A weak-star convergence result for orthogonal rational functions

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Abstract

We prove a weak-star convergence result for orthogonal rational functions on the interval $[-1, 1]$ which generalizes a well known result by Rakhmanov [9]. As an application we use this result to approximate a certain integral which occurs in the computation of orthogonal rational functions.

Key words: orthogonal rational functions, orthogonal polynomials, asymptotic properties

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1 Introduction

Orthogonal rational functions are becoming increasingly important in applied mathematics and especially in system identification. The transfer function in system theory is usually an unknown (rational) function and the main problem is to approximate this function given the input and output parameters. Only less than two decades ago researchers started systematically using rational basis functions in the approximation, first with one pole [16] or a pair of complex conjugate poles [17], then with a finite number of poles that are cyclically repeated [2,11,10] and finally with an arbitrary pole sequence [6–8]. Orthogonality with respect to the Lebesgue measure was considered. Later on

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these ideas were extended to the case of an arbitrary orthogonality measure, using the theory of orthogonal rational functions as described in [1]. For some recent developments, see [14,13]. This shows that the accurate computation of these functions is of great importance.

From a strictly mathematical point of view, orthogonal rational functions can be seen as generalizations of orthogonal polynomials. As in the polynomial case we distinguish between orthogonality on the unit circle and on (a subset of) the real line, in this case the interval $[-1,1]$. They are related to each other using the Joukowski transformation. To obtain results on the unit circle we make use of a connection between orthogonal rational functions and orthogonal polynomials with respect to varying measures. This relation was first proved in [5] and is given in a more general setting in chapter 9 of [1].

The main purpose of this paper is to provide a weak-star convergence result for orthogonal rational functions on the interval $[-1,1]$. This will be used in an upcoming paper [12] to estimate the error in the numerical calculation of orthogonal rational functions.

2 Preliminaries

We denote the complex plane by $\mathbb{C}$, the open unit disc by $\mathbb{D}$ and the unit circle by $\mathbb{T}$. By $\mathbb{N}$ we denote the set of natural numbers. Given a positive bounded Borel measure μ on $[0,2\pi)$, or equivalently on $\mathbb{T}$, the inner product in $L^2(\mu)$ is

$$\frac{1}{2\pi}\int_0^{2\pi} f(z)\overline{g(z)}d\mu(\theta), \quad z=e^{\mathrm{i}\theta}.$$

In what follows we assume $\mu' > 0$ a.e., where μ' is the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $[0,2\pi)$.

Given a set of complex numbers $A = \{\alpha_1, \alpha_2, \ldots\}$ compactly included in $\mathbb{D}$ (this means they are bounded away from the boundary $\mathbb{T}$), we can construct orthonormal polynomials $\phi_{n,m}(z) = \nu_{n,m} z^m + \ldots$ with respect to the varying measure $d\mu/|w_n(z)|^2$ with $w_n(z) = (z-\alpha_1)\ldots(z-\alpha_n)$ that are uniquely determined by the conditions

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \phi_{n,m}(z)\overline{z}^j \frac{d\mu(\theta)}{|w_n(z)|^2} = 0, \quad j=0,\ldots,m-1,\\
&\frac{1}{2\pi}\int_0^{2\pi} |\phi_{n,m}(z)|^2 \frac{d\mu(\theta)}{|w_n(z)|^2} = 1, \quad \nu_{n,m}>0, \quad z=e^{\mathrm{i}\theta}.
\end{aligned}$$

For each polynomial q_n of degree n we define the superstar transformation as

$q_n^*(z) = z^n\overline{q_n(1/\overline{z})}$. It follows that $|q_n^*(z)| = |q_n(z)|$ for $|z| = 1$.

