A Constructive Approach to Stabilizability and Detectability of Linear Time-Delay Systems via Complete-Type Functionals

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***Abstract*— We provide closed formulae for the construction of observers and controllers for linear time-delay systems, within the framework of Lyapunov-Krasovskii functionals of complete type. A main feature of the proposed observers and controllers is that they depend on the full state of the system, resulting in a feedback with a distributed term, and they directly emerge from the inverse of a positive definite operator defined on a Hilbert space, determined by the so-called delay Lyapunov matrix. This latter feature brings the proposed approach on par with well-established results for systems without delays, in which Lyapunov matrices serve not only for system analysis but also for control synthesis.**

***Index Terms*— Detectability, Stabilizability, Lyapunov-Krasovskii Functionals.**

I. Introduction

Stabilization and state observation are fundamental problems in control theory. For time-delay systems both have been approached by different methodologies since the 70s; see, for instance, the early work by [1]–[4] and, more recently, [5]–[7]. For linear systems without delays, the analysis of stabilizability and detectability can be carried out by Lyapunov matrices. Considering the system

$$\begin{aligned}\dot{x}(t) =&A_0x(t) + Bu(t),\\ y(t) =&Cx(t),\end{aligned} \tag{1}$$

the following result is well known.

Theorem 1: [8, Ch. 14 and Ch. 16] Assume that (1) satisfies the Lyapunov condition, i.e. both s_0 and $-s_0$ are not eigenvalues of A_0 simultaneously. The following holds true:

1) System (1) is detectable if and only if, for some positive definite matrix Q, there exists a positive definite matrix P such that
$$A_0^{\mathsf{T}}P + PA_0 = C^{\mathsf{T}}C - Q. \tag{2}$$
Moreover, an observer can be constructed as
$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + L(C\hat{x}(t) - y(t)) + Bu(t),$$
where $L = -lP^{-1}C^{\mathsf{T}}$, with $l \geq \frac{1}{2}$ and $\hat{x}$ is the estimate state.
2) System (1) is stabilizable if and only if, for some positive definite matrix Q, there exists a positive definite matrix P such that
$$PA_0^{\mathsf{T}} + A_0P = BB^{\mathsf{T}} - Q.$$
Moreover, the controller $u(t) = -kB^{\mathsf{T}}P^{-1}x(t)$, with $k \geq \frac{1}{2}$, stabilizes (1).

Notice that the second statement of Theorem 1 is the dual version of the first one. Indeed, the Lyapunov matrix involved in the stabilization problem is typically referred to as the dual Lyapunov matrix.

A remarkable point in the above is that matrix P characterizes an observer or a controller. In this paper, we adopt the ideas behind Theorem 1 for controller and observer design of linear time-delay systems of the form

$$\begin{aligned}\dot{x}(t) =&A_0x(t) + A_1x(t-h) + Bu(t),\ t \geq 0,\\ y(t) =&Cx(t),\end{aligned} \tag{3}$$

with matrices A_0 and A_1 in $\mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{n\times m}$ and delay $h > 0$. Without loosing generality, we assume that matrix C is of full row rank and matrix B of full column rank. The problem poses fundamental questions originating from the infinite dimensional nature of time-delay systems that have scarcely been explored in spite of their relevance: What should be the analogous to matrix P? What should be the dual version? Which form should the controller and observer have? We find (partial) answers within the framework of Lyapunov-Krasovskii functionals of complete type [9]. An important feature of this class of functionals is that they are determined by the delay Lyapunov matrix, which has paved the way in the last two decades for extending standard results from systems without delays to those with delays.

Essential to the contribution is the definition of a linear bounded operator on a Hilbert space, $\mathcal{P}$, formally defined in Section II. The operator $\mathcal{P}$ rises from the complete type functional of time-delay system (3) studied in abstract form and is characterized by the delay Lyapunov matrix. In this context, it is such an operator that plays the role of matrix P: As matrix P depends on (A_0, C) (resp. (A_0, B)) and Q, operator $\mathcal{P}$ depends on system matrices (A_0, A_1, C) (resp. (A_0, A_1, B)) and on symmetric matrices (Q, W_1, W_2). We provide sufficient conditions for detectability and stabilizability in terms of positive definiteness of the operator and exploit the fact that it is fully characterized by the delay Lyapunov matrix to construct an observer and a controller if those conditions hold true. More specifically, whenever there exist symmetric positive definite matrices (Q, W_1, W_2) such that $\mathcal{P} \succ 0$, we can construct an observer, and, by duality, a controller.

The verification problem of the last condition, not trivial due to the infinite dimensional nature of $\mathcal{P}$, is reformulated as a Linear Matrix Inequality (LMI) problem with (Q, W_1, W_2) as decision variables. Although we share this technical aspect with most methods developed within the time-domain approach to design observers or controllers, ours is fundamentally different. While standard techniques rely on fixing the structure of the observer/controller and then finding a suitable Lyapunov functional that allows assessing stability

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of the closed-loop system (e.g. [10, Ch. 6], [11, Ch. 5]), in our proposal, the observer/controller directly emerges from the construction and structure of the operator $\mathcal{P}$. Notably, the obtained controller and observer contain the full system state and are structurally identical to those obtained from the frequency-domain approach by shifting part of the spectrum and keeping the remainder of the spectrum invariant [2], and as solution of a set of partial differential equations in the LQR problem for time-delay systems [12]. More recent approaches to the observer construction problem are based on partial integral equations [13] and special parametrization of system matrices [14]. Although both exploits the Lyapunov theory for time-delay systems, neither of them rely on the theory of complete type functionals, determined by the Lyapunov matrix [9].

Closer to our proposal are the ideas presented in [5], [15], where the stabilization problem is addressed at the level of Lyapunov operators of the system in abstract form. Particularly in [5], a dual form of Lyapunov-Krasovskii functionals is presented, and the kernel functions of the associated (dual) operator are parameterized to verify positivity via the solution of LMI. However, on the one hand, we avoid the introduction of any dual Lyapunov-Krasovskii functional to establish a stabilizing controller, but we leverage the dual of the time-delay system at the level of the functional differential equation description; on the other, we do not parameterize the kernel functions of the operator as they are already determined by the delay Lyapunov matrix, but the matrices (Q, W_1, W_2) that describe the time-derivative. In this sense, the presented results can be seen as complementary to those in [5].

The remainder of the paper is organized as follows. In the next section we provide the abstract formulation of time-delay system (3) (without input) and the operator $\mathcal{P}$, along with basic facts on complete type functionals. Section III is devoted to addressing detectability and the observer construction of the system. In Section IV, we outline a methodology in order to find matrices (Q, W_1, W_2) such that $\mathcal{P}$ is a positive definite operator. In Section V, we exploit the structure of the operator for the dual system and, from detectability results, we derive stabilizability and a controller for the system. The theoretical findings are illustrated by some numerical examples in Section VI. We end the paper with a brief discussion on open questions regarding the conservatism of the proposal and the potential of the presented ideas for different classes of systems with delays in Section VII.

