MORE TALES ABOUT TAILS?

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Abstract

In this note we show some numerical aspects of the convergence of modified approximants of a continued fraction. An explanation is given for the convergence to the correct value even if the wrong tails are used in the modification of the approximants.

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1 Introduction and notations

This note is about tails of convergent continued fractions. We consider a continued fraction of the form

$$K\left(\frac{a_n}{b_n}\right) = \frac{a_1}{\left\lceil b_1\right\rfloor} + \frac{a_2}{\left\lceil b_2\right\rfloor} + \cdots + \frac{a_k}{\left\lceil b_k\right\rfloor} + \cdots, \qquad a_k,\ b_k \in \mathbb{C},\ a_k \neq 0. \tag{1.1}$$

The n-th approximant f_n (in the classical sense) of the continued fraction (1.1) is defined by $f_n = S_n(0)$, with

$$S_n(w) = \frac{a_1}{\left\lceil b_1\right\rfloor} + \frac{a_2}{\left\lceil b_2\right\rfloor} + \cdots + \frac{a_n}{\left\lceil b_n + w\right\rfloor}.$$

We assume that the continued fraction (1.1) converges in $\mathbb{C}$. Its value f is given by

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} S_n(0). \tag{1.2}$$

$S_n(w)$ for $w \neq 0$ is called a *modified approximant* of (1.1) ([2, 4]).
The n-th tail $f^{(n)}$ of the continued fraction (1.1) is defined by

$$f^{(n)} = \frac{a_{n+1}}{\left\lceil b_{n+1}\right\rfloor} + \frac{a_{n+2}}{\left\lceil b_{n+2}\right\rfloor} + \cdots \quad .$$

The sequence $\{f^{(n)}\}$ is called *the sequence of right tails* of the given continued fraction ([5]). Obviously the numbers $f^{(n)}$ satisfy the recurrence relation

$$f^{(n)} = \frac{a_{n+1}}{b_{n+1} + f^{(n+1)}} \qquad \text{for all } n. \tag{1.3}$$

(Note that $f^{(0)} = f$.)
For a convergent continued fraction the sequence of right tails is unique.
Any other sequence $\{g^{(n)}\}$ that satisfies (1.3) is called *a sequence of wrong tails*:

$$g^{(0)} \neq f^{(0)}, \quad g^{(n)} = \frac{a_{n+1}}{b_{n+1} + g^{(n+1)}} \qquad \text{for all } n.$$

A sequence of wrong tails can be generated by choosing a value for $g^{(0)} (\neq f)$ and using

$$g^{(n)} = \frac{a_n}{g^{(n-1)}} - b_n, \qquad n = 1, 2, \ldots \tag{1.4}$$

Now it is easy to see that

$$f = S_n(f^{(n)}) \qquad \text{for all } n \tag{1.5}$$

and

$$g^{(0)} = S_n(g^{(n)}) \qquad \text{for all } n \tag{1.6}$$

for all sequences of wrong tails.
The classical way to evaluate the continued fraction (1.1) is by calculating $S_n(0)$ for a sufficiently large n. In view of (1.5) this means that we replace the n-th tail of (1.1) by 0 for some n. Now it seems (and indeed it often is) useful to replace the n-th tail by something else than 0, for instance by an approximation of the n-th tail if we can find one.
Now if we replace the n-th tail by $g^{(n)}$ where $\{g^{(n)}\}$ is a sequence of wrong tails, it is clear from (1.6) that this is not a good choice. Numerically however it seems that even with such a choice we get an approximation to the value of the continued fraction (1.1).
In the next section we want to show why this is so and we'll carry out some numerical experiments.

2 Numerical aspects of using (wrong) tails

If we define A_n and B_n by

$$\begin{array}{ll} A_{-1} = 1, \ A_0 = 0, & B_{-1} = 0, \ B_0 = 1, \\ A_n = b_n A_{n-1} + a_n A_{n-2}, & B_n = b_n B_{n-1} + a_n B_{n-2}, \quad n \geq 1 \end{array}$$

then it is easy to prove that

$$S_n(w) = \frac{A_n + wA_{n-1}}{B_n + wB_{n-1}}. \tag{2.1}$$

As a consequence of this the n-th approximant of (1.1) is given by

$$f_n = \frac{A_n}{B_n}.$$

A_n and B_n are called the *n-th numerator* and *n-th denominator* of (1.1) respectively. The second-order linear recurrence relation

$$y_n = b_n y_{n-1} + a_n y_{n-2} \quad , \qquad n = 1, 2, \ldots \tag{2.2}$$

satisfied by $\{A_n\}$ and $\{B_n\}$ plays an important role in the theory of continued fractions. It is well-known that the continued fraction (1.1) converges if and only if the recurrence relation (2.2) has a minimal solution $\{F_n\}$ with $F_0 \neq 0$ (Pincherlé, see [1]). ($\{F_n\}$ is a minimal solution of (2.2) if

