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Spectral Properties and Lyapunov Matrices of Primal - Dual Periodic Time-Delay systems, with Application to Balancing ⋆

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Abstract: In the paper we establish connections between the spectral properties of linear periodic systems with multiple delays with those of a dual system, obtained by transposition of the systems matrices and affine transformations of their arguments. The dual system also allows to introduce the dual Lyapunov matrix associated with the original system. We provide various energy interpretations of the primal-dual Lyapunov matrices, which allow us to generalize the concept of position balancing and explore its potential for model reduction.

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1. INTRODUCTION

In this paper, we consider linear time-periodic systems of the form

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{m} A_i(t)x(t-\tau_i) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, and functions $A_i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $t \mapsto A_i(t)$, $i = 0, \ldots, m$, $B : \mathbb{R} \to \mathbb{R}^{n\times n_i}$, $t \mapsto B(t)$, $C : \mathbb{R} \to \mathbb{R}^{n_o\times n}$, $t \mapsto C(t)$ are smooth and T-periodic. The delays are sorted in increasing order such that

$$0 = \tau_0 \leq \tau_1 < \tau_2 < \cdots < \tau_m.$$

In what follows we refer to (1) as the *primal* system. Equations of the form (1) are suitable for modeling a variety of problems from different fields, including machine tool vibrations (Insperger and Stépán, 2000), robotics (Insperger and Stépán, 2004). The complexity, induced by the combination of periodicity and delays, makes them also of main interest from a theoretical perspective. The relevance has motivated several contributions to stability and robust stability analysis, see, e.g., Insperger and Stépán (2011); Butcher and Bobrenkov (2011); Butcher et al. (2013); Michiels and Fenzi (2019); Letyagina and Zhabko (2009); Gomez et al. (2019) and the references therein.

In the context of eigenvalue optimization for time-invariant systems and model reduction by balanced truncation, an important role is played by the dual or transposed system, described in the frequency domain by taking the pointwise transposition of the transfer matrix. As suggested by Theorem 4.3 of Michiels and Fenzi (2019), where the eigenvalue problem for the monodromy operator of (1) (with zero input) is related to a finite-dimensional nonlinear eigenvalue problem for the case where delays and period are commensurate, the natural generalization of the dual system towards linear time-periodic System (2) is described by

$$\begin{cases} \dot{z}(t) = \sum_{i=0}^{m} A_i^T(-t+\tau_i)z(t-\tau_i) + C^T(-t)\xi(t), \\ \eta(t) = B^T(-t)z(t), \end{cases} \tag{2}$$

which will be conformed by all properties and connections with (1) that will be derived in the subsequent sections. We note that the construction of (2) involves both taking the transpose of the coefficient matrices and affine transformations of their arguments. Surprisingly, the shifts in the arguments of A_i depend on i, hence, evaluating the right-hand side of (2) at a particular time-instant involves evaluating matrices A_i in an asynchronous matter. We further note that the dual of dual System (2) corresponds to the original System (1).

The dual system allows revealing relations involving the spectra and eigenfunctions of the monodromy matrices, and introducing, for the first time, the concept of dual Lyapunov matrix associated with periodic time-delay System (1), which can be related to observability, respectively controllability properties of the system. We provide some energy interpretations of the primal and dual Lyapunov matrix, which play a key role in generalizing the position balancing approach for time-invariant delay systems, proposed in Jarlebring et al. (2013). The balancing on its turn provides a natural way to obtain reduced models by truncation. As these reduced models are also in the form of a periodic time-delay system, the reduction approach is structure exploiting.

In Section 2 we analyze the spectral properties of (1) and (2). We introduce pairs of (primal-dual) Lyapunov matrices and their energy interpretations in Section 3 and Section 4, respectively. Section 5 is devoted to exploring the potential of model reduction by position balancing. Finally, some concluding remarks are given in Section 6.

2. SPECTRAL PROPERTIES AND STABILITY

As we investigate internal stability, we consider system (1) with zero input. In order to define a forward solution, in

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general, a function segment over a time-interval of length τ_m is required. More precisely, for any initial function $\varphi \in X$, where $X := \mathcal{C}([-\tau_m, 0], \mathbb{C}^n)$, with $\mathcal{C}([-\tau_m, 0], \mathbb{C}^n)$ denoting the space of $\mathbb{C}^n$-valued continuous function on $[-\tau_m, 0]$, and $t_0 \in \mathbb{R}$, the initial value problem

$$\begin{cases} \dot{x}(t) = \sum_{j=0}^{m} A_j(t) x(t-\tau_j), & t \in [t_0, \infty), \\ x(t) = \varphi(t - t_0), & t \in [t_0 - \tau_m, t_0], \end{cases} \tag{3}$$

has a unique forward solution, which we denote by $x(t; t_0, \varphi)$. The corresponding state at time t, $t \geq t_0$, i.e. the minimal information to continue the solution, is denoted by $x_t(\cdot; t_0, \varphi) \in X$, defined by

$$x_t(\vartheta; t_0, \varphi) = x(t + \vartheta; t_0, \varphi), \quad \vartheta \in [-\tau_m, 0].$$

