Linear prediction: mathematics and engineering

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Abstract

We present an introduction to some aspects of digital signal processing and time series analysis which are not always covered in classical textbooks. One of the objectives is to illustrate how mathematics and engineering can be combined in a fruitful interplay, resulting in many new techniques and methods in many different fields. We shall illustrate how the prediction problem is related to linear algebra, orthogonal polynomials, classical interpolation problems, inverse scattering theory, Toeplitz operators, network theory etc.

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1 Introduction

It is impossible to trace exactly the origin of mathematics. Counting was probably the first mathematical activity the human race ever performed. Anyway, it is for sure that mathematics was first

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introduced because it was needed to solve very practical problems or to make abstract models to understand physical phenomena. In this sense, one could say that mathematics developed from the residue of "applied sciences" like physics, astronomy, engineering etc. Gradually mathematics started to live its own life. The long and rich history of the artifact π started from a practical problem of finding the area of a circle. But, does the perfect circle of mathematics exist in real life? The very concept of real numbers is a mathematical construction, but do we ever meet a real number in our world? Nobody asks these questions anymore. We are familiar with real numbers now and we use them without thinking. It would be hard to imagine how we could have built up a technological world like ours if we never would have had real numbers.
Yet people somewhere deep in the Amazon forest lived for centuries without knowing about real numbers. So where does mathematics become abstract and where is it applied? It is hard to tell. Sometimes mathematicians deteriorate and play with mathematics for fun. They do not want to solve a real life problem, but it happens that after a while mathematical results that were developed in a pure abstract framework all of a sudden become useful in solving very practical problems. When Julia studied the iteration of functions, he probably never dreamed that there would ever exist a company exploiting iterated function systems on a commercial basis. When mathematicians developed the theory of multiresolution, they didn't know that it would be the framework to study wavelets which are very useful for image processing. Even the Euclidean algorithm can be traced in the coding system of a compact disk. Thus, when mathematicians and engineers at a certain point in the evolution seem to be working on completely unrelated matters, they sometimes learn about each others results and when joining efforts, they come up with a new marvellous application.
The examples that I have given above are all related in some way to the processing of digital signals. I want to illustrate in this paper that signal processing is a very good example of the permanent cross-fertilisation between several branches of mathematics and engineering.
Signal processing is a wide and rich branch of engineering - or should I say mathematics - where complex analysis, linear algebra, stochastic processes, electrical engineering, Fourier analysis, measure theory and many others, naturally meet each other.
There is a constant interlacing of these disciplines and abstract results are focussing on the application while new levels of abstraction emerge from the practical problems.
As a consequence of this permanent attraction and repulsion of those different disciplines, there exists a biblical confusion in the terminology used in each of them. After all, if an engineer talks about a continuous time signal, the mathematician would call this a function of a real variable, and if the mathematician uses the term sequence, the engineer will probably see it as a function of a discrete variable and he calls it a digital discrete time signal etc. This is an unfortunate situation because in our time, where the quality of a scientist is sometimes measured by the number of papers he produces, there is an overproduction of scientific literature, so that it becomes impossible to read and understand everything related to your field of interest. Consequently the same results are rediscovered over and over again in the different disciplines. This is often not a measure for the quality of the researcher, but it is certainly a measure for the quality of the result. In this paper, we shall use the mathematical terminology most of the time, but we shall also bring a modest contribution in using the terminology from the other fields and thus in clearing up some of the confusion that might exist for those who are not familiar with the material.
Of course, digital signal processing is too wide a subject to include everything related to it. For example, we shall not elaborate on the discrete or fast Fourier transform, which is certainly a major accomplishment that is intimately related to digital signal processing.
We shall restrict ourselves to the prediction and modelling of signals.

2 Discrete signals and digital filters

As we mentioned in the introduction, any function of time can be seen as a signal. If we sample such a function $s(t)$ at equally spaced time moments, which are T units of time apart, we get a *time series* or a *discrete time signal* $s(nT)$. The interval T is called the *sampling period* and $1/T$ is the *sampling frequency*. We shall abbreviate $s(nT)$ as s_n. Most of the practical signals are real, we shall, with no extra complication assume however that s_n is a complex number.
The *time domain* ℓ is the set of all possible signals:

$$\ell = \{s = (s_n) : s_n \in \mathbb{C}, n \in \mathbb{Z}\}.$$

The *shift operator* or *time delay operator* Z maps a signal into another signal that is delayed one time unit.

$$(Zs)_n = s_{n-1}, \quad s \in \ell, \quad n \in \mathbb{Z}.$$

Let ℓ_D and ℓ_R be two subsets of the time domain, then a linear operator $T : \ell_D \to \ell_R$ is called *shift invariant* if it commutes with $Z : ZT = TZ$. Note that then also $Z^m T = TZ^m$, $\forall m \in \mathbb{Z}$. A *(digital) filter* is such a shift invariant operator which we assume to be linear[1]. The signal $\delta = (\delta_{n0})$ is called the *unit pulse* (at time $n = 0$) and $h = T\delta$ is called the *impulse response* of the filter T. Let ℓ_p be the classical spaces of complex sequences (ℓ_p is a Banach space when $p \geq 1$):

$$\begin{aligned} \ell_p &= \left\{ s \in \ell : \|s\|_p = \left(\sum_k |s_k|^p \right)^{1/p} < \infty \right\}, \quad 0 < p < \infty \\ \ell_\infty &= \left\{ s \in \ell : \|s\|_\infty = \sup_k |s_k| < \infty \right\}. \end{aligned}$$

Then for $h \in \ell_1$ and $u \in \ell_\infty$, the convolution $s = u * h = h * u$ defined by

$$s_n = \sum_m u_m h_{n-m}, \quad n \in \mathbb{Z}$$

exists and the operator $T : u \to h * u$ is a digital filter from ℓ_∞ to ℓ_∞ whose impulse response is h. In fact, any filter from ℓ_∞ to ℓ_∞ can be represented in this way because we may represent any signal $s = (s_n)$ as

$$s = \sum_m s_m (Z^m \delta)$$

and hence, if $T\delta = h$, then at least formally

$$\begin{aligned} Ts &= \sum_m s_m (TZ^m \delta) = \sum_m s_m Z^m T\delta \\ &= \sum_m s_m Z^m h = \sum_{n,m} s_m h_{n-m} Z^n \delta \\ &= h * s. \end{aligned}$$

If $s \in \ell_p$, then $\|s\|$ is a measure of the "magnitude" of the signal s. In the special case that $s \in \ell_2$, then we call $\|s\|^2$ the *energy* of the signal s.
Most of the time one is interested in building *stable* filters. By this we shall mean that a "finite signal" ($\|u\| < \infty$) will be transformed by the filter into a "finite signal" ($\|Tu\| < \infty$). There are several notions of stability and our definition corresponds to what is usually called *bounded input*

[1]The linearity of the filter is a long-standing typical scientific simplification of the real life situation. Of course one never has a completely linear situation. It is only very recently that one has discovered the mathematically much more difficult, but so much more fascinating world of nonlinear dynamical systems. It becomes now possible to analyse and model chaotic behaviour that has previously been pushed aside.

– *bounded output (BIBO) stability.* Thus in ℓ_∞, BIBO stability means that the filter maps $u \in \ell_\infty$ to $s = Tu \in \ell_\infty$, i.e., if all $|u_n| < \infty$ and $s = Tu$, then also all $|s_n| < \infty$. A necessary and sufficient condition for this is that the impulse response is in ℓ_1.

A stable filter for ℓ_2 signals maps signals with finite energy into signals with finite energy. For simplicity,[2] we shall work with signals from ℓ_2.

A signal s will be called *causal* if $s_n = 0$ for $n < 0$. If we agree to call $n = 0$ the *present*, $n < 0$ the *past* and $n > 0$ the *future*, then we can say that a causal signal has a blank past or a blank history. It is a natural property of a filter to be *causal.* We mean by this that if the *input* signal for the filter is causal, then the *output* signal should also be causal. Thus a causal filter can give no output before there has been any input. The impulse response is causal if and only if the filter is causal.

The counterpart of the time domain is the *frequency domain.* It consists of all the *z-transforms* of the signals in the time domain. The z-transform is defined as the formal power series[3]

$$S(z) = \sum_n s_n z^n.$$

The z-transform of the impulse response is called the *transfer function* of the filter.

Note that the shift operator in the time domain corresponds with a multiplication by z in the frequency domain.

$$(Zs)_n = s_{n-1} \Leftrightarrow (Zs)(z) = zS(z).$$

For this reason, we shall abuse z as a notation for the shift operator itself, i.e., we shall use $zs_n = s_{n-1}$. Note that if the z-transform actually converges for some $|z| = 1$, then it will represent a function of the complex variable z which is defined at least in the open unit disk. For example when $s \in \ell_p$, then $S \in L_p$ (the classical Lebesgue space on the unit circle) and these spaces are isometric. The signal and its z-transform are a Fourier transform pair.

The *autocorrelation* function[4] r of a signal $s \in \ell_2$ is defined as the sequence $r = (r_n)$ with

$$r_n = \sum_k s_{n+k}\bar{s}_k, \quad n \in \mathbb{Z}.$$

Note that $r_0 = \|s\|^2 = \|S\|^2$ is the energy of s; the first norm is the ℓ_2-norm of the signal s and the second norm is the L_2-norm of its z-transform. The z-transform of the autocorrelation function of the signal s is

$$R_s(z) = \sum_n r_n z^n = S(z)S_*(z)$$

where $S_*(z) = \overline{S(1/\bar{z})}$. This $R_s(z)$ is called the *energy spectrum* or *power spectrum* of the signal, or just its *spectrum* for short. Note that when the series converge[5] on the unit circle, then

$$R_s(e^{i\omega}) = |S(e^{i\omega})|^2 \geq 0.$$

When $s \in \ell_p$ is causal, then $S(z) = \sum_0^\infty s_n z^n$ has an analytical extension in the open unit disk $\mathbb{D}$ and it belongs to the Hardy space H_p. We shall also need the weighted L_2 space of the unit circle. It is a Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(t)}g(t)d\mu(t), \quad d\mu(t) = R_s(t)d\omega, \quad t = e^{i\omega}. \tag{2.1}$$

[2] ℓ_2 is particularly attractive because we then work in a Hilbert space.

[3] In the engineering literature one usually uses z^{-1} instead of z. We use here the convention of the mathematical literature. We shall also use the convention that a capital letter shall denote the z-transform of the signal represented by the corresponding lower case letter.

[4] A function of the discrete variable n.

[5] Also here is a difference between the engineering and the mathematical literature. Mathematicians use i to indicate the imaginary unit, while engineers use j.

The measure μ is called the *spectral measure*.
The time domain equivalent of this space is the Hilbert space $\mathbb{L}_2$ spanned by the signal and its shifts: $z^k s$, $k \in \mathbb{Z}$. The Gram matrix there has elements

$$r_{kl} = \left\langle z^k s, z^l s \right\rangle = \sum_n \bar{s}_{n-k} s_{n-l} = r_{k-l}$$

which is the same as in the previous L_2 space, where the elements of the Gram matrix are

$$r_{kl} = \left\langle z^k, z^l \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^{l-k} R_s(t) d\omega = r_{k-l}$$

too. These L_2 and $\mathbb{L}_2$ spaces are isomorphic and we can freely switch between both.
The relation between applied topics such as stochastic processes, signal processing, time series analysis (or whatever name has been invented for it) on one hand and Toeplitz matrices and operators, convolution equations etc. on the other hand is known for some time and it is at least partially responsible for the interest of several mathematicians in these topics. See for a classical example [34] and for one of the more recent books [6].

3 Stochastic processes

Due to components in the signal such as measurement noise, a signal is often considered as a stochastic entity. Thus also statisticians have their contribution to signal processing and in connection with this field, many statistical terms are used. The term autocorrelation function is an example of that. So we shall start all over and arrive at the same mathematical framework, namely the weighted L_2 space we had at the end of the previous section.
Consider a discrete stochastic process $s = (s_n)$ where each s_n is a complex stochastic variable in some probability space $(\Sigma, \mathcal{B}, m)$ where m is a probability measure for Σ with Borel sets $\mathcal{B}$. Note that for simplicity we assumed that the s_n are equally distributed (m does not depend on n). The *mean* or expected value of s_n is defined by

$$\mathcal{E}[s_n] = \int_\Sigma s_n(\sigma) dm(\sigma).$$

Its variance is $\mathcal{E}[|s_n|^2]$. By standard techniques we can construct a Hilbert space $\mathbb{L}_2$ of linear combinations $\sum_n c_n s_n$ with inner product

$$\langle u, v \rangle = \int_\Sigma \overline{u(\sigma)} v(\sigma) dm(\sigma). \tag{3.1}$$

A process $s = (s_n)$ is called *stationary* if the *autocorrelation* function

$$r(m,n) = \mathcal{E}[\bar{s}_m s_n] = \langle s_m, s_n \rangle, \quad m, n \in \mathbb{Z}$$

satisfies

$$r(m,n) = r(m+k, n+k), \quad m, n, k \in \mathbb{Z}.$$

Thus, in the stationary case, the autocorrelation function does not depend on two discrete variables m and n separately, but it is only a function of their difference. We shall then denote it with only one index:

$$r_n = r(n+k, k) = \langle s_n, s_0 \rangle.$$

As before, we call in that case $r_0 = \|s\|^2 = \|s_n\|^2$ the *energy* of the process s. In general $r(m,n) = \overline{r(n,m)}$, so the infinite matrix of the autocorrelation is a Gram matrix for $\mathbb{L}_2$ and thus positive

definite. In the stationary case, it is a Toeplitz matrix $T = [r_{i-j}]$. The Fourier transform of the autocorrelation function will again be called the (*power*) *spectrum* of the stochastic process

$$R(e^{i\omega}) = \sum_n r_n e^{in\omega}$$

and the measure defined by $d\mu = R(t)d\omega$, $t = e^{i\omega}$ is the spectral measure. It is used to define the weighted L_2 space with inner product defined by (2.1). This space is isomorphic to the space defined in (3.1). The isomorphism between the two spaces $s_n \leftrightarrow e^{in\omega}$ is called the *Kolmogorov* isomorphism.

The stochastic equivalent of a unit pulse is (unit) *white noise*. It is a zero mean stationary process with autocorrelation $r_n = \delta_{n0}$. The spectrum of white noise is the constant 1, exactly like in the case of a unit pulse.

White noise means that the process (signal) does not have any memory. What you get at time n does not have any influence on (is uncorrelated with) what you get at any other instance $m \neq n$. This is characterised by the flatness of the spectrum, it contains no preference for any frequency whatsoever. It can be simulated by a uniform random number generator.

As before, we shall use z to denote a time delay, e.g.,

$$A(z)s_n = \sum_{k=0}^{n} a_k s_{n-k} \quad \text{for} \quad A(z) = \sum_{k=0}^{n} a_k z^k.$$

4 Linear prediction

In general, a signal is produced by a system, which we shall assume to be linear. Some input signal u, will be fed to the system which is characterised by a filter, to produce some output signal s. One can often observe the output signal and sometimes one also knows the input. For example, when analysing the acoustic properties of a concert hall, one can emit a chirp, that is a signal that sweeps through all the frequencies of interest and record the response at some particular place in the hall, or to study the mechanical properties of a car coach, one applies a certain vibration and a sensor at another place will measure the response. In principle, one can then find the transfer function of the system as $H(z) = S(z)/U(z)$, or one can find the spectrum of the filter as the ratio of the spectra of the output and input signals: $R_h(e^{i\omega}) = R_s(e^{i\omega})/R_u(e^{i\omega})$. In other applications, we can not measure the input. For example, if we want to analyse human speech, it is very difficult to measure the input (the vibration of the vocal chords) and we can only measure the pressure wave which produces the sound which is shaped by the form of the vocal tract, i.e., the complex system of the nose and the mouth [43]. Also in many medical, economical or seismic applications, the input is unknown. In the latter case, we shall assume that the output is produced by a filter, driven by white noise or a pulse (train), i.e., an information-less signal[6]. The *prediction problem* is then to find a model for the filter, supposing the signal corresponds to its impulse response. In the ideal situation we should be able to predict the outcome s_n, given its complete history $\{s_{n-k}, k = 1, 2, \ldots\}$. Since this is an infinite dimensional problem that is computationally unrealistic, we want to find an approximate solution by considering only a finite dimensional subset of the past. A simple choice is to predict s_n linearly from $\{s_{n-1}, s_{n-2}, \ldots, s_{n-p}\}$ for some finite p. Thus we construct (the best possible) estimate $\hat{s}_n$ for s_n by computing the *prediction coefficients* $a_1, \ldots, a_p$ in

$$\hat{s}_n = -a_1 s_{n-1} - a_2 s_{n-2} \ldots - a_p s_{n-p}.$$

[6] It is possible to relate this to the notion of entropy, a concept not only important in physics, but also in information processing.

We have chosen the minus sign, because it is now easy to write the *prediction error* at time n as

$$e_n = s_n - \hat{s}_n = \sum_{k=0}^{p} a_k s_{n-k}, \quad a_0 = 1$$

or, defining the *predictor (polynomial)*

$$A(z) = \sum_{k=0}^{p} a_k z^k, \quad a_0 = 1,$$

we can write

$$e_n = A(z)s_n \quad \text{or} \quad E(z) = A(z)S(z).$$

The simplest possible choice in which we want to minimise the error is to minimise its energy. So we have to solve a least squares problem by trying to find

$$E_p = \min \|e\|^2$$

For the deterministic case we have $\|e\|^2 = \sum_n |e_n|^2$, and for the stochastic case $\|e\|^2 = \mathcal{E}[|e_n|^2]$.