$$\lim_{n\to\infty} \frac{F_n}{K_n} = 0 \tag{2.3}$$

for some solution $\{K_n\}$ of (2.2) linearly independent of $\{F_n\}$.)
A modified approximant $S_n(w)$ of (1.1) is related to a solution of the recurrence relation (2.2) in the following way:
Let $\{H_n\}$ be a solution of (2.2) with

$$-\frac{H_n}{H_{n-1}} = w.$$

(Such a solution always exists: We can construct one by choosing

$$H_{n-1} = 1, \; H_n = -w$$

and using (2.2) to calculate $H_{-1}, \ldots, H_{n-2}$ (note that $a_n \neq 0$) and $H_{n+1}, H_{n+2}, \ldots$.)
Then it is easy to prove that

$$S_n(w) = -\frac{H_0}{H_{-1}}$$

We start with (2.1):

$$S_n\left(-\frac{H_n}{H_{n-1}}\right) = \frac{A_n - \frac{H_n}{H_{n-1}} A_{n-1}}{B_n - \frac{H_n}{H_{n-1}} B_{n-1}} = \frac{H_{n-1}A_n - H_n A_{n-1}}{H_{n-1}B_n - H_n B_{n-1}}.$$

Using the recurrence relation (2.2) leads to

$$S_n\left(-\frac{H_n}{H_{n-1}}\right) = \frac{(-a_n)\cdot(-a_{n-1})\cdots(-a_1)\cdot(H_{-1}A_0 - H_0 A_{-1})}{(-a_n)\cdot(-a_{n-1})\cdots(-a_1)\cdot(H_{-1}B_0 - B_0 A_{-1})} = \frac{H_{-1}\cdot 0 - H_0 \cdot 1}{H_{-1}\cdot 1 - H_0 \cdot 0}$$

and this is what we had to prove.
Note that to each sequence of (wrong or right) tails of the continued fraction (1.1) corresponds a solution of the recurrence (2.2). If $\{g^{(n)}\}$ is a sequence of wrong tails, then

$$g^{(n)} = -\frac{G_n}{G_{n-1}}$$

with $\{G_n\}$ a solution of (2.2) which is not minimal. For the sequence of right tails $\{f^{(n)}\}$ we have

$$f^{(n)} = -\frac{F_n}{F_{n-1}}$$

with $\{F_n\}$ a minimal solution of (2.2). (This is a consequence of the fact that a minimal solution is essentially unique.)
Although it seems that using wrong tails to calculate the value of a continued fraction isn't such a good idea, numerical experiments show that in many cases there is no problem:

$$S_n(g^{(n)}) \to f^{(0)} \quad \text{numerically} \quad \text{if} \quad n \to \infty.$$

Let us try to understand what happens. To a sequence of wrong tails $\{g^{(n)}\}$ corresponds a solution $\{G_n\}$ of (2.2). We won't be using the exact value of $g^{(n)}$ since we can't avoid rounding errors. So we really will be calculating $S_n(g^{(n)}(1+\varepsilon))$ with ε very small (of the size of the machine precision). Now let $\{H_n\}$ be a solution of (2.2) with

$$-\frac{H_n}{H_{n-1}} = g^{(n)}(1+\varepsilon). \tag{2.4}$$

Assuming that other errors introduced during the calculation cancel out, we will have

$$S_n\left(g^{(n)}(1+\varepsilon)\right) = -\frac{H_0}{H_{-1}}. \tag{2.5}$$

The solution $\{H_n\}$ of (2.2) can be written as a linear combination of $\{G_n\}$ and the minimal solution $\{F_n\}$ of (2.2):

$$H_n = \alpha \cdot F_n + \beta \cdot G_n \quad \text{for all } n$$

with $\alpha \neq 0$.
Hence we get from (2.4):

$$-\frac{\alpha \cdot F_n + \beta \cdot G_n}{\alpha \cdot F_{n-1} + \beta \cdot G_{n-1}} = -\frac{G_n}{G_{n-1}}(1+\varepsilon)$$

or equivalently, with $C = \beta/\alpha$

$$-\frac{F_n + C \cdot G_n}{F_{n-1} + C \cdot G_{n-1}} = -\frac{G_n}{G_{n-1}}(1+\varepsilon).$$

Solving this equation for C gives us:

$$C = \frac{F_n G_{n-1} - F_{n-1} G_n - \varepsilon F_{n-1} G_n}{\varepsilon G_{n-1} G_n}.$$

This can be rewritten as:

$$C = \frac{1}{\varepsilon}\left(\frac{F_n}{G_n} - (1+\varepsilon)\frac{F_{n-1}}{G_{n-1}}\right). \tag{2.6}$$

