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# ORTHOGONALITY AND QUADRATURE ON THE UNIT CIRCLE

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ABSTRACT: In a recent paper, Jones et al. (see [1]) deal with a classical topic, namely "Quadratures on the unit circle" from a new point of view. They introduce the concept of "a sequence of para-orthogonal polynomials" with respect to a quasi-definite functional, and making use of this they characterize those formulas with a "maximal domain of validity" in certain subspaces of the linear space of the Laurent polynomials. The aim of our work is obtaining different expressions for the error in these quadratures and giving some results about convergence.

**Keywords:** Distributions, Quadratures, Szegö polynomials, para-orthogonality.
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## 1. INTRODUCTION.

In order to estimate the integral

$$I_\psi\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})\, d\psi(\theta) \qquad (1.1)$$

$\psi$ being a distribution function, that is, a real-valued, bounded, non-decreasing function defined on $[-\pi,\pi]$, we shall make use of "linear quadrature formulas" of the form,

$$I_n\{f\} = \sum_{j=1}^{n} \lambda_j f(\xi_j) \qquad (1.2)$$

where the nodes $\{\xi_j\}$ must lie on the unit circle $\Gamma = \{ z: |z|=1 \}$ and the weights $\{\lambda_j\}$ will be determined by imposing that

$$E_n\{f\} = I_\psi\{f\} - I_n\{f\} = 0 \qquad (1.3)$$

holds for any function $f$ belonging to an "appropiate" function space $S_p$ of dimension $p$ "as large as possible".
In order to characterize those spaces $S_p$, we recall that for every pair $(p,q)$ of integers, with $p \le q$, $\Lambda_{p,q}$ will denote the linear space of all Laurent polynomials (L-polynomials)

$$L(z) = \sum_{j=p}^{q} \alpha_j z^j$$

We write $\Lambda$ for the linear space of all L-polynomials (see [2], [3]), $\Pi$ for the space of all polynomials and $\Pi_n$ $(= \Lambda_{0,n})$ for the space of all polynomials of degree at most $n$.
Now, $\Lambda_{-p,q}$ ($p$ and $q$ being non-negative integers depending on $n$) will be called "a domain of validity" for the formula (1.2) if (1.3) holds true for any $f$ in $\Lambda_{-p,q}$. Furthermore, $\Lambda_{-p,q}$ will be called "a maximal domain of validity" when $p=q$ and $\Lambda_{-(p+1),q}$ or $\Lambda_{-p,q+1}$ can not be a domain of validity.
In [1] it was proved that $\Lambda_{-(n-1),n-1}$ represents a "maximal domain of validity" for the formula (1.2) with the nodes on $\Gamma$. In this case,

$$E_n\{f\} = 0, \ \forall f \in \Lambda_{-(n-1),(n-1)}$$

or equivalently

$$E_n\{e^{ik\theta}\} = 0, \ -(n-1) \le k \le n-1$$

gives rise to a system with $2n-1$ equations and $2n$ unknowns, yielding a one-parameter family of quadrature formulas $I_n\{f\} = \sum_{j=1}^{n} \lambda_j f(\xi_j)$ satisfying,

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i) $\xi_i \neq \xi_j$ and $|\xi_j| = 1$ for $j = 1,2,\ldots,n$

ii) $\lambda_j > 0$ for $j = 1,2,\ldots,n$

It seems appropiate to call such a formula "an n-point Szegö quadrature formula with respect to $\psi$", or in short a "Szegö formula".

When obtaining these formulas the concepts of sequences of polynomials para-orthogonal and k-invariant play a fundamental role (see [1] for more details)

For the sake of completeness we shall briefly recall that a sequence $\{X_n\}$ will be called para-orthogonal w.r.t. the distribution $\psi$ if for each $n \geq 0$, $X_n(z)$ is a polynomial of degree n satisfying $\langle X_n,1\rangle_\psi \neq 0$; $\langle X_n,z^m\rangle_\psi = 0$; $1 \leq m \leq n-1$ and $\langle X_n,z^n\rangle_\psi \neq 0$ where

$$\langle f(z),g(z)\rangle_\psi = \int_{-\pi}^{\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\psi(\theta)$$

On the other hand, for $k \in \mathbb{C}$ ; $k \neq 0$ a polynomial $X(z)$ is called k-invariant if $X^*(z) = k\,X(z)$, for all $z \in \mathbb{C}$ where $X^*(z) = z^{\deg X}\,\overline{X}(1/z)$.

