Computing the robust H-infinity norm of time-delay LTI systems with real-valued and structured uncertainties ⋆

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Abstract: In this paper a new method to calculate the robust (worst-case) $\mathcal{H}_\infty$-norm of a linear time-invariant time-delay system with an arbitrary number of delays with uncertainties both in the system matrices and the delay parameters is presented. The proposed approach fully exploits the real-valued and structured nature of the uncertainties. As in real-life applications, uncertainties can influence more than one system matrix. The algorithm utilizes the relation between the robust $\mathcal{H}_\infty$-norm and the pseudo-spectrum of an associated non-linear (delay) eigenvalue problem with both real and complex perturbations. More precisely it uses the Newton-bisection method to find the robust complex distance to instability of this eigenvalue problem which will be shown to be equal to the reciprocal of the robust $\mathcal{H}_\infty$-norm.

Keywords: Time delay systems - Uncertain dynamic systems - Robust performance - H-infinity norm - Distance to instability - Pseudo spectrum

1. INTRODUCTION

The $\mathcal{H}_\infty$-norm is an important performance measure for the control of dynamical systems as it quantifies the biggest possible energy gain of the system over all energy-bounded disturbance signals (Doyle (2002)). The $\mathcal{H}_\infty$-norm of an exponentially stable system can, using Parseval's theorem, also be interpreted as the largest value of the frequency response measured in spectral norm. In other words, it measures how well exogenous disturbances are suppressed by the system. Controller design methods to minimize the $\mathcal{H}_\infty$-norm of the closed-loop system are therefore plenty: e.g. Doyle et al. (1989), Wu et al. (2004) (based on Riccati equations and Linear Matrix Inequalities), and Gumussoy and Michiels (2011) (via a direct optimisation approach).

In real-life scenarios the mathematical model often deviates from the physical system it describes, due to unmodeled (non-linear) behaviour, model order reductions or uncertain parameters. A typical solution to deal with these deviations is to consider a family of systems. One could for example use a nominal system with additional uncertainties to account for the differences between model and reality (Hinrichsen and Pritchard (2005)). These uncertainties are typically real-valued (as the model of the system itself is real-valued), structured (only a certain parameter or group of parameters is affected) and bounded. For such a family of models it is often desirable to quantify (analysis) or optimise (controller design) the worst possible behaviour over all realisation. In the context of $\mathcal{H}_\infty$-control this led to the notions of robust performance (Zhou and Doyle (1998)) and the robust $\mathcal{H}_\infty$-norm. The latter is for example examined in Xie and de Souza Carlos (1992) and Ji et al. (2006), which present controller design methods, based on Riccati equations and linear matrix inequalities, that guarantee that the disturbance attenuation is below a specified level. This paper in contrast will focus on quantifying the robust $\mathcal{H}_\infty$-norm (analysis) and will consider linear time invariant (LTI) time-delay systems with an arbitrary number of delays and uncertainties both in system matrices and in the time delays. Such systems are for example considered in Borgioli and Michiels (2018), where an algorithm to compute the robust spectral abscissa of these systems is presented.

The standard methods to calculate the $\mathcal{H}_\infty$-norm are based on the level set algorithm presented in Boyd et al. (1989). For time-delay systems see for example Gumussoy and Michiels (2011). In each iteration step these methods need to check if the spectrum of an associated Hamiltonian eigenvalue problem contains strictly imaginary (ie. real part equal to zero) eigenvalues. For large systems this operation is however often too costly. Therefore new techniques that overcome this problem for (singular) LTI-systems without delays, were presented in Guglielmi and Lubich (2013) and Benner and Voigt (2014). These methods are based on the relation between the $\mathcal{H}_\infty$-norm and the complex distance to instability (also known as the spectral radius (Hinrichsen and Pritchard (2005))) of a corresponding matrix pencil. More precisely the $\mathcal{H}_\infty$-norm of $T(j\omega) = C\left(j\omega E - A\right)^{-1} B$ is equal to the reciprocal of the complex distance to instability of $M(\lambda, \Delta) = \lambda E - A - B\Delta C$ (Benner and Voigt (2014)). These methods obtain a significant speed-up for systems with a large and sparse state matrix as only the rightmost eigenvalues of the aforementioned matrix pencil are needed (which can be calculated using iterative methods such as presented in Meerbergen et al. (1994)).