Notation: The spaces of square integrable and absolutely continuous functions from $[a, b]$ to $\mathbb{C}^n$ are denoted by $L_2([a, b], \mathbb{C}^n)$ and $AC([a, b], \mathbb{C}^n)$, respectively. The symbol $\otimes$ denotes the Kronecker product, vec stands for the vectorization of a matrix and mat for the conversion of a vector of appropriate length into a square matrix (column-wise). The symbol $\star$ indicates the complex conjugate of a vector, while $'$ the derivative of a function w.r.t. its argument. The rest of notation remains standard.

II. Preliminaries

Since we obtain the stabilizability results by duality of detectability, we lay the basis for system (3) without input, i.e.

$$\begin{aligned} \dot{x}(t) =& A_0 x(t) + A_1 x(t-h),\ t \geq 0, \\ y(t) =& Cx(t). \end{aligned} \tag{4}$$

Let us define the Hilbert space

$$M_2 := \mathbb{C}^n \times L_2([-h, 0], \mathbb{C}^n),$$

equipped with the inner product

$$\langle \varphi, \psi \rangle := \varphi_0^\star \psi_0 + \int_{-h}^{0} \Phi^\star(\theta)\Psi(\theta)d\theta,$$

where

$$\varphi = \begin{pmatrix} \varphi_0 \\ \Phi(\cdot) \end{pmatrix},\ \psi = \begin{pmatrix} \psi_0 \\ \Psi(\cdot) \end{pmatrix} \in M_2,$$

and the corresponding norm $\|\cdot\|$.

System (4) can be expressed as a differential equation in abstract form. Indeed, define the operator $\mathcal{T}$ on M_2:

$$\mathcal{T}(t) \begin{pmatrix} \varphi_0 \\ \Phi(\cdot) \end{pmatrix} = x_t(\varphi) := \begin{pmatrix} x(t, \varphi) \\ X(t, \cdot, \varphi) \end{pmatrix},\ t \geq 0,$$

where $X(t, \cdot, \varphi) = x(t + \cdot, \varphi)$ and $x(\cdot, \varphi)$ is the solution of (4) with initial condition φ at $t = 0$. The operator $\mathcal{T}$ is the C_0-semigroup on M_2 corresponding to system (4). Let us define $\mathcal{A}$ as the infinitesimal generator of this C_0-semigroup,

$$\mathcal{A} \begin{pmatrix} \varphi_0 \\ \Phi(\cdot) \end{pmatrix} := \begin{pmatrix} A_0\varphi_0 + A_1\Phi(-h) \\ \Phi'(\cdot) \end{pmatrix},$$

with the domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} \varphi_0 \\ \Phi(\cdot) \end{pmatrix} \in M_2 : \Phi(\cdot) \in AC\left([-h, 0], \mathbb{C}^n\right), \right. \\ \left. \Phi'(\cdot) \in L_2([-h, 0), \mathbb{C}^n), \varphi_0 = \Phi(0) \right\}.$$

In addition, if we define operator $\mathcal{C} : M_2 \to \mathbb{C}^p$ by

$$\mathcal{C} \begin{pmatrix} \varphi_0 \\ \Phi(\cdot) \end{pmatrix} = C\varphi_0,$$

we can reformulate system (4) as

$$\begin{aligned} \frac{dx_t}{dt} =& \mathcal{A}x_t, \\ y(t) =& \mathcal{C}x_t, \end{aligned} \tag{5}$$

where $x_t \in M_2$, $t \geq 0$. For $\varphi \in D(\mathcal{A})$, the solution $x_t(\varphi)$ of (5) corresponds to a solution $x(t, \varphi)$ satisfying (4). Conversely, every solution of (4) induces a trajectory $x_t(\varphi)$ that satisfies (5); [16, Ch. 2].

For given symmetric matrices W, W_1, $W_2 \in \mathbb{R}^{n\times n}$ the *complete type functional* for (4) is given by [9, Ch. 2]

$$\begin{aligned} v(\varphi) =& \varphi_0^\star U(0)\varphi_0 + \varphi_0^\star \int_{-h}^{0} U(-h-\theta)A_1\Phi(\theta)d\theta \\ &+ \int_{-h}^{0} \Phi^\star(\theta)A_1^\mathsf{T} U(h+\theta)d\theta\varphi_0 \\ &+ \int_{-h}^{0} \Phi^\star(\theta_1)A_1^\mathsf{T} \int_{-h}^{0} U(\theta_1-\theta_2)A_1\Phi(\theta_2)d\theta_2 d\theta_1 \\ &+ \int_{-h}^{0} \Phi^\star(\theta)(W_1 + (h+\theta)W_2)\Phi(\theta)d\theta,\ \varphi \in M_2, \end{aligned} \tag{6}$$

where $U : [-h, h] \to \mathbb{R}^{n\times n}$ is a continuous matrix valued function, known as *delay Lyapunov matrix* associated with W [9, Ch. 2], which satisfies the dynamic property

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + U(\tau - h)A_1,\ \tau \in (0, h), \tag{7}$$

symmetry property

$$U^\mathsf{T}(\tau) = U(-\tau),\ \tau \in (-h, h), \tag{8}$$

and algebraic property

$$A_0^{\mathsf{T}}U(0)+U(0)A_0+A_1^{\mathsf{T}}U(h)+U(-h)A_1=-W. \quad (9)$$

Definition 1: System (4) satisfies the Lyapunov condition if its spectrum (the set of eigenvalues) contains no pair $\{s_0,-s_0\}$.

If the Lyapunov condition is satisfied, the delay Lyapunov matrix is uniquely defined by conditions (7)-(9) and, by vectorization of these expressions, can be computed as

$$U(\tau)=\mathrm{mat}\left(\begin{pmatrix}I & 0\end{pmatrix}e^{L\tau}\left(M_0+M_1e^{-Lh}\right)^{-1}\begin{pmatrix}0\\ -\mathrm{w}\end{pmatrix}\right) \quad (10)$$

for $\tau\in[0,h]$, with $\mathrm{w}=\mathrm{vec}(W)$,

$$M_0=\begin{pmatrix}I & 0\\ I\otimes A_0^{\mathsf{T}}+A_0^{\mathsf{T}}\otimes I & A_1^{\mathsf{T}}\otimes I\end{pmatrix},\ M_1=\begin{pmatrix}0 & -I\\ I\otimes A_1^{\mathsf{T}} & 0\end{pmatrix}$$

and

$$L=\begin{pmatrix}A_0^{\mathsf{T}}\otimes I & A_1^{\mathsf{T}}\otimes I\\ -I\otimes A_1^{\mathsf{T}} & -I\otimes A_0^{\mathsf{T}}\end{pmatrix}.$$