4.1 Deterministic case

Let us define the column vectors $s = (s_n)$ (the signal), $e = (e_n)$ (the prediction error) and $a = (a_i)_{i=1}^p$ (the predictor coefficients) and also the $\infty \times p$ matrix S with columns $zs, z^2 s, \ldots, z^p s$. Then we have to find a such that $e = s + Sa$ is as small as possible. Minimising $\|e\|$ corresponds to finding a least squares solution of the $\infty \times p$ system $Sa = -s$. Its solution is given by the normal equations

$$(S^H S)a = -S^H s,$$

where $S^H S$ is a $p \times p$ matrix and $S^H s$ a $p \times 1$ vector.
Thus, explicitly

$$\sum_{k=1}^{p} \left(\sum_n s_{n-k} \bar{s}_{n-i} \right) a_k = -\sum_n s_n \bar{s}_{n-i}, \quad i = 1, \ldots, p. \tag{4.1}$$

In geometrical terms, we project s onto the span of $zs, z^2 s, \ldots, z^p s$, the column space of S. The projection is orthogonal in the space ℓ^2. By the Pythagoras theorem we find for the least squared error

$$\begin{aligned} E_p &= \|s + Sa\|^2 = \|s\|^2 - \|Sa\|^2 \\ &= \|s\|^2 - a^H S^H S a = \|s\|^2 + s^H S a \\ &= \sum_n |s_n|^2 + \sum_{k=1}^{p} a_k \left(\sum_n \bar{s}_n s_{n-k} \right). \end{aligned}$$

Using the definition of the autocorrelation function, this reduces to

$$E_p = r_0 + \sum_{k=1}^{p} r_{-k} a_k$$

and the system (4.1) defining a_k is a Toeplitz system

$$\sum_{k=1}^{p} r_{i-k} a_k = -r_i, \quad i = 1, \ldots, p.$$

Moreover, the Toeplitz matrix $T_p = [r_{i-k}]$ of order $p+1$ is hermitian and positive definite.
Defining the predictor in this way corresponds to what is known as the *autocorrelation method.*
There are however some practical problems with this method. One can of course never sum over all $n \in \mathbb{Z}$ in real life.
One has to make the signal finite in some way. In mathematical terms, we place a window over the signal. This means that we replace s by s' where

$$s'_n = s_n w_n$$

and $w = (w_n)$ is the *window* which is only nonzero in a finite interval for example for $0 \leq n \leq N-1$. We then still have a Toeplitz system to solve, but the autocorrelation function r_k for s has to be replaced by the autocorrelation function r'_k for s'.
There is another way around the infinite sums. One can solve instead of the infinite system $Sa = -s$ only a finite subset of the equations. For example solve in least squares sense

$$\sum_{k=0}^{p} a_k s_{n-k} = 0, \quad a_0 = 1, \quad n = 0, 1, \ldots, N-1, \quad (N < \infty)$$

The normal equations then become

$$\sum_{k=1}^{p} r_{ik} a_k = -r_{i0}, \quad i = 1, \ldots, p$$

where

$$r_{ik} = \sum_{n=0}^{N-1} \bar{s}_{n-i} s_{n-k}.$$

Now the matrix of the normal equations is no longer Toeplitz and its solution will be somewhat more complex from an algorithmic point of view. It has still a special structure though as the product of two Toeplitz matrices. This method is sometimes called the *covariance method*, a term that is somewhat regrettable, because it is not related to the statistical term. Of course one can combine this with the windowing technique and at the same time do minimisation over only a finite set. For example, we can first assume that $s_n = 0$ for $n < 0$ which corresponds to a half-infinite window, and then apply the covariance method to the *prewindowed* signal.

4.2 The stochastic case

Similar situations can be obtained in the stochastic case. Here we have to minimise

$$\|e\|^2 = \mathcal{E}[|e_n|^2] = \mathcal{E}\left[\left|s_n + \sum_{k=1}^{p} a_k s_{n-k}\right|^2\right]$$

giving the normal equations

$$\sum_{k=1}^{p} a_k \mathcal{E}[s_{n-k}\bar{s}_{n-i}] = -\mathcal{E}[s_n \bar{s}_{n-i}], \quad 1 \leq i \leq p$$

with least squared error

$$E_p = \mathcal{E}[|s_n|^2] + \sum_{k=1}^{p} a_k \mathcal{E}[\bar{s}_n s_{n-k}].$$

If we have a stationary process, then $r_{i-k} = \mathcal{E}[s_{n-k}\bar{s}_{n-i}]$ does not depend on n and the normal equations reduce to a Toeplitz system, known as the set of *Yule-Walker equations.* In the non-stationary case however, we are stuck with the dependence upon n and we do not have a Toeplitz system. As in the deterministic case we can consider processes that are only non-stationary in a mild way. They are in a sense "close to stationary". The matrix of the normal equations will then be "close to Toeplitz". We shall elaborate on that in later sections.

5 The filter model

In the previous section we have constructed a prediction filter. When the signal s is given as input to the system with transfer function $A(z)$, then the output will be the prediction error, i.e.,

$$A(z)S(z) = E(z) \quad \text{or} \quad A(z)s_n = e_n, \quad n \in \mathbb{Z}$$

or explicitly

$$e_n = \sum_{k=0}^{p} a_k s_{n-k} = s_n + \sum_{k=1}^{p} a_k s_{n-k} = s_n - \hat{s}_n.$$

The signal s can be regenerated by inverting the filter. This is simple in mathematical terms:

$$S(z) = \frac{1}{A(z)} E(z).$$

Thus we have to build a filter with transfer function, $1/A(z)$, feed it with the error signal e and it will reproduce the signal s. But we have said before that we want to model the signal as the impulse response of a system. There are good practical reasons for this. If, for example, the signal s is a speech signal to be transmitted over a telephone line, we can do this by digitising the continuous signal and transmitting it.
When doing this, we are bounded by Shannon's sampling theorem. I.e., if we want to include a certain frequency band in the sampled signal, we have to sample it sufficiently often (more than the Nyquist frequency), which results in a heavy data transmission load.
However, when doing the previous analysis, the predictive filter $A(z)$, will absorb all the relevant information of the signal and the prediction error will only contain the noise component. The predictor coefficients are a much more compact way of representing a stationary time slice of the speech signal than the sampled signal itself. We obtain an important *data reduction*. By multiplexing, it is then possible to transmit many more messages over the same physical telephone line. If we would be forced to send the error signal as well, this advantage would be completely lost because if we have to send the error signal, we could as well send the original signal.
Thus our problem is to replace the input e by some other signal e', such that the output s' of the filter $1/A(z)$ gives an approximation of the signal s. In the frequency domain this is expressed as

$$S'(z) = \frac{1}{A(z)} E'(z).$$

The signal should be easy to generate at the receiving end of the transmission line so that it needs not be transmitted.
We choose e' to be Gu with u an impulse (deterministic case) or white noise (stochastic case) and G some scaling factor. Thus $E'(z) = G$, with G a positive constant. This is a reasonable choice since, if the predictor does a good job, then the error will mainly consist of some residual or noise that has a flat spectrum.
Suppose that $E'(z) = GU(z)$, then we can also express the modelled signal as

$$S'(z) = \frac{G}{A(z)} U(z) = \frac{G}{A(z)}.$$

The constant $G > 0$ is called the *gain* of the filter $G/A(z)$. The problem now is to find a reasonable choice for G. We shall define it by requiring that the energy of the simulated signal s' equals the energy of the true signal s. Thus $\|s\|^2 = \|s'\|^2$, i.e., $r_0 = r'_0$.
We already know that

$$r_0 = E_p - \sum_{k=1}^{p} r_{-k} a_k.$$

On the other hand, in the deterministic case,

$$s'_n = \begin{cases} 0 & n < 0 \\ G & n = 0 \\ -\sum_{k=1}^{p} a_k s'_{n-k} & n > 0, \end{cases}$$

and thus we also have (using a notation similar to section 4.1)

$$S'a = -s'$$

with

$$S' = \begin{bmatrix} s'_0 & & \\ s'_1 & s'_0 & \\ s'_2 & s'_1 & \ddots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad s' = \begin{bmatrix} s'_1 \\ s'_2 \\ s'_3 \\ \vdots \end{bmatrix}$$

so that

$$S'^H S'a = -S'^H s'$$

with

$$[S'^H S'] = [r'_{i-k}]_{i,k=1}^{p} \quad \text{and} \quad S'^H s' = [r'_i]_{i=1}^{p},$$

giving

$$r'_i = -\sum_{k=1}^{p} a_k r'_{i-k}, \quad 1 \le i \le p$$

while for $i = 0$ we have

$$\begin{aligned} r'_0 &= |s'_0|^2 + \sum_{n=1}^{\infty} |s'_n|^2 = G^2 + s'^H s' \\ &= G^2 - s'^H S'a = G^2 - \sum_{k=1}^{p} a_k r'_{-k}. \end{aligned}$$

It needs some extra manipulations, which basically involves the proof of the Gohberg-Semencul formula for the inversion of Toeplitz matrices (see later), to find that

$$G^2 = \|e\|^2 = E_p$$

implies that $r_0 = r'_0$ and then also $r_i = r'_i$, $-p \le i \le p$.
The same result can be obtained for stationary stochastic processes. Thus, as a conclusion, we can say that when we have computed the predictor $A(z)$ and the residual energy of the prediction error, $E_p = G^2$, then we can build a filter with transfer function $G/A(z)$, which, when given an impulse response (or white noise) as input, will produce an approximation of the signal. There are several ways in which we can express the criterion of approximation of s by s'. We have constructed the approximation s' for s such that the spectral function

$$R'(z) = R_{s'}(z) = \frac{G^2}{A(z)A_*(z)} = \sum_{-\infty}^{\infty} r'_k z^k = S'(z)S'_*(z)$$

matches the spectral function

$$R(z) = R_s(z) = \sum_{-\infty}^{\infty} r_k z^k = S(z)S_*(z)$$

such that $r_k = r'_k$, $k = -p, \ldots, p$. The approximant is the ratio of a trigonometric polynomial of degree zero over a trigonometric polynomial of degree p. Therefore, it is a $(0/p)$ Laurent-Padé approximant [9], or because of the symmetry and the relation between Chebyshev polynomials on an interval and powers of z on the unit circle, it has also been called a $(0/p)$ Chebyshev-Padé approximant.
Using standard techniques which map the unit circle onto the interval $[-1, 1]$, by setting $x = \cos\omega$, $z = e^{j\omega}$, we can reformulate the previous result into an approximant by truncated Fourier-Chebyshev series in the variable x.
Another way of writing the approximation is obtained by noting that

$$\frac{R(e^{i\omega})}{R'(e^{i\omega})} = \frac{|E(e^{i\omega})|^2}{G^2}$$

so that

$$E_p = G^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{i\omega})|^2 d\omega = \frac{G^2}{2\pi}\int_{-\pi}^{\pi} \frac{R(e^{i\omega})}{R'(e^{i\omega})} d\omega,$$

or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R(e^{i\omega})}{R'(e^{i\omega})} = 1.$$

Thus, on the average, the ratio of the given spectrum and the approximating spectrum equals 1. There is also an interpretation where the approximation is optimal in least squares sense, of the so called *spectral factor*. This relates the whole theory to the problem of spectral factorization, another fundamental problem in signal processing and linear systems. However, we shall not elaborate on that aspect in these notes.
Let's return to the model. A modelling filter of the form $G/A(z)$ is only a special case of a more general filter of the form

$$H(z) = G\frac{B(z)}{A(z)} = G\frac{1 + \sum_{l=1}^{q} b_l z^l}{1 + \sum_{k=1}^{p} a_k z^k}.$$

This general model is called a *pole-zero model* since it has both poles and zeros which in general do not correspond to the trivial choice zero or infinity. In stochastic terms, it is called an ARMA (*autoregressive-moving average*) model. When $a_k = 0$, $1 \leq k \leq p$, we get a *moving average* model and when $b_l = 0$, $1 \leq l \leq q$, we have an *autoregressive* model, which is the one we obtained earlier. The meaning of zeros and poles in the transfer function near the unit circle corresponds to dips and peaks in the power spectrum. A zero will more or less eliminate certain frequencies (they are called *transmission zeros*) and poles will enforce certain frequencies (*transmission modes*).
These are of course important data for analysing the resonance in e.g. a mechanical structure.
Such more general ARMA models could be obtained by approximating the power spectrum (or more precisely, its Fourier series) in a more general (q/p) Chebyshev-Padé sense. There is also another way, by fixing the transmission zeros, as we shall see later.

6 Toeplitz systems and orthogonal polynomials

For the autocorrelation method/stationary case, we can write the normal equations in the form:

$$\begin{bmatrix} r_0 & r_{-1} & \cdots & r_{-p} \\ r_1 & r_0 & \ddots & \\ \vdots & \ddots & \ddots & \\ r_p & & & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} E_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or equivalently (use $r_{-k} = \bar{r}_k$)

$$\begin{bmatrix} r_0 & r_{-1} & \dots & r_{-p} \\ r_1 & r_0 & \ddots & \\ \vdots & \ddots & \ddots & \\ r_p & & & r_0 \end{bmatrix} \begin{bmatrix} \bar{a}_p \\ \vdots \\ \bar{a}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_p \end{bmatrix}$$

where $E_p = r_0 + \sum_{k=1}^{p} r_{-k} a_k = r_0 + \sum_{k=1}^{p} r_k \bar{a}_k$.
These equations give the solution of the optimisation problem for the prediction error: $E_p = \min \|A(z)s\|^2$, which in the frequency domain can be formulated as

$$E_p = \min_{a_1,\dots,a_p} \left\| 1 + \sum_{k=1}^{p} a_k z^k \right\|^2 = \|A_p\|^2$$

where $A_p = \sum_{k=0}^{p} a_k z^k$, $a_0 = 1$, and the norm is taken in the L_2 space for the spectral measure[7] $d\mu = R(e^{i\omega})d\omega$.
If A_p represents the optimal solution, then

$$\begin{aligned} E_p = \|A_p(z)\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{p} a_k e^{i\omega k} \right|^2 R(e^{i\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{p} \bar{a}_{p-k} e^{i\omega k} \right|^2 R(e^{i\omega}) d\omega \\ &= \|\varphi_p(z)\|^2 \end{aligned}$$

where $\varphi_p(z) = z^p + \bar{a}_1 z + \dots + \bar{a}_p = z^p \overline{A_p(1/\bar{z})} = z^p A_{p*}(z)$. In the sequel, we shall denote this transformation as $\varphi_p(z) = A_p^*(z)$, while the substar conjugate represents a reflection in the unit circle: $f_*(z) = \overline{f(1/\bar{z})}$ (cfr. Schwarz reflection principle [11]).
It is easy to see that in this L_2 space

$$\left\langle z^j, \varphi_p \right\rangle = \sum_{k=0}^{p} \bar{a}_{p-k} r_{j-k}.$$

Thus the previous Toeplitz system expresses that

$$\begin{aligned} \left\langle z^j, \varphi_p \right\rangle &= 0 \quad \text{for} \quad j = 0, 1, \dots, p-1 \\ &= E_p \quad \text{for} \quad j = p. \end{aligned}$$

In other words φ_p is an orthogonal (monic) polynomial with minimal norm

$$\langle z^p, \varphi_p \rangle = \|\varphi_p\|^2 = E_p.$$

Thus we may conclude that our prediction problem is equivalent with the construction of an orthogonal polynomial in an appropriate Hilbert space. This is consistent with the result that in a Hilbert space, the monic polynomial of minimal norm is the orthogonal one.

[7] In the stochastic case this is an application of the Kolmogorov isomorphism, in the deterministic case, we can use the fact that the r_k are the Fourier coefficients of R.

7 Construction of the predictor

If the predictor can be found as the superstar conjugate of an orthogonal polynomial, this will immediately introduce the Gram-Schmidt procedure into the picture. This is indeed a standard procedure for transforming the basis $\{1, z, \ldots, z^p\}$ into an orthogonal one $\{\varphi_0, \varphi_1, \ldots, \varphi_p\}$. In general, this procedure involves $O(p^3)$ operations. However in this case, the Gram-matrix has a Toeplitz structure. This will result in a much faster algorithm which requires only $O(p^2)$ operations. It is usually called the Levinson algorithm after N. Levinson [42], although it has been reinvented many times in different contexts.
Szegő studied this same algorithm as the recurrence relation for polynomials on the unit circle [53]. One can also derive the result with purely linear algebra arguments.
When setting

$$[T_p]_{ij} = r_{i-j}; \quad i, j = 0, 1, \ldots, p, \quad r_{-i} = \bar{r}_i$$

and $A_p = [1 \quad a_1 \ldots a_p]^T$ such that

$$T_p A_p = [E_p \quad 0 \ldots 0]^T = E_p e_0$$

and $A_p^* = [\bar{a}_p \ldots \bar{a}_1 \quad 1]^T$, then as long as $E_{p-1} \neq 0$, it holds that

$$T_p A_p = E_p e_0, \quad p = 0, 1, 2, \ldots$$

if these A_k are computed by the algorithm

$$\begin{array}{l} E_0 = r_0 \\ \textbf{for } p = 1, 2, \ldots \\ \left| \begin{array}{l} \eta_{p-1} = [r_p \ldots r_1] A_{p-1} \\ \rho_p = \eta_{p-1} / E_{p-1} \\ A_p = [A_{p-1}^T 0]^T - \rho_p [0 A_{p-1}^{*T}]^T \\ E_p = (1 - |\rho_p|^2) E_{p-1} \end{array} \right. \end{array}$$