The translation along the solutions is described by the solution operator $\mathcal{T}(t_1, t_0) : X \to X$, parametrized by $t_0 \in \mathbb{R}$, $t_1 \in \mathbb{R}_+$ and defined through the relation

$$\mathcal{T}(t_1, t_0)\, \varphi = x_{t_0+t_1}(\cdot; t_0, \varphi), \quad \varphi \in X.$$

It can be shown that the spectrum of operator $\mathcal{T}(T, t_0)$ (recall that T is the period of functions A_j), is an at most countable compact set in the complex plane, with zero as only possible accumulation point. The spectrum is independent of the choice of t_0 and all its nonzero elements are eigenvalues. Operator $\mathcal{T}(T, 0)$ is called the monodromy operator and denoted by $\mathscr{U}$ in what follows. Hence, we have

$$\mathscr{U}\varphi = x_T(\cdot; 0, \varphi), \quad \varphi \in X.$$

The nonzero eigenvalues of the monodromy operator are called Floquet multipliers of (1). By definition they satisfy the infinite-dimensional linear eigenvalue problem

$$\mathscr{U}\varphi = \mu\, \varphi, \qquad \mu \in \mathbb{C},\ \varphi \in X \setminus \{0\}. \tag{4}$$

As the Floquet multipliers determine the growth/decay of solutions of (1) in time-intervals of length T and the system is T-periodic, they are important for stability assessment. In particular, the zero solution of (1) is uniformly exponentially stable if and only if all Floquet multipliers have modulus strictly smaller than one. For more results on Floquet theory for (3).

The following theorem relates the spectra of the monodromy operators corresponding to (1) and (2).

Theorem 1. Let $\mathscr{U}$, respectively $\mathscr{U}_D$, be the monodromy operator corresponding to (1), respectively (2). Then their spectra satisfy $\sigma(\mathscr{U}) \setminus \{0\} = \sigma(\mathscr{U}_D) \setminus \{0\}$.

Proof. From Theorems 2.2, 4.1 and 4.3 in Michiels and Fenzi (2019), a one-to-one correspondence between the eigenvalues of $\mathscr{U}$ and $\mathscr{U}_D$ can be established for the special case where the numbers $(T, \tau_1, \ldots, \tau_m)$ are commensurate. Since small delay perturbations correspond to compact perturbations on the monodromy operator and the set of nonnegative commensurate $(m+1)$-tuples is dense in $\mathbb{R}_+^{m+1}$, this result carries over to the general case.

In the remainder of this section we strengthen Theorem 1 for two special cases of (1)-(2). In the case of *commensurate delays and period*, more precisely, under the condition that there exist real number $\Delta > 0$, integers N and n_j, for $j = 1, \ldots, m$, such that

$$T = N\Delta, \quad \tau_j = n_j \Delta, \quad j = 1, \ldots, m,$$

it is shown in Michiels and Fenzi (2019) that the Floquet multipliers of (1) coincide with the nonzero solutions of a finite-dimensional nonlinear eigenvalue problem of the form

$$M(\mu)\, v = 0,\ \mu \in \mathbb{C},\ v \in \mathbb{C}^{Nn} \setminus \{0\}, \tag{5}$$

where function $M : \mathbb{C} \to \mathbb{C}^{Nn \times Nn}$ is analytic in $\mathbb{C} \setminus \{0\}$ and, for a specified value of μ, evaluating the left-hand side of (5) involves solving an initial value problem. A pair (μ, v) satisfying (5) is called a right eigenpair of M, and v a right eigenvector corresponding to μ. A left eigenpair $(\mu, u) \in \mathbb{C} \times \mathbb{C}^{Nn} \setminus \{0\}$ satisfies $u^* M(\mu) = 0$, and u is called a left eigenvector of M corresponding to μ.

If μ is an eigenvalue of (5), then it follows from (Michiels and Fenzi, 2019, Theorem 2.2) that a right eigenvector can be obtained from stacking samples of the eigenfunction φ of $\mathscr{U}$, corresponding to Floquet multiplier μ. At the same time, according to (Michiels and Fenzi, 2019, Theorem 4.3) a left eigenvector of (5) can be obtained by stacking samples of eigenfunction ψ of operator $\mathscr{U}_D$, corresponding to Floquet multiplier $\bar{\mu}$, i.e. such that $\mathcal{U}_D \psi = \bar{\mu}\psi$. We refer to Michiels and Fenzi (2019) for the details of these results. Finally, we consider the *delay-free case*, where $\mathscr{U}$ and $\mathscr{U}_D$ are $n \times n$ matrices.