By construction, functional (6) satisfies

$$\frac{d}{dt}v(x_t)=-w(x_t),$$

with

$$w(\varphi)=\varphi_0^{\star}W_0\varphi_0+\Phi^{\star}(-h)W_1\Phi(-h)+\int_{-h}^{0}\Phi^{\star}(\theta)W_2\Phi(\theta)d\theta,$$

where $W_0=W-W_1-hW_2$. We define the linear bounded self-adjoint operator $\mathcal{P}:M_2\to M_2$ by

$$\mathcal{P}\varphi:=\begin{pmatrix}\mathcal{F}\varphi(0)\\ A_1^{\mathsf{T}}\mathcal{F}\varphi(\cdot+h)+\mathcal{W}(\cdot+h)\Phi(\cdot)\end{pmatrix}, \quad (11)$$

with

$$\mathcal{F}\varphi(\theta)=U(\theta)\varphi_0+\int_{-h}^{0}U(\theta-h-s)A_1\Phi(s)ds,\ \theta\in[0,h],$$

and $\mathcal{W}(\theta)=(W_1+\theta W_2)$, $\theta\in[0,h]$. The operator $\mathcal{P}$ is well defined if system (4) satisfies the Lyapunov condition. Note that we can express functional (6) in the form $v(\varphi)=\langle\varphi,\mathcal{P}\varphi\rangle$, from which it follows that

$$\frac{d}{dt}v(x_t)=\langle\mathcal{A}x_t,\mathcal{P}x_t\rangle+\langle\mathcal{P}x_t,\mathcal{A}x_t\rangle=-w(x_t). \quad (12)$$

III. Detectability and observer construction

We present detectability results in terms of operator $\mathcal{P}$ and provide a closed formula for the construction of a Luenberger type observer in the next two subsections.

A. Detectability in terms of $\mathcal{P}$

Definition 2: [16, Def. 8.1.1] System (4) is detectable if there exists an operator $\mathcal{L}:\mathbb{C}^p\to M_2$ such that $\mathcal{A}+\mathcal{LC}$ generates an exponentially stable C_0-semigroup on M_2.

The following result was introduced in [17], relying on spectral properties of detectable systems [18]. We present a new proof aligned with the purpose of this note.

Theorem 2: Let system (4) satisfy the Lyapunov condition. If there exist positive definite matrices (Q,W_1,W_2) such that $\mathcal{P}\succ 0$, where $\mathcal{P}$ is defined by (11) with the Lyapunov matrix $U(\cdot)$ associated with $W=Q-C^{\mathsf{T}}C+W_1+hW_2$, then system (4) is detectable. Moreover, we can compute operator $\mathcal{L}$ in Definition 2 as $\mathcal{L}=-l\mathcal{P}^{-1}\mathcal{C}^{\star}$, with $l\geq\frac{1}{2}$ and $\mathcal{C}^{\star}:\mathbb{C}^p\to M_2$ defined by

$$\mathcal{C}^{\star}\xi=\begin{pmatrix}C^{\mathsf{T}}\xi\\ 0\end{pmatrix},\ \xi\in\mathbb{C}^p.$$

Proof: Let $\mathcal{A}+\mathcal{LC}$ be the infinitesimal generator of the C_0-semigroup $\mathcal{T}_{cl}$. Consider the Lyapunov functional

$$v(e_t)=\langle e_t,\mathcal{P}e_t\rangle,\ e_t=\mathcal{T}_{cl}(t)e_0=\begin{pmatrix}e(t)\\ E(t,\cdot)\end{pmatrix}$$

with an arbitrary $e_0\in D(\mathcal{A})$. The time derivative of this functional satisfies

$$\begin{aligned}\frac{d}{dt}v(e_t)=&\langle(\mathcal{A}+\mathcal{LC})e_t,\mathcal{P}e_t\rangle+\langle\mathcal{P}e_t,(\mathcal{A}+\mathcal{LC})e_t\rangle\\ =&\langle\mathcal{A}e_t,\mathcal{P}e_t\rangle+\langle\mathcal{P}e_t,\mathcal{A}e_t\rangle+\langle\mathcal{LC}e_t,\mathcal{P}e_t\rangle+\langle\mathcal{P}e_t,\mathcal{LC}e_t\rangle\\ \leq&e^{\star}(t)C^{\mathsf{T}}Ce(t)-\gamma\|e_t\|^2+\langle\mathcal{LC}e_t,\mathcal{P}e_t\rangle+\langle\mathcal{P}e_t,\mathcal{LC}e_t\rangle,\end{aligned}$$

where we used the right-hand side identity of (12) and the definition of e_t for deducing the inequality of the last line, and $\gamma=\min\{\lambda_{\min}(Q),\lambda_{\min}(W_2)\}$. If $\mathcal{L}=-l\mathcal{P}^{-1}\mathcal{C}^{\star}$, we get

$$\begin{aligned}&e^{\star}(t)C^{\mathsf{T}}Ce(t)+\langle\mathcal{LC}e_t,\mathcal{P}e_t\rangle\\ &\qquad+\langle\mathcal{P}e_t,\mathcal{LC}e_t\rangle=-(2l-1)e^{\star}(t)C^{\mathsf{T}}Ce(t)\leq 0\end{aligned}$$

and

$$\frac{d}{dt}v(e_t)\leq-\gamma\|e_t\|^2,\ t\geq 0,$$

from which it follows that

$$\gamma\int_0^t\|e_\tau\|^2d\tau\leq v(e_0),\ t\geq 0,$$

and in turn, as $D(\mathcal{A})$ is dense in M_2 (cf. proof of Theorem 4.1.3 in [16]),

$$\int_0^{\infty}\|\mathcal{T}_{cl}(\tau)e_0\|^2d\tau\leq\frac{\gamma_z}{\gamma},\ \gamma_z=v(e_0)=\langle e_0,\mathcal{P}e_0\rangle>0,$$

for any nonzero $e_0\in M_2$. Finally, exponential stability of $\mathcal{T}_{cl}$ follows from the above inequality and [16, Lemma 4.1.2]. ■

B. Construction of the observer

We assume that the conditions of Theorem 2 are satisfied. An observer for (5) can then be constructed as

$$\frac{d}{dt}z_t=\mathcal{A}z_t+\mathcal{L}(\mathcal{C}z_t-y(t)), \quad (13)$$

with z_t representing the state estimate and $\mathcal{L}=-l\mathcal{P}^{-1}\mathcal{C}^{\star}$. The observer error $e_t:=z_t-x_t$ satisfies

$$\frac{d}{dt}e_t=(\mathcal{A}+\mathcal{LC})e_t$$

and $\mathcal{A}+\mathcal{LC}$ generates an exponentially stable C_0-semigroup. An explicit expression for operator $\mathcal{L}$ in terms of the delay Lyapunov matrix $U(\cdot)$ is derived in the Appendix, namely

$$\mathcal{L}q=\begin{pmatrix}-lG_0C^{\mathsf{T}}q\\ lG(\cdot+h)C^{\mathsf{T}}q\end{pmatrix},\ q\in\mathbb{C}^p, \quad (14)$$

where G is the solution of the Fredholm integral equation of the second kind

$$G(\theta)=F(\theta)+\int_0^h K(\theta,s)G(s)ds,\ \theta\in[0,h], \quad (15)$$

with

$$\begin{aligned}F(\theta)=&\mathcal{W}^{-1}(\theta)A_1^{\mathsf{T}}U(\theta)U^{-1}(0),\\ K(\theta,s)=&\mathcal{W}^{-1}(\theta)\\ &\cdot A_1^{\mathsf{T}}\left(U(\theta)U^{-1}(0)U(-s)-U(\theta-s)\right)A_1,\end{aligned} \quad (16)$$

and G_0 is defined as

$$G_0 = U^{-1}(0)\left(I + \int_0^h U(-s)A_1 G(s)ds\right). \tag{17}$$

Note that the solution of (15) is unique as operator $\mathcal{P}$ is positive definite. Moreover, letting $R(\cdot,\cdot)$ be the resolvent of this integral equation, it can be expressed as[1]

$$G(\theta) = F(\theta) + \int_0^h R(\theta, s)F(s)ds,\ \theta \in [0, h]. \tag{18}$$