With the sequence A and the measure μ we can also associate a set of orthonormal rational functions as follows. Define the Blaschke factors

$$\zeta_n(z) = -\frac{\overline{\alpha}_n}{|\alpha_n|}\frac{z-\alpha_n}{1-\overline{\alpha}_n z}, \quad n = 1, 2, \ldots$$

(for $\alpha_n = 0$ the constant factor in front is taken equal to 1) and the basis functions

$$B_0 = 1, \quad B_n(z) = \zeta_n(z)B_{n-1}(z), \quad n = 1, 2, \ldots$$

Then the space $\mathcal{L}_n$ of rational functions with poles in $\{\alpha_1, \ldots, \alpha_n\}$ is defined as

$$\mathcal{L}_n = \text{span}\{B_0, \ldots, B_n\}.$$

Orthonormalizing this basis we obtain the orthonormal rational functions $\{\phi_n\}$ where we take the leading coefficient κ_n in the expansion $\phi_n(z) = \kappa_n B_n(z) + \ldots$ to be positive. Note that we obtain the orthonormal Szegő polynomials if $\alpha_k = 0$ for all k. If we write $\phi_n(z) = g_n(z)/w_n^*(z)$ then we have the following relation between the orthogonal rational functions and the orthogonal polynomials with respect to the varying measures. For the proof we refer to [5] or for a more general formulation to [1].

Lemma 1 *Let $\phi_{n,m}$ and ϕ_n be as defined above. Then the following relation holds,*

$$g_n(z) = \frac{1}{z-\alpha_n}\frac{\tau_n}{\nu_{n,n}}\left(z\phi_{n,n}(z) - \alpha_n\frac{\phi_{n,n}(\alpha_n)}{\phi_{n,n}^*(\alpha_n)}\phi_{n,n}^*(z)\right)$$

where $g_n = \tau_n z^n + \ldots$ is the numerator polynomial of $\phi_n(z) = g_n(z)/w_n^*(z)$.

We also define a superstar transformation for the rational functions as follows. If f_n is a function in $\mathcal{L}_n$ then $f_n^*(z) = B_n(z)\overline{f_n(1/\overline{z})}$. Again we have $|f_n^*(z)| = |f_n(z)|$ for $|z| = 1$. It should be clear from the context if the superstar refers to the polynomial superstar transformation or the rational superstar transformation. If all $\alpha_k = 0$ then both transformations are the same. Note that for any $f = p_n/w_n^* \in \mathcal{L}_n$ we have

$$f^*(z) = \eta_n\frac{p_n^*(z)}{w_n^*(z)}, \tag{1}$$

where $\eta_n = \prod_{k=1}^n(-\overline{\alpha}_k/|\alpha_k|)$.

3 Auxiliary results

Before we can state and prove our main theorems, we will need several well known results about the asymptotic behaviour of orthogonal polynomials and

rational functions. We assume that μ and A are as before, i.e. $\mu' > 0$ a.e. and $|\alpha_k| \leq \rho < 1$ for all k.

If μ_n and μ are positive Borel measures on $[0, 2\pi)$ we denote by $\mu_n \xrightarrow{*} \mu$ the weak convergence of μ_n to μ as $n \to \infty$. That means that for every continuous 2π-periodic function f on $[0, 2\pi)$

$$\lim_{n\to\infty} \int_0^{2\pi} f(\theta) d\mu_n(\theta) = \int_0^{2\pi} f(\theta) d\mu(\theta).$$

The same definition is given for complex Borel measures of bounded variation.

In what follows, local uniform convergence in a region Ω will mean uniform convergence on compact subsets of Ω.

First we state several results about orthogonal polynomials with respect to varying measures. The following theorem can be found in [3].

Theorem 2 *For every integer k we have*

$$\lim_{n\to\infty} \frac{\phi_{n,n+k}(z)}{\phi^*_{n,n+k}(z)} = 0,$$

locally uniformly in $\mathbb{D}$.

We will use the next lemma when we make the transition from the unit circle to the interval $[-1, 1]$. It is completely analogous to the one for the case of a fixed measure as described in [9].

Lemma 3 *For every integer k we have*

$$\frac{\phi^2_{n,n+k}(z)}{z^{n+k}} \frac{d\mu(\theta)}{|w_n(z)|^2} \xrightarrow{*} 0, \quad z = e^{\mathbf{i}\theta}.$$

PROOF. We only briefly outline the proof, since it is exactly the same as in [9], apart from the obvious modifications for the case of a varying measure.

Use the first statement in theorem 2 and the Poisson representation for harmonic functions to find that

$$\frac{\phi_{n,n+k}(z)}{\phi^*_{n,n+k}(z)} d\theta \xrightarrow{*} 0, \quad z = e^{\mathbf{i}\theta}$$

for every integer k.