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This paper will extend these methods to systems with time-delays and structured, real-valued uncertainties. The main differences are that the associated eigenvalue problem becomes non-linear (due to the delays) and has besides a complex- also real-valued perturbations (due to the real-valued uncertainties on the system). After introducing the needed concepts, notations, and assumptions in Section 2, the relation between the robust $\mathcal{H}_\infty$-norm and the robust complex distance to instability of a corresponding delay eigenvalue problem (DEP) will be examined in Section 3. Subsequently a Newton-bisection method to compute the robust $\mathcal{H}_\infty$-norm, is presented in Section 4. Section 5 gives a short example of the proposed method. Finally, Section 6 gives an outlook to further work.

2. CONCEPTS, NOTATIONS AND ASSUMPTIONS

This section will introduce the necessary concepts, notations, and assumptions. First the considered family of systems is defined and an expression for the robust $\mathcal{H}_\infty$-norm is given. Subsequently, we introduce the associated DEP and define related concepts such as pseudo-spectrum, pseudo-spectral abscissa, and the robust complex distance to instability. Finally some preliminary lemmas are given.

2.1 Considered system and robust $\mathcal{H}_\infty$-norm

In this paper we consider families of linear time-invariant state-space models with time delays and uncertainties on both the system matrices and the delays. The focus lies on systems without direct feed-through (although section 6 briefly discusses how to generalise the presented methods to systems with direct feed-through). The uncertainties on the system matrices are real-valued, structured and bounded in Frobenius norm and can influence multiple matrices. Structure is obtained using left and right shape matrices. The uncertainties on the time-delays are limited in absolute value.

$$\begin{cases} \dot{x}(t) = \sum_{k=0}^{K} \left(A_k + \sum_{l=1}^{L} \sum_{s=1}^{S_l^{A_k}} G_{l,s}^{A_k} \delta_l H_{l,s}^{A_k} \right) x(t-(\tau_k+\delta_{\tau_k})) + \\ \qquad \sum_{k=0}^{K} \left(B_k + \sum_{l=1}^{L} \sum_{s=1}^{S_l^{B_k}} G_{l,s}^{B_k} \delta_l H_{l,s}^{B_k} \right) w(t-(\tau_k+\delta_{\tau_k})) \\ z(t) = \sum_{k=0}^{K} \left(C_k + \sum_{l=1}^{L} \sum_{s=1}^{S_l^{C_k}} G_{l,s}^{C_k} \delta_l H_{l,s}^{C_k} \right) x(t-(\tau_k+\delta_{\tau_k})) \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$ the internal state; $w(t) \in \mathbb{R}^m$ the exogenous input; $z(t) \in \mathbb{R}^p$ the exogenous output; A_k, B_k, and C_k real system matrices of appropriate dimension; $\tau_k \in \mathbb{R}^+$ the time-delays with $\tau_0 = 0$; L the number of matrix uncertainties; $\delta_l \in \mathbb{R}^{q_l \times r_l}$ with $\|\delta_l\|_F \leq \overline{\delta_l}$ for $l = 1..L$; $G_{l,s}^{A_k}$, $G_{l,s}^{B_k}$, $G_{l,s}^{C_k}$, $H_{l,s}^{A_k}$, $H_{l,s}^{B_k}$, and $H_{l,s}^{C_k}$ real-valued shape matrices of appropriate dimension; and $\delta_{\tau_k} \in \mathbb{R}$ the uncertainties on the delays with $|\delta_{\tau_k}| \leq \overline{\delta_{\tau_k}}$ (to avoid non-causal systems we assume $\overline{\delta_{\tau_k}} \leq \tau_k$). For sake of readability, we will introduce following notation $\delta = (\delta_1, .., \delta_L, \delta_{\tau_1}, .., \delta_{\tau_K})$ and $\mathbb{R}^\star = \mathbb{R}^{q_1 \times r_1} \times .. \times \mathbb{R}^{q_K \times r_K} \times \mathbb{R} \times .. \times \mathbb{R}$. With the latter we will associate the following weighted infinity norm:

$$\|\delta\|_{\bar{\delta}} := \left\| \left[\frac{\|\delta_1\|_F}{\overline{\delta_1}} \; .. \; \frac{\|\delta_L\|_F}{\overline{\delta_L}} \; \frac{|\delta\tau_1|}{\overline{\delta_{\tau_1}}} \; .. \; \frac{|\delta\tau_K|}{\overline{\delta_{\tau_K}}} \right]^T \right\|_\infty .$$

The condition on the size of the allowable perturbations now reduces to $\|\delta\|_{\bar{\delta}} \leq 1$.

In the remainder of this work we will make the following assumption:

Assumption 1. system (1) is internally asymptotically stable for all allowable perturbations.