In what follows, we derive an alternative expression of the observer dynamics in terms of system description (4) rather than (5). It follows from (14) that equation (13) with

$$z_t = \begin{pmatrix} z(t) \\ Z(t,\cdot) \end{pmatrix}$$

can be written as the following ODE-PDE system:

$$\begin{aligned} \dot{z}(t) =& A_0 z(t) + A_1 Z(t,-h) - lG_0 C^{\mathsf{T}} C(z(t) - x(t)), \\ Z_t(t,\theta) =& Z_\theta(t,\theta) + lG(\theta + h)C^{\mathsf{T}} C(z(t) - x(t)), \\ Z(t,0) =& z(t). \end{aligned} \tag{19}$$

By the change of variables $\xi = t$ and $\eta = t + \theta$, and $Z(t,\theta) = \hat{Z}(\xi,\eta)$, we obtain $Z_t = \hat{Z}_\xi + \hat{Z}_\eta$ and $Z_\theta = \hat{Z}_\eta$. Hence, we get

$$\hat{Z}_\xi(\xi,\eta) = lG(\eta + h - \xi)C^{\mathsf{T}} C(z(\xi) - x(\xi)),$$

and by integration from ξ to η,

$$\hat{Z}(\xi,\eta) = \hat{Z}(\eta,\eta) + l\int_\eta^\xi G(\eta + h - s)C^{\mathsf{T}} C(z(s) - x(s))ds,$$

or, equivalently,

$$Z(t,\theta) = Z(t+\theta,0) + l\int_{t+\theta}^t G(t+\theta+h-s)C^{\mathsf{T}} C(z(s) - x(s))ds.$$

As $Z(t+\theta,0) = z(t+\theta)$, we finally arrive at

$$Z(t,-h) = z(t-h) + l\int_{t-h}^t G(t-s)C^{\mathsf{T}} C(z(s) - x(s))ds.$$

Substituting this expression in the first equation of (19) yields

$$\begin{aligned} \dot{z}(t) = A_0 z(t) + A_1 z(t-h) - lG_0 C^{\mathsf{T}} C(z(t) - x(t)) \\ + lA_1 \int_{-h}^0 G(-s)C^{\mathsf{T}} C(z(t+s) - x(t+s))ds \end{aligned} \tag{20}$$

as counterpart of (13). Setting $e(t) := z(t) - x(t)$ we get, from (20) and (4), the observer error dynamics

$$\begin{aligned} \dot{e}(t) = (A_0 - lG_0 C^{\mathsf{T}} C)e(t) + A_1 e(t-h) \\ + lA_1 \int_{-h}^0 G(-s)C^{\mathsf{T}} Ce(t+s)ds. \end{aligned} \tag{21}$$

The next result is a direct consequence of Theorem 2 and formula (21).

Corollary 1: Let $Q \succ 0$, $W_1 \succ 0$, $W_2 \succ 0$, and $\mathcal{P}$ be a positive definite operator defined by (11) with the Lyapunov matrix $U(\cdot)$ associated with $W = Q - C^{\mathsf{T}} C + W_1 + hW_2$. Then system (21) with $l \geq \frac{1}{2}$ is exponentially stable and, equivalently, the spectrum is confined to the open left half plane.

[1]For a deeper study on integral equations and operator theory the reader is referred to [19] and [20].

IV. On positive definiteness of $\mathcal{P}$

As previously shown, positive definiteness of the operator $\mathcal{P}$ allows assessing detectability and constructing an observer. However, while the result is of main interest, how to find matrices (Q, W_1, W_2) such that $\mathcal{P} \succ 0$ turns out to be a non-trivial problem. We address it in this section.

Using Theorem 1 we first demonstrate the existence of (Q, W_1, W_2) such that $\mathcal{P} \succ 0$ for (at least) some time-delay systems of the form (4).

Proposition 1: Let system (4) with $A_1 \to \epsilon A_1$ satisfy the Lyapunov condition and let system (1) be detectable. For sufficiently small ϵ, there exist positive definite matrices (Q, W_1, W_2) such that $\mathcal{P} \succ 0$.

Proof: Let $U(\cdot)$ be the delay Lyapunov matrix of system (4) associated with $W = Q - C^{\mathsf{T}} C + \rho(W_1 + hW_2)$, where ρ is a positive number. Notice that

$$v(\varphi) \to \varphi_0^\star U(0)\varphi_0 + \rho\int_{-h}^0 \Phi^\star(\theta)(W_1 + (h+\theta)W_2)\Phi(\theta)d\theta \tag{22}$$

as $\epsilon \to 0$. By (8), matrix $U(0)$ is symmetric, and by (9),

$$\begin{aligned} &A_0^{\mathsf{T}} U(0) + U(0)A_0 \\ &\quad = C^{\mathsf{T}} C - Q - \rho(W_1 + hW_2) - \epsilon(A_1^{\mathsf{T}} U(h) + U(h)^{\mathsf{T}} A_1). \end{aligned}$$

As $\epsilon \to 0$ and $\rho \to 0$, $U(0) \to P$ satisfying (2), and since the pair (A_0, C) is detectable, it follows from Theorem 1 that $U(0) \succ 0$ and, by (22), that $\mathcal{P} \succ 0$. ∎

For general systems of the form (4) the problem is more difficult. In order to make the adopted approach more comprehensible, we present it first for delay-free systems (1). The problem of finding a positive definite matrix P satisfying (2) can be formulated as the LMI

$$A_0^{\mathsf{T}} P + PA_0 < C^{\mathsf{T}} C,$$

but we consider instead the problem of determining the pair (P, Q) satisfying (2). For stability tests, a common practice is to fix Q and then find a positive definite matrix P satisfying the Lyapunov equation. Here, in contrast, Q is set as a variable to be found as well. Let

$$\{E_1, \ldots, E_m\},\ m = \frac{n(n+1)}{2},$$

be a basis of the space of all real symmetric n-by-n matrices. Then we can express

$$Q = \sum_{j=1}^m q_j E_j, \tag{23}$$

with q_j the coefficients of the expansion of Q in this basis. Under the Lyapunov condition vectorization of (2) yields[2]

$$\mathrm{vec}(P) = \left(I \otimes A_0^{\mathsf{T}} + A_0^{\mathsf{T}} \otimes I\right)^{-1}\left(\mathrm{vec}(C^{\mathsf{T}} C) - \mathrm{vec}(Q)\right).$$