Theorem 6.2 in [1] assures that a para-orthogonal and k-invariant polynomial $X(z)$ of degree n has exactly n simple zeros on the unit circle.

## 2. INTERPOLATORY QUADRATURE FORMULAS ON THE UNIT CIRCLE

Assume now $\{\xi_j\}$ given on $\Gamma$ such that $\xi_i \neq \xi_j$ $(i \neq j)$ so that (1.3) holds for $f \in \Lambda_{-p,q}$ where p and q are non-negative integers with $p+q = n-1$. Under these conditions the weights $\{\lambda_j\}$ are uniquely determined from an algebraic point of view. That is, $\{\lambda_j\}$ are the unique solution of the linear system $E_n\{e^{ik\theta}\} = 0$ with $-p \leq k \leq q$. On the other hand, let $R_{p,q}(z)$ be the interpolating L-polynomial to $f(z)$ at the nodes $\{\xi_j\}$ ( $R_{p,q}$ in $\Lambda_{-p,q}$ ). We can write

$$R_{p,q} = \sum_{i=1}^{n} L_{i,n}(z)\, f(\xi_i)$$

where

$$L_{i,n}(z) = \frac{V_n(z)}{(z-\xi_i)\,V_n'(\xi_i)} \quad \text{and}$$

$$V_n(z) = z^{-p}\,N_n(z) = z^{-p}\prod_{j=1}^{n}(z-\xi_j)$$

so that

$$\int_{-\pi}^{\pi} R_{p,q}(e^{i\theta})\,d\psi(\theta) = \sum_{j=1}^{n}\left[\int_{-\pi}^{\pi} L_{j,n}(e^{i\theta})\,d\psi(\theta)\right] f(\xi_j)$$

will provide a formula like (1.2) which will be called "of interpolatory type" in $\Lambda_{-p,q}$. The following statement can be easily proved, "A formula (1.2) has a domain of validity $\Lambda_{-p,q}$ with $p+q = n-1$, if and only if, it is of interpolatory type in this subspace".

Observe that we have made no assumptions about the nodes $\{\xi_j\}$ (except they are distinct and on $\Gamma$). Let us see, how to choose these, in order to get a maximal domain of validity i.e. $\Lambda_{-(n-1),n-1}$.

Indeed, in [4] the second author and M. Camacho proved the following

THEOREM 1.

$I_n(f) = \sum_{j=1}^{n} \lambda_j f(\xi_j)$ represents an n-point Szegö quadrature formula with respect to $\psi$ iff

a) $I_n(f)$ is of interpolatory type in $\Lambda_{-p,q}$ with p and q being arbiratry non-negative integers satisfying $p+q = n-1$.

b) The nodal polynomial $N_n(z) = \prod_{j=1}^{n}(z-\xi_j)$ is para-orthogonal and k-invariant for some $k \in \mathbb{C}$ with $|k| = 1$.

Remark:

In [1] only the particular case $p = 0$ $(q = n-1)$ i.e. $\Lambda_{-p,q} = \Lambda_{0,n-1} = \Pi_{n-1}$, is considered. That is, the interpolating process in Theorem 1 is carried out in the subspace of the usual polynomials of degree $n-1$ at most and Laurent polynomials are not taken into account.

In the next two sections aspects concerning the error and the convergence for Szegö quadrature formulas will be studied.