This is easily checked by straightforward computation. This algorithm works for hermitian positive definite Toeplitz matrices with a special right hand side. Many generalisations have been designed of this algorithm to solve efficiently general Toeplitz systems with arbitrary right hand side. The method assumes that the matrix is strongly non-singular. This means that all its leading sub-matrices are non-singular, which is reflected here by the condition that $E_{p-1} \neq 0$. However there are ways around this condition. When the leading sub-matrices are exactly singular, several techniques were discussed in [9]. See also [35]. A recent result is the so called look-ahead strategy, which will give numerically stable algorithms even when the leading sub-matrices are nearly singular [13].
In fact the present state of research goes far beyond Toeplitz matrices and it is now possible to find fast algorithms for all kinds of structured systems of equations, block Toeplitz systems, systems close to Toeplitz etc. The origin of this research is certainly to be traced in the efforts put into the solution of digital signal processing problems.
Let's return to the problem at hand. Because the E_k are prediction errors, which form a non-negative non-increasing sequence, we have

$$0 \leq E_p \leq E_{p-1} \leq \ldots \leq E_0 = r_0.$$

Thus we shall only come into computational difficulties when $E_p \approx 0$, i.e., when our approximation is good enough and then we shall only be happy to stop.
The previous property of the prediction errors and the relation $E_p = (1 - |\rho_p|^2) E_{p-1}$ implies that $|\rho_p| \leq 1$ and $|\rho_p| = 1$ means that $E_p = 0$. These coefficients ρ_k are crucial in our analysis. They were given many names, depending on the context where the algorithm is placed. The most wide

spread now is *reflection coefficients* (derived from scattering theory), *partial correlation coefficients* or PARCOR (from their stochastic interpretation), *Szegő coefficients* (from the theory of orthogonal polynomials) or *Schur coefficients* (because of the relation with the analysis by I. Schur of bounded analytic functions in the unit disk).
Let's have a look at the "mysterious" A_p^* that showed up in the algorithm. If we represent by $A_p(z)$ and $A_p^*(z) = \varphi_p(z)$ the polynomials whose coefficients are found in A_p and A_p^* respectively, then we already know that

$$e_n = A_p(z)s_n = s_n + a_1 s_{n-1} + \cdots + a_p s_{n-p}$$

with A_p the predictor of order p and e_n the prediction error.
Defining

$$f_n = A_p^*(z)s_n = s_{n-p} + \bar{a}_1 s_{n-p+1} + \cdots + \bar{a}_p s_n,$$

we can also call A_p^* a "predictor" of s_{n-p} in terms of its future. It is a predictor if we reverse the time axis.
Therefore A_p is usually referred to as the *forward* predictor and e_n as the *forward* prediction error while A_p^* is the *backward* predictor and f_n the *backward* prediction error. In a stochastic context, (f_n) and (e_n) are called the backward and forward *innovation processes*.
Note that $E_p = \|f\|^2$ where $\|f\|^2 = \sum_n |f_n|^2$ in the deterministic case and $\|f\|^2 = \|f_n\|^2 = \mathcal{E}[|f_n|^2]$ in the (stationary) stochastic case. The coefficients $\bar{a}_1, \ldots, \bar{a}_p$ are computed so as to minimise this norm. In the frequency domain, $\varphi_p(z) = A_p^*(z)$ is orthogonal to $\text{span}\{1, z, \ldots, z^{p-1}\}$ while $\varphi_p^* = A_p(z)$ is orthogonal to $\text{span}\{z, z^2, \ldots, z^p\}$. In the corresponding time domain terminology, we see that this means that the backward prediction error f is orthogonal to $\text{span}\{s, zs, \ldots, z^{p-1}s\}$ while the forward prediction error e is orthogonal to $\text{span}\{zs, z^2 s, \ldots, z^p s\}$. Since the order p of predictor will be changing, we shall now write $e(p) = (e_n(p))$ to mean the forward prediction error for a filter of order p, and similarly the notation $f(p) = (f_n(p))$ is used for the backward prediction error of order p. Using the orthogonality, we can write in the frequency domain

$$\begin{aligned} \rho_p = \frac{\eta_{p-1}}{E_{p-1}} &= \frac{\langle z^p, A_{p-1}\rangle}{\|A_{p-1}\|^2} \\ &= \frac{\left\langle z\varphi_{p-1}, \varphi_{p-1}^*\right\rangle}{\|\varphi_{p-1}\|\|\varphi_{p-1}^*\|} \end{aligned}$$

which in the time domain becomes

$$\rho_p = \frac{\langle zf(p-1), e(p-1)\rangle}{\|f(p-1)\|\|e(p-1)\|} = \frac{\left\langle zA_{p-1}^*(z)s, A_{p-1}(z)s\right\rangle}{\|f(p-1)\|\|e(p-1)\|}$$

where

$$A_{p-1}(z) = 1 + \cdots \quad \text{and} \quad zA_{p-1}^*(z) = z^p + \cdots$$

If the numerator were $\langle z^p s, s\rangle$, we would find r_p, the autocorrelation coefficient of s instead of ρ_p. In the numerator of ρ_p, however, s and $z^p s$ are replaced by their projecting vectors onto $\text{span}\{zs, z^2 s, \ldots, z^{p-1}s\}$. This explains the term *partial* correlation coefficient (PARCOR) for ρ_p.
Note that in terms of the predictors we can write

$$\begin{bmatrix} A_i^*(z) \\ A_i(z) \end{bmatrix} = \begin{bmatrix} 1 & -\bar{\rho}_i \\ -\rho_i & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{i-1}^*(z) \\ A_{i-1}(z) \end{bmatrix} = \vartheta_i(z) \begin{bmatrix} A_{i-1}^*(z) \\ A_{i-1}(z) \end{bmatrix}. \tag{7.1}$$

This recurrence relation is reflected in the circuit given in Figure 1. It is called a *lattice realization* of section i of the filter. Thus we can build the filter as a *cascade* of such sections and we do not need to know the prediction coefficients a_i explicitly. It is sufficient to know the reflection

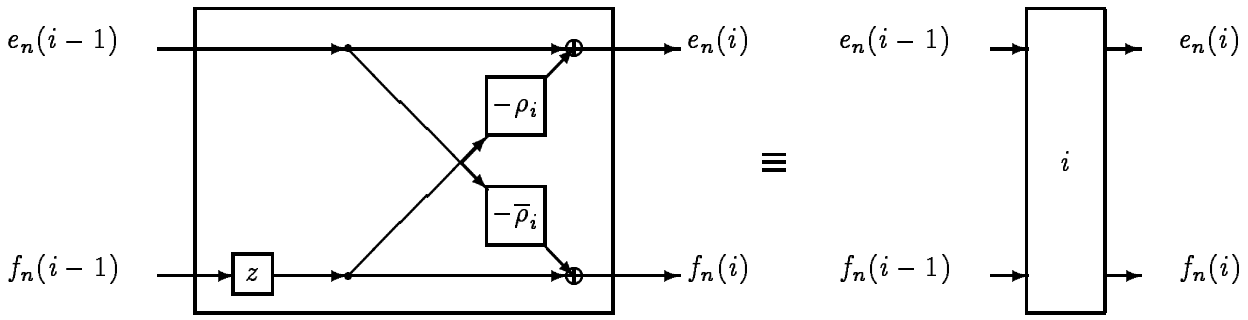


Figure 1: Lattice filter

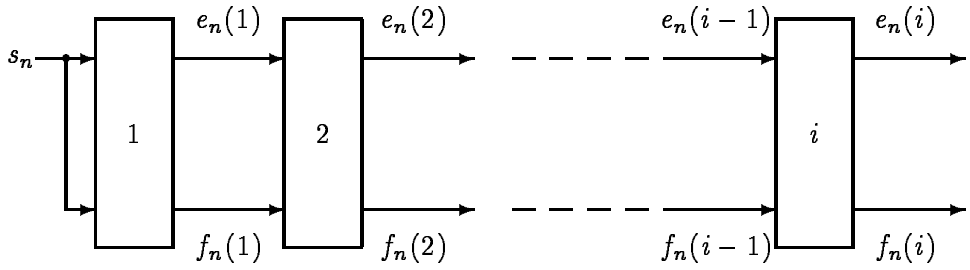


Figure 2: Whitening filter

coefficients ρ_i.
The *whitening* filter, which does the analysis of the signal is realized with such blocks as in Figure 2.
The *modelling* filter, which synthesises the signal has the realization of Figure 3, where each block is similar to block $\bar{i}$ which is the inverse of the circuit in Figure 1. It is represented in Figure 4. Note that the normalised predictors $A'_i(z)$, which generate

$$e'_n(i) = \frac{e_n(i)}{G_i} = \frac{A_i(z)}{G_i} s_n = A'_i(z) s_n; \quad G_i^2 = E_i$$

are given by the recurrence

$$\begin{bmatrix} A'^*_i(z) \\ A'_i(z) \end{bmatrix} = \frac{1}{\sqrt{1-|\rho_i|^2}} \begin{bmatrix} 1 & -\bar{\rho}_i \\ -\rho_i & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A'^*_{i-1}(z) \\ A'_{i-1}(z) \end{bmatrix} = \theta_i(z) \begin{bmatrix} A'^*_{i-1}(z) \\ A'_{i-1}(z) \end{bmatrix}.$$

8 Example: seismic signal processing

The mathematical quantities given in the previous section all have some physical meaning [25, 48]. We shall illustrate this with a seismic signal, but this could as well have been some other scattering problem. For example, instead of a shock wave travelling through the earth surface, we could think of the scattering of the money stream through the different socio-economic layers. Money travels down, for example by government investment, and there is an upward flow by tax-paying. The scattering medium could as well have been the vocal tract for the analysis of a speech signal. If we disregard the role of the nose, we can model the vocal tract as a sequence of coaxial cylindrical

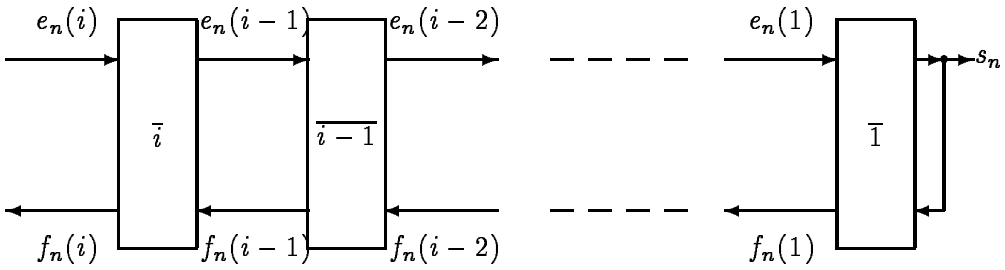


Figure 3: Modelling filter

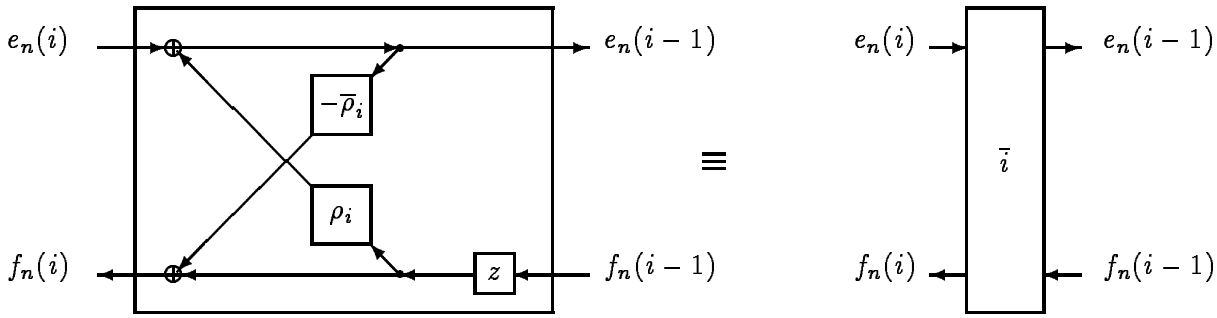


Figure 4: Inverse lattice filter

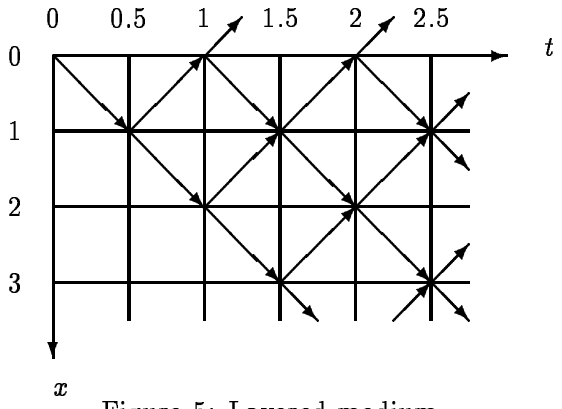


Figure 5: Layered medium

sections with different diameter. In all cases, the terminology may be different, but the principles remain the same.
So let us consider a layered medium like the earth surface. A wave is propagating through it. At every interface between two layers, the wave is partially reflected and partially transmitted, according to the relative physical properties of the two layers meeting there. The wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

has two particular solutions

$$\begin{aligned} u(x,t) &= \tilde{e}\left(t + \frac{x}{c}\right) \quad \text{(ascending wave)} \\ u(x,t) &= \tilde{f}\left(t - \frac{x}{c}\right) \quad \text{(descending wave)} \end{aligned}$$

We shall consider discrete versions of these two waves. Note that we normalised the time axis, so that it takes half a time unit for a wave to travel through each layer. Thus at integer times, a scattered wave will reach the surface producing the data we observe. See figure 5. To describe the whole process, we use the following notations.

$$\begin{aligned} \tilde{f}'_n(i) &= \text{descending wave, emerging from boundary } i \text{ at time } n \\ \tilde{f}_n(i) &= \text{descending wave, reaching boundary } i+1 \text{ at time } n \\ \tilde{e}_n(i) &= \text{ascending wave, emerging from boundary } i+1 \text{ at time } n \\ \tilde{e}'_n(i) &= \text{ascending wave, reaching boundary } i \text{ at time } n \end{aligned}$$

These are illustrated in Figure 6. We suppose also that the medium does not add or absorb energy (it is passive and lossless). Using again z as the delay operator: $z^{1/2}s_n(\cdot) = s_{n-\frac{1}{2}}(\cdot)$, we have in the homogeneous layer i:

$$\begin{bmatrix} \tilde{e}_n(i) \\ \tilde{f}_n(i) \end{bmatrix} = \begin{bmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{bmatrix} \begin{bmatrix} \tilde{e}'_n(i) \\ \tilde{f}'_n(i) \end{bmatrix}$$

which says that e.g., signal $\tilde{f}'_n(i)$ which emerges from boundary i at time n will reach boundary $i+1$ unaltered, half a time unit later, where according to our notation, it is called $\tilde{f}_n(i)$.
The interaction at the boundary, connecting the outgoing waves $O = (\tilde{f}'_n(i+1), \tilde{e}_n(i))$ with the incoming waves $I = (\tilde{f}_n(i), \tilde{e}'_n(i+1))$, is described by

$$\begin{bmatrix} \tilde{f}'_n(i+1) \\ \tilde{e}_n(i) \end{bmatrix} = \Sigma_{i+1} \begin{bmatrix} \tilde{f}_n(i) \\ \tilde{e}'_n(i+1) \end{bmatrix}$$

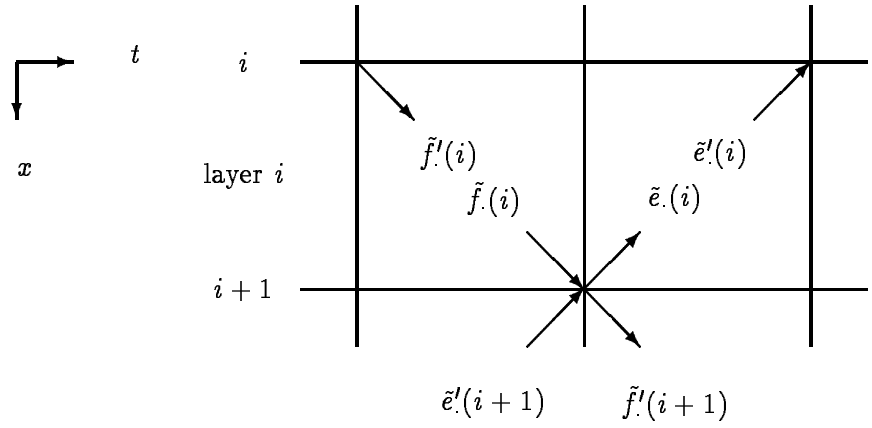


Figure 6: Notation of interacting waves

where Σ_{i+1} is called the *scattering matrix*. The conservation of energy requires that

$$\|O\|^2 = |\tilde{f}'_n(i+1)|^2 + |\tilde{e}_n(i)|^2 = |\tilde{f}_n(i)|^2 + |\tilde{e}'_n(i+1)|^2 = \|I\|^2.$$

This implies that Σ_{i+1} should be a unitary matrix: $\Sigma_{i+1}^H \Sigma_{i+1} = I$, or

$$\Sigma_{i+1} = \begin{bmatrix} c_{i+1} & -\bar{s}_{i+1} \\ s_{i+1} & c_{i+1} \end{bmatrix}, \quad |c_{i+1}|^2 + |s_{i+1}|^2 = 1, \quad c_{i+1} \in \mathbb{R}.$$

There are some physical reasons for choosing c_{i+1} real which we do not discuss here. The scattering matrix is interpreted as follows:

$$\begin{aligned} \tilde{f}'_n(i+1) &= \underbrace{c_{i+1}\tilde{f}_n(i)}_{\text{transmitted}} + \underbrace{(-\bar{s}_{i+1}\tilde{e}'_n(i+1))}_{\text{reflected}} \\ \tilde{e}_n(i) &= \underbrace{s_{i+1}\tilde{f}_n(i)}_{\text{reflected}} + \underbrace{c_{i+1}\tilde{e}'_n(i+1)}_{\text{transmitted}}. \end{aligned}$$

It is computationally simpler to work with the *chain scattering matrix* θ_{i+1} instead, which is defined by the relation connecting waves of layer i and layer $i+1$ as

$$\begin{bmatrix} \tilde{f}'_n(i+1) \\ \tilde{e}'_n(i+1) \end{bmatrix} = \theta_{i+1} \begin{bmatrix} \tilde{f}_n(i) \\ \tilde{e}_n(i) \end{bmatrix}.$$

A simple computation shows that

$$\theta_{i+1} = \frac{1}{c_{i+1}} \begin{bmatrix} 1 & -\bar{s}_{i+1} \\ -s_{i+1} & 1 \end{bmatrix} = \frac{1}{\sqrt{1-|\rho_{i+1}|^2}} \begin{bmatrix} 1 & -\bar{\rho}_{i+1} \\ -\rho_{i+1} & 1 \end{bmatrix}, \quad \rho_{i+1} = s_{i+1}.$$

The chain scattering matrix is not unitary in the classical sense, but it is still J-unitary, which means that it is unitary with respect to the indefinite metric

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.1}$$

which means that $\theta_{i+1}^H J \theta_{i+1} = J$. Like unitary matrices can be represented by sines and cosines, a J-unitary matrix is likewise represented by cosh and sinh. We now work in a hyperbolic space. To describe the complete effect of a layer we now join the two parts:

$$\begin{bmatrix} \tilde{f}'_n(i+1) \\ \tilde{e}'_n(i+1) \end{bmatrix} = \theta_{i+1} \begin{bmatrix} \tilde{f}_n(i) \\ \tilde{e}_n(i) \end{bmatrix} = \theta_{i+1} \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \begin{bmatrix} \tilde{f}'_n(i) \\ \tilde{e}'_n(i) \end{bmatrix}.$$

When we switch to the time-delayed signals[8]

$$e'_n(i) = z^{i/2}\tilde{e}'_n(i) \quad \text{and} \quad f'_n(i) = z^{i/2}\tilde{f}'_n(i),$$

this reduces to

$$\begin{bmatrix} f'_n(i+1) \\ e'_n(i+1) \end{bmatrix} = \frac{1}{\sqrt{1-|\rho_{i+1}|^2}} \begin{bmatrix} 1 & -\bar{\rho}_{i+1} \\ -\rho_{i+1} & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f'_n(i) \\ e'_n(i) \end{bmatrix} = \theta_{i+1}(z) \begin{bmatrix} f'_n(i) \\ e'_n(i) \end{bmatrix}.$$

If you recall from the end of section 7 that the normalized prediction errors were given by $e'_n(i) = A'_i(z)s_n$ and $f'_n(i) = A'^*_i(z)s_n$, then the previous recurrence will be readily recognised as the recurrence for these prediction errors and the ρ_i are the associated reflection coefficients. From our interpretation of the scattering matrix, it is now clear why $\rho_i = s_i$ is called a reflection coefficient. The previous analysis suggests in principle the following method to get a model of the earth surface (without digging) which is of crucial interest to oil companies. We apply an impulse as input at the surface (e.g., an explosion) and measure the response that is observed at the surface. When the Levinson algorithm computes the reflection coefficients of this signal, it actually computes the relative densities at successive depths. This may disclose the presence of a fluid (oil?) or a gas bubble.