Remark: In (1) a single uncertainty can influence a system matrix using multiple shape matrices. For example the following structured uncertainty

$$\begin{bmatrix} \star_1 + \star_2 & 0 \\ 0 & \star_1 \end{bmatrix}$$

with $\left\| [\star_1 \; \star_2]^T \right\|_F \leq \bar{\delta}$, can be represented by

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix} [1 \; 0] + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix} [0 \; 1] .$$

To ease future derivations and to highlight the relation with the methods in Guglielmi and Lubich (2011) and Borgioli and Michiels (2018), we will first transform (1) to a system of delay differential algebraic equations (DDAE) by introducing slack variables (γ_w and γ_z).

$$\begin{cases} E^e \dot{x}^e(t) &= \sum_{k=0}^{K} \left(A_k^e + \sum_{l=1}^{L} \sum_{s=1}^{S_L^k} G_{l,s}^k \delta_l H_{l,s}^k \right) x^e(t-(\tau_k+\delta\tau_k)) \\ &\quad + B^e w(t) \\ z(t) &= C^e x^e(t) \end{cases} \tag{2}$$

with $x^e = \begin{bmatrix} x \\ \gamma_w \\ \gamma_z \end{bmatrix}$, $E^e = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A_0^e = \begin{bmatrix} A_0 & B_0 & 0 \\ 0 & -I_m & 0 \\ C_0 & 0 & -I_p \end{bmatrix}$,

$A_k^e = \begin{bmatrix} A_k & B_k & 0 \\ 0 & 0 & 0 \\ C_k & 0 & 0 \end{bmatrix}$, $B^e \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}$, $C^e = [0 \; 0 \; I_p]$,

$G_{l,s}^k$ and $H_{l,s}^k$ extended shape matrices such that the correct block in A_k^e is perturbed, and $S_L^k = S_L^{A_k} + S_L^{B_k} + S_L^{C_k}$. The transfer function of this extended system is:

$$T^e(s;\delta) := C^e \left(F(s;\delta) \right)^{-1} B^e$$

with $F(s;\delta) := sE^e - \left(\tilde{A}_0^e + \sum_{k=1}^{K} \tilde{A}_k^e e^{-s\tilde{\tau}_k} \right)$ where

$$\tilde{A}_k^e := A_k^e + \sum_{l=1}^{L} \sum_{s=1}^{S_L^k} G_{l,s}^k \delta_l H_{l,s}^k$$

and $\tilde{\tau}_k := \tau_k + \delta_{\tau_k}$.

2.2 Robust $\mathcal{H}_\infty$-norm

Under Assumption 1, the $\mathcal{H}_\infty$-norm of (2) is for a given realisation of the uncertainties equal to:

$$\|T^e(j\omega,\delta)\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_1 \left(T^e(j\omega,\delta) \right)$$

with $\sigma_1(\cdot)$ the largest singular value (Gu et al. (2003)). The robust $\mathcal{H}_\infty$-norm is defined as the supremum over all possible uncertainties:

$$\|T^e(j\omega,\delta)\|_\infty^{\bar{\delta}} := \sup_{\substack{\delta\in\mathbb{R}^\star \\ \|\delta\|_{\bar{\delta}}\leq 1}} \|T^e(j\omega,\delta)\|_\infty.$$

2.3 Associated delay eigenvalue problem

As mentioned in the introduction, the proposed method will calculate the robust $\mathcal{H}_\infty$-norm of (2) using its relation with the pseudo-spectrum of a retarded DEP with both real- and complex-valued uncertainties. Based on the results of Benner and Voigt (2014) mentioned in the introduction, one expects that characteristic matrix of this DEP has the following form

$$M(\lambda,\delta,\Delta) := \underbrace{\lambda E^e - \tilde{A}_0^e - \sum_{k=1}^{K} \tilde{A}_k^e e^{-\lambda\tilde{\tau}_k}}_{F(\lambda,\delta):=M(\lambda,\delta,0)} - B^e\Delta C^e \quad (3)$$

with E^e, $\tilde{A}_k^e$, B^e, C^e and $\tilde{\tau}_k$ as defined previously and $\Delta\in\mathbb{C}^{m\times p}$.
The $(\bar{\delta},\epsilon)$-pseudo-spectrum of (3) is defined as:

$$\Lambda\left(M,\bar{\delta},\epsilon\right) := \bigcup_{\substack{\Delta\in\mathbb{C}^{m\times p} \\ \|\Delta\|_2\leq\epsilon}} \bigcup_{\substack{\delta\in\mathbb{R}^* \\ \|\delta\|_{\bar{\delta}}\leq 1}} \{\lambda\in\mathbb{C} : \det\left(M(\lambda,\delta,\Delta)\right)=0\}. \quad (4)$$