## 3. ERROR

We are now interested in giving different expressions of the error term

$$E_n(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\,d\psi(\theta) - \sum_{j=1}^{n}\lambda_j f(\xi_j) =$$

$$= I_\psi(f) - I_n(f),$$

$I_n(f)$ being a Szegö quadrature formula. In this case (see [1]) the nodes are the zeros of the polynomial $B_n(w,z) = \rho_n(z) + w\,\rho_n^*(z)$, where $\rho_n(z)$ represents the n-th Szegö polynomial w.r.t. the distribution $\psi$ and $\rho_n^*(z)$ its corresponding reciprocal polynomial, that is,

$$\rho_n^*(z) = z^n\,\overline{\rho}_n(z^{-1})$$

and w being a free parameter (recall that one had a one-parameter family of quadrature formulas) such that $|w|=1$. Let us write,

$$\hat{B}_n(w,z) = \frac{B_n(w,z)}{1 + w\,\overline{\delta}_n} \qquad (\delta_n = \rho_n(0)) \qquad (3.1)$$

and introduce

$$\sigma_n(z) = \int_{-\pi}^{\pi} \frac{\hat{B}_n(w, e^{i\theta})}{z - e^{i\theta}} d\psi(\theta)$$

Assuming that f is a holomorphic function on a domain D containing $\Gamma$, one has

THEOREM 2.

$$E_n\{f\} = \frac{1}{2\pi i} \int_{\partial D} \frac{\sigma_n(z)}{\hat{B}_n(w,z)} f(z)\, dz$$

($\partial D$ = boundary of D)

Proof

By Theorem 1, $E_n\{f\} = I_\psi\{f - P_n\}$ where $P_n$ represents the interpolating polynomial of degree n-1 at most of the function f at the nodes $\{\xi_j\}$. Now, by taking into account the integral representation for the remainder in the interpolating polynomial the proof can be obtained. ■

When D is given by $\{z \in \mathbb{C} : |z| < R\}$ $(R > 1)$, then one gets inmediately

COROLLARY 1.

$|E_n\{f\}| \le \gamma_n \max_{z \in \Gamma_R} |f(z)|$, where $\gamma_n$ represents an upper bound of $\left| \frac{\sigma_n(z)}{\hat{B}_n(w,z)} \right|$ on $\Gamma_R = \{z \in \mathbb{C}: |z| = R\}$.

Next, let us assume that f is analytic on the annulus $r < |z| < R$ $(r < 1 < R)$ so that we can write

$$f(z) = \sum_{-\infty}^{+\infty} a_j z^j \; ; \; r < |z| < R \qquad (3.2)$$

Then, by Theorem 2 and (3.2) one has

COROLLARY 2.

$$E_n\{f\} = \sum_{\nu=0}^{\infty} [a_{n+\nu}\, r_{n,\nu} + a_{-n-\nu}\, \hat{r}_{n,\nu}] \qquad (3.3)$$

where $r_{n,\nu} = E_n\{t^{n+\nu}\}$ and $\hat{r}_{n,\nu} = E_n\{t^{-n-\nu}\}$, $\nu = 0,1,\ldots$

In order to give a simpler expression for $E_n$, we assume that f is hermitian, that is, $a_{-j} = \bar{a}_j$; for $j = 0,1,\ldots$ Since $I_n\{f\}$ is a Szegö quadrature formula with respect to a distribution $\psi$ we always have

$$\hat{r}_{n,\nu} = \bar{r}_{n,\nu} \text{ for } \nu = 0,1,\ldots$$

Thus, from (3.3)

$$E_n\{f\} = 2 \sum_{\nu=0}^{\infty} \mathrm{Re}\, [a_{n+\nu}\, r_{n,\nu}]$$

Now, by using Schwartz's inequality provided that $\{r_{n,\nu}\} \in l_2$, one can deduce

$$|E_n\{f\}| \le 2 \left[ \sum_{\nu=0}^{\infty} |a_{n+\nu}|^2 \right]^{1/2} \left[ \sum_{\nu=0}^{\infty} |r_{n,\nu}|^2 \right]^{1/2} \qquad (3.4)$$

On the other hand,

$$a_j = \frac{1}{2\pi i} \int_{|z|=\rho} f(z)\, z^{-(j+1)}\, dz, \; j = 0, \pm 1, \ldots$$

with $r < \rho < R$.