The expansion (23) gives rise to a corresponding affine expression of P, namely $P = P_0 + \sum_{j=1}^m q_j P_j$, with

$$P_0 = \mathrm{mat}\left(\left(I \otimes A_0^{\mathsf{T}} + A_0^{\mathsf{T}} \otimes I\right)^{-1} \mathrm{vec}(C^{\mathsf{T}} C)\right)$$

and, for $j = 1, \ldots, m$,

$$P_j = -\mathrm{mat}\left(\left(I \otimes A_0^{\mathsf{T}} + A_0^{\mathsf{T}} \otimes I\right)^{-1} \mathrm{vec}(E_j)\right).$$

[2]For any matrices B, C and X of suitable dimensions, it holds that $\mathrm{vec}(BXC) = \left(C^{\mathsf{T}} \otimes B\right)\mathrm{vec}(X)$ [21, Ch. 7].

Finding positive definite matrices (P, Q) satisfying (2) thus reduces to the following LMI problem:

$$\begin{aligned}&\text{Find } q_j,\ j=1,\dots,m,\ \text{ s.t.}\\ &P = P_0 + \sum_{j=1}^{m} q_j P_j \succ 0,\ \ Q = \sum_{j=1}^{m} q_j E_j \succ 0.\end{aligned}$$

This problem has a solution whenever system (4) is detectable.

To adopt the previous idea in the context of Theorem 2, let us expand Q as in (23) and W_i as

$$W_i = \sum_{j=1}^{m} w_j^{(i)} E_j,\ i=1,2,$$

and collect all coefficients by

$$\chi_j := \begin{cases} q_j, & j=1,\dots,m,\\ w^{(1)}_{j-m}, & j=m+1,\dots,2m,\\ w^{(2)}_{j-2m}, & j=2m+1,\dots,3m.\end{cases}$$

This parametrization of (Q, W_1, W_2) results in an affine parametrization of the corresponding delay Lyapunov matrix, namely

$$U(\tau) = U_0(\tau) + \sum_{j=1}^{3m} \chi_j U_j(\tau),\ \tau\in[0,h], \tag{24}$$

where (10) leads to

$$\begin{aligned}&\operatorname{vec}\big(U_j(\tau)\big)\\ &= -\begin{pmatrix} I & 0\end{pmatrix} e^{L\tau}\left(M_0 + M_1 e^{-Lh}\right)^{-1}\begin{pmatrix}0\\ \operatorname{vec}(X_j)\end{pmatrix}\end{aligned} \tag{25}$$

for $j=0,\dots,3m$ with

$$X_j := \begin{cases} -C^{\mathsf T}C, & j=0,\\ E_j, & j=1,\dots,m,\\ E_{j-m}, & j=m+1,\dots,2m,\\ hE_{j-2m}, & j=2m+1,\dots,3m.\end{cases} \tag{26}$$

Substituting (24) into complete type functional (6) gives rise to an affine parametrization of $v(\cdot)$ in the unknown coefficients χ_j. Finally, we need to express positive definiteness of this functional in a computationally tractable way. For this, we adopt a discrete version of the complete type functional proposed in [22], [23], which takes inspiration from [24]:

$$\begin{aligned}v^{(N)}(\varphi) = &\int_0^1 \begin{pmatrix}\varphi_0^\star & \tilde\Phi^\star(\alpha)\end{pmatrix}\begin{pmatrix}U(0) & \mathcal{Q}^{(N)}\\ \star & \mathcal{R}^{(N)}\end{pmatrix}\begin{pmatrix}\varphi_0\\ \tilde\Phi(\alpha)\end{pmatrix} d\alpha\\ &+\int_{-h}^{0}\Phi^\star(\theta)(W_1+(h+\theta)W_2)\Phi(\theta)d\theta,\end{aligned} \tag{27}$$

with

$$\begin{aligned}\mathcal{Q}^{(N)} &:= \begin{pmatrix}U^{\mathsf T}(N\Delta)A_1 & \cdots & U^{\mathsf T}(\Delta)A_1 & U^{\mathsf T}(0)A_1\end{pmatrix},\\ \mathcal{R}^{(N)} &:= \left[A_1^{\mathsf T}U((q-p)\Delta)A_1\right]_{p,q=0}^{N},\end{aligned}$$

$\Delta := h/N$ and suitable vector function $\tilde\Phi(\alpha)$. Here, $\left[A_1^{\mathsf T}U((q-p)\Delta)A_1\right]_{p,q=0}^{N}$ defines a square block matrix whose block at position (p,q), with indices starting from $(0,0)$, is given by $A_1^{\mathsf T}U((q-p)\Delta)A_1$. It is important to note that (27) is obtained from (6) and (24) by replacing the kernels of the integrals by piecewise linear approximations on intervals of length Δ.

We can now express the positivity test in terms of a semidefinite program.

Proposition 2: Let system (4) satisfy the Lyapunov condition, U_j and X_j, $j=0,\dots,3m$, be given by (25)-(26), and N be a positive integer. If there exist χ_j, $j=1,\dots,3m$, such that

$$\begin{aligned}&P_0^{(N)} + \sum_{j=1}^{3m}\chi_j P_j^{(N)} \succ 0,\\ &\sum_{j=1}^{m} X_j\chi_j \succ 0,\ \sum_{j=m+1}^{2m} X_j\chi_j \succ 0,\ \sum_{j=2m+1}^{3m} X_j\chi_j \succ 0,\end{aligned}$$

where

$$\begin{aligned}P_j^{(N)} &:= \begin{pmatrix}U_j(0) & \mathcal{Q}_j^{(N)}\\ \star & \mathcal{R}_j^{(N)} + S_j\end{pmatrix},\\ \mathcal{Q}_j^{(N)} &:= \begin{pmatrix}U_j^{\mathsf T}(N\Delta)A_1 & \cdots & U_j^{\mathsf T}(\Delta)A_1 & U_j^{\mathsf T}(0)A_1\end{pmatrix},\\ \mathcal{R}_j^{(N)} &:= \left[A_1^{\mathsf T}U_j((p-q)\Delta)A_1\right]_{p,q=0}^{N},\ j=0,\dots,3m,\end{aligned}$$

and

$$S_j = \frac{1}{h}\cdot \begin{cases}0, & j=0,\dots,m,\\ \operatorname{diag}\left\{NX_j,\ NX_j,\ \cdots,\ NX_j\right\}, & j=m+1,\dots,2m,\\ \operatorname{diag}\left\{NX_j,\ (N-1)X_j,\ \cdots,\ 0\right\}, & j=2m+1,\dots,3m,\end{cases}$$

then functional (27) corresponding to

$$Q = \sum_{j=1}^{m} X_j\chi_j \succ 0,$$

$$W_1 = \sum_{j=m+1}^{2m} X_j\chi_j \succ 0,\ W_2 = \frac{1}{h}\sum_{j=2m+1}^{3m} X_j\chi_j \succ 0$$

is positive definite.