Proposition 2. For the case $m = 0$ in (1) and (2), it holds that $\mathscr{U}_D = \mathscr{U}^*$.

Proof. In the delay-free case, it is possible to express $\mathscr{U} = K(T, 0)$, with K the fundamental solution defined through $K(t, s) = 0$ for $t < s$, $K(s, s) = I$ and

$$\frac{\partial}{\partial t} K(t, s) = A(t) K(t, s), \quad t \geq s,$$

which induces the property

$$\frac{\partial}{\partial s} K(t, s) = -K(t, s) A(s), \quad t \geq s.$$

We observe that the fundamental solution K_D of the dual system satisfies $K_D(t, s) = K^T(-s, -t)$, which on its turn leads to $\mathscr{U} = K(T, 0) = K_D^T(0, -T) = K_D^T(T, 0) = \mathscr{U}_D^T$, from which the assertion follows.

3. LYAPUNOV MATRICES

We recall the concept of Lyapunov matrix of system (1) from Zhabko and Letyagina (2009), and introduce the dual Lyapunov matrix, which is associated with system (2). These Lyapunov matrices are the cornerstone of the results developed in sections 4 and 5.

Throughout the section and the remainder of the paper we assume that System (1), and thus (2), are exponentially stable, which is equivalent to the property that all Floquet multipliers are located in the open unit disk.

The definition of the Lyapunov matrix of System (1) relies on the so-called *fundamental matrix*, which generalizes the concept of fundamental solution for delay-free systems. The fundamental matrix of (1), which we denote by K, is the function $K : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$, $(t, s) \to K(t, s)$, satisfying (Halanay, 1966),

$$\frac{\partial}{\partial t} K(t, s) = A_0(t) K(t, s) + \sum_{i=1}^{m} A_i(t) K(t - \tau_i, s), \quad t \geq s, \tag{6}$$

with $\frac{\partial K}{\partial t}$ denoting the right-hand derivative of K with respect to t, as well as

$$K(t, s) = 0, \text{ for } t < s \text{ and } K(s, s) = I.$$

The fundamental matrix associated with the dual System (2) is denoted by K_D. The following lemma states it can be expressed as a function of K.

Lemma 3. The equality $K_D(t,s) = K^T(-s,-t)$ holds for all $(t,s) \in \mathbb{R}^2$.

Proof. The case where $t \leq s$, which implies $-s \leq -t$, is trivial. Therefore, we restrict ourselves to the case where $t > s$ in the remainder of the proof.
Besides equation (6), function K also satisfies (see Letyagina and Zhabko (2009))

$$-\frac{\partial}{\partial s}K(t,s) = \sum_{i=0}^{m} K(t,s+\tau_i)A_i(s+\tau_i), \quad t \geq s,$$

which implies

$$\frac{\partial}{\partial t}K(-s,-t) = \sum_{i=0}^{m} K(-s,-t+\tau_i)A_i(-t+\tau_i), \quad -s \geq -t$$

and

$$\frac{\partial}{\partial t}K^T(-s,-t) = \sum_{i=0}^{m} A_i^T(-t+\tau_i)K^T(-s,-(t-\tau_i)), \quad t \geq s.$$

The assertion follows from the definition of K_D.

The Lyapunov matrix of System (1), associated with a smooth T-periodic function $W : \mathbb{R} \to \mathbb{R}^{n\times n}, \ t \mapsto W(t)$ satisfying $W(t) \geq 0$ for all t, is defined as

$$U(s_1,s_2) := \int_{-\infty}^{\infty} K^T(t,s_1)W(t)K(t,s_2)dt.$$

Similarly as in Letyagina and Zhabko (2009), where the case of constant W is considered, it can be easily shown that U satisfies the following properties:

(1) the dynamic property: for $s_2 > s_1$ we have

$$\begin{aligned}\frac{\partial}{\partial s_1}U(s_1,s_2) &= -\textstyle\sum_{i=0}^{m} A_i^T(s_1+\tau_i)U(s_1+\tau_i,s_2),\\ \frac{\partial}{\partial s_2}U(s_1,s_2) &= -\textstyle\sum_{i=0}^{m} U(s_1,s_2+\tau_i)A_i(s_2+\tau_i)\\ &\quad -K^T(s_2,s_1)W(s_2);\end{aligned} \tag{7}$$