Then write

$$\frac{\phi_{n,n+k}(z)}{\phi^*_{n,n+k}(z)} = \frac{\phi^2_{n,n+k}(z)}{z^{n+k}|\phi_{n,n+k}(z)|^2}, \quad z = e^{\mathbf{i}\theta}$$

and use

$$\lim_{n\to\infty}\left|\frac{\phi_{n,n+k+m}(z)}{\phi_{n,n+k}(z)}\right|^2 = 1, \quad z = e^{\mathbf{i}\theta}$$

for every integer k and for every $m \in \mathbb{N}$ and

$$\int_0^{2\pi} z^j |\phi_{n,m}(z)|^{-2} d\theta = \int_0^{2\pi} z^j \frac{d\mu(\theta)}{|w_n(z)|^2}, \quad j = 0, \pm 1, \ldots, \pm m, \quad z = e^{\mathbf{i}\theta}. \quad (2)$$

Both statements can be found in [4]. Combining all the previous formulas we can prove the lemma. □

We conclude this section with some results about the convergence of the orthogonal rational functions $\{\phi_n\}$. For the proofs we refer to chapter 9 of [1].

Theorem 4 *Let $P(z,t)$ denote the Poisson kernel,*

$$P(z,t) = \frac{1-|t|^2}{|z-t|^2}, \quad z = e^{\mathbf{i}\theta}$$

then we have

$$\frac{|\phi_n(z)|^2}{P(z,\alpha_n)}\mu'(\theta)d\theta \xrightarrow{*} d\theta$$

and locally uniformly in $\mathbb{D}$

$$\lim_{n\to\infty}\frac{\phi_n(z)}{\phi_n^*(z)} = 0.$$

4 Some lemmas for the unit circle

Using the results from the previous section we can now prove some convergence results for orthogonal polynomials with respect to varying measures and for orthogonal rational functions on the unit circle. In the rest of this paper we will use the notation $a_n \approx b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$.

Lemma 5 *Let A, μ, ϕ_n and $\phi_{n,n}$ be as defined above. Then we have*

$$\left|\frac{\phi_n^*(0)}{\phi_{n,n}^*(0)}\right| \approx \sqrt{1-|\alpha_n|^2}.$$

PROOF. Let $k_n(z,w)$ denote the reproducing kernel for $\mathcal{L}_n$. Then it follows from [1] that

$$k_n(z,z) = \frac{|\phi^*_{n,n}(z)|^2 - |z|^2|\phi_{n,n}(z)|^2}{(1-|z|^2)|w^*_n(z)|^2},$$
$$= \frac{|\phi^*_n(z)|^2 - |\zeta_n(z)|^2|\phi_n(z)|^2}{1-|\zeta_n(z)|^2}.$$

Substituting zero for z and using theorem 4 together with the definition of $\zeta_n(z)$ proves the lemma. $\square$

The next result extends the first statement of theorem 4.

Lemma 6 *Let $P(z,t)$ denote the Poisson kernel,*

$$P(z,t) = \frac{1-|t|^2}{|z-t|^2}, \quad z = e^{\mathbf{i}\theta}$$

then we have

$$\frac{|\phi_n(z)|^2}{P(z,\alpha_n)} d\mu(\theta) \xrightarrow{*} d\theta.$$

PROOF. We begin by proving that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{|\phi_n(z)|^2}{P(z,\alpha_n)} d\mu(\theta) = 1, \quad z = e^{\mathbf{i}\theta}. \tag{3}$$

Let $k_n(z,w)$ denote the reproducing kernel for $\mathcal{L}_n$,

$$k_n(z,w) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)}$$

and define $K_n(z) = k_n(z,0)/\sqrt{k_n(0,0)}$ then it follows from the orthogonality of the ϕ_k that

$$\frac{1}{2\pi} \int_0^{2\pi} |K_n(z)|^2 d\mu(\theta) = 1. \tag{4}$$

Theorem 3.1.3 in [1] states that the functions $\{\phi_n\}$ satisfy the following Christoffel-Darboux relation,

$$k_n(z,w) = \frac{\phi^*_n(z)\overline{\phi^*_n(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}.$$