Proof: Let $P^{(N)} := P_0^{(N)} + \sum_{j=1}^{3m}\chi_j P_j^{(N)}$ and define $\mathcal{Q}^{(N)}$ and $\mathcal{R}^{(N)}$ in a similar way. Then we can express

$$P^{(N)} = \begin{pmatrix}U(0) & \mathcal{Q}^{(N)}\\ \star & \mathcal{R}^{(N)} + S\end{pmatrix},$$

with

$$S = \frac{1}{\Delta}\operatorname{diag}\left\{W_1 + hW_2,\ W_1 + (N-1)\Delta W_2,\ \cdots,\ W_1\right\}.$$

In [22] it is shown that

$$v^{(N)}(\varphi) \ge \int_0^1 \begin{pmatrix}\varphi_0^\star & \tilde\Phi^\star(\alpha)\end{pmatrix} P^{(N)}\begin{pmatrix}\varphi_0\\ \tilde\Phi(\alpha)\end{pmatrix} d\alpha.$$

Hence, the imposed condition $P^{(N)} \succ 0$ implies that $v^{(N)}(\cdot)$ is positive definite. ∎

The approximation of $v(\cdot)$ by $v^{(N)}(\cdot)$ is an example of *late lumping*: we do not discretize system (4) beforehand, but we only make the approximation in the final step where we need to express positivity of the functional. Since $v^{(N)}(\cdot)$ relies on approximating the integral kernels in (6) with piecewise linear functions, the parameter N must be large enough to ensure accuracy; see [23, Thm 9]. Another approach to find suitable matrices (Q, W_1, W_2) consists of discretizing first system (4) and formulate the LMI problem as in the delay free case [25].

V. Dual results: Stabilizability

We take into account the input term in (3) and study the (spectral) stabilizability of

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t). \tag{28}$$

Deriving the counterparts of the detectability condition and observer construction directly from the abstract formulation (5) is hard, due to the lack of interpretation of the adjoint operator $\mathcal{A}^\star$ in terms of generator of a C_0-semigroup associated with a dynamical time-delay system. Instead, we infer the generalization from the *dual system* of (28), described by

$$\dot{x}_d(t) = A_0^\mathsf{T} x_d(t) + A_1^\mathsf{T} x_d(t-h), \quad y_d(t) = B^\mathsf{T} x_d(t). \tag{29}$$

Lemma 1: System (28) is stabilizable if and only if system (29) is detectable.

Proof: The lemma is a consequence of the fact that (28) is stabilizable if and only if [16, Thm. 8.2.5]

$$\operatorname{rank}\begin{pmatrix}\lambda I - A_0 - A_1 e^{-\lambda h} & B\end{pmatrix} = n$$

for all $\lambda \in \{\lambda \in \mathbb{C} : \Re(\lambda) \geq 0\}$, and the property

$$\operatorname{rank}\begin{pmatrix}\lambda I - A_0 - A_1 e^{-\lambda h} & B\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\lambda I - A_0^\mathsf{T} - A_1^\mathsf{T} e^{-\lambda h} \\ B^\mathsf{T}\end{pmatrix},$$

which result in the necessary and sufficient detectability condition for (29) [16, Thm. 8.2.5]. ∎

Associated with symmetric matrices W^{ad}, W_1^{ad}, $W_2^{\mathrm{ad}} \in \mathbb{R}^{n\times n}$, we now introduce the operator $\mathcal{P}_{\mathrm{ad}}$ on M_2,

$$\mathcal{P}_{\mathrm{ad}}\varphi := \begin{pmatrix}\mathcal{F}_{\mathrm{ad}}\varphi(0) \\ A_1\mathcal{F}_{\mathrm{ad}}\varphi(\cdot+h) + \mathcal{W}^{\mathrm{ad}}(\cdot+h)\Phi(\cdot)\end{pmatrix},$$

with

$$\mathcal{F}_{\mathrm{ad}}\varphi(\theta) = V(\theta)\varphi_0 + \int_{-h}^{0} V(\theta-h-s)A_1^\mathsf{T}\Phi(s)ds,\ \theta\in[0,h],$$

and $\mathcal{W}^{\mathrm{ad}}(\theta) = (W_1^{\mathrm{ad}} + \theta W_2^{\mathrm{ad}})$, $\theta \in [0,h]$. Here, matrix function $V(\cdot)$ is a dual delay Lyapunov matrix satisfying

$$\begin{gathered}\frac{d}{d\tau}V(\tau) = V(\tau)A_0^\mathsf{T} + V(\tau-h)A_1^\mathsf{T},\ \tau\in(0,h),\\ V^\mathsf{T}(\tau) = V(-\tau),\ \tau\in(-h,h),\\ A_0V(0) + V(0)A_0^\mathsf{T} + A_1V(h) + V(-h)A_1^\mathsf{T} = -W^{\mathrm{ad}}.\end{gathered}$$

Note that operator $\mathcal{P}_{\mathrm{ad}}$ corresponds to $\mathcal{P}$ with the change of variables $A_0 \to A_0^\mathsf{T}$ and $A_1 \to A_1^\mathsf{T}$, i.e. it is the operator defined in Section II associated with the dual system (29). We can now state the main result of the section.

Theorem 3: Assume that there exist positive definite matrices $(Q^{\mathrm{ad}}, W_1^{\mathrm{ad}}, W_2^{\mathrm{ad}})$ such that $\mathcal{P}_{\mathrm{ad}} \succ 0$, constructed with $W^{\mathrm{ad}} = Q^{\mathrm{ad}} - BB^\mathsf{T} + W_1^{\mathrm{ad}} + hW_2^{\mathrm{ad}}$. Then system (28) in closed loop with the control law

$$u(t) = -kB^\mathsf{T}G_{0_{\mathrm{ad}}}^\mathsf{T}x(t) + k\int_{-h}^{0} B^\mathsf{T}G_{\mathrm{ad}}^\mathsf{T}(-\theta)A_1x(t+\theta)d\theta, \tag{30}$$

$k \geq \frac{1}{2}$, is exponentially stable. Here, G_{ad} solves the Fredholm integral equation

$$G_{\mathrm{ad}}(\theta) = F_{\mathrm{ad}}(\theta) + \int_0^h K_{\mathrm{ad}}(\theta,s)G_{\mathrm{ad}}(s)ds,\ \theta\in[0,h],$$

with

$$\begin{aligned}F_{\mathrm{ad}}(\theta) &= \left(\mathcal{W}^{\mathrm{ad}}(\theta)\right)^{-1}A_1V(\theta)V^{-1}(0),\\ K_{\mathrm{ad}}(\theta,s) &= \left(\mathcal{W}^{\mathrm{ad}}(\theta)\right)^{-1}\\ &\quad\cdot A_1\left(V(\theta)V^{-1}(0)V(-s) - V(\theta-s)\right)A_1^\mathsf{T},\end{aligned}$$

and

$$G_{0_{\mathrm{ad}}} = V^{-1}(0)\left(I + \int_0^h V(-s)A_1^\mathsf{T}G_{\mathrm{ad}}(s)ds\right).$$