Setting $M(R) = \sup_{|z|=R} |f(z)|$ $(R>1)$, then

$$|a_j| \le M(R)\, R^{-j} \quad \text{(Cauchy's inequality)} \qquad (3.5)$$

From (3.4) and (3.5), we conclude

THEOREM 3.

$$|E_n\{f\}| \le \frac{M(R)}{R^{n-1}\,(R^2-1)^{1/2}} \left[ \sum_{\nu=0}^{\infty} |r_{n,\nu}|^2 \right]^{1/2}$$

Let us now give an alternative approach to the Szegö quadrature via Hermite Interpolation Laurent polynomial (compare with [6]). In general, given the nodes $\xi_1, \xi_2, \ldots, \xi_n$ such that $\xi_i \neq \xi_j$ $(i \neq j)$ and $\xi_i \neq 0$, then a unique L-polynomial $\hat{Q}(z)$ in $\Lambda_{-(n-1),n-1}$ $(n > 1)$ exists, so that

$$\begin{aligned} \hat{Q}(\xi_i) &= f(\xi_i); \; 1 \le i \le n \\ Q'(\xi_i) &= f'(\xi_i); \; 1 \le i \le n-1 \end{aligned} \qquad (3.6)$$

Furthermore (see [7]) we can write

$$\hat{Q}(z) = \sum_{i=1}^{n} \hat{L}_{i,0}(z)\, f(\xi_i) + \sum_{i=1}^{n-1} \hat{L}_{i,1}(z)\, f'(\xi_i) \qquad (3.7)$$

where

$$\hat{L}_{i,0}(z) = \left(\frac{\xi_i}{z}\right)^{n-1} \left( L_{i,0}(z) + \frac{(n-1) L_{i,1}(z)}{\xi_i} \right)$$

for $1 \le i \le n-1$, and

$$\hat{L}_{n,0}(z) = \left(\frac{\xi_n}{z}\right)^{n-1} L_{n,0}(z)$$

whereas

$$\hat{L}_{i,1}(z) = \left(\frac{\xi_i}{z}\right)^{n-1} L_{i,1}(z), \text{ if } 1 \le i \le n-1$$

Here, $L_{i,0}(z)$ $(1 \le i \le n)$ and $L_{i,1}(z)$ $(1 \le i \le n-1)$ are polynomials of degree n-1 at most satisfying the respective interpolation conditions

$$L_{i,0}(\xi_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$L'_{i,0}(\xi_j) = 0; \; 1 \le j \le n-1$$

and

$$L_{1,1}(\xi_j) = 0;\ 1 \le j \le n$$

$$L'_{1,1}(\xi_j) = \delta_{1j};\ 1 \le j \le n-1$$

From (3.7), one gets

$$\int_{-\pi}^{\pi} \hat{Q}(e^{i\theta})\, d\psi(\theta) =$$

$$= \sum_{j=1}^{n} w_j f(\xi_1) + \sum_{j=1}^{n-1} w'_j f'(\xi_1) = \hat{I}_n\{f\} \qquad (3.8)$$

where $w_j = \int_{-\pi}^{\pi} \hat{L}_{j,0}(e^{i\theta}) d\psi(\theta)$ and

$$w'_j = \int_{-\pi}^{\pi} \hat{L}_{j,1}(e^{i\theta}) d\psi(\theta)$$

Therefore, $\hat{I}_n\{f\}$ can be regarded as a quadrature formula which makes use of values of the function f and its derivative. Clearly, $\hat{I}_n\{f\}$ has a domain of validity $\Lambda_{-(n-1),n-1}$. However, an adequate choice of the nodes can greatly simplify formula (3.8). Indeed, when $\{\xi_j\}$ are taken as the zeros of a polynomial $B_n(z)$ of degree n, which is para-orthogonal and k-invariant, one has.

THEOREM 4.

$\hat{I}_n\{f\}$ given by (3.8) reduces to the corresponding Szegö quadrature formula $I_n\{f\}$ with nodes $\{\xi_j\}$.