Together with the definition of $K_n(z)$ this yields after some computations

$$|K_n(z)| = \frac{\left|1 - |\alpha_n| \frac{\overline{\phi_n(0)}}{\phi^*_n(0)} \frac{\phi_n(z)}{\phi^*_n(z)} \zeta_n(z)\right|}{\sqrt{1-|\alpha_n|^2 \left|\frac{\phi_n(0)}{\phi^*_n(0)}\right|^2}} \frac{|1-\overline{\alpha}_n z|}{\sqrt{1-|\alpha_n|^2}} |\phi^*_n(z)|$$

so if $z \in \mathbb{T}$ we get

$$\frac{|\phi_n^*(z)|^2}{P(z,\alpha_n)} = |K_n(z)|^2 \frac{1-|\alpha_n|^2\left|\frac{\phi_n(0)}{\phi_n^*(0)}\right|^2}{\left|1-|\alpha_n|\frac{\overline{\phi_n(0)}}{\overline{\phi_n^*(0)}}\frac{\phi_n(z)}{\phi_n^*(z)}\zeta_n(z)\right|^2}.$$

Integrating both sides this yields using (4)

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{P(z,\alpha_n)}d\mu(\theta) \leq \max_{z\in\mathbb{T}}\frac{1-|\alpha_n|^2\left|\frac{\phi_n(0)}{\phi_n^*(0)}\right|^2}{\left|1-|\alpha_n|\frac{\overline{\phi_n(0)}}{\overline{\phi_n^*(0)}}\frac{\phi_n(z)}{\phi_n^*(z)}\zeta_n(z)\right|^2}.$$

Because $|\alpha_k| \leq \rho < 1$ for all k and $|\phi_n(z)/\phi_n^*(z)| < 1$ for $z \in \mathbb{D}$ (this follows from corollary 3.1.4 in [1]), the maximum in the right hand side will be finite. Using the second statement in theorem 4 we obtain

$$\limsup_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{P(z,\alpha_n)}d\mu(\theta) \leq 1$$

but because of the first statement in that theorem we also have

$$\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{P(z,\alpha_n)}d\mu(\theta) \geq 1$$

so that (3) follows.

Because the unit sphere in the dual space of the space of 2π-periodic continuous functions on $[0,2\pi)$ is compact in the weak star topology we can extract from any sequence of natural numbers a subsequence $\Gamma \subset \mathbb{N}$ such that $|\phi_n(z)|^2/P(z,\alpha_n)d\mu(\theta) \xrightarrow{*} dg(\theta)$ as $n \to \infty, n \in \Gamma$, where $dg(\theta)$ is a positive Borel measure and $\int dg = 2\pi$ (the last statement follows from equation (3) which we just proved). The rest of the proof is now exactly as in the proof of lemma 2 in [9] and since it is rather technical, we do not repeat it. The key element is that we also have

$$\frac{P(z,\alpha_n)}{|\phi_n(z)|^2}d\theta \xrightarrow{*} d\mu(\theta), \quad z = e^{\mathbf{i}\theta}.$$

This is theorem 9.7.1 in [1]. Using this formula we can prove that dg is the Lebesgue measure on $[0,2\pi)$. We refer to [9] for more details. □

Now let us prove the last lemma in this section.

Lemma 7 *Let $P(z,t)$ denote the Poisson kernel and $B_n(z)$ the Blaschke product as defined in section 2. Then we have*

$$\frac{\phi_n^2(z)}{P(z,\alpha_n)B_n(z)}d\mu(\theta) \xrightarrow{*} 0, \quad z = e^{\mathbf{i}\theta}.$$

PROOF. Take $z = e^{\mathrm{i}\theta}$ and let $\phi_n = g_n/w_n^*$ then from the definition of $B_n(z)$ and $P(z,\alpha_n)$ we have

$$\frac{\phi_n^2(z)}{P(z,\alpha_n)B_n(z)} = \frac{g_n^2(z)(1-\overline{\alpha}_n z)(z-\alpha_n)}{\eta_n z^{n+1}(1-|\alpha_n|^2)|w_n^*(z)|^2}.$$