This analysis gives the solution of an *inverse scattering problem.* Rather than computing the scattering effect of a known medium, the problem here is to find a model for the medium, given its effect. Such problems are usually ill conditioned which means that the resulting model is not well defined in terms of the available data.

Of course we have presented here only a very elementary picture of current geophysical signal processing techniques. More recent problems dealt with in this context can be found in [5].

In seismic processing, computation time is no problem, but when using the same technique for digital speech transmission, the computations have to be performed in real time. The computations should then be done by specially designed hardware. A CORDIC chip (coordinate rotation for digital computing) can perform such a hyperbolic rotation in a few clock cycles. The concept dates back to 1959 [54]. See also [51].

9 Orthogonal polynomials

We have seen before that the backward predictors $A^*_n = \varphi_n$ are monic polynomials of degree n orthogonal with respect to the spectral measure supported on the unit circle of the complex plane. For a recent survey see [17].

These polynomials were extensively studied by G. Szegő [53] and are therefore often called *Szegő polynomials.* Geronimus studied many of their properties [28]. These polynomials satisfy the recurrence (we use the orthonormal polynomials $\phi_k = \varphi_k/\|\varphi_k\|$. Recall $\|\varphi_k\|^2 = G_k^2 = E_k$).

$$\begin{bmatrix} \phi_n(z) \\ \phi^*_n(z) \end{bmatrix} = \theta_n(z) \begin{bmatrix} \phi_{n-1}(z) \\ \phi^*_{n-1}(z) \end{bmatrix}, \quad \phi_0 = \phi^*_0 = 1/\sqrt{r_0}$$

with

$$\theta_n(z) = \theta_n \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{1-|\varphi_n|^2}} \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$$

which is somewhat different from the three-term recurrence relation for polynomials on an interval. For the monic polynomials φ_n we have

$$\begin{bmatrix} \varphi_n(z) \\ \varphi^*_n(z) \end{bmatrix} = \vartheta_n(z) \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi^*_{n-1}(z) \end{bmatrix}, \quad \varphi_0 = \varphi^*_0 = 1$$

[8] The effect of waves at depth i will be observed at the surface exactly $i/2$ time units later.

with

$$\vartheta_n(z) = \vartheta_n \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

When the Levinson algorithm is translated in terms of these polynomials, one finds the recurrence algorithm as it was proposed by Szegő. The reflection coefficients are given by

$$\rho_i = \langle z\phi_{i-1}, \phi_{i-1}^* \rangle = \frac{\langle z\varphi_{i-1}, \varphi_{i-1}^* \rangle}{\|\varphi_{i-1}\|^2}$$

and that is why these are also called Szegő coefficients.
For these polynomials, one can show the Christoffel-Darboux type relations [53]

$$\begin{aligned} \sum_{k=0}^{n} \phi_k(x)\overline{\phi(y)} &= \frac{\phi_n^*(x)\overline{\phi_n^*(y)} - x\bar{y}\phi_n(x)\phi_n(y)}{1 - x\bar{y}} \\ &= \frac{\phi_{n+1}^*(x)\overline{\phi_{n+1}^*(y)} - \phi_{n+1}(x)\overline{\phi_{n+1}(y)}}{1 - x\bar{y}}. \end{aligned}$$

The left hand side is the reproducing kernel $k_n(x, y)$ for the space Π_n of polynomials of degree $\leq n$, i.e.,

$$\langle k_n(x, y), p_n(x) \rangle = p_n(y), \quad \forall p_n \in \Pi_n.$$

This kernel will show up again in later sections.
As an introduction of the Schur algorithm, we define the function $\Omega(z)$ for $z \in \mathbb{D} = \{z : |z| < 1\}$ in terms of the spectral measure as

$$\Omega(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(t, z) R(t) d\omega, \quad D(t, z) = \frac{t + z}{t - z}, \quad t = e^{i\omega}. \tag{9.1}$$

This function is a *positive real* function which means that in $\mathbb{D}$ it is analytic and its real part $\Re\Omega(z) \geq 0$. These functions were studied by Carathéodory and Fejér [12] and we shall therefore say that it belongs to the Carathéodory class $\mathcal{C} : \Omega \in \mathcal{C}$. Complex analysis shows that Ω has a non-tangential limit to the unit circle which satisfies

$$\Re\Omega(e^{i\omega}) = R(e^{i\omega}).$$

The formulas will simplify considerably if we assume that $R(t)$ is normalised such that $r_0 = 1$ hence we shall do this from now on.
This function $\Omega(z)$ can be approximated by the ratio of the orthonormal polynomials ϕ_n and the so called polynomials of the second kind ψ_n, which are defined by (we set $t = e^{i\omega}$)

$$\psi_n(z) = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} D(t, z)[\phi_n(t) - \phi_n(z)] R(t) d\omega & , \quad n \geq 1 \\ \sqrt{r_0} = 1 & , \quad n = 0 \end{cases}$$

These ψ_n are polynomials of degree n which form a solution of the recurrence

$$\begin{bmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{bmatrix} = \theta_n(z) \begin{bmatrix} \psi_{n-1}(z) \\ -\psi_{n-1}^*(z) \end{bmatrix}, \quad \psi_0 = \psi_0^* = 1$$

which means that they satisfy the same recurrence as the ϕ_n's, but with a sign change for the reflection coefficients. Using the definitions, it is not difficult to prove that [53, 28]

$$\phi_n(z)\Omega(z) + \psi_n(z) = \begin{cases} \Omega(z) + 1 & , \quad n = 0 \\ z^n g(z) & , \quad n > 0 \quad , \quad g \text{ analytic in } \mathbb{D} \end{cases}$$

$$\phi_n^*(z)\Omega(z) - \psi_n^*(z) = \begin{cases} \Omega(z) - 1 & , \quad n = 0 \\ z^{n+1}h(z) & , \quad n > 0 \end{cases} \quad , \quad h \text{ analytic in } \mathbb{D}.$$

The latter relations show that both the rational functions $-\psi_n(z)/\phi_n(z)$ and $\psi_n^*(z)/\phi_n^*(z)$ are approximants for $\Omega(z)$ since they are Padé type approximants in $z = 0$.

Joining the recurrences for ϕ_n and ψ_n shows that they are collected in the chain scattering matrix. We have indeed

$$\begin{bmatrix} \phi_n & \psi_n \\ \phi_n^* & -\psi_n^* \end{bmatrix} = \theta_n(z) \begin{bmatrix} \phi_{n-1} & \psi_{n-1} \\ \phi_{n-1}^* & -\psi_{n-1}^* \end{bmatrix}$$

with $\theta_n(z)$ as above and with initial conditions $\phi_0 = \psi_0 = 1$. Thus we get

$$\Theta_n(z) = \theta_n(z) \ldots \theta_1(z) = \frac{1}{2} \begin{bmatrix} \phi_n + \psi_n & \phi_n - \psi_n \\ \phi_n^* - \psi_n^* & \phi_n^* + \psi_n^* \end{bmatrix}.$$

By taking determinants, we get

$$\phi_n(z)\psi_n^* + \psi_n(z)\phi_n^*(z) = z^n.$$

10 J-inner matrices

The matrix

$$\theta_n(z) = \frac{1}{\sqrt{1 - |\rho_n|^2}} \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$$

is an example of a J-inner matrix, which means that it is a generalisation of the concept of an inner function in complex analysis [50].

In general, a J-inner matrix θ is a matrix, depending on the complex variable z whose entries are functions in the Nevanlinna class (that is the class of functions which are the ratio of two bounded analytic functions [23, 50]), which satisfies (J is the indefinite matrix (8.1) which was used before)

$$\theta_*(z)J\theta(z) \leq J \quad \text{for } |z| \leq 1$$

with equality holding for $|z| = 1$. The substar of a matrix function is defined by

$$\begin{bmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{bmatrix}_* = \begin{bmatrix} \theta_{11*}(z) & \theta_{21*}(z) \\ \theta_{12*}(z) & \theta_{22*}(z) \end{bmatrix} = [\theta(1/\bar{z})]^H,$$

where H means the complex conjugate transpose.

The inequality between the matrices is understood in the sense of positive (semi-) definiteness. Thus $J - \theta_* J\theta$ is positive semi-definite in the unit disk.

Being J-inner is a rather stringent condition for a matrix. J-inner matrices have rather interesting properties which were studied in the work by Dewilde and Dym [21, 24] and much earlier by Potapov [46]. It can be shown for example that the product of J-inner matrices is J-inner, and if θ is J-inner, then also θ_* is J-inner, both $(\theta_{11} + \theta_{12})_*^{-1}(\theta_{11} - \theta_{12})_*$ and $(\theta_{22} + \theta_{21})^{-1}(\theta_{22} - \theta_{21})$ are in the Carathéodory class and both $(\theta_{11} + \theta_{12})_*^{-1}$ and $(\theta_{22} + \theta_{21})_*^{-1}$ are in H_2, that is, they are square integrable on the unit circle and are analytic inside the unit disk.

If we apply these properties to the J-inner matrix $\Theta_n = \theta_n(z) \ldots \theta_1(z)$, we see that ψ_n^*/ϕ_n^* is a Carathéodory function and $1/\phi_n^*$ is analytic in the unit disk. This implies that the rational approximation ψ_n^*/ϕ_n^* to the Carathéodory function $\Omega(z)$ is itself in the Carathéodory class.

Also, because ϕ_n^* is a polynomial, $1/\phi_n^*$ being analytic means that its zeros are outside the unit disk, and hence the orthonormal polynomial ϕ_n has all its zeros inside the unit disk. The latter is

a well known property of the Szegő polynomials.
The filter interpretation of this result is that the filter

$$S(z) = \frac{1}{\phi_n^*} = \frac{1}{A_n'} = \frac{G}{A_n}$$

with A_n' the normalised predictor and A_n the predictor polynomial is stable. $S(z)$ is the transfer function of a stable system.

11 Stability test

We have seen that the AR filter is stable by construction. All $|\rho_i| < 1$ which implied that the poles of the transfer function $S(z) = 1/\phi_n^*$ are all outside the unit disk. However, we already mentioned that the inverse scattering problem which is solved here is ill conditioned. The smallest perturbation on the data, which are the correlation coefficients r_k, will cause a perturbation of the result, which are the predictor coefficients, even if there are no rounding errors, which of course can not be avoided. Especially, when the application requires that the analysis of the signal has to be done in real time, the computations are performed in fixed point arithmetic with a minimum of bits. Thus, if a filter is computed, it might be worth checking whether it is really stable. In other applications, it is also necessary to check the stability of the filter, if it is not guaranteed to be theoretically stable by construction. Again the Levinson algorithm will provide here a fast method to perform this stability test. Since the reflection coefficients are completely defined in terms of φ_n (or $A_n = \varphi_n^*$) and because stability is guaranteed iff all these reflection coefficients are bounded by 1 in modulus, one can do the Levinson algorithm backwards and compute the reflection coefficients from the predictor polynomial A_n (or from φ_n). If these are all in the unit disk, we have stability. The algorithm goes as follows

$$\begin{array}{l} \bar{\rho}_n = -\varphi_n(0) \\ \textbf{for } k = n, n-1, \ldots, 2 \\ \left| \begin{array}{l} \varphi_{k-1}(z) = \frac{\varphi_k(z) + \bar{\rho}_k \varphi_k^*(z)}{z(1-|\rho_k|^2)} \\ \bar{\rho}_{k-1} = -\varphi_{k-1}(0) \end{array} \right. \end{array}$$

This algorithm is called the Lehmer-Schur, Schur-Cohn or Jury [38, 37] test. It is the analog of the Routh test [36] to check the location of the zeros of a polynomial with respect to the imaginary (or real) axis. The latter test is needed to check stability for continuous time signals.
The Routh test needs only half the computations of the Jury test. A careful investigation of the origin of this difference led recently to the development of the Split Levinson algorithm [15, 16] which divides the number of floating point operations by 2.

12 Fast (inverse) Cholesky factorisation

The orthogonality properties of the predictor polynomials A_n or of their associates, the polynomials $\varphi_n = A_n^*$, leads to a triangular factorisation of the corresponding Gram matrix, which is the Toeplitz matrix we already met in section 7. This interpretation in terms of linear algebra can be seen as follows. We note the relation for $i = 0, 1, \ldots, n$

$$\left\langle z^k, \varphi_i \right\rangle = \begin{cases} 0 & , \quad k = 0, 1, \ldots, i-1 \\ E_i & , \quad k = i \\ \times & , \quad k = i+1, \ldots, n. \end{cases}$$

Using $\varphi_i(z) = f_{ii} z^i + f_{i-1,i} z^{i-1} + \cdots + f_{0i}$, $f_{ii} = 1$ and setting $f_{ji} = 0$, $j = i+1, \ldots, n$, we get that

$$\left\langle z^k, \varphi_i \right\rangle = \sum_{j=0}^{n} \left\langle z^k, z^j \right\rangle f_{ji} = \sum_{j=0}^{n} r_{k-j} f_{ji}.$$

This leads to the matrix relation

$$T_n U_n = \begin{bmatrix} r_0 & r_{-1} & \dots & r_{-n} \\ r_1 & r_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ r_n & & \dots & r_0 \end{bmatrix} U_n = \begin{bmatrix} E_0 & & & \\ \times & E_1 & & \\ \vdots & \vdots & \ddots & \\ \times & \times & \dots & E_n \end{bmatrix} = F_n$$

where $U_n = [f_{ji}]$ has the coefficients of φ_i as its i-th column. Thus U_n is unit upper triangular, while F_n is lower triangular. The Toeplitz matrix with the correlation coefficients is denoted by T_n. Note that the order of the matrices is $(n+1)\times(n+1)$, we number the rows and columns from 0 to n.
If we now take the conjugate transpose of this relation, we get, because $T_n = T_n^H$

$$U_n^H T_n = F_n^H.$$

This is an upper triangular matrix. Hence, since then also $F_n^H U_n$ is upper triangular and

$$U_n^H T_n U_n = D_n$$

is hermitian, the latter will be both upper and lower triangular, thus D_n is a diagonal matrix with entries $E_0, E_1, \dots, E_n$.
Writing this relation in the form

$$T_n^{-1} = U_n D_n^{-1} U_n^H$$

we recognise here a unit upper triangular $\times$ diagonal $\times$ unit lower triangular factorisation of T_n^{-1}. All the elements of this factorisation are computed by the Levinson algorithm. Since T_n, and thus also T_n^{-1} is positive definite, we almost have here a Cholesky factorisation of T_n^{-1}. We say almost because a Cholesky factorisation would be a factorisation of the form unit lower triangular $\times$ diagonal $\times$ unit upper triangular. By the Toeplitz structure of T_n, this can easily be derived from the previous one as follows. Denote by $\tilde{I}$ the matrix with ones on the anti-diagonal and zeros everywhere else. Multiplying with $\tilde{I}$ from the left reverses the order of the rows, multiplication from the right reverses the order of the columns. Hence, a Toeplitz matrix satisfies $\tilde{I} T_n \tilde{I} = T_n^T$. Because $\tilde{I}^2 = I$, we also get $\tilde{I} T_n^{-1} \tilde{I} = T_n^{-T}$. This leads to

$$T_n^{-1} = (\tilde{I}\bar{U}_n\tilde{I})(\tilde{I}D_n^{-1}\tilde{I})(\tilde{I}\bar{U}_n^T\tilde{I})$$

which is a genuine Cholesky factorisation of T_n^{-1}. Note that the i-th columns of $\tilde{I}\bar{U}_n\tilde{I}$ contains the coefficients of $A_{n-i} = \varphi_{n-i}^*$.
Thus we can say that the Levinson algorithm is a fast method to compute the Cholesky factorisation of T_n^{-1}, given the entries of the Toeplitz matrix T_n. That is why it is called an *inverse* Cholesky factorisation.
An obvious question that pops to the mind now is whether it is possible to get a Cholesky factorisation of T_n itself in a fast way. After all, this is the most natural computation to be expected if T_n itself is given. It is easy to identify these factors because

$$T_n = U_n^{-H} D_n U_n^{-1} = L_n D_n L_n^H$$

with $L_n = U_n^{-H}$, unit lower triangular. The Levinson algorithm does compute U_n in a fast way, but inverting U_n would require $O(n^2)$ extra operations and we can look for a way to compute L_n directly from T_n. From their defining relations, it is easily seen that $U_n^H = D_n F_n^{-1}$ or $F_n = U_n^{-H} D_n = L_n D_n$. Now there is indeed a way of computing F_n directly from T_n which we shall show now with pure

linear algebra arguments.
Consider the vectors (V_{1k} is the k-th column of U_n)

$$\begin{aligned} V_{1k} &= [\bar{a}_k \dots \bar{a}_0 \; 0 \dots 0]^T = [A_k^{*T} \; 0 \dots 0]^T \in \mathbb{C}^{(n+1)\times 1} \\ V_{2k} &= [a_0 \dots a_k \; 0 \dots 0]^T = [A_k^T \; 0 \dots 0]^T \in \mathbb{C}^{(n+1)\times 1} \end{aligned}$$

where $\varphi_k^*(z) = \sum_{j=0}^k a_j z^j = A_k(z)$, is the predictor polynomial, then $T_n V_{1k}$ will be equal to the k-th column of F_n.
We now make use of the relation (7.1), rewritten as

$$[A_{k+1}^*(z) \;\; A_{k+1}(z)] = [A_k^*(z) \;\; A_k(z)]\vartheta_{k+1}(z)^T$$

to see that[9] $T_n[V_{1k}|V_{2k}]\vartheta_{k+1}(z)^T = T_n[V_{1,k+1}|V_{2,k+1}]$. This shows that the $(k+1)$-st column of F_n can be computed from the k-th column and some auxiliary column, viz. $T_n V_{2k}$. Now note that

$$T_n[V_{1k}|V_{2k}] \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{\eta}_k & E_k \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ E_k & \eta_k \\ \times & \times \\ \vdots & \vdots \\ \times & \times \end{bmatrix} = [Q_{1k}|Q_{2k}]$$

with

$$\begin{aligned} \eta_k &= [r_{k+1} \dots r_1][a_0 \dots a_k]^T \\ \bar{\eta}_k &= [r_{-1} \dots r_{-(k+1)}][\bar{a}_k \dots \bar{a}_0]^T \end{aligned}$$