The $(\bar{\delta},\epsilon)$-pseudo-spectral abscissa is the real part of the rightmost point of this pseudo-spectrum:

$$\alpha(M,\bar{\delta},\epsilon) := \max\left\{\Re(\lambda) : \lambda\in\Lambda\left(M,\bar{\delta},\epsilon\right)\right\}. \quad (5)$$

The robust complex distance to instability for a given $\bar{\delta}$ is equal to the smallest ϵ such that (5) is no longer strictly negative:

$$r(M,\bar{\delta}) := \inf\left\{\epsilon : \alpha(M,\bar{\delta},\epsilon)\geq 0\right\}.$$

2.4 Preliminary lemmas

Lemma 1. $F(\lambda,\delta)$, as defined in (3), is invertible for given δ and λ if and only if λ is not an eigenvalue of $\lambda I_n - \sum_{k=0}^{K}\tilde{A}_k e^{-\lambda\tilde{\tau}_k}$.

Proof.

$$F(\lambda,\delta) = \left[\begin{array}{c|cc} \lambda I_n - \sum_{k=0}^{K}\tilde{A}_k e^{-\lambda\tilde{\tau}_k} & -\sum_{k=0}^{K}\tilde{B}_k e^{-\lambda\tilde{\tau}_k} & 0 \\ \hline 0 & I_m & 0 \\ -\sum_{k=0}^{K}\tilde{C}_k e^{-\lambda\tilde{\tau}_k} & 0 & I_p \end{array}\right] \quad (6)$$

Using the subdivision in (6), F can be seen as a 2 times 2 block matrix and because

$$X := \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}$$

is always invertible, it is well known that F is invertible if and only if the Schur complement of X:

$$(F/X) = \lambda I_n - \sum_{k=0}^{K}\tilde{A}_k e^{-\lambda\tilde{\tau}_k}$$

is invertible. □

Lemma 2. The pseudo-spectral abscissa $(\alpha(M,\bar{\delta},\epsilon))$ is continuous, monotonically increasing in ϵ.

Proof. To prove continuity, the argument in Borgioli and Michiels (2018) (Section IV) can be generalised to retarded DEP with a singular leading matrix. To prove that $\alpha(M,\bar{\delta},\epsilon)$ is monotonically increasing in ϵ, the argument in Benner et al. (2012) (Section 4) can be extended to systems with delays.

Corollary 3. $r(M,\bar{\delta})$ is the unique zero-crossing of the function $\epsilon\mapsto\alpha(M,\bar{\delta},\epsilon)$ when $\alpha(M,\bar{\delta},0)<0$ and is zero otherwise.

3. EQUIVALENCE BETWEEN THE ROBUST $\mathcal{H}_\infty$-NORM AND THE COMPLEX DISTANCE TO INSTABILITY

Theorem 4. The robust $\mathcal{H}_\infty$-norm of (2) is equal to the reciprocal of the robust complex distance to instability of (3):

$$\|T^e(j\omega,\delta)\|_{\mathcal{H}_\infty}^{\bar{\delta}} = \frac{1}{r(M,\bar{\delta})}.$$

Proof. By Lemma 1 and Assumption 1, $\alpha(M,\bar{\delta},0)<0$. From Corollary 3 it follows that

$$\begin{aligned} r(M,\bar{\delta}) &= \inf_{\substack{\omega\in\mathbb{R},\Delta\in\mathbb{C}^{m\times p} \\ \|\delta\|_{\bar{\delta}}\leq 1}} \{\|\Delta\|_2 : \det\left(M(\jmath\omega,\delta,\Delta)\right)=0\} \\ &= \inf_{\substack{\omega\in\mathbb{R} \\ \Delta\in\mathbb{C}^{m\times p} \\ \|\delta\|_{\bar{\delta}}\leq 1}} \{\|\Delta\|_2 : \det\left(F(\jmath\omega,\delta)-B^e\Delta C^e\right)=0\}. \end{aligned}$$

From Assumption 1 and Lemma 1 it follows that $F(\jmath\omega,\delta)$ is invertible for all $\omega\in\mathbb{R}$ and $\delta\in\mathbb{R}^*$ with $\|\delta\|_{\bar{\delta}}\leq 1$:

$$r(M,\bar{\delta}) = \inf_{\substack{\omega\in\mathbb{R} \\ \Delta\in\mathbb{C}^{m\times p} \\ \|\delta\|_{\bar{\delta}}\leq 1}} \left\{\|\Delta\|_2 : \det(I-F(\jmath\omega,\delta)^{-1}B^e\Delta C^e)=0\right\}.$$