(2) the symmetry property:

$$U^T(s_1,s_2) = U(s_2,s_1); \tag{8}$$

(3) the periodicity property:

$$U(s_1,s_2) = U(s_1+T,s_2+T); \tag{9}$$

(4) the algebraic property:

$$\begin{aligned}\frac{d}{ds}U(s,s) = &-\sum_{i=0}^{m} A_i^T(s+\tau_i)U(s+\tau_i,s)\\ &-\sum_{i=0}^{m} U(s,s+\tau_i)A_i(s+\tau_i) - W(s).\end{aligned} \tag{10}$$

Remark 4. Theorem 4.1 of Zhabko and Letyagina (2009) states that, assuming exponential stability, the Lyapunov matrix is uniquely determined by the first equation of (7), along with equations (8)-(10), for the special case where W is a constant positive definite matrix. As W is the only inhomogeneous term in the equations, this result carries over to an arbitrary periodic function. The argument is by contradiction: if there would be distinct solutions for a particular periodic function W, then there would be distinct solutions for any constant positive function. In view of this comment, we do not consider the second equation of (7) anymore in what follows.

We now turn our attention to the dual System (2). The Lyapunov matrix of System (2), associated with a T-periodic matrix valued function W, is the matrix function $V : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$, defined as

$$V(s_1,s_2) := \int_{-\infty}^{\infty} K_D^T(t,s_1)W(t)K_D(t,s_2)dt,$$

and which we refer to as the *dual Lyapunov matrix* corresponding to (1). From (7)-(10) and a comparison of (2) and (1), it directly follows that V satisfies:

(1) the dynamic property: for $s_2 > s_1$

$$\frac{\partial}{\partial s_1}V(s_1,s_2) = -\textstyle\sum_{i=0}^{m} A_i(-s_1)V(s_1+\tau_i,s_2);$$

(2) the symmetry property:

$$V^T(s_1,s_2) = V(s_2,s_1);$$

(3) the periodicity property:

$$V(s_1,s_2) = V(s_1+T,s_2+T);$$

(4) the algebraic property:

$$\begin{aligned}\frac{d}{ds}V(s,s) = &-\sum_{i=0}^{m} A_i(-s)V(s+\tau_i,s)\\ &-\sum_{i=0}^{m} V(s,s+\tau_i)A_i^T(-s) - W(s).\end{aligned} \tag{11}$$

We are now ready to make further connections between (1) and (2), and the associated Lyapunov matrices U and V. Lemma 3 directly leads us to the following characterization.

Proposition 5. Lyapunov matrix V of dual System (2), associated with function $W(t)$, satisfies

$$V(s_1,s_2) = \int_{-\infty}^{\infty} K(-s_1,-t)W(t)K^T(-s_2,-t)dt.$$

The Lyapunov matrix $U(s_1,s_2)$ can be obtained by solving equations (7)-(10). This is a non-trivial task, as they are 2D partial differential equations with periodic coefficients and delays (see Letyagina and Zhabko (2009) and Zhabko and Letyagina (2009)). In addition, function $(s_1,s_2) \mapsto U(s_1,s_2)$ is in general a non-smooth function, inheriting this property from the fundamental matrix. However, for $m = 1$ and $T = \tau_1$ function U is smooth in the region

$$\mathcal{D} := \{(s_1,s_2) \in \mathbb{R}^2 : s_2 \in [0,\tau_1], \ s_2 - s_1 \in [0,\tau_1]\}.$$

In Gomez et al. (2019), a numerical scheme for computing the Lyapunov matrix on the domain $\mathcal{D}$ based on the formulation and solution of a delay-free partial differential equations with boundary conditions is proposed. From its definition, the dual Lyapunov matrix V can be computed as the Lyapunov matrix corresponding to system (2).
A generally applicable but less efficient approach, which extends the results presented in Section 2 of Michiels and Zhou (2019) to periodic systems, consists of inferring an approximation of U from the Lyapunov matrix associated with a spectral discretization of the delay equation. This approach requires solving a standard delay-free periodic Lyapunov equation of increased dimensions.

4. ENERGY INTERPRETATIONS

In this section, we give interpretations of the one-parameter families of matrices $U(s,s)$ and $V(s,s)$, associated with systems (1) and (2), in terms of energy. They are at the basis of the reduction approach outlined in Section 5.