With lemma 1 and some calculations we obtain

$$\frac{\phi_n^2(z)}{P(z,\alpha_n)B_n(z)} = \frac{1}{1-|\alpha_n|^2}\frac{\tau_n^2}{\nu_{n,n}^2}\frac{1-\overline{\alpha}_n z}{z-\alpha_n}\frac{1}{\eta_n|w_n^*(z)|^2}\left(\frac{\phi_{n,n}^2(z)}{z^{n-1}}\right.$$
$$\left.-2\frac{\phi_{n,n}(\alpha_n)}{\phi_{n,n}^*(\alpha_n)}\alpha_n|\phi_{n,n}(z)|^2 + \left(\frac{\phi_{n,n}(\alpha_n)}{\phi_{n,n}^*(\alpha_n)}\alpha_n\right)^2 z^{n-1}\overline{\phi_{n,n}^2(z)}\right).$$

Because of theorem 2 it suffices to show that

$$\frac{1}{1-|\alpha_n|^2}\frac{\tau_n^2}{\nu_{n,n}^2}\frac{1}{\eta_n}\frac{z(1-\overline{\alpha}_n z)}{z-\alpha_n}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2} \xrightarrow{*} 0$$

but since it follows from lemma 5 that the constant factor in this expression is bounded (it tends to 1 in modulus), we can ignore it. It thus remains to prove that

$$\frac{z(1-\overline{\alpha}_n z)}{z-\alpha_n}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2} \xrightarrow{*} 0.$$

First expand $z(1-\overline{\alpha}_n z)/(z-\alpha_n)$ in a Taylor series as follows

$$\frac{z(1-\overline{\alpha}_n z)}{z-\alpha_n} = 1-|\alpha_n|^2-\overline{\alpha}_n z + (1-|\alpha_n|^2)\alpha_n\sum_{l=0}^{\infty}\frac{\alpha_n^l}{z^{l+1}}.$$

This series will converge because $|z|=1$ and $|\alpha_k| \le \rho < 1$ for all k.

Because the trigonometric polynomials are dense in the space of 2π-periodic continuous functions on $[0,2\pi)$, the theorem will be proved if we can show that

$$\lim_{n\to\infty}\int_0^{2\pi} z^k\frac{z(1-\overline{\alpha}_n z)}{z-\alpha_n}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2} = 0$$

for every integer k. Using the Taylor series expansion we may write

$$\begin{aligned}
\left|\int_0^{2\pi} z^k \frac{z(1-\overline{\alpha}_n z)}{z-\alpha_n}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right| &\le (1-|\alpha_n|^2)\left|\int_0^{2\pi} z^k \frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right| \\
&\quad + |\alpha_n|\left|\int_0^{2\pi} z^{k+1}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right| \\
&\quad + (1-|\alpha_n|^2)\sum_{l=0}^{\infty}|\alpha_n|^{l+1}\left|\int_0^{2\pi} z^{k-l-1}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right|, \\
&\le \left|\int_0^{2\pi} z^k \frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right| + \rho\left|\int_0^{2\pi} z^{k+1}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right| \\
&\quad + \sum_{l=0}^{\infty}\rho^{l+1}\left|\int_0^{2\pi} z^{k-l-1}\frac{\phi_{n,n}^2(z)}{z^n}\frac{d\mu(\theta)}{|w_n^*(z)|^2}\right|.
\end{aligned}$$

Taking limits on both sides and using lemma 3 then proves the lemma (note that each integral in the series is less than one in modulus). □

5 Transition to the interval

Just as in the polynomial case we can relate the orthogonal rational functions on $[-1,1]$ to those on the unit circle. First let us define the spaces of rational functions on the interval. In this section we use x as the independent variable for the rational functions on $[-1,1]$ and z for the functions on the unit circle.

The real line is denoted by $\mathbb{R}$ and the extended real line by $\hat{\mathbb{R}} = \mathbb{R}\cup\infty$. For the interval we use $I = [-1,1]$ and its complement with respect to the real line is $\mathbb{R}^I = \hat{\mathbb{R}} \setminus I$. Similarly we define its complement with respect to the Riemann sphere, $\mathbb{C}^I = (\mathbb{C}\cup\infty)\setminus I$. Now let there be given a finite positive Borel measure λ on I satisfying $\lambda' > 0$ a.e. and a sequence of poles $B = \{\beta_1, \beta_2, \ldots\}$ such that $B \subset \mathbb{R}^I$ and $|\beta_k| \ge \rho > 1$ for all k. Note that we may have poles at infinity.