Using Schur's determinant lemma we find

$$\begin{aligned} r(M,\bar{\delta}) &= \inf_{\substack{\omega\in\mathbb{R} \\ \Delta\in\mathbb{C}^{m\times p} \\ \|\delta\|_{\bar{\delta}}\leq 1}} \left\{\|\Delta\|_2 : \det(I-C^eF(\jmath\omega,\delta)^{-1}B^e\Delta)=0\right\} \\ &= \inf_{\substack{\omega\in\mathbb{R} \\ \Delta\in\mathbb{C}^{m\times p} \\ \|\delta\|_{\bar{\delta}}\leq 1}} \{\|\Delta\|_2 : \det(I-T^e(\jmath\omega,\delta)\Delta)=0\}. \end{aligned}$$

It is well known that $1/\sigma_1(A)vu^H\in\operatorname{argmin}_{\Delta\in\mathbb{C}^{m\times p}}\{\|\Delta\|_2 : \det(I-A\Delta)=0\}$ with u and v respectively the left and right singular vectors of norm one corresponding to $\sigma_1(A)$.

$$\begin{aligned} r(M,\bar{\delta}) &= \inf_{\substack{\omega\in\mathbb{R} \\ \|\delta\|_{\bar{\delta}}\leq 1}} \frac{1}{\sigma_1\left(T^e(\jmath\omega,\delta)\right)} \\ &= \left(\sup_{\substack{\delta\in\mathbb{R}^\star \\ \|\delta\|_{\bar{\delta}}\leq 1}} \sup_{\omega\in\mathbb{R}} \sigma_1\left(T^e(\jmath\omega,\delta)\right)\right)^{-1}. \end{aligned}$$

□

Corollary 5. Theorem 4 remains valid if we restrict the complex permutation Δ in (3) to the manifold of rank 1 matrices.

4. ALGORITHM FOR ROBUST $\mathcal{H}_\infty$-NORM

Theorem 4 showed that computing the robust $\mathcal{H}_\infty$-norm of system (2) is equivalent to computing the robust complex

distance to instability of (3). The approach presented in this section will use this relation.

Inspired by Corollary 3, we will employ a Newton-Bisection method to find the root of the pseudo-spectral abscissa. This rootfinding method combines the robustness of the bisection method (if the initial search bracket contains a unique root then the algorithm will converge to this root), with the fast (quadratic when close to the root) convergence of the Newton method (Guglielmi et al. (2013)). A reference implementation of this algorithm can be found in Press et al. (1996). In each iteration step it requires the $(\bar{\delta}, \epsilon)$-pseudo-spectral abscissa and its derivative with respect to ϵ. Therefore we will first present an approach to calculate $\alpha(M, \bar{\delta}, \epsilon)$ for given values of ϵ and $\bar{\delta}$. Subsequently we show that $d\alpha(M, \bar{\delta}, \epsilon)/d\epsilon$ can be calculate cheaply from this result. Finally the complete algorithm to calculate the $\mathcal{H}_\infty$-norm is summarized in Section 4.2.

4.1 Calculating the pseudo-spectral abscissa for given values of ϵ and $\bar{\delta}$

To calculate the $(\bar{\delta}, \epsilon)$-pseudo-spectral abscissa of (3) we combine the methods proposed in Guglielmi and Lubich (2011) and Borgioli and Michiels (2018). The strategy of these methods is to define a path through the space of possible perturbations, $(\delta(\theta), \Delta(\theta))$, along which the real part of the rightmost eigenvalue (λ_{RM}) monotonically increases and that converges to a (local) rightmost point of the $(\bar{\delta}, \epsilon)$-pseudo-spectrum. To define this path we start with stating the derivative of $\Re(\lambda_{RM})$ with respect to θ:

$$\frac{d\Re\left(\lambda_{RM}(\theta)\right)}{d\theta} = -\Re\left(\frac{\phi^H\left(\frac{\partial M(\lambda_{RM};\delta,\Delta)}{\partial\theta}\right)\psi}{\phi^H\left(\frac{\partial M(\lambda_{RM};\delta,\Delta)}{\partial\lambda}\right)\psi}\right)$$

