The smoothed spectral abscissa for robust controller design

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1 Introduction

Stability optimization of linear and nonlinear continuous-time dynamic systems is both a highly relevant and a difficult task. The optimization parameters often stem from a feedback controller, which can be used to optimize either a performance criterion or the asymptotic stability around a certain steady state. When also robustness against perturbations of the system must be taken into account, the resulting optimization problem becomes even more challenging.

Assuming an adequate parameterization of the desired feedback controller is available, the problem of finding a suitable steady state along with a stabilizing feedback controller can essentially be transformed into a nonlinear programming problem. By collecting all optimization variables in a vector x, we can summarize the described stability optimization problem as

$$\min_x \ \Phi_{\textbf{stab}}(A(x)), \quad \text{s.t.} \quad g(x) = 0,\, h(x) \leq 0,$$

where $A(x)$ is the system matrix depending smoothly on x and the function $\Phi_{\textbf{stab}}(\cdot)$ shall express our desire to optimize stability, under the given constraints. In the field of linear output feedback control, the closed-loop system matrix $A(x)$ will typically be of the form $A+BKC$, with A the open-loop system matrix, B and C the input and output matrices, and K containing the controller parameters x to be optimized.

2 Spectral abscissa minimisation

The most straightforward choice for the objective function $\Phi_{\textbf{stab}}$ is related to the eigenvalues of A, namely the spectral abscissa $\alpha(A)$. This value is defined as the real part of the rightmost eigenvalue of the spectrum $\Lambda(A) = \{z \in \mathbb{C} \,|\, \det(zI-A) = 0\}$, that is, $\alpha(A) := \sup\{\Re(z) \,|\, z \in \Lambda(A)\}$. However, it is well known that the minimization of the spectral abscissa function $\alpha(A)$ gives rise to very difficult optimization problems, since $\alpha(A)$ is not everywhere differentiable, and even not everywhere Lipschitz. Moreover, the spectral abscissa is also known to perform quite poorly in terms of robustness against parameter uncertainties. A tiny perturbation or disturbance to a parameter of a system that was optimized in the spectral abscissa can possibly lead to instability.

3 The smoothed spectral abscissa

We therefore propose a new stability measure, namely the *smoothed spectral abscissa* $\alpha_\varepsilon(A)$, which is based on the inversion of a relaxed H_2-type cost function. It alleviates the problem of nonsmoothness, and has at the same time certain beneficial robustness properties. A regularization parameter ε allows to tune the degree of smoothness. For ε approaching zero, the smoothed spectral abscissa $\alpha_\varepsilon(A)$ converges towards the nonsmooth spectral abscissa from above, so that $\alpha_\varepsilon(A) \leq 0$ guarantees asymptotic stability. Evaluation of the smoothed spectral abscissa and its derivatives w.r.t. the matrix parameters can be performed – by using an adjoint differentiation technique – at the cost of solving only a single primal-dual Lyapunov equation pair

$$\begin{aligned} 0 &= (A-sI)P + P(A-sI)^{\mathrm{T}} + UU^{\mathrm{T}}, \\ 0 &= (A-sI)^{\mathrm{T}}Q + Q(A-sI) + V^{\mathrm{T}}V. \end{aligned}$$

for s. This allows for an efficient integration into a derivative based optimization framework.

4 Optimization formulations

Two optimization problems are considered. A first variant is to simply choose a fixed $\varepsilon > 0$ and then solve

$$\min_x \ \alpha_\varepsilon(A(x)), \quad \text{s.t.} \quad g(x) = 0,\, h(x) \leq 0.$$

Should this problem not result in a negative optimum for the chosen ε, then one can try again with a smaller ε.

As the choice for ε of α_ε is somewhat arbitrary, we might alternatively search for the largest ε so that the stability certificate $\alpha_\varepsilon(A) \leq 0$ still holds. Because $\alpha_\varepsilon(A)$ becomes smoother with increasing values for $\varepsilon > 0$, we can expect a larger ε to yield a more robust optimal controller. This translates into the following optimization problem

$$\max_{x,\varepsilon} \ \varepsilon \quad \text{s.t.} \quad \alpha_\varepsilon(A(x)) \leq 0 \ \text{ and } \ g(x) = 0,\, h(x) \leq 0.$$

Note that this second problem can be shown to be equivalent to minimizing the H_2-norm of the systems transfer function over all feasible parameters x.

In both cases additional equality and inequality constraints on the variables can be naturally taken into account in the optimization problem.