Define factors

$$Z_n(x) = \frac{x}{1 - x/\beta_n}, \quad n = 1, 2, \ldots$$

and basis functions

$$b_0 = 1, \quad b_n(x) = b_{n-1}(x)Z_n(x), \quad n = 1, 2, \ldots$$

Then we define the space of rational functions with poles in $\{\beta_1, \ldots, \beta_n\}$ as

$$\mathcal{L}_n^- = \text{span}\{b_0, \ldots, b_n\}.$$

Orthonormalizing this basis with respect to λ we obtain orthogonal rational functions $\{\varphi_0, \ldots, \varphi_n\}$ where we choose the leading coefficient κ_n^- in the expansion $\varphi_n(x) = \kappa_n^- b_n(x) + \ldots$ to be positive. Note that we recover the polynomial situation if we take all poles at infinity.

We denote the Joukowski transform $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ to the cut Riemann sphere $\mathbb{C}^I$ and the unit circle $\mathbb{T}$ to the interval I. The inverse mapping is denoted by $z = J^{-1}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \mathbb{C}^I$. To the sequence $B = \{\beta_1, \beta_2, \ldots\} \subset \mathbb{R}^I$ we associate a sequence $A = \{\alpha_1, \alpha_2, \ldots\} \subset I$ such that $\alpha_k = J^{-1}(\beta_k)$, and a sequence $\hat{A} = \{\hat{\alpha}_1, \hat{\alpha}_2, \ldots\}$ such that $\hat{\alpha}_{2k} = \hat{\alpha}_{2k-1} = \alpha_k$. The corresponding Blaschke products and orthogonal functions are denoted by a hat. Obviously $\hat{B}_{2k} = (B_k)^2$.

Next define the measure μ on $\mathbb{T}$ as

$$\mu(E) = \lambda(\{\cos\theta, \theta \in E \cap [0, \pi)\}) + \lambda(\{\cos\theta, \theta \in E \cap [\pi, 2\pi)\}).$$

for every Borel measurable set E. This implies that for every $d\lambda$-measurable function f we have

$$\int_{-1}^{1} f(x) d\lambda(x) = \frac{1}{2} \int_0^{2\pi} f(\cos\theta) d\mu(\theta). \tag{5}$$

Then if $\{\hat{\phi}_n\}$ is the orthonormal set associated with $(\hat{A}, \mu)$, we have the following theorem from [15] (note that we have to double the multiplicity of every pole).

Theorem 8 *Let $\{\varphi_n\}$ be a set of orthonormal rational functions on I and $\{\hat{\phi}_n\}$ the corresponding set of functions orthogonal on $\mathbb{T}$ with poles and measure as defined above, then they are related by*

$$\varphi_n(x) = \eta_n (2\pi)^{-\frac{1}{2}} \left\{ 1 + \frac{\hat{\phi}_{2n}(\alpha_n)}{\hat{\phi}^*_{2n}(\alpha_n)} \right\}^{-\frac{1}{2}} \frac{\hat{\phi}_{2n}(z) + \hat{\phi}^*_{2n}(z)}{B_n(z)}$$

where $x = J(z)$.

Recall that $\eta_n = \prod_{k=1}^{n} (-\alpha_k / |\alpha_k|)$. In [15] this factor does not appear because the Blaschke products are defined in a different way.

Using this relation and the results from the previous sections we can easily prove our main theorem.

Theorem 9 *With the notation introduced above we have*

$$\frac{1 - x/\beta_n}{\sqrt{1 - 1/\beta_n^2}} \varphi_n^2(x) d\lambda(x) \xrightarrow{*} \frac{1}{\pi} \frac{dx}{\sqrt{1 - x^2}}.$$

PROOF. First note that

$$\frac{1 - x/\beta_n}{\sqrt{1 - 1/\beta_n^2}} = P(z, \alpha_n)^{-1}$$

where $P(z,t)$ is the Poisson kernel, $x = J(z)$ and $\beta_n = J(\alpha_n)$.

