044362
260	0.950312375790	0.950000000000	0.950312226581
268	0.950312350608	0.950000000000	0.950312312204
269	0.950312347461	0.950000000000	0.950312320159
350	0.950312092732	0.950000000000	0.950312287246
400	0.950311935698	0.950000000000	0.950312134303
447	0.950311788231	0.950000000000	0.950311987034
448	0.950311785095	0.950000000000	0.950311983896

Table 4a.

ν	$\xi_\nu^{(2)}$	$w_\nu^{(2)}$	$\tilde{w}_\nu^{(2)}$
100	-1.950346405495	-1.950000000000	-1.949534263528
200	-1.950346056315	-1.950000000000	-1.950339523047
250	-1.950345881989	-1.950000000000	-1.950345479923
260	-1.950345847144	-1.950000000000	-1.950345680072
268	-1.950345819274	-1.950000000000	-1.950345774045
269	-1.950345815791	-1.950000000000	-1.950345782770
350	-1.950345533862	-1.950000000000	-1.950345744762
400	-1.950345360062	-1.950000000000	-1.950345575462
447	-1.950345196849	-1.950000000000	-1.950345412469
448	-1.950345193378	-1.950000000000	-1.950345408997

Table 4b.

Example 3 :

Consider the ordinary continued fraction

$$\begin{cases} a_k^{(1)} = -1.1 \\ b_k = 2.1 - (\frac{4}{10})^k . \end{cases}$$

The exact value of this continued fraction is (all digits correct) :

$$\xi_0^{(1)} = -2.84481852131856.$$

The results are given in tables 5 and 6.

ν	$S_\nu^{(1)}(0)$	$S_\nu^{(1)}(w_\nu^{(1)})$	$S_\nu^{(1)}(\tilde{w}_\nu^{(1)})$
20	-2.574878293036	-2.844818503381	-2.833807812186
32	-2.761091769957	-2.844818521318	-2.843702294428
33	-2.768789141648	-2.844818521319	-2.843896046267
50	-2.829911383947	-2.844818521319	-2.844782418074
100	-2.844691811928	-2.844818521319	-2.844818518699
150	-2.844817441958	-2.844818521319	-2.844818521318
155	-2.844817851121	-2.844818521319	-2.844818521318
156	-2.844817912048	-2.844818521319	-2.844818521319
200	-2.844818512124	-2.844818521319	-2.844818521319
250	-2.844818521240	-2.844818521319	-2.844818521319
300	-2.844818521318	-2.844818521319	-2.844818521319
324	-2.844818521318	-2.844818521319	-2.844818521319
325	-2.844818521319	-2.844818521319	-2.844818521319

Table 5.

ν	$\xi_\nu^{(1)}$	$w_\nu^{(1)}$	$\tilde{w}_\nu^{(1)}$
20	-1.000000006283	-1.000000000000	-0.995995485280
32	-1.000000000000	-1.000000000000	-0.998757931155
33	-1.000000000000	-1.000000000000	-0.998872120054
50	-1.000000000000	-1.000000000000	-0.999778855735
100	-1.000000000000	-1.000000000000	-0.999998120293
150	-1.000000000000	-1.000000000000	-0.999999983988
155	-1.000000000000	-1.000000000000	-0.999999990058
156	-1.000000000000	-1.000000000000	-0.999999990962
200	-1.000000000000	-1.000000000000	-0.999999999864
250	-1.000000000000	-1.000000000000	-0.999999999999
300	-1.000000000000	-1.000000000000	-1.000000000000
324	-1.000000000000	-1.000000000000	-1.000000000000
325	-1.000000000000	-1.000000000000	-1.000000000000

Table 6.

References

[1] BRUIN, Marcel G. de and Lisa JACOBSEN : Modification of generalised continued fractions I, to appear.

[2] LEVRIE, Paul and Adhemar BULTHEEL : Modification of Generalized Continued Fractions Associated with Recurrence Relations of Perron-Kreuser Type, Department of Computer Science K.U.Leuven, Report TW 78, 1986.