(thus $\bar{\eta}_k$ is the complex conjugate of η_k) and

$$\begin{aligned} E_k = \|\varphi_k\|^2 &= [r_k \dots r_0][\bar{a}_k \dots \bar{a}_0]^T \\ &= [r_0 \dots r_{-k}][a_0 \dots a_k]^T. \end{aligned}$$

These quantities η_k and E_k are precisely the same as the ones that showed up in the Levinson algorithm and the reflection coefficient is $\rho_{k+1} = \eta_k / E_k$. It is now easy to verify that

$$\begin{aligned} T_n[Q_{1k}|Q_{2k}] \begin{bmatrix} 1 & -\rho_{k+1} \\ -\bar{\rho}_{k+1} & 1 \end{bmatrix} &= T_n[Q_{1k}|Q_{2k}]\vartheta_{k+1}^T \\ &= \begin{bmatrix} 0 & E_{k+1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ E_{k+1} & 0 \\ \times & \eta_{k+1} \\ \times & \times \\ \vdots & \vdots \\ \times & \times \end{bmatrix} = [P_{1,k+1}|P_{2,k+1}], \quad (k+1 \text{ zeros}). \end{aligned}$$

[9] z represents here a down shift operation, which is compatible with its meaning of delay operator or multiplication with the variable z.

The first column of the right hand side is the $(k+1)$-st column of F_n. This gives in principle the algorithm we are looking for since we have here again some η_{k+1} and some E_{k+1} which will define $\rho_{k+2} = \eta_{k+1}/E_{k+1}$ and another step can be performed.
As you may see from these calculations, it is not necessary to keep all the zeros in the algorithm to compute the reflection coefficients, i.e. we may keep from the vectors $[P_{1k}|P_{2k}]$ only the elements[10]

$$[\Delta_{1k}|\Delta_{2k}] = [P_{1k}|P_{2k}]z^{-k} = \begin{bmatrix} E_k & 0 \\ \times & \eta_k \\ \vdots & \vdots \\ \times & \times \end{bmatrix}$$

It is then not difficult to see that these reduced vectors satisfy the recurrence relation

$$\begin{aligned} [\Delta_{1,k+1}|\Delta_{2,k+1}] &= [\Delta_{1k}|\Delta_{2k}] \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \vartheta_{k+1}^T \cdot z^{-1} \\ &= [\Delta_{1k}|\Delta_{2k}] \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \vartheta_{k+1}^T \end{aligned}$$

where the reflection coefficients $\rho_{k+1} = \eta_k/E_k$ can be found from the elements E_k and η_k which are the top elements of the reduced vectors.
The initial conditions for this recurrence are

$$[\Delta_{10}|\Delta_{20}] = \begin{bmatrix} r_0 & 0 \\ r_1 & r_1 \\ \vdots & \vdots \\ r_n & r_n \end{bmatrix}.$$

An operation count on this algorithm will reveal that it requires $O(n^2)$ flops. So what we got here is a fast Cholesky factorisation algorithm for Toeplitz matrices. This algorithm is not new. It is a finite version of the algorithm that was already given by I. Schur in 1917 [52]. It was formulated in terms of complex functions and may not immediately be recognised as a Cholesky factorisation algorithm. We shall explain this in the next section. To prepare this, we let n tend to infinity and set

$$D_k(z) = [D_{1k}(z)|D_{2k}(z)] = z^{k-1}[1 \; z \; z^2 \ldots][\Delta_{1k}|\Delta_{2k}] \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that the recurrence becomes

$$D_k(z) = [D_{1k}(z)|D_{2k}(z)] = [D_{1,k-1}(z)|D_{2,k-1}(z)] \begin{bmatrix} 1 & -\rho_k \\ -\bar{\rho}_k & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

The initial conditions are

$$D_0(z) = \left[\; \sum_{k=0}^{\infty} r_k z^k \; \middle| \; \sum_{k=1}^{\infty} r_k z^{k-1} \; \right] = \left[\; \tfrac{1}{2}(\Omega(z) + r_0) \; \middle| \; \tfrac{1}{2z}(\Omega(z) - r_0) \; \right]. \qquad (12.1)$$

We explain in the next section how these can lead to Schur functions.

[10] By the operation z^{-1} we cut away the top element of a vector.

13 The Schur algorithm

With the notation introduced in the previous section, we define

$$\Gamma_k(z) = \frac{D_{2k}(z)}{D_{1k}(z)}.$$

We get for $\Gamma_k(z)$ the recurrence

$$\begin{aligned}\Gamma_k(z) &= \frac{D_{2k}(z)}{D_{1k}(z)} = \frac{-D_{1,k-1}(z)\rho_k + D_{2,k-1}(z)}{z(D_{1,k-1}(z) - \bar{\rho}_k D_{2,k-1}(z))} \\ &= \frac{1}{z}\frac{\Gamma_{k-1}(z) - \rho_k}{1 - \bar{\rho}_k\Gamma_{k-1}(z)}\end{aligned}$$

with initial condition

$$\Gamma_0(z) = \left(\sum_{k=1}^{\infty} r_k z^{k-1}\right) / \left(\sum_{k=0}^{\infty} r_k z^k\right)$$

The reflection coefficient is given by $\rho_k = \Gamma_{k-1}(0)$.
We first show that Γ_0 belongs to the class of Schur functions. This class $\mathcal{S}$ is the class of bounded analytic functions, i.e.

$$\mathcal{S} = \{\Gamma : |\Gamma(z)| \leq 1 \text{ in } |z| \leq 1 \text{ and analytic in } |z| < 1\}.$$

We already know that $\Omega(z)$ of (9.1) is a Carathéodory function. Now, using the expansion of the kernel $D(t,z) = (t+z)/(t-z)$ for $|z| < 1$, we get

$$\begin{aligned}\Omega(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)R(t)d\omega, \quad t = e^{i\omega} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + 2\sum_{k=1}^{\infty} t^{-k}z^k\right)R(t)d\omega \\ &= r_0 + 2\sum_{k=1}^{\infty} r_k z^k.\end{aligned}$$

Its real part is

$$\begin{aligned}\Re\Omega(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re D(t,z)R(t)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K(t,z)R(t)d\omega\end{aligned}$$

where

$$K(t,z) = \Re D(t,z) = \frac{1-|z|^2}{|t-z|^2}$$

is the Poisson kernel. In $|z| < 1$, $K(t,z)$ is positive and also $R(t)$ is positive, so that the real part of $\Omega(z)$ is indeed positive. Now we recall the initial conditions (12.1), to see that

$$\Gamma_0(z) = \frac{1}{z}\frac{\Omega(z) - r_0}{\Omega(z) + r_0}.$$

But

$$\left|\frac{\Omega - r_0}{\Omega + r_0}\right|^2 = \frac{|\Omega|^2 - 2r_0\Re\Omega + r_0^2}{|\Omega|^2 + 2r_0\Re\Omega + r_0^2} \leq 1$$

since $r_0 \Re\Omega \geq 0$. Thus $(\Omega - r_0)/(\Omega + r_0) \in \mathcal{S}$. Because it is zero for $z = 0$, we can divide out z, and it will still be in $\mathcal{S}$, as follows from the maximum modulus principle. Thus $\Gamma_0 \in \mathcal{S}$. All the subsequent Γ_k are also members of $\mathcal{S}$ because the Moebius transformation

$$z \to \frac{z - \alpha}{1 - \bar{\alpha} z}, \quad |\alpha| < 1$$

is a bijection of the unit disk, hence it will transform a Schur function into a Schur function. Moreover each $\alpha = \rho_k$ is chosen such that this Schur function is zero for $z = 0$, hence Γ_k will be in $\mathcal{S}$ if Γ_{k-1} is in $\mathcal{S}$ and $\rho_k = \Gamma_{k-1}(0) \in \mathbb{D}$. I. Schur developed this continued fraction-like algorithm to check whether a given function Γ_0 is in $\mathcal{S}$. He proved the theorem.

Theorem 13.1 *Let Γ_0 be a given function. Define recursively*

$$\Gamma_k(z) = \frac{1}{z} \frac{\Gamma_{k-1}(z) - \rho_k}{1 - \bar{\rho}_k \Gamma_{k-1}} \quad , \quad \rho_k = \Gamma_{k-1}(0).$$

Then $\Gamma_0 \in \mathcal{S}$ iff $|\rho_k| < 1$ for $k = 1, 2, \ldots$ or $|\rho_k| < 1$ for $k = 1, 2, \ldots, n-1$ while $|\rho_n| = 1$ and $\rho_{n+k} = 0$, $k = 1, 2, \ldots$. In the latter case, Γ_0 is a Blaschke product of degree n:

$$\Gamma_0 = c \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \quad , \quad |\alpha_i| < 1, \quad i = 1, \ldots, n, \quad |c| = 1.$$

Thus the algorithm of the previous section for the D_{ik}, $i = 1, 2$ is a homogeneous form of the Schur algorithm, i.e. it is decoupled into a recurrence for the numerator and denominator separately.
The Schur functions appearing here have a scattering interpretation. Recall from section 8 the relation

$$\begin{bmatrix} f'_n(i+1) \\ e'_n(i+1) \end{bmatrix} = \theta_{i+1}(z) \begin{bmatrix} f'_n(i) \\ e'_n(i) \end{bmatrix}.$$

$e'_n(i)$ is the wave reflected by the medium below depth i and $f'_n(i)$ the wave transmitted to that part of the medium. The n refers to the moment in time. Taking the z-transforms leads to

$$\begin{bmatrix} F'_{i+1}(z) \\ E'_{i+1}(z) \end{bmatrix} = \theta_{i+1}(z) \begin{bmatrix} F'_i(z) \\ E'_i(z) \end{bmatrix}.$$

The ratio of the output over the input $\Gamma'_i(z) = E'_i(z)/F'_i(z)$ is the transfer function of the scattering medium below depth i. It is called the *scattering function* of the medium. You can easily check that it follows from these relations that

$$\Gamma'_{i+1}(z) = \frac{\Gamma'_i(z) - \rho_{i+1} z}{z - \bar{\rho}_{i+1} \Gamma'_i(z)}.$$

We recover the previous Schur recursion by setting or setting $\Gamma'_i(z) = z\Gamma_i(z)$, so that

$$\Gamma_{i+1}(z) = \frac{1}{z} \frac{\Gamma_i(z) - \rho_{i+1}}{1 - \bar{\rho}_{i+1} \Gamma_i(z)}.$$

Thus $\Gamma_0(z) = z^{-1}\Gamma'_0(z)$ as defined above is a scattering function for the medium with input wave F_0 and output wave E_0 given by

$$F_0(z) = \sum_{k=0}^{\infty} r_k z^k, \quad E_0(z) = \sum_{k=1}^{\infty} r_k z^{k-1}$$

while $\Gamma_i(z)$ is the scattering function for the medium below layer i.
To read more about this, one may consult the book [30] which is devoted to the work of Schur in this context. See also [20] and for more recent developments and a much higher level of abstraction, consult [4].

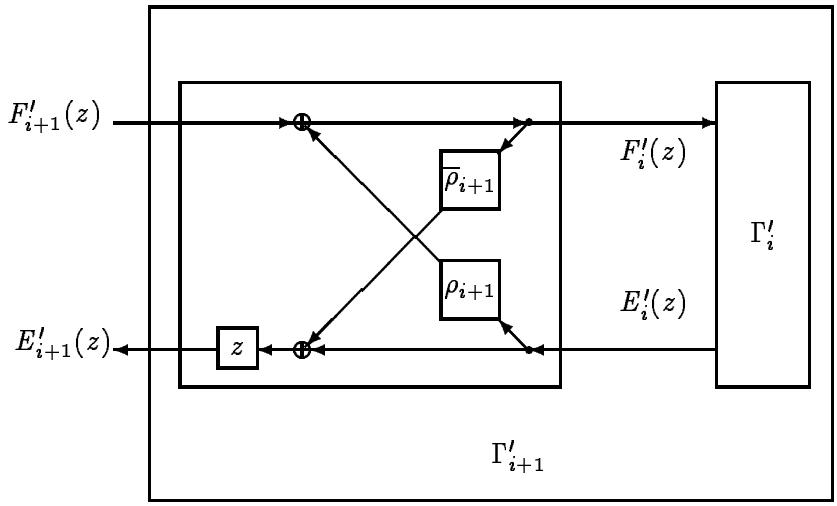


Figure 7: Scattering function and Schur recurrence

14 The Gohberg-Semencul formula

We have seen that the Toeplitz matrix $T_n = [r_{i-j}]$ is the Gram matrix for the subspace of polynomials of degree at most n. When we arrange the coefficients of the orthogonal polynomials as the columns of the unit upper triangular matrix U_n, then the orthogonality is expressed by

$$U_n^H T_n U_n = D_n$$

with $D_n = \mathrm{diag}(E_0, \ldots, E_n)$ and $E_k = \|\varphi_k\|^2$. The reproducing kernel for this subspace is then given by

$$k_n(z,w) = [1 \; z \; \ldots \; z^n] T_n^{-1} [1 \; w \; \ldots \; w^n]^H.$$

Now

$$\begin{aligned} k_n(z,w) &= [1 \; z \; \ldots \; z^n] U_n D_n^{-1} U_n^H [1 \; w \; \ldots \; w^n]^H \\ &= \sum_{k=0}^{n} \frac{\varphi_k(z)\overline{\varphi_k(w)}}{\|\varphi_k\|^2}. \end{aligned}$$

Using the Christoffel-Darboux relation, we have

$$(1 - z\bar{w}) k_n(z,w) = [\overline{\varphi_n^*(w)}\varphi_n^*(z) - z\bar{w}\varphi_n(z)\overline{\varphi_n(w)}]/E_n$$

which in matrix notation corresponds to

$$[1 \; z \; \ldots \; z^{n+1}] \left\{ \left[\begin{array}{c|c} T_n^{-1} & 0 \\ \hline 0 & 0 \end{array} \right] - \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & T_n^{-1} \end{array} \right] \right\} [1 \; w \; \ldots \; w^{n+1}]^H =$$

$$= \frac{1}{E_n} [1 \; z \; \ldots \; z^{n+1}] \left\{ \begin{bmatrix} A_n \\ 0 \end{bmatrix} [A_n^H \; 0] - \begin{bmatrix} 0 \\ A_n^* \end{bmatrix} [0 \; A_n^{*H}] \right\} [1 \; w \; \ldots \; w^{n+1}]^H$$

where $A_n^*(z) = \varphi_n(z) = [1 \; z \; \ldots \; z^n] A_n^*$ and $\varphi_n^*(z) = A_n(z) = [1 \; \ldots \; z^n] A_n$.
This leads to the recurrence relation

$$[T_n^{-1}]_{i+1,j+1} = [T_n^{-1}]_{i,j} + \frac{1}{E_n}[a_{i,n}\bar{a}_{j,n} - a_{n-i,n}\bar{a}_{n-j,n}]$$

where $A_n = [a_{0,n} \; \ldots \; a_{n,n}]^T$, $a_{0,n} = 1$ and $A_n^* = [\bar{a}_{n,n} \; \ldots \; \bar{a}_{0,n}]^T$.
These can again be recollected in the formula

$$T_n^{-1} = \frac{1}{E_n}\{L(A_n)L(A_n)^T - L(zA_n^*)L(zA_n^*)^T\}$$

where

$$L(A_n) = \begin{bmatrix} a_{0,n} & & & \\ a_{1,n} & a_{0,n} & & \\ \vdots & \vdots & \ddots & \\ a_{n,n} & a_{n-1,n} & \dots & a_{0,n} \end{bmatrix}$$

and

$$L(zA_n^*) = \begin{bmatrix} 0 & & & \\ a_{n,n} & 0 & & \\ \vdots & & \ddots & \\ a_{1,n} & a_{2,n} & \dots & 0 \end{bmatrix}$$

are lower triangular Toeplitz matrices.
Similarly, the second form of the Christoffel-Darboux relation leads to the formula

$$T_n^{-1} = \frac{1}{E_{n+1}}\{L(A'_{n+1})L(A'_{n+1})^T - L(A'^*_{n+1})L(A'^*_{n+1})^T\}$$

where

$$L(A'_{n+1}) = \begin{bmatrix} a_{0,n+1} & & & \\ a_{1,n+1} & a_{0,n+1} & & \\ \vdots & \vdots & \ddots & \\ a_{n,n+1} & a_{n-1,n+1} & \dots & a_{0,n+1} \end{bmatrix}$$

and

$$L(A'^*_{n+1}) = \begin{bmatrix} a_{n+1,n+1} & & & \\ a_{n,n+1} & a_{n+1,n+1} & & \\ \vdots & \vdots & \ddots & \\ a_{1,n+1} & a_{2,n+1} & \dots & a_{n+1,n+1} \end{bmatrix}$$

These two formulas for the inverse of a Toeplitz matrix were derived by Gohberg and Semencul [32]. These are two considerations to be connected to these formulas. First we remark that although the inverse of a Toeplitz matrix is not Toeplitz, it is characterised by the same number of parameters as the Toeplitz matrix itself. In fact this observation holds for non-hermitian Toeplitz matrices too. We haven't done this, but using biorthogonal polynomials for a non-hermitian Toeplitz matrix, the Christoffel-Darboux relations have direct non-symmetric generalisations and a non-symmetric formula of Gohberg-Semencul type also holds for T_n^{-1}. Second, it is easy to do computations with the inverse of a Toeplitz matrix because the matrix-vector multiplication of a lower triangular Toeplitz matrix with a vector can be done by the fast Fourier transform in $O(n \log_2 n)$ operations [3]. Thus the multiplication $T_n^{-1}X$ can be performed by FFT's when the inverse is known in the Gohberg-Semencul form. In fact this observation led to the recognition of the matrix T_n^{-1} as a matrix with low displacement rank to which we shall come back later.