For a given input $u \in L_2([t_0,\infty),\mathbb{R}^{n_i})$, where the notation $L_2([t_0,\infty),\mathbb{R}^{n_i})$ represents the space of $\mathbb{R}^{n_i}$-valued squared integrable functions on $[t_0,\infty)$, we denote the solution of (1) with initial condition $\varphi \in X$ at time t_0 by $x(t;t_0,\varphi,u)$ and $y(t;t_0,\varphi,u)$. The variation of constants formula for (1) reads as (see Bellman and Cooke (1963))

$$x(t;t_0,\varphi,u) = K(t,t_0)\varphi(0) + \sum_{j=1}^{m}\int_{-\tau_j}^{0} K(t,t_0+\xi+\tau_j)A_j(t_0+\xi+\tau_j)\varphi(\xi)d\xi + \int_{t_0}^{t} K(t,\xi)B(\xi)u(\xi)d\xi, \quad (12)$$

and is key in generalizing the results by (Jarlebring et al., 2013, Section 3) from time-invariant to time-periodic delay systems.

Let us now initialize (1) at time s with

$$x(t) = \varphi(t-s) := \begin{cases} 0, & t \in [s-\tau_m,\ s), \\ q, & t = s, \end{cases} \quad (13)$$

and consider the emanating solution for zero input. We define the output energy, associated with $q \in \mathbb{R}^n$ and horizon $T_s > 0$, as

$$E_p(q,s,T_s) := \int_s^{s+T_s} \|y(t;s,\varphi,0)\|^2 dt. \quad (14)$$

It follows from formula (12) that

$$y(t;s,\varphi,0) = C(t)x(t;s,\varphi,0) = C(t)K(t,s)q,$$

and

$$\begin{aligned}\lim_{T_s\to\infty} E_p(q,s,T_s) &= \int_s^{\infty} \|y(t;s,\varphi,0)\|^2 dt \\ &= q^T \int_s^{\infty} K^T(t,s)C^T(t)C(t)K(t,s)dt\ q \\ &= q^T U(s,s)q,\end{aligned}$$

with $U(s_1,s_2)$ the Lyapunov matrix of (1) associated with $W(t) = C^T(t)C(t)$.

Similarly, we can initialize dual System (2) at time s with

$$z(t) = \varphi(t-s) := \begin{cases} 0, & t \in [s-\tau_m,\ s), \\ q, & t = s, \end{cases} \quad (15)$$

and define the output energy of dual System (2), associated with $q \in \mathbb{R}^n$ and horizon T_s, as

$$E_d(q,s,T_s) := \int_s^{s+T_s} \|\eta(t;s,\varphi,0)\|^2 dt,$$

with $\eta(t;s,\varphi,u)$ the output corresponding to (15). By following similar arguments as for the primal system, we obtain

$$\lim_{T_s\to\infty} E_d(q,s,T_s) = q^T V(s,s)q,$$

where $V(s_1,s_2)$ is the Lyapunov matrix of (2) associated with $W(t) = B(-t)B^T(t)$.

Let us now characterize the minimal energy required in the input to reach a given $q \in \mathbb{R}^n$ at time s when the system is at rest at time $s-T_s$, with $T_s > 0$:

$$E_r(q,s,T_s) := \min_{\substack{u\in L_2([s-T_s,s],\mathbb{R}^{n_i}) \\ x(s;s-T_s,0,u)=q}} \int_{s-T_s}^{s} \|u(t)\|^2 dt. \quad (16)$$

In the next lemma, we provide the control u that minimizes $E_r(q,s,T_s)$.

Lemma 6. Let System (1) be exponentially stable and define

$$P(\alpha,s) := \int_{s-\alpha}^{s} K(s,\xi)B(\xi)B^T(\xi)K^T(s,\xi)d\xi.$$

If $q \in \mathrm{Im}P(T_s,s)$, then the unique minimizer of the right-hand side of (16) is

$$u_{\mathrm{opt}}(t) := B^T(t)K^T(s,t)P^{\dagger}(T_s,s)q$$

and, in addition, we have

$$E_r(q,s,T_s) = q^T P^{\dagger}(T_s,s)q, \quad (17)$$

where the symbol † denotes the Moore-Penrose inverse.

Proof. The arguments are similar as in the proof of Lemma 2 in Jarlebring et al. (2013). By the variation of constants formula (12), we have

$$x(s,s-T_s,0,u) = \int_{s-T_s}^{s} K(s,\xi)B(\xi)u(\xi)d\xi.$$

Notice first that u_{opt} allows to reach q. Indeed, by direct substitution of u_{opt} into the previous expression we get

$$\begin{aligned}x(s,s-T_s,0,u_{\mathrm{opt}}) &= \int_{s-T_s}^{s} K(s,\xi)B(\xi)B^T(\xi)K^T(s,\xi)d\xi \\ &\quad \times P^{\dagger}(T_s,s)q = P(T_s,s)P^{\dagger}(T_s,s)q = q.\end{aligned}$$