Proof

Write $I_n\{f\} = \sum_{j=1}^{n} \lambda_j f(\xi_j)$ for the Szegö formula. First, we shall prove that the coefficients $w'_j$ are equal to zero for $j = 1,2,\ldots,n-1$

Indeed, $w'_j = \int_{-\pi}^{\pi} \hat{L}_{j,1}(e^{i\theta}) d\psi(\theta)$, where

$$\hat{L}_{j,1}(z) = (\xi_j/z)^{n-1}(z-\xi_j) \prod_{\substack{i=1 \\ j\neq i}}^{n-1} \left(\frac{z-\xi_i}{\xi_j-\xi_i}\right)^2 \left(\frac{z-\xi_n}{\xi_j-\xi_n}\right)$$

Hence, we can write $w'_j = k_j \int_{-\pi}^{\pi} B_n(e^{i\theta}) M_j(e^{i\theta}) d\psi(\theta)$ ($k_j$ being a constant) with

$$M_j(z) = \frac{1}{z^{n-1}} \prod_{\substack{i=1 \\ j\neq i}}^{n-1} \left(\frac{z-\xi_i}{\xi_j-\xi_i}\right) = \frac{S(z)}{z^{n-1}};$$

where $S \in \Pi_{n-2}$. Therefore,

$$M_j(e^{i\theta}) = \sum_{k=1}^{n-1} m_{k,j}\, e^{-ik\theta},$$

and consequently

$$w'_j = k_j \sum_{k=1}^{n-1} m_{k,j} \langle B_n(z), z^k \rangle_{\psi} = 0 \text{ for } j = 1,2,\ldots n$$

due to the para-orthogonality property of the polynomial $B_n(z)$.

Let us see now that $w_j = \lambda_j$; $j = 1,2,\ldots,n$. Indeed, both of the formulas $I_n\{f\}$ and $\hat{I}_n\{f\}$ are valid in $\Lambda_{-(n-1),n-1}$. Let F be an element in this subspace, then

$$\int_{-\pi}^{\pi} F(e^{i\theta})\, d\psi(\theta) = I_n\{F\} = \hat{I}_n\{F\},$$

that is,

$$\sum_{j=1}^{n} w_j F(\xi_j) = \sum_{j=1}^{n} \lambda_j F(\xi_j)$$

In particular, taking $F = \hat{L}_{j,0}$, because of $\hat{L}_{j,0}(\xi_k) = \delta_{jk}$, then one concludes that $w_j = \lambda_j$; $j=1,2,\ldots n$ ■

In short a Szegö quadrature formula $I_n\{f\}$ will be given by

$$I_n\{f\} = I_{\psi}\{\hat{Q}_i\} \text{ for any } i = 1,2,\ldots,n$$

and where $\hat{Q}_i(z)$ represents the Laurent polynomial in $\Lambda_{-(n-1),n-1}$ satisfying the interpolation conditions,

$$\hat{Q}_i(\xi_j) = f(\xi_j); \qquad 1 \le j \le n \qquad (3.9)$$

$$\hat{Q}'_i(\xi_j) = f'(\xi_j);\ 1 \le j \le n (j \neq i) \qquad (3.10)$$

$\{\xi_j\}$ being the zeros of the polynomial

$$B_n(w,z) = \rho_n(z) + w\, \rho_n^*(z) \quad (|w| = 1)$$

From this approach, we can deduce another expression for the error $E_n\{f\}$ with less restrictive conditions on the function f(z), namely

THEOREM 5.

$$E_n\{f\} = \int_{-\pi}^{\pi} \frac{[\hat{B}_n(e^{i\theta})]^2 e^{-(n-1)i\theta}}{e^{-i\theta} - \xi_k}\, g_{n,k}[\ldots,e^{i\theta}]\, d\psi(\theta)$$

for any integer k: $1 \le k \le n$ and where $g_{n,k}[\ldots,z]$ denotes the (2n-1)-th divided differences of the function $g_n(z) = z^{n-1} f(z)$ formed with the nodes $\xi_j$ each taken twice if $j \neq k$ and once if $j=k$.

Here, $\hat{B}_n(z)$ represents the corresponding monic polynomial given by (3.1).

Proof.