15 The general pole-zero model

The construction of a general rational ARMA model is a nonlinear problem, whereas the AR model we have studied before could be generated by solving a linear least squares problem, because the numerator of the model is just a constant. However, if we have some idea about the location of the transmission zeros $1/\bar{\alpha}_i$, $|\alpha_i| < 1$, then we can fix these and propose a filter of the form

$$H(z) = G\frac{(1-\bar{\alpha}_1 z)\dots(1-\bar{\alpha}_n z)}{A_n(z)} = \frac{G}{D_n(z)}$$

where

$$D_n(z) = \frac{A_n(z)}{\pi_n(z)}, \quad \pi_n(z) = \prod_{k=1}^{n}(1 - \bar{\alpha}_k z), \quad A_n(z) \in \Pi_n$$

Now define

$$\varphi_n(z) = \frac{A_n^*(z)}{\pi_n(z)},$$

then both φ_n and D_n are in the space

$$\mathcal{L}_n = \left\{ \frac{p_n}{\pi_n} : p_n \in \Pi_n, \pi_n(z) = \prod_{k=1}^{n}(1 - \bar{\alpha}_k z) \right\}$$

with prescribed poles $1/\bar{\alpha}_k$, all outside the unit disk.
Defining the Blaschke factors

$$\zeta_0 = 1; \quad \zeta_k = \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad k = 1, 2, \ldots, n$$

and the finite Blaschke products

$$B_0 = 1; \quad B_k = B_{k-1}\zeta_k, \quad k = 1, 2, \ldots, n$$

it is clear that

$$\mathcal{L}_n = \text{span}\{\zeta_k, k = 0, \ldots, n\} = \text{span}\{B_k; k = 0, \ldots, n\}.$$

This space is a natural generalisation of the polynomials. If all $\alpha_k = 0$, then $\mathcal{L}_n = \Pi_n$. The substar conjugate is defined as before, the superstar conjugate for an element $D_n \in \mathcal{L}_n$ is defined by

$$D_n^*(z) = B_n(z)D_{n*}(z) = B_n(z)\overline{D_n(1/\bar{z})}.$$

The prediction problem we considered before for a stochastic process (s_n) was (for the stationary case) to predict e.g., s_0 from its past, i.e., from $\text{span}\{s_k : k = -1, -2, \ldots\}$. Thus to make $\|s_0 - \sum_{k=1}^{\infty} h_k s_{-k}\|$ minimal with respect to the $h_1, h_2, \ldots$.
Under the Kolmogorov isomorphism, this is equivalent with finding the minimum of $\|H\|$ in the weighted Hilbert space L_2 where

$$H(z) = 1 + \sum_{k=1}^{\infty} h_k z^k.$$

Thus we have to solve the optimisation problem

$$\min\{\|H\| : H \in H_2, H(0) = 1\}.$$

In the AR model, this problem was solved by successively solving the problem over the subspaces of polynomials $\Pi_n \subset H_2$, $n = 0, 1, 2, \ldots$. However, it is perfectly possible to consider other finite dimensional subspaces of growing dimension. For example by solving the problem for $H \in \mathcal{L}_n$, $n = 0, 1, \ldots$. Thus we consider

$$\min\{\|f\|, f \in \mathcal{L}_n, f(0) = 1\}.$$

This optimisation problem is also classical in complex interpolation theory. To formulate its solution efficiently, we introduce the concept of a *reproducing kernel*, which we already met in section 9.
Let $\mathcal{H}$ be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$, then we call $k(z, w)$ a reproducing kernel for $\mathcal{H}$ if for all w

$$\forall f \in \mathcal{H} : \langle k(z, w), f(z) \rangle = f(w).$$

It is not difficult to prove that if φ_k, $k = 0, 1, 2, \ldots$ is an orthogonal basis for $\mathcal{H}$, then

$$k(z, w) = \sum_k \frac{\varphi_k(z)\overline{\varphi_k(w)}}{\|\varphi_k\|^2}$$

is a reproducing kernel. Note that $k(w, w) > 0$.
We now have the following result

Theorem 15.1 *For given w, the optimisation problem*

$$\min\{\|f\|^2, f \in \mathcal{H}, f(w) = 1\}$$

in the separable Hilbert space $\mathcal{H}$, with reproducing kernel k is solved by setting

$$f(z) = g_w(z) = \frac{k(z, w)}{k(w, w)}$$

and the minimum is $1/k(w, w)$.

Proof. It is clear that $g_w(w) = 1$ and $\|g_w\|^2 = 1/k(w, w)$. Now let $f = \sum_k a_k\varphi_k$ be an arbitrary function in $\mathcal{H}$ such that $f(w) = 1$. Without loss of generality we assume that the basis $\{\varphi_k\}$ is orthonormal, then we have to find

$$\min\{\|f\|^2 = \sum_k |a_k|^2 : \sum_k a_k\varphi_k(w) = 1\}$$

By Cauchy-Schwarz inequality,

$$1 = \left|\sum_k a_k\varphi_k(w)\right|^2 \leq \left(\sum_k |a_k|^2\right)\left(\sum_k |\varphi_k(w)|^2\right)$$

which implies

$$\frac{1}{k(w, w)} = \frac{1}{\sum_k |\varphi_k(w)|^2} \leq \sum_k |a_k|^2.$$

The lower bound is reached by choosing $a_k = \overline{\varphi_k(w)}/k(w, w)$. This proves the theorem. □

This theorem gives a framework in which the AR and ARMA models can both be solved in a similar way. In the case of an ARMA model with fixed zeros, one has to minimise over $\mathcal{L}_n$ and by finding a recurrence relation for an orthogonal basis $\varphi_0, \varphi_1, \ldots, \varphi_n$, one can reconstruct the reproducing kernel $k_n(z, w)$ and then get the approximate solution of the prediction problem as $Gk_n(0, 0)/k_n(z, 0)$. However, since $\|f\| = \|f^*\|$, we prefer as predictor

$$\frac{Gk_n(0, 0)}{k_n^*(z, 0)}$$

instead because, like in the polynomial case, this one has all its zeros outside the unit disk and thus gives a stable filter. Because we have chosen the zeros outside the unit disk, this filter will also be a minimal phase filter. This means that of all filters with the same magnitude characteristic, the range of the phase angle will be minimal.
One can show that the orthogonal basis φ_k, obtained by orthogonalisation of the basis $B_0, B_1, \ldots, B_n$ satisfies a recurrence which immediately generalises the Szegö recurrence.
It is

$$\begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix}$$

with

$$\rho_n = \frac{\langle \varphi_{n-1}, \zeta_n \varphi_{n-1} \rangle}{\|\varphi_{n-1}\|^2}$$

and the prediction errors is $G^2 = E_n = \|\varphi_n\|^2$.
The Schur algorithm generalises to

$$\Gamma_{k+1}(z) = \frac{1}{\zeta_{k+1}(z)} \frac{\Gamma_k(z) - \rho_{k+1}}{1 - \bar{\rho}_{k+1}\Gamma_k(z)} \quad , \quad \rho_{k+1} = \Gamma_k(\alpha_{k+1}).$$

This is in fact the algorithm studied by Nevanlinna and Pick [44, 45] and it follows that all the Γ_k are in the Schur class if all $|\rho_k| < 1$.
The theory of the orthogonal rational functions φ_k has been studied recently as a generalisation of the Szegő polynomials [10].

16 Relation with classical problems in complex analysis

The connection of the prediction problem with the work of Schur, relates the analysis also with many other classical interpolation problems in complex analysis. The problem raised by Carathéodory and Schur is the following: Given a polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$$

what are the necessary and sufficient conditions for the existence of an extension

$$\Gamma(z) = a_0 + \cdots + a_{n-1} z^{n-1} + a_n z^n + \cdots \in H_\infty$$

such that $\|\Gamma\|_\infty \leq 1$, i.e., Γ belongs to the Schur class $\Gamma \in \mathcal{S}$, and give a description of all its solutions.
The answer is given by the Schur-algorithm.
One sets $\Gamma_0 = P(z)$ and computes for $i = 1, 2, \ldots, n$

$$\Gamma_k(z) = \frac{1}{z} \frac{\Gamma_{k-1}(z) - \rho_k}{1 - \bar{\rho}_k \Gamma_{k-1}(z)} \quad , \quad \rho_k = \Gamma_{k-1}(0).$$

All solutions are then given by

$$\Gamma(z) = \zeta_0^{-1}(\zeta_1^{-1}(\ldots \zeta_{n-1}^{-1}(z\Gamma_n(z))\ldots))$$

for arbitrary $\Gamma_n \in H_\infty$, $\|\Gamma_n\| \leq 1$ where ζ_k represents the Moebius transformation $\zeta_k(z) = \frac{z-\rho_k}{1-\bar{\rho}_k z}$, hence $\zeta_k^{-1}(z) = \frac{z+\rho_k}{1+\bar{\rho}_k z}$ under the conditions of the theorem of Schur for the parameters ρ_k. I.e., one should have all $|\rho_i| < 1$, $1 \leq i \leq n$ or $|\rho_j| < 1$, $1 \leq j < i$, $|\rho_i| = 1$ and $\rho_j = 0$, $i < j \leq n$.
Otherwise there is no solution.
Using a Cayley transform which maps the unit disk onto the right half plane, this can be reformulated in terms of positive real functions, i.e. in terms of functions from class $\mathcal{C}$.
The previous problem was generalised by Pick and Nevanlinna who stated the following problem: Given $\{c_i\}_{i=1}^n$ a set of complex numbers and $\{\alpha_i\}_{i=1}^n$ a set of complex numbers in $\mathbb{D}$, find necessary and sufficient conditions for the existence of a function $\Gamma \in \mathcal{S}$ such that

$$\Gamma(\alpha_i) = c_i, \quad 1 \leq i \leq n.$$

The solution to this problem is similar to the Schur result, except that the Schur algorithm has to be replaced by the Pick-Nevanlinna algorithm, i.e.

$$\Gamma_k(z) = \frac{1}{\zeta_k(z)} \frac{\Gamma_{k-1}(z) - \rho_k}{1 - \bar{\rho}_k \Gamma_{k-1}(z)}, \quad \rho_k = \Gamma_{k-1}(\alpha_k), \quad \zeta_k(z) = \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}.$$

17 Order and time updates

So far, we have restricted our attention to order updates, i.e., we have found efficient algorithms to construct a predictor of order $p+1$, given the predictor of order p and an extra autocorrelation coefficient r_{p+1}. In applications like speech processing, where the computations have to be performed in real time as the speech signal becomes available, the predictor is changing with time, and thus also the reflection coefficients will depend on time.
We shall not give all the technical details of the algorithms, but to give an idea, we consider the following general framework for the prediction problem, which, after what we have seen before, will not be difficult to understand.
Consider a complex Hilbert space of vectors y_i, with inner product whose Gram matrix has elements $r_{ij} = \langle y_i, y_j \rangle$. You can think of y_0 as a given signal $s = (s_n)$ (deterministic or stochastic) and $y_i = z^i s$ is the time delayed signal. For the inner product you can use $\langle y_i, y_j \rangle = y_i^H y_j$ in the deterministic case (the vectors y_i are here considered as columns of samples). In the stochastic case you can think of the inner product as $\langle y_i, y_j \rangle = \mathcal{E}[y_i^H y_j]$. However, we now allow each s_n to be a row vector of size p. Hence the "inner product" is $p \times p$ matrix valued. Of course, using the Kolmogorov isomorphism, you can also think of the Hilbert space as a space of complex functions.
The orthogonal projection $\hat{y}_{t|U}$ of y_t on the subspace spanned by $U = \{y_{t-n}, \ldots, y_{t-1}\}$ is the predictor of y_t from that subspace. The forward prediction error is

$$e_{n,t} = y_t - \hat{y}_{t|U}$$

As you can expect, we shall also need a backward predictor $\hat{y}_{t-n-1|U}$ with backward prediction error

$$f_{n,t-1} = y_{t-n-1} - \hat{y}_{t-n-1|U}.$$

For an order update, we have to find

$$\hat{y}_{t|\{U, y_{t-n-1}\}} = \hat{y}_{t|U} + \hat{y}_{t|f_{n,t-1}} = \hat{y}_{t|U} + \bar{f}_{n,t-1} \langle \bar{f}_{n,t-1}, y_t \rangle$$

where we use a bar to denote normalised vectors $\bar{y} = y \|y\|^{-1}$. So we get

$$\begin{aligned} e_{n+1,t} &= e_{n,t} - \bar{f}_{n,t-1} \langle \bar{f}_{n,t-1}, y_t \rangle \\ &= (\bar{e}_{n,t} - \bar{f}_{n,t-1} \langle \bar{f}_{n,t-1}, \bar{e}_{n,t} \rangle) \|e_{n,t}\| \\ &= (\bar{e}_{n,t} - \bar{f}_{n,t-1} \rho_{n+1,t}) \|e_{n,t}\|, \quad \rho_{n+1,t} = \langle \bar{f}_{n,t-1}, \bar{e}_{n,t} \rangle . \end{aligned}$$

We used $f_{n,t-1} \perp U$. The $\rho_{n+1,t}$ will now be the reflection coefficients. The least squared error can be found as

$$\|e_{n+1,t}\|^2 = \|e_{n,t}\|^H (I - \rho_{n+1,t}^H \rho_{n+1,t}) \|e_{n,t}\|.$$

Similar computations for the backward error lead to

$$\begin{aligned} f_{n+1,t} &= (\bar{f}_{n,t-1} - \bar{e}_{n,t} \rho_{n+1,t}^H) \|f_{n,t-1}\| \\ \|f_{n+1,t}\|^2 &= \|f_{n,t-1}\|^H (I - \rho_{n+1,t} \rho_{n+1,t}^H) \|f_{n,t-1}\|. \end{aligned}$$

The initial conditions are $e_{0,t} = y_t = f_{0,t}$.
It is immediately clear by the Cauchy-Schwarz inequality that

$$\|\rho_{n+1,t}\| \leq \|\bar{f}_{n,t-1}\| \|\bar{e}_{n,t}\| = I$$

Thus the reflection coefficients are still bounded by I.
For a time update, we shall have to project onto a subspace of the same dimension, but shifted in time. I.e., we have to go from

$$U = \{y_{t-n-1}, \ldots, y_{t-1}\} \to \{y_{t-n}, \ldots, y_t\}.$$

Thus we have to add a vector y_t and delete a vector y_{t-n-1}. Thus we have to perform two steps: an update and a down-date. We do not elaborate this further. More details can be found in [39].