Suppose now that there exists another control $\tilde{u}$ such that $x(s,s-T_s,0,\tilde{u}) = q$, then

$$\begin{aligned}&x(s,s-T_s,0,\tilde{u}) - x(s,s-T_s,0,u_{\mathrm{opt}}) \\ &\qquad = \int_{s-T_s}^{s} K(s,\xi)B(\xi)\left(\tilde{u}(\xi) - u_{\mathrm{opt}}(\xi)\right)d\xi,\end{aligned}$$

which implies that

$$\int_{s-T_s}^{s} u_{\mathrm{opt}}^T(\xi)\left(\tilde{u}(\xi) - u_{\mathrm{opt}}(\xi)\right)d\xi = 0.$$

It follows from the previous equality that

$$\|\tilde{u}\|_{L_2}^2 = \|u_{\mathrm{opt}} + \tilde{u} - u_{\mathrm{opt}}\|_{L_2}^2 = \|\tilde{u} - u_{\mathrm{opt}}\|_{L_2}^2 + \|u_{\mathrm{opt}}\|_{L_2}^2,$$

therefore $\|\tilde{u}\|_{L_2} \geq \|u_{\mathrm{opt}}\|_{L_2}$, with $\|\cdot\|_{L_2}$ denoting the norm induced by the L_2-inner product. Thus, u_{opt} is the unique global minimizer of (16).

Finally, expression (17) is deduced from

$$\begin{aligned}&\int_{s-T_s}^{s} \|u_{\mathrm{opt}}(t)\|^2 dt \\ &= q^T(P^{\dagger}(T_s,s))^T \int_{s-T_s}^{s} K(s,t)B(t)B^T(t)K^T(s,t)\ dt \\ &\qquad \times P^{\dagger}(T_s,s)q = q^T P^{\dagger}(T_s,s)q.\end{aligned}$$

We observe from Lemma 3 that

$$\begin{aligned}\lim_{\alpha\to+\infty} P(\alpha,s) &= \int_{-\infty}^{s} K(s,\xi)B(\xi)B^T(\xi)K^T(s,\xi)d\xi \\ &= \int_{-s}^{\infty} K_D^T(\xi,-s)B(-\xi)B^T(-\xi)K_D(\xi,-s)d\xi \\ &= V(-s,-s) = V(T-s,T-s);\end{aligned}$$

hence, it is possible to characterize $E_r(q,s,T_s)$, for $T_s \to \infty$ in terms of the dual Lyapunov matrix V as

$$\lim_{T_s\to\infty} E_r(q,s,T_s) = q^T V^{\dagger}(T-s,T-s)q$$

for $q \in \mathrm{Im}V(T-s,T-s)$. The previous results are summarized in the following theorem.

Theorem 7. Let System (1) be exponentially stable. The following expressions hold for any $q \in \mathbb{R}^n$ and $s \in \mathbb{R}$:

$$E_{p,o}(q,s) := \lim_{T_s\to\infty} E_p(q,s,T_s) = q^T U(s,s) q, \tag{18}$$

$$E_{d,o}(q,s) := \lim_{T_s\to\infty} E_d(q,s,T_s) = q^T V(s,s) q \tag{19}$$

and

$$E_{p,r}(q,s) := \lim_{T_s\to\infty} E_r(q,s,T_s)$$
$$= \begin{cases} q^T V^\dagger(T-s,T-s)q & \text{if } q \in \mathrm{Im}V(T-s,T-s), \\ +\infty, & \text{otherwise} \end{cases}$$

where $U(s_1,s_2)$ and $V(s_1,s_2)$ are Lyapunov matrices of systems (1) and (2) associated with $C^T(t)C(t)$ and $B(-t)B^T(-t)$, respectively.

Note that Lyapunov matrices $U(s,s)$ and $V(s,s)$ play the role of infinite reachability and observability Grammians of periodic systems with delays. With q a unit vector, $E_{p,o}(q,s)$ can be interpreted as a measure of how well pseudo-state q, parameterizing initial condition (13) is observable in the output. At the same time the measures $E_{d,o}(q,s)$ and $E_{p,r}(q,s)$ give an indication of the reachability of vectors in direction q.

5. STRUCTURE PRESERVING MODEL REDUCTION BY POSITION BALANCING

The starting point is formed by the energy interpretations (18)-(19). Since both $E_{p,o}(q,s)$ and $E_{d,o}(q,s)$ are periodic in s, it is natural to look at the averaged values,

$$E_{p,\mathrm{av}}(q) := q^T U_{\mathrm{av}} q \tag{20}$$

and

$$E_{d,\mathrm{av}}(q) := q^T V_{\mathrm{av}} q, \tag{21}$$

with

$$U_{\mathrm{av}} := \frac{1}{T}\int_0^T U(s,s)ds, \text{ and } V_{\mathrm{av}} := \frac{1}{T}\int_0^T V(s,s)ds.$$