Take $P \in \Pi_{2n-2}$ such that

$$P(\xi_j) = g_n(\xi_j) \qquad 1 \le j \le n, \text{ and}$$
$$P'(\xi_j) = g_n'(\xi_j) \qquad j=1,2,\ldots n \ (j\neq k)$$

One has

$$g_n(z) - P(z) = \frac{[\hat{B}_n(z)]^2}{z - \xi_k}\, g_{n,k}[\ldots,z]$$

Therefore

$$f(z) - z^{-n+1}P(z) = \frac{z^{-n+1}[\hat{B}_n(z)]^2}{z - \xi_k}\, g_{n,k}[\ldots,z)$$

Clearly, $z^{-n+1}P(z)$ coincides with the unique L-polynomial in $\Lambda_{-(n-1),n-1}$ satisfying the above interpolation conditions. Now, from this the proof follows readily. ■

## 4. CONVERGENCE.

Assume f a complex function defined on the unit circle Γ. We can write

$$f(e^{i\theta}) = f_1(\theta) + if_2(\theta)$$

where $f_j(\theta)$ (j=1,2) are real valued functions defined on the interval $[-\pi,\pi]$.
Clearly f is a continuous function if and only if, $f_j$ (j=1,2) are continuous.
In order to give a first result on convergence of the sequence $I_n(f)$ as n tends to infinity we need the following

THEOREM 6.

Let $f:\Gamma \longrightarrow \mathbb{C}$ be a continuous function and $\varepsilon > 0$. Then, there exists an L-polynomial $T_N$ in $\Lambda_{-N,N}$ so that

$$|f(z) - T_N(z)| < \varepsilon,$$

with $z = e^{i\theta}$ and $\theta \in [-\pi,\pi]$

Proof.

Writing $f(e^{i\theta}) = f_1(\theta) + if_2(\theta)$, then by Weierstrass Approximation Theorem, there exist $T_1(\theta)$ and $T_2(\theta)$, trigonometric polynomials such that

$$|f_1(\theta) - T_1(\theta)| < \varepsilon/\sqrt{2},$$

for any $\theta \in [-\pi,\pi]$ and i = 1,2.

Making use of the known relations,

$$\cos k\theta = \frac{e^{ik\theta} + e^{-ik\theta}}{2}$$
$$\sin k\theta = \frac{e^{ik\theta} - e^{-ik\theta}}{2i} \qquad (4.1)$$

we can write $T_1(\theta) + iT_2(\theta) = \sum_{-N}^{N} \alpha_k z^k \in \Lambda_{-N,N}$

with $z = e^{i\theta}$ and $N = \max(n_1,n_2)$ $n_1$, $n_2$ being the degrees of $T_1$ and $T_2$ respectively.

Finally, if $z = e^{i\theta}$, then

$$|f(z) - T_N(z)|^2 = |f_1(\theta) - T_1(\theta)|^2 +$$
$$+ |f_2(\theta) - T_2(\theta)|^2 < \varepsilon^2 \ ■$$

This leads to the following

THEOREM 7.

Let f be a continuous function on Γ, then

$$\lim_{n\to\infty} I_n(f) = I_\psi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\, d\psi(\theta)$$

where $I_n(f)$ represents a Szegö quadrature formula for n = 1,2,...

Proof.
Let ε be a given real positive number. Take

$$\varepsilon' = \frac{\varepsilon}{2c_0} > 0 \text{ where } c_0 = \int_{-\pi}^{\pi} d\psi(\theta).$$

By Theorem 6, there exists $T_N(z)$ in $\Lambda_{-N,N}$ such that

$$|f(e^{i\theta}) - T_N(e^{i\theta})| < \varepsilon'; \ \forall\ \theta \in [-\pi,\pi].$$

Assume $n > N$ and write now $I_n(f) = \sum_{j=1}^{n} \lambda_j^{(n)} f(\xi_j^{(n)})$, then

$I_\psi(f) - I_n(f) = I_\psi(f - T_N) + I_n(T_N - f)$, hence

$$|I_\psi(f) - I_n(f)| \le \int_{-\pi}^{\pi} |f(e^{i\theta}) - T_N(e^{i\theta})|\, d\psi(\theta) +$$
$$+ \sum_{j=1}^{n} \lambda_j^{(n)} |f(\xi_j^{(n)}) - T_N(\xi_j^{(n)})| < 2c_0\varepsilon' = \varepsilon$$

(Recall that $\lambda_j^{(n)} > 0$ ; j = 1,2,...,n and

$$\sum_{j=1}^{n} \lambda_j^{(n)} = c_0) \ ■$$

In case f were only an integrable function on Γ, we must refer to the following property of the Riemann-Stieltjes integral (see [8]).

