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The magic theorem. A greatly-expanded, much-abridged edition of the symmetries of things. (English) Zbl 08079774

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In their appraised SoT book [*J.H. Conway* et al., The symmetries of things. Wellesley, MA: A K Peters (2008; Zbl 1173.00001)] the authors gave a general treatment of the classification of symmetries of repeated patterns. There were 3 parts: (I) "Symmetries of finite objects and plane repeated patterns" enumerates and classifies them by introducing the orbifold signature notation, which culminates in a proof of the Magic Theorem (MT) which is a general statement classifying the patterns, implying for example that there are exactly 17 different wallpaper patterns to fill the plane. (II) "Color symmetry, group theory, and tilings", introduces the group theory and topology that underlies the previous findings. (III) "Repeating patterns in other spaces", deals with hyperbolic and 3D geometries.

What started as the preparation for a second edition of the SoT gradually grew into a stand-alone version of part I that is worked out by the remaining two authors after Conway passed away in 2020. The titles of the 9 chapters in part I of the SoT are kept in this version, but since the focus is on the introduction of the orbifold notation and the MT, some parts of the original text could be removed which left some room for even more illustrations, more exercises with solutions, and more constructing, cutting and folding projects to enjoy. There is only the chapter 10, that is giving a hint of "the bigger picture" which means introducing some elements of hyperbolic geometry by just using pictures, summarizing some elements of what was much more elaborated in parts II and III of the SoT.

This is a rather intuitive approach to the theory of symmetries of repeated patterns mainly in the plane or on a sphere in Euclidean (or hyperbolic) geometry, avoiding the underlying group theory as much as possible. The result is a marvelous picture book with some exercises and tasks that stimulate the reader to enjoy the beauty of symmetries.

A pattern is identified by a fundamental region, i.e., a decorated (polygonal) tile that will generate the whole pattern by applying a group of selected (glide) reflections, rotations, and translations. Conversely, such a tile is obtained by identifying it with all its orbital images generated by the symmetries of the pattern. Hence the name orbifold and the pattern can be characterized, by denoting its pattern generating symmetries. This avoids the topological definition of an orbifold as a locally finite group quotient of a differentiable manifold, which is only hinted at in chapter 9.

The authors have their own terminology: kaleidoscopic for reflections along lines (they can be simulated using mirrors and are denoted by * followed by integers indicating the number of line reflections at the different vertices of the fundamental region); gyrational for rotation around a point (the order of the different rotations are enumerated before the *); miracles (successive orbital point can be connected without crossing a reflection line like for glide reflections which are denoted as x); and wonder-rings (pure translations that come in pairs since they define a grid and they are denoted as o). Hence, each pattern (i.e. symmetry group) can be identified by its signature which is a string consisting of the symbols *, x, o, and the symbols for the natural numbers larger than 1.

The Magic Theorem allows to identify all the patterns possible in the considered space. Its proof relies on assigning a "cost" to the pattern by summing the costs assigning to each symbol. The MT then states for example that the only possible patterns of the plane must have a cost 2, implying there are only 17 different wallpaper patterns of the plane. For the spherical patterns the total cost is less, giving 14 possibilities, and for frieze patterns, an extra symbol ∞ is introduce characterizing it as an infinitely repeated linear pattern, and the cost of a frieze is again 2, leaving 7 possible friezes.

To prove the previous results, first a connection is made between surface maps and the Euler characteristic $\chi = V - E + F$ where (V, E, F) = number of (vertices, edges, faces), which equals 2 for the sphere and polyhedra, 1 for the disk and polygons, 0 for the torus and Möbius strip, etc.). Surfaces can be classified using a signature-like string notation with associated cost, which can be evaluated to give the characteristic. It is only in the penultimate chapter 9 that orbifolds are analysed in a different

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context, identifying them as topological surfaces. For example a rotation corresponds to a cone, a frieze to a cylinder, etc. This also explains where the costs of the symbols come from, since these costs were previously just stated without explanation. This shows that the MT is just a disguised statement about the Euler characteristic of the orbifold. Along the way the reader learns about Heersch types, graphs, polyhedra, complicated topological surfaces, etc.

The chapters end with a section "Where are we" to summarize the essence that has been learned and sometimes adding some additional examples, exercises, or extra information. In an appendix of the book a table is added showing the equivalence of the orbifold notation and other notations for the patterns that were proposed by other authors.

There is a web page `themagictheorem.com` with a lot of extra illustrations and tasks related to this book that help to understand the underlying structure and to appreciates the beauty of these mathematical patterns.

Reviewer: Adhemar Bultheel (Leuven)

MSC:

00A08 Recreational mathematics
00A05 Mathematics in general
51-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to geometry
53-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to differential geometry
20-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to group theory

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