Let us now, for given nonsingular matrix $\mathcal{T} \in \mathbb{R}^{n\times n}$, apply similarity transformation

$$\bar{x}(t) = \mathcal{T}^{-1}x(t) \tag{22}$$

to (1) and, at the same time, similarity transformation

$$\bar{z}(t) = \mathcal{T}^T z(t) \tag{23}$$

to (2). They lead to a transformed pair of primal-dual systems

$$\begin{cases} \dot{\bar{x}}(t) = \sum_{i=0}^m \bar{A}_i(t)\bar{x}(t-\tau_i) + \bar{B}(t)u(t), \\ y(t) = \bar{C}(t)x(t), \end{cases} \tag{24}$$

and

$$\begin{cases} \dot{\bar{z}}(t) = \sum_{i=0}^m \bar{A}_i^T(-t+\tau_i)\bar{z}(t-\tau_i) + \bar{C}^T(-t)\xi(t), \\ \eta(t) = \bar{B}^T(-t)z(t), \end{cases} \tag{25}$$

with $\bar{B}(t) = \mathcal{T}^{-1}B(t)$, $\bar{C}(t) = C(t)\mathcal{T}$ and

$$\bar{A}_i(t) = \mathcal{T}^{-1}A_i(t)\mathcal{T}, \quad i = 0,\ldots,m.$$

For Systems (24)-(25) the Lyapunov matrices can be expressed as

$$\bar{U}(s_1,s_2) = \mathcal{T}^T U(s_1,s_2)\mathcal{T}, \quad \bar{V}(s_1,s_2) = \mathcal{T}^{-1}V(s_1,s_2)\mathcal{T}^{-T},$$

while the kernels of (20)-(21) become

$$\bar{U}_{\mathrm{av}} = \mathcal{T}^T U_{\mathrm{av}}\mathcal{T}, \quad \bar{V}_{\mathrm{av}} = \mathcal{T}^{-1}V_{\mathrm{av}}\mathcal{T}^{-T}.$$

Note that the eigenvalues of the product $\bar{U}_{\mathrm{av}}\bar{V}_{\mathrm{av}}$ are independent of $\mathcal{T}$. Inspired by the approach of Jarlebring et al. (2013) for balanced truncation of non-periodic delay systems, and the energy interpretations related to expressions (20)-(21), we call System (1) *position-balanced* if

$$\bar{U}_{\mathrm{av}} = \bar{V}_{\mathrm{av}} = \Sigma, \tag{26}$$

with $\Sigma \geq 0$ a diagonal matrix with its elements in non-increasing order. The term position balancing stems from the property that expressions (20)-(21) only characterize partial state $p = x(t) = x_t(0) \in \mathbb{R}^n$; see Jarlebring et al. (2013) and references therein for an analogy with position balancing of second-order systems.

Given that $U(s,s) \geq 0$ and $V(s,s) \geq 0$ for all s, following from Theorem 7, balancing is possible under a very mild condition.

Proposition 8. Assume that $U_{\mathrm{av}} > 0$ and $V_{\mathrm{av}} > 0$. Let factorizations $U_{\mathrm{av}} = R^T R$ and $V_{\mathrm{av}} = S^T S$ correspond to Cholesky factorizations, and let $\mathcal{U}\Sigma\mathcal{V}^T$ be a singular value decomposition of RS^T. Then the choice

$$\mathcal{T} = S^T\mathcal{V}\Sigma^{-\frac{1}{2}}, \quad \mathcal{T}^{-1} = \Sigma^{-\frac{1}{2}}\mathcal{U}^T R$$

in (22)-(23) induces property (26), i.e. the transformed system is position-balanced.

If the system has been position-balanced, then the diagonal elements of Σ give an indication about the importance of the components of transformed state variable $\bar{x}$ (the canonical directions), with respect to the input-output behavior of the system. Partitioning

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

with the diagonal elements of Σ_2 preferably small compared to the diagonal elements of Σ_1, and making a corresponding partition of the state variable $\bar{x} = [\bar{x}_1^T \ \bar{x}_2^T]^T$ and the system matrices,

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i,11} & \bar{A}_{i,12} \\ \bar{A}_{i,21} & \bar{A}_{i,22} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \ \bar{C} = [\bar{C}_1 \ \ \bar{C}_2],$$

lead us to the following reduced-order model,

$$\begin{cases} \dot{\bar{x}}_1(t) = \sum_{i=0}^m \bar{A}_{i,11}(t)\,\bar{x}_1(t-\tau_i) + \bar{B}_1(t)\,u(t), \\ y(t) = \bar{C}_1(t)\,\bar{x}_1(t). \end{cases}$$