THEOREM 8.

Let the real-valued function g(x) be bounded on $[-\pi,\pi]$, α(x) non-decreasing and let the Riemann-Stieljes integral $\int_{-\pi}^{\pi} g(x)d\alpha(x)$ exist. For every ε > 0, there exist trigonometric polynomials p(x) and P(x) such that:

$p(x) \le g(x) \le P(x)$ and $\int_{-\pi}^{\pi} [P(x) - p(x)]d\alpha(x) < \varepsilon$

By applying this Theorem, we obtain

THEOREM 9.
Let the complex-valued function f(z) be integrable on Γ with respect to dψ. Then

$$\lim_{n\to\infty} I_n(f) = I_\psi(f)$$

($I_n\{f\}$ being as in Theorem 7)

Proof.

Write $f(e^{i\theta}) = f_1(\theta) + if_2(\theta)$. Here $f_1$ and $f_2$ can be considered as two real-valued functions defined on $\Gamma$ or equivalently on $[-\pi,\pi]$. We have

$$I_\psi\{f\} = \int_{-\pi}^{\pi} f_1(\theta)\, d\psi(\theta) + i \int_{-\pi}^{\pi} f_2(\theta)\, d\psi(\theta) =$$

$$= I_\psi\{f_1\} + i\, I_\psi\{f_2\}.$$

Likewise

$$I_n\{f\} = I_n\{f_1\} + i\, I_n\{f_2\}$$

Now, given $\varepsilon > 0$, by Theorem 8, there exist trigonometric polynomials $p(\theta)$ and $P(\theta)$ such that

$$p(\theta) \le f_1(\theta) \le P(\theta) \text{ and } \int_{-\pi}^{\pi} [P(\theta) - p(\theta)]d\psi(\theta) < \varepsilon \tag{4.2}$$

From (4.1) we can write

$$p(\theta) = \sum_{-N}^{N} \alpha_k z^k = T_N(z) \in \Lambda_{-N,N} \text{ and}$$

$$P(\theta) = \sum_{-M}^{M} \beta_k z^k \in \Lambda_{-M,M} \text{ with } z = e^{i\theta}.$$

Assume $n > \max\{N,M\}$, then from (4.2) we can conclude that

$$\left| \int_{-\pi}^{\pi} f_1(\theta)\, d\psi(\theta) - \sum_{j=1}^{n} \lambda_j^{(n)} f_1(\xi_j^{(n)}) \right| < \varepsilon \tag{4.3}$$

In effect, it follows from (4.2) that

$$\int_{-\pi}^{\pi} [T_M(e^{i\theta}) - f_1(\theta)]\, d\psi(\theta) < \varepsilon \text{ and}$$

$$\int_{-\pi}^{\pi} [f_1(\theta) - T_N(e^{i\theta})]\, d\psi(\theta) < \varepsilon$$

Hence,

$$\int_{-\pi}^{\pi} f_1(\theta)\, d\psi(\theta) - \varepsilon <$$

$$< \int_{-\pi}^{\pi} f_1(\theta)\, d\psi(\theta) - \int_{-\pi}^{\pi} [f_1(\theta) - T_N(e^{i\theta})]\, d\psi(\theta) =$$

$$= I_\psi\{T_N\} = I_n\{T_N\} \le I_n\{f_1\} \le I_n\{T_M\} = I_\psi\{T_M\} =$$

$$= I_\psi\{f_1\} + I_\psi\{T_M - f_1\} < \int_{-\pi}^{\pi} f_1(\theta)\, d\psi(\theta) + \varepsilon$$

Similarly,

$$\left| \int_{-\pi}^{\pi} f_2(\theta)\, d\psi(\theta) - \sum_{j=1}^{n} \lambda_j^{(n)} f_2(\xi_j^{(n)}) \right| < \varepsilon \tag{4.4}$$

Finally, by (4.3) and (4.4) the proof can be completed ■

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