Borgioli and Michiels (2018). By pluggin in (3) one finds:

$$\begin{aligned}\frac{d\Re\left(\lambda_{RM}(\theta)\right)}{d\theta} = \frac{1}{\xi}\Bigg(&\Re\left(\left\langle B^{eT}\phi\psi^H C^{eT}, \dot{\Delta}\right\rangle_F\right)\\ &+\sum_{l=1}^{L}\left\langle\sum_{k=0}^{K}\sum_{s=1}^{S_l^k} {G_{l,s}^k}^T \Phi\Gamma_k\Psi^T {H_{l,s}^k}^T, \dot{\delta}_l\right\rangle_F\\ &-\sum_{k=1}^{K}\Re\left(\lambda_{RM}e^{-\lambda_{RM}\tilde{\tau}_k}\left(\phi^H\tilde{A}_k^e\psi\right)\right)\dot{\delta\tau}_k\Bigg)\end{aligned} \tag{7}$$

where the dependency on θ is omitted in the right hand side and

$$\begin{aligned}\xi &= \phi^H\left(E^e + \sum_{k=1}^{K}\tilde{\tau}_k\tilde{A}_k^e e^{-\lambda_{RM}\tilde{\tau}_k}\right)\psi; \qquad (8)\\ \Phi &= \left[\Re\left(\phi\right)\ \Im\left(\phi\right)\right];\\ \Psi &= \left[\Re\left(\psi\right)\ \Im\left(\psi\right)\right];\\ \Gamma_k &= \begin{bmatrix}\Re(e^{-\lambda_{RM}\tilde{\tau}_k}) & -\Im(e^{-\lambda_{RM}\tilde{\tau}_k})\\ \Im(e^{-\lambda_{RM}\tilde{\tau}_k}) & \Re(e^{-\lambda_{RM}\tilde{\tau}_k})\end{bmatrix};\end{aligned}$$

λ_{RM} the rightmost eigenvalue of (3); ϕ and ψ its corresponding left and right eigenvector normalised such that ξ is real and strictly positive; $\dot{\Delta}$, $\dot{\delta}_l$ and $\dot{\delta\tau}_k$ the derivates of respectively Δ, δ_l and $\delta\tau_k$ with respect to θ; and $\langle X, Y\rangle_F = \sum_{i,j}\bar{A}_{i,j}B_{i,j}$ the complex Frobenius inner product.

Before defining the resulting paths, we note that we can limit the search space for the complex perturbation Δ to the manifold of rank one matrices of norm ϵ (see Benner and Voigt (2014); Guglielmi and Lubich (2011); Guglielmi et al. (2013)):

$$\Delta(\theta) = \epsilon u(\theta)v(\theta)^H$$

with $u(\theta) \in \mathbb{C}^m$, $v(\theta) \in \mathbb{C}^p$ and $\|u(\theta)\|_2 = \|v(\theta)\|_2 = 1$. This statement can easily be verified using a similar argument as in Corolarry 5. In Borgioli and Michiels (2018) a similar low rank structure for the real-valued matrix perturbation was discussed; however, in this paper we will not exploit this low rank structure because the targeted real uncertainties are scalar or low dimensional for which working with the low rank factorisation is not advantageous.

Lets consider the scaled projection of the gradient flow (the steepest ascend direction) on the manifold of permitted permutations:

$$\begin{aligned}\dot{u} &= \frac{\jmath}{2}\Im\Big(u^H B^{eT}\phi\psi^H C^{eT}v\Big)u + (I - uu^H)B^{eT}\phi\psi^H C^{eT}v\\ \dot{v} &= \frac{\jmath}{2}\Im\left(v^H C^e\psi\phi^H B^e u\right)v + (I - vv^H)C^e\psi\phi^H B^e u\\ \dot{\delta}_l &= \begin{cases}\Theta_l - \dfrac{\langle\delta_l, \Theta_l\rangle_F\,\delta_l}{\|\delta_l\|_F^2} & \|\delta_l\|_F = \bar{\delta}_l \text{ and } \langle\delta_l, \Theta_l\rangle_F > 0\\ \Theta_l & \text{otherwise}\end{cases}\\ \dot{\delta\tau}_k &= \begin{cases}0 & |\delta\tau_k| = \bar{\delta\tau}_k \text{ and } \beta_k * \delta\tau_k > 0\\ \beta_k & \text{otherwise}\end{cases}\end{aligned} \tag{9}$$

with

$$\begin{aligned}\Theta_l &= \sum_{k=1}^{K}\sum_{s=1}^{S_l^k} {G_{l,s}^k}^T\Phi\Gamma_k\Psi^T {H_{l,s}^k}^T\\ \beta_k &= -\Re\left(\lambda_{RM}e^{-\lambda_{RM}\tilde{\tau}_k}\left(\psi^H\tilde{A}_k^e\phi\right)\right)\end{aligned}$$

where we again omitted the dependency on θ. By plugging in this path into (7), one observes that $\dfrac{d\Re\left(\lambda_{RM}(\theta)\right)}{d\theta} \geq 0$ (ie. the real part of the rightmost eigenvalue monotonically increases along this path).