Proof: Constructing an observer for the dual system (29) with the approach of Section III, starting from operator $\mathcal{P}_{\mathrm{ad}}$, results in the error system

$$\begin{aligned}\dot{e}(t) =& (A_0^\mathsf{T} - kG_{0_{\mathrm{ad}}}BB^\mathsf{T})e(t) + A_1^\mathsf{T}e(t-h)\\ &+ kA_1^\mathsf{T}\int_{-h}^{0} G_{\mathrm{ad}}(-s)BB^\mathsf{T}e(t+s)ds,\end{aligned} \tag{31}$$

which is exponentially stable for $k \geq \frac{1}{2}$ by Corollary 1; see (21). The dual of error system (31) is precisely the closed-loop system of (28) and (30), and the proof is concluded by the fact that if λ is an eigenvalue of (31), then $\lambda^\star$ is an eigenvalue of (28) with the feedback (30). ∎

Finally, the observer (20) and controller (30) can be combined in an observer-based controller. Exponential stability then follows from a separation principle in the frequency domain, as observer and controller keep the system linear time-invariant.

Corollary 2: Assume that there exist positive definite matrices (Q, W_1, W_2) and $(Q^{\mathrm{ad}}, W_1^{\mathrm{ad}}, W_2^{\mathrm{ad}})$ such that $\mathcal{P} \succ 0$ and $\mathcal{P}_{\mathrm{ad}} \succ 0$, constructed with $W = Q - C^\mathsf{T}C + W_1 + hW_2$ and $W^{\mathrm{ad}} = Q^{\mathrm{ad}} - BB^\mathsf{T} + W_1^{\mathrm{ad}} + hW_2^{\mathrm{ad}}$, respectively. Then system (3) in closed loop with

$$u(t) = -kB^\mathsf{T}G_{0_{\mathrm{ad}}}^\mathsf{T}z(t) + k\int_{-h}^{0} B^\mathsf{T}G_{\mathrm{ad}}^\mathsf{T}(-\theta)A_1z(t+\theta)d\theta,$$

$k \geq \frac{1}{2}$, is exponentially stable; here, z satisfies

$$\begin{aligned}\dot{z}(t) &= A_0z(t) + A_1z(t-h) - lG_0C^\mathsf{T}C(z(t) - x(t))\\ &+ lA_1\int_{-h}^{0} G(-s)C^\mathsf{T}C(z(t+s) - x(t+s))ds + Bu(t),\ l \geq \frac{1}{2}.\end{aligned}$$

VI. Examples

We illustrate the results by constructing an observer and a controller for both a detectable and a stabilizable system. The simulation (and implementation) requires the solution of integral equation (15), which is solved by discretization of the kernel (Nyström method, [19, Ch. 4]). We use the trapezoidal quadrature rule for approximating the integral equations solution with weights equal to the integration step, $\Delta_{int} = 0.001$. The latter has not to be necessarily the same as Δ from Proposition 2. A safe implementation of the observer and controller via approximation of the integrals is guaranteed for sufficiently small integration step [26, Th. 2]. We consider unstable systems to illustrate the proposal outlined in Section IV.

Example 1: Let us consider a detectable system (4) with delay $h = 1$, and matrices

$$A_0 = \begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix},\ A_1 = \begin{pmatrix}4 & 2\\ 0 & 0\end{pmatrix},\ \text{and}\ C = \begin{pmatrix}1 & 0\end{pmatrix}.$$

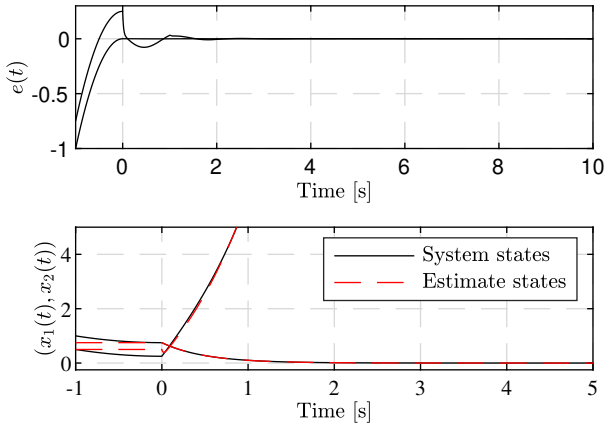


Fig. 1. Estimate error of the state of Example 1 (top) and system and estimate states (bottom).

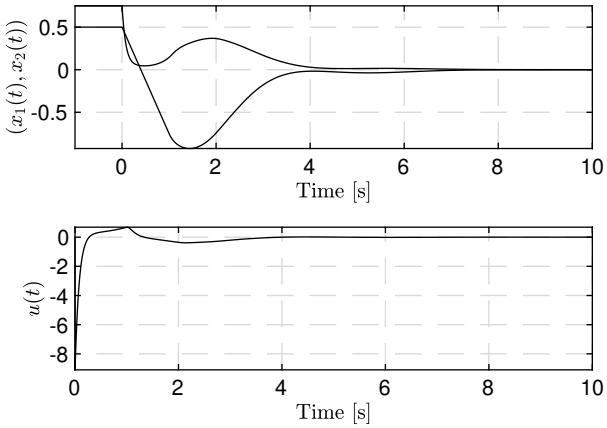


Fig. 2. System states of Example 2 (top) and control signal (bottom).

We set $N = 20$ and find that $P^{(N)} \succ 0$ (i.e. $v^{(N)}(\cdot)$ is positive definite)[3] for matrices

$$Q = \begin{pmatrix} 0.5400 & 0.0319 \\ 0.0319 & 13.0594 \end{pmatrix}, \; W_1 = \begin{pmatrix} 0.3378 & 0.0005 \\ 0.0005 & 0.3400 \end{pmatrix},$$
$$W_2 = \begin{pmatrix} 0.2046 & 0.0020 \\ 0.0020 & 0.2346 \end{pmatrix}.$$

The observer is constructed according to (20) with gain $l = 1$. For initial conditions

$$\varphi(\theta) = \begin{pmatrix} \frac{1}{4} + \theta^2 \\ \frac{3}{4} + \theta^2 \end{pmatrix}, \; \hat{\varphi}(\theta) = \frac{1}{4}\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \; \theta \in [-1, 0].$$

The estimation error, simulated according to (21), and the estimate states are depicted in Figure 1.