Note that C will be small if n is large as a consequence of (2.3). Hence what we really get in (2.5) is:

$$S_n\left(g^{(n)}(1+\varepsilon)\right) = -\frac{F_0 + C \cdot G_0}{F_{-1} + C \cdot G_{-1}} \approx -\frac{F_0}{F_{-1}} = f^{(0)}.$$

Example 1. The sequence of Bessel functions of the first kind $\{F_n = J_{n-1}(1)\}$ is a minimal solution of the recurrence relation ([1])

$$y_n = 2n y_{n-1} - y_{n-2} \ , \quad n = 1, 2, \ldots$$

We know from the Perron-Kreuser theorem that

$$\frac{F_n}{F_{n-1}} \sim \frac{1}{2n} \ , \qquad \frac{G_n}{G_{n-1}} \sim 2n \quad \text{for } n \to \infty$$

for $\{G_n\}$ not a minimal solution of this recurrence relation. Furthermore we know that

$$\frac{F_n}{G_n} \sim \frac{1}{(2n)!} \quad \text{when } n \to \infty.$$

Hence we get from (2.6):

$$C = \frac{1}{\varepsilon}\left(\frac{1}{(2n)!} - (1+\varepsilon)\frac{1}{(2n-2)!}\right).$$

We've calculated C for some values of n, under the assumption that $\varepsilon = 10^{-16}$ (double precision):

$$\begin{aligned} n = 5 \quad &\rightarrow \quad C = -2 \cdot 10^{11} \\ n = 10 \quad &\rightarrow \quad C = -2 \cdot 10^{0} \\ n = 15 \quad &\rightarrow \quad C = -3 \cdot 10^{-14} \end{aligned}$$

If we use the sequence of wrong tails $\{g^{(n)}\}$ with $g^{(0)} = 1$ to approximate the value of the continued fraction, we find:

$$\begin{aligned} n = 10 \quad &\rightarrow \quad \text{approx. (2.5)} \quad = 0.54 \\ n = 15 \quad &\rightarrow \quad \text{approx. (2.5)} \quad = 0.5750809150043070 \\ &\qquad\quad \text{exact value} \quad = 0.5750809150043060. \end{aligned}$$

The approximation $S_n(w)$ in (2.5) is computed using backward recursion:

$$q_n = w, \quad q_{i-1} = \frac{a_i}{b_i + q_i} \text{ for } i = n, n-1, \ldots, 1$$

Example 2. The periodic continued fraction

$$\cfrac{1}{1} + \cfrac{1}{1} + \cdots + \cfrac{1}{1} + \cdots$$

converges to the value $-\frac{1-\sqrt{5}}{2}$. The associated recurrence relation

$$y_n = y_{n-1} + y_{n-2} \ , \quad n = 1, 2, \ldots$$

has the solutions $\{F_n = (\frac{1-\sqrt{5}}{2})^n\}$ and $\{G_n = (\frac{1+\sqrt{5}}{2})^n\}$. Let us use the sequence of wrong tails associated with this $\{G_n\}$: $g^{(n)} = -\frac{1+\sqrt{5}}{2}$.
In this case we have for C:

$$C = \frac{1}{\varepsilon}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}} - (1+\varepsilon)\right)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{(n-1)}$$

and, with $\varepsilon = 10^{-16}$:

$$\begin{aligned} n = 25 &\rightarrow C = -1 \cdot 10^{6} \\ n = 50 &\rightarrow C = \ \ 5 \cdot 10^{-5} \\ n = 75 &\rightarrow C = -2 \cdot 10^{-15} \end{aligned}$$

$$\begin{aligned} n = 50 &\rightarrow \text{approx. } (2.5) = 0.61812 \\ n = 75 &\rightarrow \text{approx. } (2.5) = 0.6180339887498918 \\ &\quad\ \ \text{exact value} = 0.6180339887498948. \end{aligned}$$

It is interesting to note that the linear fractional transformation associated with this continued fraction has two fixed points:

the *attractive* fixed point: $-(\frac{1-\sqrt{5}}{2})$
the *repulsive* fixed point: $-(\frac{1+\sqrt{5}}{2})$.

(see [3]). The calculations above show that those names are well-chosen.