The specified ordinary differential equation (ODE) is solved numerically using a forward Euler discretization with an adaptive step size. Unlike in traditional ODE-solvers the step size is not chosen to fulfil a specified error bound, but to guarantee that real part of the rightmost eigenvalue monotonically increases. In each discretization step the rightmost eigenvalues of (3) are calculate using the method presented in Wu and Michiels (2012). The path following algorithm is stopped when it is converged to a locally rightmost point. This can detected by monitoring the increase of the real part of the rightmost eigenvalue or the step size. Because the algorithm can get stuck in a locally rightmost point, we initialise the ODE starting from several eigenvalues.

Once the $(\bar{\delta},\epsilon)$-pseudo-spectral abscissa is known, its derivative with respect to ϵ can easily be calculate with almost no additional cost, in the following way:

$$\frac{d\alpha(M, \bar{\delta}, \epsilon)}{d\epsilon} = \frac{\Re\left(\phi^{\star H}B^e u^\star v^{\star H}C^e\psi^\star\right)}{\xi^\star} \tag{10}$$

with $\phi^\star$ and $\psi^\star$ respectively the left and right eigenvectors associated with the rightmost eigenvalue, $\xi^\star$ as in (8) evaluated at rightmost point of the pseudo-spectrum, and $\epsilon u^\star v^{\star H}$ the complex perturbation associated with the

rightmost point of the pseudo-spectrum. See for example Borgioli and Michiels (2018).

4.2 Algorithm for the robust $\mathcal{H}_\infty$-norm

Algorithm 1 Computation of $\|T^e(\jmath\omega,\delta)\|^{\bar{\delta}}_{\mathcal{H}_\infty}$ for given $\bar{\delta}$ using the Newton-Bisection method

Input: $\epsilon^{(0)}_{lb}$, $\epsilon^{(0)}_{ub}$ with $\epsilon^{(0)}_{lb} < \epsilon^{(0)}_{ub}$

Compute $\alpha_0 := \alpha(M,\bar{\delta},0)$
if $\alpha_0 \geq 0$ **then**
    Give error. System is not asymptotically internally stable.
end if
Verify that $\alpha^{(0)}_{ub} := \alpha(M,\bar{\delta},\epsilon^{(0)}_{ub}) > 0$ and $\alpha^{(0)}_{lb} := \alpha(M,\bar{\delta},\epsilon^{(0)}_{lb}) < 0$. If not give an error.
$i \leftarrow 1$, $\epsilon^{(1)} \leftarrow \frac{\epsilon^{(0)}_{lb}+\epsilon^{(0)}_{ub}}{2}$
Compute $\alpha^{(1)} := \alpha(M,\bar{\delta},\epsilon^{(1)})$ using the approach described in Section 4.1 and $d\alpha^{(1)} := \frac{d\alpha}{d\epsilon}(M,\bar{\delta},\epsilon^{(1)})$ using (10).
loop
    Calculate $\epsilon^{(i+1)}_{\text{Newton}} := \epsilon^{(i)} - \frac{\alpha^{(i)}}{d\alpha^{(i)}}$.
    if $\epsilon^{(i+1)}_{\text{Newton}}$ lies outside $\left[\epsilon^{(i)}_{lb},\epsilon^{(i)}_{ub}\right]$ or the change of $\alpha^{(i)}$ was insufficient **then**
        $\epsilon^{(i+1)} \leftarrow \frac{\epsilon^{(i)}_{lb}+\epsilon^{(i)}_{ub}}{2}$
    else
        $\epsilon^{(i+1)} \leftarrow \epsilon^{(i+1)}_{\text{Newton}}$
    end if
    if $|\epsilon^{(i+1)} - \epsilon^{(i)}|$ is below a specified threshold **then**
        return $\|T^e(\jmath\omega,\delta)\|^{\bar{\delta}}_{\mathcal{H}_\infty} := 1/\epsilon^{(i+1)}$
    end if
    Compute $\alpha^{(i+1)}$ and $d\alpha^{(i+1)}$
    if $\alpha^{(i+1)} < 0$ **then**
        $\epsilon^{(i+1)}_{lb} \leftarrow \epsilon^{(i+1)}$ and $\epsilon^{(i+1)}_{ub} \leftarrow \epsilon^{(i)}_{ub}$
    else
        $\epsilon^{(i+1)}_{ub} \leftarrow \epsilon^{(i+1)}$ and $\epsilon^{(i+1)}_{lb} \leftarrow \epsilon^{(i)}_{lb}$
    end if
    $i \leftarrow i+1$
end loop

Algorithm 1 gives the pseudo-code to compute the robust $\mathcal{H}_\infty$-norm for a given $\bar{\delta}$ using the Newton-bisection method. In each iteration step it uses the approach described in Section 4.1 to compute the pseudo-spectral abscissa and its derivative with respect to ϵ. The method must be initialised with an initial bracket which contains the zero-crossing of $\alpha(\epsilon;M,\bar{\delta})$. This interval is subsequently updated using the Newton-bisection method which takes Newton steps to update the estimate unless these lie outside the interval or when the decrease in absolute function value is insufficient; in such cases a bisection step is taken. This is repeated until the change on $\epsilon^{(i)}$ falls below a predefined threshold.