Example 2: We consider now a stabilizable system (28) with delay $h = 1$, and matrices

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} -1 & -1 \\ 0 & -0.5 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To find matrices $(Q^{\text{ad}}, W_1^{\text{ad}}, W_2^{\text{ad}})$, we can use Proposition 2 by substituting $A_0 \to A_0^{\mathsf{T}}$, $A_1 \to A_1^{\mathsf{T}}$ and $C \to B^{\mathsf{T}}$. We set $N = 20$ and find matrices

$$Q^{\text{ad}} = \begin{pmatrix} 0.2944 & -0.0054 \\ -0.0054 & 0.6139 \end{pmatrix}, \; W_1^{\text{ad}} = \begin{pmatrix} 0.1499 & -0.0001 \\ -0.0001 & 0.1546 \end{pmatrix},$$
$$W_2^{\text{ad}} = \begin{pmatrix} 0.0959 & -0.0025 \\ -0.0025 & 0.1103 \end{pmatrix}.$$

Figure 2 illustrates the simulation results of the closed-loop system with the controller constructed according to (30) with gain $k = 1$ and initial condition $\varphi(\theta) = \frac{1}{4}\begin{pmatrix} 2 & 3 \end{pmatrix}^{\mathsf{T}}$, $\theta \in [-1, 0]$.

VII. Conclusion and discussion

We presented the construction of an observer and a controller for linear time-delay systems of the form (3) based on the inversion of a positive definite operator defined in a Hilbert space, determined by the so-called delay Lyapunov matrix. The construction is applicable under the assumption that the system satisfies the Lyapunov condition, as in the delay-free case. The ideas can be straightforwardly extended to the multiple delay case in the state, and shed light on control synthesis of other classes of systems, e.g. distributed, neutral and integral, where the theory of Lyapunov-Krasovskii functionals of complete type is already established [9], [27], [28].

An open problem is to assess how close the sufficient condition $\mathcal{P} \succ 0$ in Theorem 2 is to a necessary condition for detectability (resp. stabilizability). Indeed, although the system in Example 1 is delay-independent detectable, no solution to the LMI in Proposition 2 is found when $h > 1$. The source of conservatism might be originated from the structure of the operator by itself. By construction, we have

$$\langle \mathcal{A}\varphi, \mathcal{P}\varphi \rangle + \langle \mathcal{P}\varphi, \mathcal{A}\varphi \rangle + \varphi_0^\star Q \varphi_0 + \Phi^\star(-h) W_1 \Phi(-h) + \int_{-h}^{0} \Phi^\star(\theta) W_2 \Phi(\theta) d\theta \leq \langle \varphi, C^\star C \varphi \rangle.$$

On the one hand, the terms involving Q and W_2 correspond to a positive definite functional on M_2 with a specific "block diagonal" structure, which may induce conservatism, even though the term with W_1 offers some additional degrees of freedom in the design. On the other hand, it is not clear whether (2), as a necessary condition, can be generalized to an operator setting. To see this, consider an observable delay-free system. The argument for the solvability of (2) with $P \succ 0$, $Q \succ 0$ is as follows: take $\ell > 0$ sufficiently large such that $-A_0 - \ell I$ is Hurwitz. Then P is positive definite, which satisfies

$$P(-A_0 - \ell I) + (-A_0 - \ell I)^{\mathsf{T}} P + C^{\mathsf{T}} C = 0,$$

equivalently, $A_0^{\mathsf{T}} P + P A_0 = C^{\mathsf{T}} C - 2\ell P$. However, this argument cannot be generalized to delay systems as $-\mathcal{A}$ cannot be associated with a flow of solutions forward in time.

VIII. Acknowledgment

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Appendix

A. Construction of operator $\mathcal{L}$

Let $\mathcal{P}$ be defined as in (11) and assume $\mathcal{P} \succ 0$. To compute the operator $\mathcal{P}^{-1}\mathcal{C}^*$, we need to find

$$\psi = \mathcal{P}^{-1}\mathcal{C}^* q, \; q \in \mathbb{C}^p,$$

[3] Although no formula exists for choosing N, one may start with a small value, and then validate stability of the closed-loop system using established computational tools for systems with distributed delays. If needed, N can be adjusted accordingly.

or equivalently, solve the following equation with respect to ψ:

$$\mathcal{P}\psi = \begin{pmatrix} C^\mathsf{T} q \\ 0 \end{pmatrix}.$$

This equation is equivalent to the system

$$\begin{aligned} C^\mathsf{T} q =& U(0)\psi_0 + \int_{-h}^{0} U(-h-s)A_1\Psi(s)ds \\ 0 =& \left(W_1 + (h+\theta)W_2\right)\Psi(\theta) + A_1^\mathsf{T} U(h+\theta)\psi_0 \\ &+ A_1^\mathsf{T}\int_{-h}^{0} U(\theta-s)A_1\Psi(s)ds,\ \theta\in[-h,0]. \end{aligned}$$

As operator $\mathcal{P}$ is positive definite, we have $U(0)\succ 0$. Therefore, we can reduce the system to the integral equation

$$\begin{aligned} &(W_1 + (h+\theta)W_2)\Psi(\theta) \\ &\quad - A_1^\mathsf{T} U(h+\theta)U^{-1}(0)\left(\int_{-h}^{0} U(-h-s)A_1\Psi(s)ds - C^\mathsf{T} q\right) \\ &\quad + A_1^\mathsf{T}\int_{-h}^{0} U(\theta-s)A_1\Psi(s)ds = 0,\ \theta\in[-h,0], \end{aligned}$$

with respect to Ψ. Some trivial simplifications lead to the form

$$\Psi(\theta) = -F(\theta+h)C^\mathsf{T} q + \int_{-h}^{0} K(\theta+h, s+h)\Psi(s)ds$$

for $\theta\in[-h,0]$, where F and K are given by (16). Introducing the new function $f(\cdot) = \Psi(\cdot - h)$, we obtain the integral equation

$$f(\theta) = -F(\theta)C^\mathsf{T} q + \int_{0}^{h} K(\theta, s)f(s)ds,\ \theta\in[0,h],$$

with respect to f, with solution

$$f(\theta) = -\left(F(\theta) + \int_{0}^{h} R(\theta,s)F(s)ds\right)C^\mathsf{T} q,\ \theta\in[0,h],$$

where R is the resolvent, being independent of q. An explicit formula for the operator $\mathcal{P}^{-1}\mathcal{C}^*$ is now obtained:

$$\begin{aligned} \mathcal{P}^{-1}\mathcal{C}^* q &= \begin{pmatrix} \psi_0 \\ \Psi(\cdot) \end{pmatrix} = \begin{pmatrix} \psi_0 \\ f(\cdot+h) \end{pmatrix} \\ &= \begin{pmatrix} U^{-1}(0)\left(C^\mathsf{T} q - \int_0^h U(-s)A_1 f(s)ds\right) \\ f(\cdot+h) \end{pmatrix} \\ &= \begin{pmatrix} G_0 C^\mathsf{T} q \\ -G(\cdot+h)C^\mathsf{T} q \end{pmatrix}, \end{aligned}$$

with G_0 and G defined in (17) and (18), respectively. The expression (14) for $\mathcal{L}$ follows immediately.

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