Remark 9. The history of the state is assumed zero in deriving energies (20)-(21), whereas $E_r(p,s)$ determines the input energy to reach a point in $\mathbb{R}^n$ at time s, not considering the shape of the solution in the interval $[t-s,\ s)$. Hence, position balancing is based on interpretations of pseudo-states, parameterized by $p \in \mathbb{R}^n$, rather than elements of state space X. Accordingly, a transformation of the form (22) in $\mathbb{R}^n$ is constrained when it is extended to a transformation in X (for instance, it implies $\bar{\varphi}(s) = \mathcal{T}^{-1}\varphi(s)$, $s \in [-\tau_m,\ 0]$, with $\mathcal{T}$ independent of s). However, a transformation in $\mathbb{R}^n$, the "physical" space in which the trajectories of (1) reside, has the advantage that the reduction approach is structure preserving, i.e. the reduced model is also in the form of a periodic time-delay system.

We conclude the section with a proof-of-concept case-study for the above reduction approach.

Example 10. We consider System (1) with $n = 2$, $T = 2\pi$, $m = 1$, $\tau_1 = 1$,

$$A_0(t) = \begin{bmatrix} \sin(2\,t) + \frac{5\,\cos(t)}{8} - 4 & \frac{5\,\cos(t)}{8} - \sin(2\,t) + 1 \\ \frac{3\,\cos(t)}{8} + 1 & \frac{3\,\cos(t)}{8} - 4 \end{bmatrix},$$
$$A_1(t) = \begin{bmatrix} \frac{\sin(t)}{2} & -\frac{\cos(t)}{2} \\ \frac{\cos(t)}{2} & \frac{\sin(t)}{2} \end{bmatrix} \tag{27}$$

and

$$B(t) = \begin{bmatrix} \frac{\cos(t)}{2} + \frac{\sin(t)}{8} + 1 \\ \frac{\cos(t)}{2} - \frac{\sin(t)}{8} + 1 \end{bmatrix}, \quad C(t) = [-1 + 2\cos(t) \;\; -1].$$

In this case we compute the Lyapunov matrices from the solution of standard delay-free periodic Lyapunov equations applied to a spectral discretization of the system (see the discussion in Section 3). Solving the associated periodic Lyapunov equations yields

$$U_{\rm av} = \begin{bmatrix} 1.86 & 0.769 \\ 7.68 & 0.746 \end{bmatrix}, \quad V_{\rm av} = \begin{bmatrix} 1.22 & 1.16 \\ 1.16 & 1.14 \end{bmatrix}.$$

In agreement with the statement of Proposition 8, the transformation

$$\bar{x}(t) = \mathcal{T}^{-1} x(t),$$

with

$$\mathcal{T} = \begin{bmatrix} -0.741 & -0.230 \\ -0.713 & 0.402 \end{bmatrix},$$

results in balanced (pseudo)-Grammians $\bar{U}_{\rm av}$ and $\bar{V}_{\rm av}$, equal to

$$\Sigma = \begin{bmatrix} 2.21 & 0 \\ 0 & 0.0768 \end{bmatrix}.$$

The diagonal elements of Σ indicate the potential for reduction to a first order system. Truncating the balanced system to the first canonical direction leads us to the reduced model

$$\begin{cases} \bar{x}_1(t) = (0.0248\sin(2t) + 1.06\cos(t) - 3.01)\;\bar{x}_1(t) \\ \qquad\quad +(0.5\sin(t) - 0.126\;\cos(t))\;\bar{x}_1(t-1) \\ \qquad\quad +(-0.681\;\cos(t) - 0.0463\sin(t) - 1.36)\;u(t), \\ \;\; y(t) = (1.46 - 1.49\;\cos(t))\,\bar{x}_1(t). \end{cases} \tag{28}$$

Denoting by $\mathcal{G}$, respectively $\mathcal{G}_r$, the input-output map of (27) and (28), we obtain

$$\|\mathcal{G}\|_{\mathcal{H}_2} = 0.677, \;\; \|\mathcal{G}_r\|_{\mathcal{H}_2} = 0.681,$$

while the error on the input-output map is characterized by

$$\|\mathcal{G} - \mathcal{G}_r\|_{\mathcal{H}_2} = 0.048,$$

where the $\mathcal{H}_2$ norm of the input-output maps are computed via the methodology presented in the extended version of this paper (Michiels and Gomez, 2020). Hence, the balancing procedure is able to extract the (reduced) dynamics, mainly responsible for the input-output behaviour of the system. ◇

6. CONCLUSIONS

We established connections between (1) and (2) in terms of spectra and Lyapunov matrices, and showed how the concepts of position balancing and balanced truncation can be generalized. Their applicability to real-life applications however depends on algorithms for computing Lyapunov matrices for large-scale problems, which is a topic of future research.

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