18 Low displacement rank matrices

We have seen that the computations involved in the autocorrelation method, or, for stochastic signals, in the stationary case led to computations with matrices that are Toeplitz. In other cases, the matrices were not Toeplitz anymore, but they still had a special structure. The matrices of order $(n+1)$ can be represented by less than $(n+1)^2$ parameters. The inverse of a Toeplitz matrix is an example. It can be represented by the same number of parameters as the Toeplitz matrix itself. Therefore it has in a sense the same complexity as a Toeplitz matrix and thus computations with such matrices can be performed more efficiently than with a general matrix. Typically, the inversion of a general matrix requires $O(n^3)$ operations. A Toeplitz matrix needs only $O(n^2)$ operations. There is a whole range of matrices whose complexity is somewhere in between these two. Their inversion requires $O(\alpha n^2)$ operations with $1 \leq \alpha \leq n$. It are certainly the applications in digital filtering that led to the discovery of the concept of matrices with low displacement rank in linear algebra [41, 40]. We introduce this concept for matrices which are close-to-Toeplitz, but this is only a special case of a whole range of matrices with a special structure like matrices of Hankel, Bezout, Vandermonde or Loewner type [35].
Define Z to be the down shift matrix

$$Z = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}.$$

Next define a *displacement operator* ∂ which acts on a matrix T as

$$\partial T = T - ZTZ^T.$$

For example, when T is a Toeplitz matrix, then

$$\partial T = \begin{bmatrix} r_0 & r_{-1} & \dots & r_{-n} \\ r_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ r_n & 0 & \dots & 0 \end{bmatrix}.$$

In this case rank $\partial T = \alpha \leq 2$. For a general matrix T of order $(n+1)$ is rank $\partial T \leq n+1$. We shall call α the *displacement rank* of the matrix T.
Let T be positive definite and hermitian, then ∂T is still hermitian (but not positive definite). Thus the eigenvalues of ∂T are real. Suppose p of them are positive and q of them are negative, then the displacement rank of T is $\alpha = p + q$, and we can write

$$\partial T = \Delta \Sigma \Delta^H, \quad \Sigma = I_p \oplus -I_q, \quad \Delta \in \mathbb{C}^{(n+1)\times\alpha}.$$

The couple $\{\Sigma, \Delta\}$ is called a *generator* (for T). The choice of Σ is unique when it is of the correct order $\alpha = p + q$. The choice of Δ however is not unique, because for any Σ-unitary matrix U, (i.e. $U^H \Sigma U = U \Sigma U^H = \Sigma$), we have

$$\partial T = (\Delta U)\Sigma(\Delta U)^H = \Delta \Sigma \Delta^H.$$

Thus $\{\Sigma, \Delta U\}$ will be a generator if $\{\Sigma, \Delta\}$ is one.
We call the couple (p, q) the inertia of ∂T : $\operatorname{In} \partial T = (p, q)$. For example, if T is Toeplitz, one can easily check that

$$\partial T = \Delta \Sigma \Delta^H, \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Delta^H = \frac{1}{\sqrt{r_0}} \begin{bmatrix} r_0 & r_{-1} & \dots & r_{-n} \\ 0 & -r_{-1} & \dots & -r_{-n} \end{bmatrix}$$

(recall T is hermitian, thus $r_{-k} = \bar{r}_k$).
We have seen that the inverse of a Toeplitz matrix was of the same complexity as the Toeplitz matrix itself. This generalises readily to

Theorem 18.1 *Let $\tilde{I}$ denote the reversal operator (1's on the anti-diagonal), then* $\mathrm{In}(\partial T) = \mathrm{In}[\partial(\tilde{I}T^{-1}\tilde{I})]$ *when T is regular and hermitian. This implies that the displacement rank of T and $\tilde{I}T^{-1}\tilde{I}$ are the same.*

Proof. Note that

$$\begin{aligned}\begin{bmatrix} T & Z \\ Z^T & T^{-1} \end{bmatrix} &= \begin{bmatrix} I & 0 \\ Z^TT^{-1} & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-1} - Z^TT^{-1}Z \end{bmatrix} \begin{bmatrix} I & T^{-1}Z \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & ZT \\ 0 & I \end{bmatrix} \begin{bmatrix} T - ZTZ^T & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ TZ^T & I \end{bmatrix}.\end{aligned}$$

By Sylvester's law, the inertia are preserved under congruence transformations. Hence

$$\mathrm{In}(T) + \mathrm{In}(T^{-1} - Z^TT^{-1}Z) = \mathrm{In}(T^{-1}) + \mathrm{In}(T - ZTZ^T).$$

But $T = TT^{-1}T$, so that $\mathrm{In}(T) = \mathrm{In}(T^{-1})$. Thus

$$\begin{aligned}\mathrm{In}(\partial T) &= \mathrm{In}(T - ZTZ^T) \\ &= \mathrm{In}(T^{-1} - Z^TT^{-1}Z) \\ &= \mathrm{In}[\partial(\tilde{I}T^{-1}\tilde{I})].\end{aligned}$$

For the last step we used $\tilde{I}Z^T\tilde{I} = Z$. The theorem is proved. □

Another interesting property is the following generalisation of the Gohberg-Semencul formula.

Theorem 18.2 *The equation in T, given the x_i, $i = 1, \ldots, p+q$*

$$\partial T = \sum_{k=1}^{p} x_i x_i^H - \sum_{k=p+1}^{p+q} x_k x_k^H$$

has a unique solution

$$T = \sum_{k=1}^{p} L(x_k)L(x_k)^H - \sum_{k=p+1}^{p+q} L(x_k)L(x_k)^H$$

where $L(x)$ is a lower triangular Toeplitz matrix whose first column is x.

Proof. Suppose $p = 1$, $q = 0$, then (note $Z^n = 0$ when Z is the down shift operator of order n)

$$\begin{aligned}\partial T &= xx^H \text{ has a solution} \\ T &= \sum_{k=0}^{\infty} Z^i x x^H (Z^i)^H = L(x)L(x)^H.\end{aligned}$$

This solution is unique because, if there were two solutions T_1 and T_2, then $\partial T_1 = \partial T_2$ implies by induction

$$T_1 - T_2 = Z(T_1 - T_2)Z^T = Z^2(T_1 - T_2)(Z^2)^T = \cdots = Z^n(T_1 - T_2)(Z^n)^T = 0$$

and thus $T_1 = T_2$. Since ∂ is a linear operator, the result also holds for general p and q. □

These matrices of low displacement rank play an important role in linear prediction. We already know that in the autocorrelation method, the Gram matrix is Toeplitz since it has elements

$$\langle y_i, y_j \rangle = y_i^H y_j = r_{i-j}.$$

We now illustrate how in the case of a prewindowed signal with the prediction error minimised over a finite interval leads to a matrix with low displacement rank. For the prewindowed case, we suppose that for the signal s we have $s_k = 0$ for $k < 0$. Therefore

$$T_n = Y_n^H Y_n \quad \text{with} \quad Y_n = \begin{bmatrix} s_0 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ s_{n-1} & \dots & s_0 & 0 \\ s_n & \dots & s_1 & s_0 \\ \vdots & & \vdots & \vdots \end{bmatrix}.$$

If we minimise over the finite time interval $[0, N]$, then Y_n is replaced by

$$Y_n = \begin{bmatrix} s_0 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ s_{n-1} & \dots & s_0 & 0 \\ s_n & \dots & s_1 & s_0 \\ \vdots & & \vdots & \vdots \\ s_N & \dots & s_{N-n+1} & s_{N-n} \end{bmatrix} = \begin{bmatrix} y_0 & y_1 & \dots & y_n \end{bmatrix}$$

and the normal equations become

$$T_n [a_n \dots a_1 \ 1]^H = [0 \dots 0 \ E_n]^H, \quad T_n = Y_n^H Y_n.$$

This T_n is not Toeplitz (unless $s_{N-n+1} = \dots = s_N = 0$). It has however a low displacement rank, which we can see as follows. Using $y_t = [0 \dots 0 \ s_0 \dots s_{N-t}]^T$ of length $N+1$, it is not difficult to see that

$$\begin{aligned} \partial T_n &= \begin{bmatrix} r_{00} & r_{01} & \dots & r_{0n} \\ r_{10} & & & \\ \vdots & & R_1 & \\ r_{n0} & & & \end{bmatrix} \\ R_1 &= [y_t^H y_r - y_{t-1}^H y_{r-1}]_{t,r=1,\dots,n} = [-\bar{s}_{N-t+1} s_{N-r+1}]_{t,r=1,\dots,n} \\ &= -\tilde{x}\tilde{x}^H, \quad \tilde{x} = [s_N \ s_{N-1} \ \dots \ s_{N-n+1}]^H. \end{aligned}$$

Thus

$$\partial T_n = \begin{bmatrix} r'_{00} & 0 & 0 \\ r'_{10} & -r'_{10} & \bar{s}_N \\ \vdots & \vdots & \vdots \\ r'_{n0} & -r'_{n0} & \bar{s}_{N-n+1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} r'_{00} & r'_{01} & \dots & r'_{0n} \\ 0 & -r'_{01} & \dots & -r'_{0n} \\ 0 & s_N & \dots & s_{N-n+1} \end{bmatrix}, \quad (r'_{ij} = r_{ij}/\sqrt{r_{00}})$$

which shows that the displacement rank is 3 and the generator is known.
In the covariance method,

$$Y_n = \begin{bmatrix} s_0 & s_{-1} & \dots & s_{-n} \\ \vdots & \vdots & & \vdots \\ s_N & s_{N-1} & \dots & s_{N-n} \end{bmatrix} \quad \text{and} \quad T_n = Y_n^H Y_n.$$

Just like in the prewindowed case, we find that ∂T_n has the same form, but now with

$$\begin{aligned} R_1 &= [y_t^H y_r - y_{t-1}^H y_{r-1}]_{t,r=1,\ldots,n} = [\bar{s}_{-t}s_{-r} - \bar{s}_{N-t+1}s_{N-r+1}]_{t,r=1,\ldots,n} \\ &= \tilde{x}_2\tilde{x}_2^H - \tilde{x}_4\tilde{x}_4^H, \quad \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} s_{-1} & s_{-2} & \ldots & s_{-n} \\ s_N & s_{N-1} & \ldots & s_{N-n+1} \end{bmatrix}^H. \end{aligned}$$

Thus the displacement rank is 4 and

$$\begin{aligned} \partial T_n &= \left[\, x_1 \,\middle|\, x_2 \,\middle|\, x_3 \,\middle|\, x_4 \,\right] \Sigma \left[\, x_1 \,\middle|\, x_2 \,\middle|\, x_3 \,\middle|\, x_4 \,\right]^H \\ x_1 &= [r_{00} \;\; r_{10} \ldots r_{n0}]^T/\sqrt{r_{00}}, \quad x_2 = [0 \;\; s_{-1} \ldots s_{-n}]^H \\ x_3 &= -[0 \;\; r_{10} \ldots r_{n0}]^T/\sqrt{r_{00}}, \quad x_4 = [0 \;\; s_N \ldots s_{N-n+1}]^H \\ \Sigma &= \mathrm{diag}(1, 1, -1, -1). \end{aligned}$$

19 Generating functions

To give generalisations of the Schur and Levinson algorithm, when the signal is α-stationary (which means that its correlation matrix has displacement rank α), we introduce the corresponding functions in the frequency domain. When $T = [r_{ij}]$ is an infinite hermitian (correlation) matrix, then we call

$$R(z, w) = \sum_{i,j} z^i \bar{w}^j r_{ij} = [1 \;\; z \;\; \ldots] T [1 \;\; w \ldots]^H$$

a *generating function* for (the autocorrelation function) T. When the matrix T is positive definite, we say that the generating function R is a *spectral function.*
The effect of the displacement operator on T translates into

$$\partial R(z, w) = R(z, w) - z\bar{w}R(z, w) = (1 - z\bar{w})R(z, w)$$

If $\partial T = \Delta\Sigma\Delta^H$ where the rows of Δ are denoted as $g_0, g_1, \ldots \in \mathbb{C}^{1\times\alpha}$, then

$$(1 - z\bar{w})R(z, w) = \Delta(z)\Sigma\Delta^H(w), \quad \text{with } \Delta(z) = [1 \;\; z \;\; \ldots]\Delta = \sum_{k=0}^{\infty} g_k z^k.$$

Note that for a Toeplitz matrix we get

$$\begin{aligned} \Delta(z) &= \left[\, r_0 + \textstyle\sum_{k=1}^{\infty} r_k z^k \,\middle|\, -\sum_{k=1}^{\infty} r_k z^k \,\right] / \sqrt{r_0} \\ &= \frac{1}{2}\left[\, \Omega_0 + r_0 \,\middle|\, -(\Omega_0 - r_0) \,\right] / \sqrt{r_0}, \quad \Omega_0(z) = r_0 + 2\sum_{k=1}^{\infty} r_k z^k \in \mathcal{C}. \end{aligned}$$

Recall that for a matrix T subdivided as

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \text{ regular},$$

we call $S = D - CA^{-1}B$ the Schur complement of A (in T).
Now, since

$$T = \begin{bmatrix} r_{00} \\ r_{10} \\ \vdots \end{bmatrix} r_{00}^{-1} [r_{00} \;\; r_{01} \ldots] + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & R_1 & \\ 0 & & \end{bmatrix}$$

we see that R_1 is the Schur complement of r_{00}.
The following lemma is due to Schur.

Lemma 19.1 *The matrix $T = [r_{ij}]$ is positive definite iff $r_{00} > 0$ and its Schur complement R_1 is positive definite.*

This lemma can be rephrased as: $R(z,w)$ is a spectral function if $R(0,0) = r_{00} > 0$ and the generating function of the Schur complement R_1 is also a spectral function. It is easy to see that the generating function of the Schur complement R_1 is given by

$$R_1(z,w) = \frac{R(z,w) - R(z,0)R(0,0)^{-1}R(0,w)}{z\bar{w}}.$$

Note that in the generator $\{\Sigma, \Delta\}$, $\Delta(z)$ represents a Schur function in the following sense: Set

$$\Delta(z) = [\underbrace{\Delta_1(z)}_{1} \quad \underbrace{\Delta_2(z)}_{\alpha-1}],$$

then, denoting $\Sigma = 1 \oplus -S$,

$$\begin{aligned} 0 < R(z,z) &= \Delta(z)\Sigma\Delta^H(z) \\ &= [\Delta_1(z) \;\; \Delta_2(z)] \begin{bmatrix} 1 & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} \Delta_1^H(z) \\ \Delta_2^H(z) \end{bmatrix} \\ &= |\Delta_1(z)|^2 - \Delta_2(z)S\Delta_2^H(z). \end{aligned}$$

Hence, because

$$\Gamma(z) = \frac{\Delta_2(z)S\Delta_2^H(z)}{|\Delta_1(z)|^2} < 1$$

$\Gamma(z)$ is a Schur function if $\Delta_1(z) \neq 0$ in $|z| < 1$.
We now have all the elements to formulate a generalisation of the Schur algorithm in the next section.

20 Generalised Schur algorithm

Let $R(z,w)$ be a spectral function for a (positive definite) autocorrelation matrix T with displacement rank α. Thus

$$\partial T = \Delta\Sigma\Delta^H \quad \text{or} \quad (1 - z\bar{w})R(z,w) = \Delta(z)\Sigma\Delta^H(w)$$

with $\{\Sigma, \Delta\}$ a generator for T. Exploiting the freedom we have in the choice of Δ, we can always assume that the first row g_0 of Δ is of the form $\Delta(0) = g_0 = \sigma_0[1\ 0 \ldots 0]$ with $\sigma_0 = \sqrt{r_{00}} > 0$ we call this the *normalised form* of the generator.
If the columns of such a normalised Δ are given by $x_1, \ldots, x_\alpha$, and if we define

$$\tilde{R}_1 = R_0 - x_1 x_1^H = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & R_1 & \\ 0 & & \end{bmatrix}, \quad \text{with } R_0 = T$$

then $\tilde{R}_1$ has generator $\{\Sigma, \tilde{\Delta}_1\}$ where $\tilde{\Delta}_1 = [Zx_1 | x_2 | \ldots | x_\alpha]$.
This is easily seen as follows ($\Delta_0 = \Delta$)

$$\begin{aligned} \partial\tilde{R}_1 = \tilde{R}_1 - Z\tilde{R}_1 Z^T &= [R_0 - x_1 x_1^H] - Z[R_0 - x_1 x_1^H]Z^T \\ &= R_0 - ZR_0Z^T - x_1 x_1^H + Zx_1 x_1^H Z^T \\ &= \Delta_0\Sigma\Delta_0^H - x_1 x_1^H + Zx_1 x_1^H Z^T \\ &= \tilde{\Delta}_1\Sigma\tilde{\Delta}_1^H \end{aligned}$$

Note that in general $\tilde{\Delta}_1$ is not normalised.
This however shows how we can find from a normalised generator $\{\Sigma, \Delta_0\}$ for $T = R_0$, a generator $\{\Sigma, \Delta_1'\}$ for the Schur complement R_1 of r_{00}. It is simply given in the frequency domain by

$$\Delta_1'(z) = \Delta_0(z) \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix}^{-1}.$$

Before we can apply this recursively, we should be able to replace this with a Σ-unitary transformation θ_1 by

$$\Delta_1(z) = \Delta_1'(z)\theta_1^{-1}$$

such that $\Delta_1(z)$ is normalised. This transformation θ_1 will turn out to be

$$\theta_1 = \begin{bmatrix} (1 - K_1 S K_1^H)^{-1/2} & 0 \\ 0 & (S - K_1^H K_1)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & -K_1 S \\ -K_1^H & I \end{bmatrix}$$

where $\Sigma = 1 \oplus (-S)$ and $K_1 = \sigma_0^{-1}\Gamma_0 S$, and

$$\Delta_0 = \left[\begin{array}{c|c} \sigma_0 & 0 \ldots 0 \\ \times & -\Gamma_0 \\ \vdots & \vdots \end{array}\right].$$

We first have to show that θ_1 is indeed Σ-unitary.
This follows by simple checking of $\theta_1 \Sigma \theta_1^H = \Sigma$.
Next we check that

$$\Delta_1 = \begin{bmatrix} \sigma_0 & -\Gamma_0 \\ \times & \times \ldots \times \\ \vdots & \vdots \end{bmatrix} \quad \theta_1^{-1} = \begin{bmatrix} \sigma_1 & 0 \ldots 0 \\ \times & -\Gamma_1 \\ \vdots & \vdots \end{bmatrix}.$$

The inverse of a Σ-unitary matrix θ is given by $\theta^{-1} = \Sigma \theta^H \Sigma$ so that

$$[\sigma_0 \mid \quad \Gamma_0 \quad]\theta^{-1} = [\sigma_1 \mid \times \ldots \times]$$

with

$$\times \ldots \times = (-\sigma_0 K_1 + \Gamma_0 S)(I - K_1^H K_1)^{-1/2}(-S) = 0$$

which is indeed as we expected. To recapitulate, the generalised Schur algorithm goes as follows: Given a normalised generator $\{\Sigma, \Delta_0\}$, for $R_0 = T$, start with

$$\Delta_0 = \begin{bmatrix} \sigma_0 & 0 \ldots 0 \\ \times & -\Gamma_0 \\ \vdots & \vdots \end{bmatrix} \quad , \quad \Sigma = 1 \oplus (-S).$$

Then compute recursively (in the frequency domain)

$$\Delta_k(z) = \Delta_{k-1}(z) \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix}^{-1} \quad \theta_k^{-1} = \left[\begin{array}{c|c} \sigma_k & 0 \ldots 0 \\ \times & -\Gamma_k \\ \vdots & \vdots \end{array}\right]$$

with

$$\theta_k = \begin{bmatrix} (1 - K_k S K_k^H)^{-1/2} & 0 \\ 0 & (S - K_k^H K_k)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & -K_k S \\ -K_k^H & I \end{bmatrix}, \quad K_k = \frac{\Gamma_{k-1} S}{\sigma_{k-1}} \qquad (20.1)$$

where

$$[\sigma_{k-1} \mid \ -\Gamma_{k-1}] = \Delta_k'(0), \quad \Delta_k'(z) = \Delta_{k-1}(z) \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix}^{-1}.$$

The $K_k \in \mathbb{C}^{1\times(\alpha-1)}$ are the generalisations of the reflection coefficients. The generating function of the Schur complement R_k is given by

$$R_k(z,w) = \frac{\Delta_k(z)\Sigma\Delta_k(w)^H}{1-z\bar{w}}$$

and one can easily check that

$$R_k(0,0) = \sigma_{k-1}^2 - \Gamma_{k-1}S\Gamma_{k-1}^H = \sigma_{k-1}^2(1-K_kSK_k^H).$$

If $R_0(z,w) = R(z,w)$ is a spectral function, then by Schur's lemma, also $R_1(z,w)$, and by induction all $R_k(z,w)$ have to be spectral functions, hence $R_k(0,0) > 0$, or

$$1 - K_kSK_k^H > 0.$$

In the case of a Toeplitz matrix, $K_k = \rho_k$ and $S = 1$, so that this relation then implies that $|\rho_k| < 1$. The previous inequality can thus be seen as a generalisation of this classical property.
An operation count will reveal that this generalised Schur algorithm will need $O(\alpha n^2)$ operations to perform n steps if the infinite vectors are restricted to their first $(n+1)$ components, which are the only data needed to perform these n steps when the generator is known. As we have shown before, this generator is in specific cases often cheap to compute.