5. EXAMPLE

We will consider the following simple LTI time-delay system with uncertainties δ_1 and δ_{τ_1}:

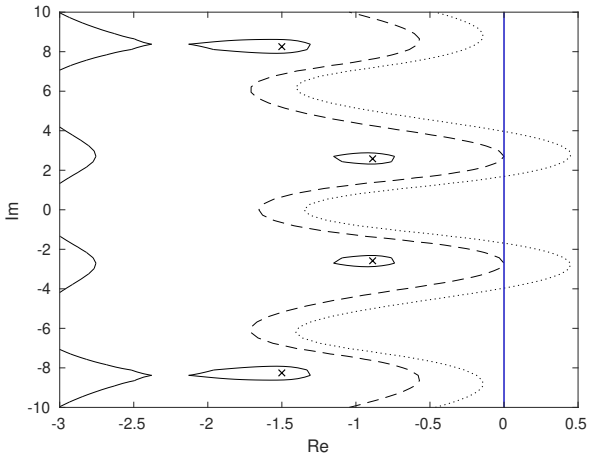

Fig. 1. Rightmost poles of system (11) (marked with $\times$) and pseudo-spectrum of the associated delay eigenvalue problem for $\bar{\delta}_1 = 0.1$, $\bar{\delta\tau}_1 = 0.05$, and ϵ respectively equal to 0 (solid), 1/11.3622 (dashed), and 2/11.3622 (dotted).

$$\begin{cases} \dot{x}(t) = (-5+\delta_1)x(t) - 2x(t-(1+\delta_{\tau_1}))+ \\ \qquad 4w(t) - (2+\delta_1)w(t-(1+\delta_{\tau_1}) \\ z(t) = (2+\delta_1)x(t) - 5x(t-(1+\delta_{\tau_1})) \end{cases} \tag{11}$$

with $\bar{\delta}_1 = 0.1$ and $\overline{\delta_{\tau_1}} = 0.05$. The $\mathcal{H}_\infty$-norm of the system without uncertainties is equal to 10.9263 and is attained at $\omega = \pm 2.7061$. To use Algorithm 1 we must specify an initial bracket that contains the robust complex distance to instability. As lower bound we can use 0. As upper bound we take 1/10.9263 as the $\mathcal{H}_\infty$-norm is smaller or equal to the robust $\mathcal{H}_\infty$-norm. Starting from this initial interval we apply the Newton-bisection method. It finds the robust complex distance to instability in 5 steps and the absolute value of α goes quadratically to zero (as expected for a Newton-type method). The resulting robust $\mathcal{H}_\infty$-norm is equal to 11.3622 and is attained at $\omega_{wc} = \pm 2.5848$.

Figure 1 plots the poles of system (11) and the pseudo-spectrum of its corresponding DEP for different values of ϵ. We see that for ϵ equal to $\left(\|T^e(j\omega,\delta)\|^{\bar{\delta}}_{\mathcal{H}_\infty}\right)^{-1}$ the pseudo-spectrum touches the imaginary axis in $j\omega_{wc}$ as expected from theorem 4.

6. OUTLOOK

In this paper we have presented an algorithm to compute the robust $\mathcal{H}_\infty$-norm of an uncertain LTI time-delay system without direct feed-through. In this final section we will discuss the difficulties of systems with direct feed-through terms.

Firstly, the corresponding (singular) DEP becomes of neutral type. This has as a consequence that for certain perturbations the DEP can become not well posed. Also the spectrum location of such eigenvalue problems differs greatly from that of retarded DEPs, as it can contain chains of eigenvalues whose imaginary part goes to infinity. A second difficulty is that the $\mathcal{H}_\infty$-norm of such systems may be sensitive to arbitrary small perturbations on the delays (as shown in Gumussoy and Michiels (2011)). In further work we will examine these difficulties in more

detail and propose an adapted algorithm to deal with such systems. This algorithm will consist of two steps: first the worst-case asymptotic ($\omega \to \infty$) behaviour is quantified; subsequently the algorithm presented in this work is used to check if there exist a finite frequency for which worst-case spectral norm of the frequency response function is larger.

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