Example 3. The periodic continued fraction

$$\frac{30\rfloor}{\lceil 1} + \frac{30\rfloor}{\lceil 1} + \cdots + \frac{30\rfloor}{\lceil 1} + \cdots$$

converges to the value $f^{(0)} = 5$. The associated recurrence relation is

$$y_n = y_{n-1} + 30y_{n-2}\ , \quad n = 1, 2, \ldots$$

and it has the solutions $\{(-5)^n\}$ and $\{6^n\}$. If we choose $G_n = 6^n$, we find for C:

$$C = \frac{1}{\varepsilon}\left(-\frac{5}{6} - (1+\varepsilon)\right)\left(-\frac{5}{6}\right)^{(n-1)}$$

and, with $\varepsilon = 10^{-16}$:

$$\begin{aligned} n = 100 &\rightarrow C = 3 \cdot 10^{8} \\ n = 200 &\rightarrow C = 3 \\ n = 300 &\rightarrow C = 3 \cdot 10^{-8} \\ n = 400 &\rightarrow C = 4 \cdot 10^{-16} \end{aligned}$$

We will use two methods to calculate $g^{(n)}$, with $g^{(0)} = -124$. Note that in this case

$$G_n = -\frac{118}{11}(-5)^{n+1} + \frac{129}{11}6^{n+1}.$$

Hence

$$g^{(n)} = -6\frac{129 - 118(-5/6)^{n+1}}{129 - 118(-5/6)^{n}}. \tag{2.7}$$

Using this expression for $g^{(n)}$ we find

$$\begin{array}{rcrl} n = 100 & \to & \text{approx. } (2.5) & = -1.240000048657465 \cdot 10^2 \\ n = 200 & \to & \text{approx. } (2.5) & = \ 1.881090218261262 \cdot 10^1 \\ n = 202 & \to & \text{approx. } (2.5) & = -6.000000000000000 \\ n = 300 & \to & \text{approx. } (2.5) & = -6.000000000000000 \\ n = 400 & \to & \text{approx. } (2.5) & = -6.000000000000000 \\ & & \text{exact value} & = \ 5.000000000000000. \end{array}$$

If we calculate $g^{(n)}$ using (1.4):

$$g^{(n)} = \frac{30}{g^{(n-1)}} - 1$$

we get

$$\begin{array}{rcrl} n = 100 & \to & \text{approx. } (2.5) & = -1.240000048657465 \cdot 10^2 \\ n = 196 & \to & \text{approx. } (2.5) & = -7.730715977770467 \\ n = 200 & \to & \text{approx. } (2.5) & = -6.000000000000002 \\ n = 300 & \to & \text{approx. } (2.5) & = -6.000000000000002 \\ n = 400 & \to & \text{approx. } (2.5) & = -6.000000000000002 \\ & & \text{exact value} & = \ 5.000000000000000. \end{array}$$

Note that convergence is (not quite) to the fixed point -6 of the l.f.t. associated with the continued fraction.
What is happening for $n \approx 200$?
It is a consequence of (2.7) and of the fact that

$$(-5/6)^{200} \approx 10^{-16}$$

that when $n \approx 200$ the sequence of tails is approaching its limiting value -6.
If we use the first method, for some value of n, $g^{(n)}$ will be equal to -6 and no rounding error will occur in the subsequent calculations.
In the second method, due to the slow convergence of the tail sequence $\{g^{(n)}\}$ to -6, we get a kind of 2-periodic effect: using (1.4) for some k, $g^{(k)}$ will be equal to -6.000000000000002 and from then on we find:

$$g^{(k+1)} = -5.999999999999998, \quad g^{(k+2)} = -6.000000000000002$$

as a consequence of rounding.
In both cases we don't have convergence to the value of the continued fraction.

Example 4. It is interesting to study the related limit-periodic continued fraction

$$\cfrac{30+0.9}{1} + \cfrac{30+0.9^2}{1} + \cdots + \cfrac{30+0.9^n}{1} + \cdots$$

for which tail sequences also converge to 5 and -6.
We've calculated $g^{(n)}$ using (1.4) with $g^{(0)} = 32.367267$. We find the following results:

$$\begin{array}{rcrl} n = 200 & \to & \text{approx. } (2.5) & = -7.864495586486798 \cdot 10^1 \\ n = 300 & \to & \text{approx. } (2.5) & = \ 5.085064169276248 \\ n = 310 & \to & \text{approx. } (2.5) & = \ 5.085066121315100 \\ n = 400 & \to & \text{approx. } (2.5) & = \ 5.085066121315100 \\ & & \text{exact value} & = \ 5.085066164924414. \end{array}$$

Note that convergence is not to the value of the continued fraction, but to a value that is quite close: The terms 0.9^n in the numerators of the c.f. introduce rounding errors into the calculations, so we expect convergence to the value of the c.f. However, these terms 0.9^n die out for $n \approx 310$:

$$0.9^{310} = 7 \cdot 10^{-15}.$$

So if we replace the tail of the c.f. by $g^{(n)}$ with $n = 400$, then rounding errors will only appear in the calculations when we get near the point $k = 310$ in the backward recursion

$$h^{(k)} = \frac{30 + 0.9^{k+1}}{1 + h^{(k+1)}}.$$

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