21 Triangular factorisation of T

The relation

$$T = R_0 = \begin{bmatrix} r_{00} \\ r_{10} \\ \vdots \end{bmatrix} r_{00}^{-1}[r_{00} \ \ r_{01}\ldots] + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & R_1 & \\ 0 & & \end{bmatrix}$$

translates in the frequency domain as

$$z\bar{w}R_1(z,w) = R_0(z,w) - R_0(z,0)[R_0(0,0)]^{-1}R_0(0,w)$$

or in general we get

$$z\bar{w}R_{i+1}(z,w) = R_i(z,w) - R_i(z,0)[R_i(0,0)]^{-1}R_i(0,w).$$

Setting

$$d_i = R_i(0,0) > 0, \quad l_i(z) = z^iR_i(z,0)d_i^{-1}, \quad l_i(z) = [1 \ z\ldots]l_i$$

we see that

$$R_0(z,w) = \sum_{i=0}^{\infty} d_il_i(z)l_i(w)^H$$

which is equivalent with the matrix relation giving a dyadic decomposition of the autocorrelation matrix T

$$T = R_0 = \sum_{i=0}^{\infty} d_il_il_i^H = LDL^H$$

where $L = [l_0 \;\; l_1 \ldots]$ is a unit lower triangular matrix and $D = \mathrm{diag}(d_0, d_1, \ldots)$.
Note that

$$\begin{aligned} z^{-i} l_i(z) &= R_i(z,0) d_i^{-1} = \Delta_i(z) \Sigma \Delta_i(0)^H d_i^{-1} \\ &= \Delta_i(z) \begin{bmatrix} 1 & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \sigma_i^{-1} = \Delta_i(z) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} / \sigma_i. \end{aligned}$$

This is also applicable for the finite dimensional case. If we take only the leading sub-matrices of order $(n+1)$ of the matrices, then it still holds that

$$T_n = L_n D_n L_n^H$$

is a Cholesky factorisation of T_n. This is computed by the generalised Schur algorithm in $O(\alpha n^2)$ operations.

22 Triangular factorisation of T^{-1}

The obvious question now is whether there is a Levinson equivalent which will compute the inverse Cholesky factorisation of T for α-stationary signals. The columns of the triangular factors in this factorisation will give the predictor coefficients. The answer is yes, but its construction is not as simple as it was for the Schur algorithm and we shall not give all the details.
With not too much effort however, we can give the generalisation of the orthogonal polynomials involved and their recurrence relations, which will thus give the inverse Cholesky factorisation of T itself.
We first recall the Schur algorithm gave us

$$\Delta_n(z) \Theta_n(z) = \Delta_0(z)$$

where (Σ, Δ_0) is a generator for $R_0 = T$ and $\Theta_n(z) = \theta_n(z)\theta_{n-1}(z)\ldots\theta_1(z)$ with $\theta_i(z)$ of the form

$$\theta_i \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix} \quad \text{and } \theta_i \text{ as before in (20.1).}$$

If we replace in that relation $\Delta_n(z)$, by its constant term $\Delta_n(0) = \sigma_n[1\ 0 \ldots 0]$, then we may expect that

$$\Delta_0^{(n)}(z) = \Delta_n(0) \Theta_n(z)$$

is in some sense an approximation for $\Delta_0(z)$. Note that $\Delta_0^{(n)}(z)$ is a polynomial of degree n in z with coefficients in $\mathbb{C}^{1 \times \alpha}$. It contains all the information of the matrix $T = R_0$ that has been extracted from it by the reflection coefficients $K_1, K_2, \ldots, K_n$. To put it in another way, $(\Sigma, \Delta_0^{(n)})$ will be the generator of a matrix $T^{(n)}$ whose reflection coefficients are given by $K_1, K_2, \ldots, K_n, 0, 0, \ldots$.
We call $\Delta_0^{(n)}$ a *canonical* generator for these reflection coefficients.
To see that we get indeed an approximation $\Delta_0^{(n)}(z)$ for $\Delta_0(z)$, we need the following lemma. For its proof we refer to [39].

Lemma 22.1 *Let (Σ, Δ') and (Σ, Δ) be two normalised generators for the matrices T' and T respectively. Then the first n reflection coefficients $K_1, \ldots, K_n$ of T and T' are the same iff there exist coefficients $A_n = (a_{0n}, \ldots, a_{nn})$ such that*

$$G'_n = L(A_n) G_n$$

where G'_n and G_n represent the first $n+1$ rows of Δ' and Δ respectively and $L(A_n)$ is a lower triangular Toeplitz matrix whose first column is A_n.

This lemma can be reformulated in the frequency domain as

$$\Delta'(z) = A_n(z)\Delta(z) + O(z^{n+1}).$$

Since $A_n(0) = 1$ because $\Delta'(0) = \Delta(0) = [\sigma_0\ 0 \ldots 0] \neq 0$, $G'_n(z)/A_n(z)$ is an order $n+1$ approximant for $\Delta(z)$. The generators $\Delta_0(z)$ and $\Delta'_0(z)$ are called congruent because they generate matrices T and T' such that their leading submatrices of order $(n+1)$ satisfy

$$T'_n = L(A_n)T_nL_n(A_n)^H$$

i.e., they are congruent.
In our case, we can take Δ' to be the canonical generator $\Delta_0^{(n)}$.
We show next that T', the matrix generated by $(\Sigma, \Delta_0^{(n)})$, has a leading submatrix T'_n of order $(n+1)$ which is of the form

$$T'_n = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & \sigma_n^2 \end{array}\right] \quad , \quad X \in \mathbb{C}^{n\times n}$$

with Cholesky factorisation

$$T'_n = L'_nD'_nL'^H_n, \quad L'_n = \begin{bmatrix} 1 & & & & \\ \times & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \times & \times & & 1 & \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

The form of T'_n will be proved if the Cholesky factor L'_n has the preset form. Therefore we have to show that, with the notation of the last section, the columns of L'_n satisfy

$$l'_i(z) = z^iR'_i(z,0)d'^{-1}_i, \quad d'_i = R'_i(0,0)$$

is a polynomial of degree $n-1$ for $i = 0, 1, \ldots, n-1$. This is indeed the case because

$$l'_i(z) = z^i\Delta_i^{(n)}(z)\begin{bmatrix}\sigma_i \\ 0 \\ \vdots \\ 0\end{bmatrix}\frac{1}{\sigma_i^2} = z^i[\sigma_n\ 0 \ldots 0]\theta_n(z)\theta_{n-1}(z)\ldots\theta_{i+1}(z)\begin{bmatrix}1 \\ 0 \\ \vdots \\ 0\end{bmatrix}\frac{1}{\sigma_i}$$

and each factor $\theta_k(z)$ will increase the degree in z by 1. We are now ready to prove that A_n is the predictor for the α-stationary process. Therefore, we have to show that it satisfies

$$[a_{nn} \ldots a_{1n}\ 1]T_n = [0 \ldots 0\ E_n]$$

(this is the conjugate transpose of the normal equations). Now, let $T_n = L_nD_nL_n^H$ be the Cholesky factorisation of T_n, then A_n is the last row of L_n^{-1} (recall that L_n is unit lower triangular). Since

$$T'_n = \left[\begin{array}{c|c} X & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0\ldots 0 & \sigma_n^2 \end{array}\right] = L'_nD'_nL'^H_n = L(A_n)L_nD_nL_n^HL(A_n)^H$$

we see that $d'_n = d_n = \sigma_n^2 = E_n$ and

$$L'_n = \begin{bmatrix} 1 & & & \\ \vdots & \ddots & & \\ \times & \ldots & 1 & \\ 0 & \ldots & 0 & 1 \end{bmatrix} = L(A_n)L_n$$

or, taking the last row:

$$[0 \ldots 0 \ 1] L_n^{-1} = [0 \ldots 0 \ 1] L(A_n) = [a_{nn} \ldots a_{1n} \ 1]$$

which means that $[a_{nn} \ldots a_{1n} \ 1]$ is indeed the last row of L_n^{-1} and thus is A_n the predictor of the process.
The interpretation also shows that $A_n^*(z) = z^n \overline{A_n(1/\bar{z})} = \varphi_n(z)$ is a monic orthogonal polynomial for the Gram matrix T_n.
To find the generalisation of the Szegő recurrence relation, which is essentially the recurrence used in the generalised Levinson algorithm, we need only a little more.
It can be shown that under rather mild conditions, there exists a generator Δ which is *admissible* [39], which means that there exists a $\lambda \in \mathbb{C}^{\alpha \times 1}$ such that $\Delta(z)\lambda = 1$. The existence of an admissible generator depends on the difference $\alpha - \delta$ where α is the displacement rank of T_n and δ is the displacement rank of T_∞. This difference can only be 0,1 or 2. If $\alpha - \delta = 2$ then a minimal[11] generator is admissible with $\lambda^H \Sigma \lambda = 0$. For example, a Toeplitz matrix is in this class with $\lambda = [1 \ \ 1]^T / \sqrt{r_0}$. If $\alpha - \delta = 1$, there exists a λ such that $\Delta(z)\lambda = 1$ and $\lambda^H \Sigma \lambda \neq 0$. If $\alpha - \delta = 0$, no minimal generator is admissible. However setting $\alpha = \delta + 2$, hence allowing α not to be minimal, one can always find λ such that $\Delta(z)\lambda = 1$.
We can now find an explicit expression for $A_n(z)$ as follows. Note that $\Delta_0^{(n)}(z) = [\sigma_n \ 0 \ldots 0]\Theta_n(z) \in \mathbb{C}_n^{1 \times \alpha}[z]$ is a polynomial of degree n, and

$$\begin{aligned} \Delta_0^{(n)}(z)\lambda = [\sigma_n \ 0 \ldots 0]\Theta_n(z)\lambda &= A_n(z)\Delta(z)\lambda + O(z^{n+1})\lambda \\ &= A_n(z). \end{aligned}$$

Thus, if we set

$$\Theta_n(z)\lambda = \begin{bmatrix} A_n'(z) \\ C_n'(z) \end{bmatrix} \in \mathbb{C}^{\alpha \times 1}[z],$$

then

$$\begin{bmatrix} A_n'(z) \\ C_n'(z) \end{bmatrix} = \theta_n(z) \begin{bmatrix} A_{n-1}'(z) \\ C_{n-1}'(z) \end{bmatrix} \quad , \quad \begin{bmatrix} A_0' \\ C_0' \end{bmatrix} = \lambda$$

gives the generalisation of the Szegő recurrence.
Indeed for the Toeplitz case we get $\lambda = [1 \ \ 1]^T / \sqrt{r_0}$ so that for $[A_n'(z) \ \ C_n'(z)] = [\phi_n^*(z) \ \ \Phi_n(z)]$

$$\begin{bmatrix} \phi_n^*(z) \\ \Phi_n(z) \end{bmatrix} = \frac{1}{\sqrt{1 - |\rho_n|^2}} \begin{bmatrix} 1 & -\rho_n \\ -\bar{\rho}_n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} \phi_{n-1}^*(z) \\ \Phi_{n-1}(z) \end{bmatrix} , \begin{bmatrix} \phi_0^* \\ \Phi_0 \end{bmatrix} = \frac{1}{\sqrt{r_0}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} \phi_n(z) \\ \Phi_n^*(z) \end{bmatrix} = \frac{1}{\sqrt{1 - |\rho_n|^2}} \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}(z) \\ \Phi_{n-1}^*(z) \end{bmatrix}$$

where

$$[\phi_n^*(z) \ \ \Phi_n^*(z)] = z^n [\overline{\phi_n(1/\bar{z})} \ \ \overline{\Phi_n(1/\bar{z})}]$$

and therefore $\phi_n = \Phi_n$ in that case, with ϕ_n the normalised Szegő polynomials.
It can be shown [39] that the Christoffel-Darboux relations are given by

$$\begin{aligned} k_n(z,w) &= \sum_{i=0}^{n} \phi_i(z)\overline{\phi_i(w)} = [1 \ z \ldots z^n] T_n^{-1} [1 \ w \ldots w^n]^H \\ &= \frac{\Phi_{n+1}^*(w)^H S \Phi_{n+1}^*(z) - \overline{\phi_{n+1}(w)}\phi_{n+1}(z) - \lambda^H \Sigma \lambda}{1 - z\bar{w}}. \end{aligned}$$

[11] Meaning with minimal α

Note the presence of the term $\lambda^H \Sigma \lambda$ which in the Toeplitz case is 0. This formula can be used to write T_n^{-1} as a sum of plus or minus a product of lower times upper triangular matrices. This is the obvious generalisation of the Gohberg-Semencul formula.
We note that this setting can be generalised in a straightforward way to give a Nevanlinna-Pick analogon. One has for example the recurrences

$$\begin{aligned} R_k(z,w) &= \frac{R_{k-1}(z,w) - R_{k-1}(z,\alpha_k) R_{k-1}(\alpha_k,\alpha_k)^{-1} R_{k-1}(\alpha_k,w)}{\zeta_k(z)\overline{\zeta_k(w)}} \\ &= \frac{\Delta_k(z) \Sigma \Delta_k(w)^H}{1 - \zeta_k(z)\overline{\zeta_k(w)}} \end{aligned}$$

where $\zeta_k(z) = \frac{z-\alpha_k}{1-\bar{\alpha}_k z}$, $\alpha_k \in \mathbb{D}$ and

$$\Delta_k(z) = \Delta_{k-1}(z) \begin{bmatrix} 1 & 0 \\ 0 & \zeta_{k-1}(z) I \end{bmatrix}^{-1} \theta_k^{-1}$$

with θ_k as before in (20.1) and the reflection coefficient K_k is now chosen to make $\Delta_k(\alpha_k)$ of the form $[\sigma_k \; 0 \ldots 0]$.
This will result in an ARMA model for α-stationary signals.

23 Recent developments

We have linked the problem of linear prediction to several mathematical fields like stochastic processes, complex analysis, orthogonal polynomials, linear algebra, Padé approximation etc.
It is impossible to give the evolution in each of these areas that took place since this link was made. We mention one of the early developments.
When a signal is n-dimensional vector valued, which happens when you measure not one but several outputs simultaneously. The autocorrelation function r_{ij} will then be matrix valued $r_{ij} \in \mathbb{C}^{n \times n}$. The autocorrelation matrix is in the stationary case a block Toeplitz matrix. The algorithms can then be generalized to block-versions [47, 7, 8, 14], giving rise to matrix valued Padé approximation, block orthogonal polynomials [22], block Nevanlinna-Pick [18] etc. The step towards the case $n = \infty$ where these quantities are operator valued is obvious for a mathematician and operator valued versions of Schur, and Pick-Nevanlinna algorithms and the classical interpolation algorithms were developed, as well as all kinds of extension problems [49]. The link with the theory of Adamjan, Arov and Krein [1, 2] who extended several of these interpolation problems in the context of infinite (block) Hankel matrices and their singular value decomposition inspired many researchers to develop a complete field of what is now known as H_∞ control or Hankel norm approximation which is important in the theory of optimal control of linear systems [29, 26, 25, 27].
Also signals which are multivariate, like for example when you sample an image, the pixels are related in x and in y direction. One gets then a whole area to be developed where you have to consider polynomials in two variables, orthogonal on a bidisk [19]. This is also related to multivariate Padé approximation and rational interpolation.
Also the linear algebra aspects have caused an avalange of publications on "structured" matrices for which fast algorithms can be designed. It has given an impulse to the development of systolic arrays [33].
All this was essentially developed during the last 15 years.
The most recent discovery, which dates from the last year is a generalization of the previous theory to arbitrary autocorrelation matrices, which do not have a finite displacement rank. The step that has to be made here is not trivial. See the collection of papers [31].

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