The sequence of norms of the measures

$$d\lambda_n(x) = \frac{1 - x/\beta_n}{\sqrt{1 - 1/\beta_n^2}}\,\varphi_n^2(x)d\lambda(x)$$

is bounded because $|\beta_k| \geq \rho > 1$ for all k and $\int_{-1}^{1} \varphi_n^2(x)d\lambda(x) = 1$ by definition. Therefore it suffices to show that for each nonnegative integer k we have

$$\lim_{n\to\infty} \int_{-1}^{1} x^k \frac{1 - x/\beta_n}{\sqrt{1 - 1/\beta_n^2}}\,\varphi_n^2(x)d\lambda(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{x^k}{\sqrt{1-x^2}}dx.$$

Using equation (5) and theorem 8 we obtain

$$\begin{aligned}
&\int_{-1}^{1} x^k \frac{1 - x/\beta_n}{\sqrt{1 - 1/\beta_n^2}}\,\varphi_n^2(x)d\lambda(x) = \\
&\quad \frac{1}{4\pi}\left\{1 + \frac{\hat{\phi}_{2n}(\alpha_n)}{\hat{\phi}^*_{2n}(\alpha_n)}\right\}^{-1} \int_0^{2\pi} \frac{\cos^k\theta}{P(z,\alpha_n)} \left(\frac{\hat{\phi}_{2n}(z) + \hat{\phi}^*_{2n}(z)}{B_n(z)}\right)^2 d\mu(\theta) \\
&= \frac{1}{2\pi}\left\{1 + \frac{\hat{\phi}_{2n}(\alpha_n)}{\hat{\phi}^*_{2n}(\alpha_n)}\right\}^{-1} \int_0^{2\pi} \frac{\cos^k\theta}{P(z,\alpha_n)} \left(|\hat{\phi}_{2n}(z)|^2 + \Re\left\{\frac{\hat{\phi}^2_{2n}(z)}{\hat{B}_{2n}(z)}\right\}\right) d\mu(\theta).
\end{aligned}$$

Taking limits and using theorem 4, lemma 6 and lemma 7 this yields

$$\lim_{n\to\infty} \int_{-1}^{1} x^k \frac{1 - x/\beta_n}{\sqrt{1 - 1/\beta_n^2}}\,\varphi_n^2(x)d\lambda(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos^k\theta d\theta,$$

which proves the theorem. □

6 Application

In [12] we discuss gaussian quadrature formulas based on orthogonal rational functions. For analytical integrands, the expression for the quadrature error involves the following function,

$$H(t) = \int_{-1}^{1} \frac{\varphi_n^2(x)(1 - x/\beta_n)}{t - x} d\lambda(x), \quad t \in \mathbb{C}^I.$$

In the above-mentioned paper we use these quadrature formulas to compute orthogonal rational functions on $[-1,1]$ and in order to estimate the accuracy we need to evaluate the function $H(t)$ at different points in the complex plane (for more details we refer to the article). The exact evaluation of $H(t)$ can be

very difficult, but since we are only interested in the order of magnitude, we may use theorem 9 to approximate this function. First note that

$$\int_{-1}^{1} \frac{dx}{(t-x)\sqrt{1-x^2}} = \frac{1}{2}\int_{0}^{2\pi} \frac{d\theta}{t-\cos\theta} = \frac{z}{\mathbf{i}} \oint_{\mathbb{T}} \frac{du}{(u-z)(1-zu)} = \frac{2\pi z}{1-z^2}$$

where the last equality follows from the residue theorem and $t = J(z)$. Using this formula, theorem 9 and some algebra then gives

$$H(t) \approx \frac{2z}{1-z^2}\frac{1-\alpha_n^2}{1+\alpha_n^2}$$

where $\alpha_n = J^{-1}(\beta_n)$. This expression can of course easily be evaluated. Also note that the approximation is exact for $t = \infty$ and $t = \beta_n$. Numerical experiments indicate that, as long as the poles do not get too close to the boundary, convergence takes place rather quickly, yielding about 4 or 5 correct digits for $n = 20$. Again we refer to [12] for more information.

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